The group ring of $SL_2(2^f)$ over 2-adic integers.

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Dedicated to Prof. K. W. Roggenkamp in occasion of his 60th birthday

Abstract: Let $R = \mathbb{Z}_2[x_{2^f-1}]$ and $G = SL_2(2^f)$. The group ring $RG$ is calculated nearly up to Morita equivalence. In particular the irreducible $RG$-lattices can be described purely combinatorially in terms of subsets of $\{1, \ldots, f\}$.

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1 Introduction

The group $G = SL_2(p^f)$ of all $2 \times 2$-matrices over the field $k$ with $p^f$ elements is one of the simplest examples of a nonabelian finite group of Lie type. Its representation theory in characteristic 0 was already investigated by I. Schur [13] and its modular representation theory is also well understood ([1], [2]). The next step is to describe the integral group ring $RG$ of $G$ when $R$ is the ring of integers in a finite extension of the field $\mathbb{Q}_l$ of $l$-adic numbers, to bring together the characteristic 0 and the characteristic $l$ information. If $l \neq p$ and $l \neq 2$ then the defect groups of the ring direct summands of $RG$ are cyclic, so $RG$ is described by the general theory of blocks with cyclic defect groups ([10], [12], [7]). For odd primes $p$ the Sylow 2-subgroups of $G$ are dihedral groups and [10], Chapter VII investigates $RG$ for $l = 2$. So the only remaining case is $l = p$, where the Sylow $p$-subgroups of $G$ are elementary abelian of rank $f$. If $f = 1$ one again has the cyclic defect case and for $f = 2$ the group ring $\mathbb{Z}_pG$ is described up to Morita equivalence in [8]. In the present paper the remaining cases $f \geq 3$ are treated for $p = 2$.

To find $kG$, one uses methods from the representation theory for groups of Lie type in defining characteristic. However these methods are not directly applicable for calculating $RG$, when $R = \mathbb{Z}_2[x_{2^f-1}]$ is the ring of integers in the unramified extension $K$ of degree $f$ of $\mathbb{Q}_2$. The new idea used in this paper is to start from the explicit presentation of $kG$ given in [6] and lift the generators of $kG$ to generators of $RG$. The explicit knowledge of $kG$ together with the decomposition numbers calculated in [4] and [5] do not seem to be sufficient to determine $RG$ up to Morita equivalence. But they give enough information to describe the inclusion patterns of the irreducible $RG$-lattices (Theorem 3.15) as well as the endomorphism rings of the projective indecomposable $RG$-lattices (Theorem 3.12). In particular it turns out that the endomorphism ring of the projective cover of the trivial $RG$-module is isomorphic to the group ring of the Sylow 2-subgroup of $G$. From Theorem
3.15 one gets the following explicit description of the projections of $RG$ onto the simple summands of $KG$. The simple $kG$-modules $M_l$ are naturally indexed with the subsets $I$ of $\{1, \ldots, f\}$ such that $\text{dim}_k(M_I) = 2^{|I|}$.

**Theorem.** Let $V$ be a simple $KG$-module of dimension $n$ and let $M_{I_1}, \ldots, M_{I_r}$ be the 2-modular constituents of $V$ of dimension $n_j := 2^{|I_j|} = \text{dim}_k(M_{I_j})$, $1 \leq j \leq r$. Then there is a basis of $V$ such that the corresponding matrix representation $\Delta_V$ satisfies

$$\Delta_V(RG) = \{(X_{ij})_{1 \leq i, j \leq r} \in R^{n \times n} \mid X_{ij} \in 2^{[i_j]} R^{n^j \times n_j}\}.$$

This is the first time that such a detailed description of an infinite series of $p$-adic group rings has been found, where not only the order but also the number of generators of the Sylow $p$-subgroups grows. It is astonishing that, though the situation gets more and more complicated, the group rings $RSL_2(2^f)$ can be described in a uniform way for all $f \geq 3$. Similar methods can be applied for $G = SL_2(p^f)$ where $p$ is odd, to get analogous information about the group ring as is given here for $p = 2$. There are additional technical difficulties for odd primes $p$ that make the basic ideas less transparent, so the case $p > 2$ will be treated in a separate paper.

This paper is intended to make one part of my habilitation thesis [9] available to a wider audience. I thank Dr. Alexander Zimmermann for pointing out to me reference [6].

## 2 Generalities.

Throughout the paper let $K$ be a finite extension of $\mathbb{Q}_p$,

$R$ its ring of integers with maximal ideal $\pi R$ and residue class field $k := R/\pi R$.

Let $A$ be a finite-dimensional semisimple $K$-algebra

$$A = \bigoplus_{t=1}^{s} A \epsilon_t \cong \bigoplus_{t=1}^{s} K_t^{|n_t| \times |n_t|}$$

where $\epsilon_1, \ldots, \epsilon_s$ are the central primitive idempotents of $A$ and $K_t$ are $K$-division algebras.

This section develops a language for describing the ring theoretic structure of group rings $RG$ of finite groups $G$ up to Morita equivalence. Group rings are certain $R$-orders $\Lambda$ in a semisimple $K$-algebra $K \otimes_R \Lambda =: A$. 

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It turns out that it is easier to describe the overorder
\[ \Gamma := \bigoplus_{i=1}^{s} \Lambda \epsilon_i \]
of \( \Lambda \), the direct sum of the projections of \( \Lambda \) into the simple components of \( A \). If the decomposition numbers of \( RG \) are \( \leq 1 \) (and \( R \) is big enough), then the order \( \Gamma \) is a so called graduated order (see Definition 2.1). Then one may describe \( \Gamma \) by purely combinatorical data, which also allows one to read off the \( \Lambda \)-lattices in the irreducible \( A \)-modules. A language for describing such graduated orders is developed in [10] and will be repeated briefly in section 2.1.

The main tool for computing \( \Lambda \) and \( \Gamma \) is the observation that group rings and their ring direct summands are symmetric orders (see Definition 2.4). This means that \( \Lambda = \Lambda^\# \) is a self dual lattice with respect to some associative symmetric bilinear form on \( A \). The knowledge of this form allows one in particular to calculate the index of \( \Lambda \) in a maximal overorder in \( A \).

Let \( J(\Lambda) \) denote the Jacobson radical of \( \Lambda \), the smallest \( \Lambda \)-ideal \( I \) in \( \Lambda \) such that \( \Lambda / I \) is semisimple. Then \( \Lambda / J(\Lambda) \) is a semisimple \( k \)-algebra and one can lift the central primitive idempotents of \( \Lambda / J(\Lambda) \) to a system \( e_1, \ldots, e_h \) of orthogonal idempotents in \( \Lambda \) with \( 1 = \sum_{i=1}^{h} e_i \) (see e.g. [11], Theorem 6.19). Then one obtains a direct sum decomposition

\[ \Lambda = \bigoplus_{i,j=1}^{h} e_i \Lambda e_j. \]

Under certain conditions one can embed the summands \( e_i \Lambda e_j \) simultaneously into a commutative \( K \)-algebra \( E \) (isomorphic to the center of \( A \)) such that the multiplication \( e_i \Lambda e_j \times e_j \Lambda e_l \to e_i \Lambda e_l \) can be performed in \( E \). This is described in Section 2.3.

2.1 Graduated orders.

Definition 2.1 An \( R \)-order \( \Gamma \subset A \) is called graduated if there are orthogonal idempotents \( e_1, \ldots, e_h \in \Gamma \) satisfying \( e_i e_j = \delta_{ij} e_i \) and \( 1 = \sum_{i=1}^{h} e_i \) such that \( e_i \Gamma e_i \) is a maximal order in \( e_i \Lambda e_i \).

Let \( \Gamma \) be a graduated order in \( A \). Then \( \Gamma \) contains the central primitive idempotents \( e_1, \ldots, e_s \in \Gamma \) and each order \( \Gamma e_i \) is a graduated order in the simple algebra \( \Lambda e_i \).

Definition 2.2 Let \( \Gamma \) be a graduated order in the simple \( K \)-algebra \( A = D^{n \times n} \) and \( \Omega \) the maximal \( R \)-order in the division algebra \( D \) with maximal ideal \( \mathcal{P} \). Then
there are \( h, n_1, \ldots, n_h \in \mathbb{N} \) \((n = n_1 + \ldots + n_h)\) and \( M = (m_{ij}) \in \mathbb{Z}_{\geq 0}^{h \times h}\) such that \( \Gamma \) is conjugated to
\[
\Lambda(\Omega, n_1, \ldots, n_h, M) :=
\{ X = (x_{ij})_{i,j=1,\ldots,h} \in D^{n \times n} \mid x_{ij} \in (\mathcal{P}^{m_{ij}})_{n_i \times n_j} \text{ for all } 1 \leq i,j \leq h \}.
\]
If \( \Gamma/J(\Gamma) \cong \bigoplus_{i=1}^{h} (\Omega/\mathcal{P})^{n_i \times n_i} \) then \( m_{ii} = 0 \), \( m_{ij} + m_{ji} > 0 \) and \( m_{ij} + m_{ji} \geq m_{il} \) for all \( 1 \leq l, i \neq j \leq h \). In this case \( M \) is called an exponent matrix of \( \Gamma \).

The \( \Gamma \)-lattices in the simple \( A \)-module can be easily described by means of its exponent matrix \( M \), see [10], Remark (II.4). One can always conjugate \( \Gamma \) such that \( m_{ij} = 0 \) for all \( 1 \leq j \leq h \).

From [10], Proposition (IV.1) one finds

**Lemma 2.3** Let \( M \in \mathbb{Z}^{h \times h} \) be an exponent matrix of a graduated \( R \)-order \( \Gamma \) in the simple \( K \)-algebra \( A \). Assume that there is an involution \( \circ : \Gamma \to \Gamma \) (i.e. an \( R \)-order antiautomorphism of order \( \leq 2 \)) fixing the central primitive idempotents of \( \Gamma/J(\Gamma) \). Then
\[
m_{ij} + m_{ji} + m_{ii} = m_{ji} + m_{ii} + m_{jl} \text{ for all } 1 \leq i,j,l \leq h.
\]
If \( M \) is normalized such that the first row of \( M \) consists of 0 only then
\[
m_{ij} + m_{ji} = m_{ji} + m_{ii} \text{ for all } 1 \leq i,j \leq h.
\]

This lemma will be applied to epimorphic images of group rings \( RG \). The natural involution \( \circ : RG \to RG \) is the \( R \)-linear map defined by \( g \mapsto g^{-1} \) for all \( g \in G \). If \( \epsilon \) is a central primitive idempotent of \( KG \) with \( \epsilon^2 = \epsilon \) such that \( \Gamma := RG\epsilon \) is a graduated order and all \( p \)-modular constituents of the character belonging to \( \epsilon \) are self dual, then \( \Gamma \) satisfies the conditions of Lemma 2.3.

### 2.2 Symmetric orders.

**Definition 2.4** An \( R \)-order \( \Lambda \) in \( A \) is called symmetric if there is a nondegenerate symmetric associative \( K \)-bilinear form \( \Phi : A \times A \to K \) such that \( \Lambda \) is self dual with respect to \( \Phi \), i.e. \( \Lambda = \Lambda^\# = \{ a \in A \mid \Phi(\Lambda, a) \subset R \} \).

One easily shows that the nondegenerate symmetric associative \( K \)-bilinear form on \( A \) are precisely the forms
\[
Tr_u : A \times A \to K, (a,b) \mapsto \sum_{t=1}^{n} tr_{red}(au\epsilon_tb)
\]
where \( u \in Z(A)^* \) and \( tr_{red} \) denotes the reduced trace of \( Au\epsilon \) to \( K \).
Example. Let $G$ be a finite group. Then $RG$ is a symmetric order in $A = KG$ with respect to $|G|^{-1}$ times the regular trace bilinear form. If $\chi_l(1)$ denotes the dimension of an absolutely irreducible constituent of the simple $KG_\ell$-module, then this associative symmetric bilinear form equals $Tr_u$, where $u = |G|^{-1} \sum_{l=1}^s \chi_l(1)e_l$.

Lemma 2.5 ([14], Proposition (1.6.2)) If $\Lambda$ is a symmetric $R$-order with respect to $\Phi$ and $e, f$ are idempotents in $\Lambda$ then $\Phi_{[(e\Lambda f) \times (f\Lambda e)]}$ is a nondegenerate $R$-bilinear pairing. In particular $e\Lambda e$ is a symmetric order.

2.3 A language for describing certain basic orders.

Let $\Delta$ be an $R$-order in $A$. In this section it is assumed that $k = R/\pi R$ is a splitting field for $k \otimes R \Delta$ and that the division algebras $K_i$ are commutative. Let $P_1, \ldots, P_h$ represent the isomorphism classes of projective indecomposable $\Delta$ right modules.

Then $\Delta$ is Morita equivalent to

$$\Lambda := \text{End}_\Delta(P_1 \oplus \ldots \oplus P_h) = \bigoplus_{i,j=1}^h \text{Hom}_\Delta(P_{i}, P_j)$$

and $\Lambda$ is a basic order in the sense that the simple $\Lambda$-modules are one dimensional vector spaces over $k$.

Since there is an idempotent $e \in \Delta$ such that $\Lambda \cong e\Delta e$, Lemma 2.5 shows that $\Lambda$ is symmetric if $\Delta$ is symmetric. Note that the module categories of $\Delta$ and $\Lambda$ are equivalent. In particular the decomposition numbers of $\Delta$ and $\Lambda$ are equal. We assume that for $1 \leq i \leq h$ the endomorphism rings $\text{End}_\Delta(P_i)$ are commutative which is equivalent to say that the decomposition numbers of $\Delta$ are $\leq 1$.

The main new idea for describing the order $\Lambda$ is to embed the $R$-lattices $\text{Hom}_\Delta(P_i, P_j)$ simultaneously for all $1 \leq i, j \leq h$ into a commutative finite-dimensional $K$-algebra $E$ such that the multiplication $\text{Hom}_\Delta(P_i, P_j) \times \text{Hom}_\Delta(P_j, P_i) \to \text{Hom}_\Delta(P_i, P_j)$ can be performed in $E$.

To this purpose let

$$V := \bigoplus_{i=1}^s V_i$$

be the sum over a system of representatives of the isomorphism classes of simple $A$-modules and

$$E := \text{End}_A(V) \cong \bigoplus_{i=1}^s K_i \cong Z(A).$$

Let $1 \leq j \leq h$. Since $\text{End}_\Delta(P_j)$ is commutative, the $A$-module $V$ has a unique $A$-submodule isomorphic to $K \otimes R P_j$ and up to isomorphism a unique $\Delta$-sublattice
isomorphic to $P_j$. For all $1 \leq j \leq h$ choose an embedding 

$$\varphi_j : P_j \rightarrow V.$$ 

Let $Q_j$ be the unique $A$-invariant complement of $K \otimes_R \varphi_j(P_j)$ in $V$, 

$$V = (K \otimes_R \varphi_j(P_j)) \oplus Q_j.$$ 

Then the $\Delta$-homomorphisms $\varphi \in \text{Hom}_\Delta(P_j, P_i)$ for $1 \leq i, j \leq h$ are considered as elements of $E$ by letting 

$$\varphi|_{Q_i} = 0.$$ 

**Definition 2.6** For $i = 1, \ldots, h$ let $\varphi_i^{-1} : V \rightarrow K \otimes R P_i$ be the right inverse of $\varphi_i$ with $\varphi_i^{-1}(Q_i) = 0$. Then for $1 \leq i, j \leq h$ there are embeddings 

$$\text{Hom}_\Delta(P_i, P_j) \rightarrow E, \ \varphi \mapsto \varphi_i^{-1} \varphi_j.$$ 

Via these embeddings $\text{Hom}_\Delta(P_i, P_j)$ is viewed as a subset 

$$\Lambda_{ij} := (\varphi_i^{-1})\text{Hom}_\Delta(P_i, P_j) \varphi_j \subset E.$$ 

**Remark 2.7** For $1 \leq i \neq j \leq h$ the endomorphism ring $\text{End}_\Delta(P_j)$ is canonically (i.e. independent of the choice of $\varphi_j$) embedded into $E$, whereas the embedding $\text{Hom}_\Delta(P_i, P_j) \rightarrow E$ depends on the choice of $\varphi_i$ and $\varphi_j$.

For simplicity we now assume that $K$ is a splitting field for $A$. Then $K_t = K$ for all $1 \leq t \leq s$ and the central primitive idempotents $e_1, \ldots, e_s$ form a canonical $K$-basis of $E$.

**Definition 2.8** Let $\varphi = \sum_{t=1}^{s} a_t \epsilon_t \in E$ with $a_t \in K$. The fractional $R$-ideal $\sum_{t=1}^{s} a_t R$ is called the norm of $\varphi$, 

$$n(\varphi) := \sum_{i=1}^{s} a_t R \subseteq K.$$ 

The norm has a certain multiplicative property.

**Remark 2.9** If $\varphi, \psi \in E \setminus \{0\}$ then 

$$n(\varphi) n(\psi) \text{ divides } n(\varphi \psi)$$

and for $i \in \mathbb{N}$ 

$$n(\varphi^i) = n(\varphi)^i.$$ 

One may characterize the unit groups of the local rings $\Lambda_{ii}$ (cf. Definition 2.6) with help of the norm.
Lemma 2.10 For $1 \leq i \leq h$ the unit group $\Lambda^*_i$ of the local ring $\Lambda_i$ is

$$\Lambda^*_i = \{ \varphi \in \Lambda_i \mid n(\varphi) = R \}.$$  

Proof. Let $x \in \Lambda_i$ with $n(x) = R$. Then $n(x^j) = R$ and hence $x^j \notin \pi \Lambda_i$ for all $j \in \mathbb{N}$. Therefore $x$ does not lie in the unique maximal ideal of $\Lambda_i$ and is a unit. The other inclusion is trivial since $n(id_{\{R\}}) = R$. □

Since for $i \neq j$ the modules $P_i$ and $P_j$ are not isomorphic one gets the following remark from Lemma 2.10.

Remark 2.11 Let $1 \leq i \neq l \leq h$, $0 \neq \varphi \in \Lambda_i \subset E$, and $0 \neq \psi \in \Lambda_l \subset E$. Then $\varphi \psi = \psi \varphi \in \Lambda_i \cap \Lambda_l$ with

$$\pi R \supseteq n(\varphi \psi).$$

Assume now that $\Delta$ (and hence $\Lambda$) is a symmetric order with respect to the associative bilinear form $Tr_u$, $u = \sum_{t=1}^s u_\epsilon \in E$, $u_\epsilon \in K$.

Some additional notation is needed: If $L \subset M$ are two $R$-lattices with $M/L \cong \bigoplus_{i=1}^t R/\pi^x R$ the index of $L$ in $M$ is the ideal $[M : L] := \pi^{x_1 + \ldots + x_t} R$.

For $1 \leq i \leq h$ let

$$c_i := \{ 1 \leq t \leq s \mid \epsilon_\epsilon P_t \neq 0 \}$$

denote the constituents of the $K \otimes R P_i$.

Lemma 2.12 If $(\psi_1, \ldots, \psi_1)$ is an $R$-basis of $\Lambda_{jj}$, then

$$\left( \prod_{i=1}^l n(\psi_i) \right)^2 \text{ divides } \prod_{t \in c_j} u_t^{-1} R.$$

Proof. Let $M := \bigoplus_{t \in c_j} R \epsilon \epsilon$ be the maximal $R$-order in $K \otimes R \Lambda_{jj}$. Then the dual of $M$ with respect to $Tr_u$ is $M^\# = \bigoplus_{t \in c_j} u_t^{-1} R \epsilon \epsilon$. Since $M^\# \subseteq \Lambda_{jj} = \Lambda_j^\# \subseteq M$ with $[\Lambda_{jj} : M^\#] = [M : \Lambda_{jj}]$ one has

$$[M : \Lambda_{jj}]^2 = [M : M^\#] = \prod_{t \in c_j} u_t^{-1} R.$$

The Lemma follows, because $\prod_{i=1}^l n(\psi_i)$ divides $[M : \Lambda_{jj}]$. □

3 The group ring $\mathbb{Z}_2[\zeta_{2^f-1}]SL_2(2^f)$.

Let $G$ be a finite group and $R$ and $k$ be as in Section 2. This section presents a method for obtaining the ring theoretic structure of the integral group ring $RG$ from the group algebra $kG$. Assume that $k$ is a splitting field for $kG$. Then
one usually describes the finite-dimensional $k$-algebra $kG$ by giving a presentation of the Morita equivalent basic algebra $\bar{\Lambda} := \text{End}_{kG}(P_1 \oplus \ldots \oplus P_h)$, where $P_1, \ldots, P_h$ are the projective indecomposable $kG$-modules. $\bar{\Lambda}$ is generated by $id_{P_i} \in \text{End}_{kG}(P_i)$ ($1 \leq i \leq h$) and preimages in $\text{Hom}_{kG}(P_j, P_i)$ of a $k$-basis of $\text{Hom}_{kG}(P_j, J(kG)P_i)/\text{Hom}_{kG}(P_j, J(kG)P_i)$ ($1 \leq i, j \leq h$) (see [3], Proposition 4.1.7) usually encoded as vertices and arrows in the Ext-quiver. This generating set can be lifted to obtain a generating set of $\Lambda := \text{End}_{RG}(P_1 \oplus \ldots \oplus P_h)$, where $P_i$ is the projective $RG$-module with $P_i/\pi P_i = P_i$ ($i = 1, \ldots, h$) and the lifts of the generators in $\text{Hom}_{kG}(P_i, P_j)$ lie in $\text{Hom}_{RG}(P_i, P_j)$ ($1 \leq i, j \leq h$). Now Remark 2.11 gives upper bounds on the norm of the basis elements of $\text{End}_{RG}(P_i)$ obtained as product of the generators. The fact that $\text{End}_{RG}(P_i)$ is a symmetric order yields lower bounds on these norms.

In the particular situation of this section upper and lower bounds coincide.

So let $3 \leq f \in \mathbb{N}$, $R$ be the ring of integers in the unramified extension $K$ of degree $f$ of $\mathbb{Q}_2$ and $k := R/2R \cong \mathbb{F}_{2^f}$ the residue class field. Let $G := SL_2(2^f)$ denote the group of $2 \times 2$-matrices over $k$ of determinant $1$. Then $(K, R, k)$ is a $2$-modular splitting system for $G$. Since the decomposition numbers of $RG$ are $\leq 1$ (cf. [5], Corollary 2.8), the order $\oplus_{i=1}^s \epsilon_i RG$, where $\epsilon_1, \ldots, \epsilon_s$ are the central primitive idempotents of $KG$, is a graduated order in $KG$, the graduated hull of $RG$. Therefore the methods of the previous section can be applied to describe $RG$. In particular the graduated hull of $RG$ has a very nice description given in Theorem 3.15.

Since an explicit presentation of $kG$ is used to obtain information on the group ring $RG$, the description of $kG$ given in [6] is repeated in the first paragraph.

### 3.1 The group algebra in characteristic 2

Steinberg’s tensor product theorem establishes a bijection between the simple $kG$-modules and the subsets of $N := \{1, \ldots, f\}$: let $M_i$ be the natural $kG$-module $k^2$ and let $F$ be the Frobenius automorphism $F : k \to k; x \mapsto x^2$ of $k$. For $i = 0, \ldots, f - 1$ one defines $M_{i+1}$ to be the set $M_i$ with scalar multiplication $am := F^i(a)m$ for $a \in k$. Then the simple $kG$-modules are the tensor products

$$M_I := \bigotimes_{i \in I} M_i$$

where $I$ runs through the subsets of $N$. Note that $\text{dim}_k(M_I) = 2^{|I|}$. In particular $M_0$ is the trivial $kG$-module and $M_N$ is the projective simple $kG$-module, and hence lies in a block of defect zero.

The projective $kG$-modules are described in [1]. The projective cover of the simple $kG$-module $M_I$ is denoted by $P_I$ ($I \subseteq N$). Then by Theorem 1 in [1]

$$P_I = M_N \otimes M_{N-I}, \text{ if } \emptyset \neq I \subseteq N$$

and

$$P_0 \otimes M_N = M_N \otimes M_N.$$
Alperin also calculates the Ext-groups between the simple $kG$-modules. Koshita [6] extends this result to give a presentation of the basic algebra that belongs to $kG$ by describing the homomorphism spaces between the projective indecomposable $kG$-modules explicitly.

If $I, I' \subseteq N$ then $I + I' := I \cup I' - I \cap I'$ denotes the symmetric difference and for elements $i, j \in N$ let $i + j \in N$ (resp. $i - j \in N$) be the element of $N$ that is congruent to $i + j \in \mathbb{Z}$ (bzw. $i - j \in \mathbb{Z}$) modulo $f$. Then one has the following theorem.

**Theorem 3.1** ([6]) Let $f \geq 3$ and

$$\overline{X} := \text{End}_{kG}(\otimes_{N \neq I \subseteq N} \overline{P_I})$$

be the basic algebra belonging to the principal block of $kG$. Let $Q$ be the quiver with vertices corresponding to the proper subsets of $N$ and arrows

$$\alpha_{i,t} : I + \{i\} \rightarrow I$$

for all $I \subseteq N$, $i \in N$, $i - 1 \notin I$.

Let $kQ$ be the path algebra of $Q$. Then the paths from $I$ to $J = I + \{i_1\} + \ldots + \{i_a\}$ are written as

$$(I|i_1, \ldots, i_a|J) := \alpha_{i_1,t+\{i_1\}} \alpha_{i_2,t+\{i_1\}+\{i_2\}} \ldots \alpha_{i_a,t},$$

$(I||I)$ is the idempotent in $kQ$ corresponding to the vertex $I$. With this notation let $X$ be the ideal of $kQ$ generated by

$$(I + \{i\} + \{j\}|j, j|I) - (I + \{i\} + \{j\}|j, i|I)$$

$$(I|i, i|I)$$

$$(I + \{i + 1\}|i + 1, i, i|I) - (I + \{i + 1\}|i, i, i + 1|I)$$

$$(I + \{i + 1\}|i, i + 1, i|I)$$

Then there is an epimorphism

$$\Psi : kQ \rightarrow \overline{X}$$

with kernel $X$ i.e. $\overline{X} \cong kQ/X$.

Proposition 3 in [6] determines a $k$-basis for $\text{End}_{kG}(\overline{P_I})$.

**Definition 3.2** For $i \notin I \subseteq N$ let

$$\omega_{1,i} := \Psi((I|j, j + 1, \ldots, i - 1, i, i - 1, \ldots, j + 1, i|I))$$

where $j := j(I, i)$ is the unique element of $N$ such that $j - 1 \notin I$ and $J := J(I, i) := \{j, j + 1, \ldots, i - 1\} \subseteq I$. If $i - 1 \notin I$, then $j := i$, $J = \emptyset$ and

$$\omega_{1,i} := \Psi((I|i, i|I)).$$

The length of $\omega_{1,i}$ is the length of the corresponding path in $Q$,

$$l(I,i) := 2(|J| + 1).$$
The elements $\omega_{1,i}$ are endomorphisms of $P_i$. Since the decomposition numbers of $RG$ are 0 or 1 ([5], Corollary 2.8), the endomorphism rings of the projective indecomposable $kG$-modules are commutative. Therefore

$$\omega_{1,T} := \prod_{i \in T} \omega_{1,i}$$

for subsets $T \subseteq N - I$ is well defined.

**Proposition 3.3** ([6], Proposition 3) Let $I \subset N$ be a proper subset of $N$. The elements $\omega_{1,T}$ where $T$ runs through the subsets of $N - I$ form a $k$-basis of $\text{End}_{kG}(\overline{P_I})$.

To describe the vector spaces $\text{Hom}_{kG}(\overline{P_H}, \overline{P_I})$ for $I, H \subset N$ one needs further elements of $kQ$.

**Definition 3.4** Let $H \subset I \subset N$, such that there is a path in $Q$ from $H$ to $I$ of length $|I - H|$. Then let $\omega_{H \subset I}$ be the image of such a path under $\Psi$ and $\omega_{I \supset H}$ be the image of the corresponding path from $I$ to $H$ of the same length.

The next lemma follows from [6], Proposition 3.

**Lemma 3.5** Let $H, I \subset N$ with $\text{Hom}_{kG}(\overline{P_H}, \overline{P_I}) \neq 0$. Then

$$\text{Hom}_{kG}(\overline{P_H}, \overline{P_I}) = \omega_{H \supset H \cap I} \text{End}_{kG}(\overline{P_{H \cap I}}) \omega_{H \cap I \subset I}.$$ 

### 3.2 The integral group ring $RG$.

Now this characteristic 2 information is lifted to the characteristic 0 situation to obtain the $R$-order $RG$ nearly up to Morita equivalence. So let

$$\Lambda := \text{End}_{RG}(\oplus_{N \neq I \subset N} P_I),$$

where $P_i$ is the projective indecomposable $RG$-module with head $M_i$ ($I \subset N$), be the basic order that is Morita equivalent to the principal block of $RG$.

As in Section 2.3 let $V$ be the sum of all irreducible $K$-modules in the principal block and $E := \text{End}_{KG}(V)$. Let $\epsilon_1, \ldots, \epsilon_s$ be the central primitive idempotents of $KG$ that belong to the principal block of $RG$. These idempotents are identified with the primitive idempotents of $E$. If $I \subset N$ then

$$c_I := \{1 \leq t \leq s \mid \epsilon_t P_I \neq 0\}$$

denotes the indices of the irreducibles $KG$-modules that occur in $K \otimes_R P_I$. The sets $c_I$ are explicitly described in [5]. In particular $c_0 = \{1, \ldots, s\}$ and hence $V \cong K \otimes_R P_0$.

Using the fact that $\Lambda$ is a symmetric order, one gets:
Lemma 3.6 Let $I \subset N$ with $|N-I| = n$. Then $|c_I| = 2^n = \dim_R(\text{End}_{RG}(P_I))$ and $\text{End}_{RG}(P_I)$ is a sublattice of $\bigoplus_{1 \in I} R_{\epsilon_I}$ of index $2^{n-1}jR$.

Proof. Let $M := \bigoplus_{1 \in I} R_{\epsilon_I}$ be the maximal $R$-order in $K \otimes_R \text{End}_{RG}(P_I)$. Now $\text{End}_{RG}(P_I)$ is a symmetric order with respect to the form $Tr_u$ where

$$u := |G|^{-1} \sum_{\epsilon \in I} \chi_I(1) \epsilon_I,$$

and $\chi_I(1)$ is the degree of the irreducible character of $G$ belonging to $\epsilon_I$. Since $\chi_I(1) \in R^*$ is odd, the dual of $M$ with respect to $Tr_u$ is of index $(2^l)^{(2^l)R}$ in $M$. Hence $[M : \text{End}_{RG}(P_I)] = \sqrt{2^{2n}R} = 2^{n-1}jR$. □

Now suitable lifts $\beta'_{I,T} \in \Lambda$ of the elements $\omega_{I,T}$ from Definition 3.2 are constructed.

Definition 3.7 (i) For $i-1 \notin I \subset N$ let $\varphi_{I,i} \in \text{Hom}_{RG}(P_{I+i}, P_I)$ be a preimage of $\bar{\Psi}(\alpha_{i,i})$.

(ii) For $i \notin I \subset N$ let $\beta'_{I,i} \in \text{End}_{RG}(P_I)$ be defined similarly as $\omega_{I,i}$ using the $\varphi_{I,i}$ instead of $\bar{\Psi}(\alpha_{i,i})$: If $i - 1 \in I$, then let $j := j(I,i)$ and $J := J(I,i)$. Then

$$\beta'_{I,i} := \varphi_{I+(j)}j \varphi_{I-(j+1)}j + 1 \cdots \varphi_{I-J+i}j \varphi_{I-J+i}j \cdots \varphi_{I-(j+1)}j + 1 \varphi_{I-j}j .$$

If $i - 1 \notin I$ then

$$\beta'_{I,i} := \varphi_{I+(i)}i \varphi_{I,i}.$$

For any subset $T \subset N-I$ define

$$\beta'_{T,I} := \prod_{i \in T} \beta'_{I,i} \in \text{End}_{RG}(P_I).$$

Since $\text{End}_{RG}(P_I)$ is commutative, the definition does not depend on the ordering of the factors.

(iii) If $H \subset I \subset N$, such that there is a path $\omega = \alpha_{i_1,j_1} \cdots \alpha_{i_l,j_l} \in Q$ from $H$ to $I$ and a path $\omega' = \alpha_{j_1,j_1} \cdots \alpha_{j_l,j_l} \in Q$ from $I$ to $H$ of length $l := |I-H|$. Then let $\beta'_{H \subset I} := \prod_{m=1}^l \varphi_{i_m,j_m}$ and $\beta'_{I \supset H} := \prod_{m=1}^l \varphi_{j_m,i_m}$.

From Proposition 3.3 one now gets immediately

Corollary 3.8 $(\beta'_{I,T}, (T \subset N-I))$ is an $R$-basis of the $R$-lattice $\text{End}_{RG}(P_I)$.

The following lemma is the crucial point in the investigation of $\Lambda$.
Lemma 3.9 For $I \subset N$ and $T \subset N - I$

$$n(\beta_{i,T}^I) = 2^{l(I,T)}$$

where $l(I, T) := \sum_{i \in T} l(I, i)/2$.

Proof. Let $i \in N - I$ and $l := l(I, i)/2$. Then

$$\beta_{i,i} = f_1 \cdots f_l g_l \cdots g_1$$

is a product of $f_j \in \text{Hom}_R(P_{i,j}, P_{i,j+1})$ and $g_j \in \text{Hom}_R(P_{i,j+1}, P_{i,j})$ for certain pairwise distinct subsets $I_1, \ldots, I_{l+1}$ of $N$. In the commutative ring $E$ this product can be evaluated as

$$\beta_{i,i} = (f_1 g_1)(f_2 g_2) \cdots (f_l g_l).$$

According to Remark 2.11 the norm $n(f_j g_j)$ is divisible by 2. Then Remark 2.9 says that $2^l$ divides $n(\beta_{i,i}^I)$ and therefore $2^{l(I,T)}$ divides $n(\beta_{i,T}^I)$.

On the other hand let $n := |N - I|$. Then

$$\sum_{T \subseteq N - I} l(I, T) = \sum_{T \subseteq N - i} \sum_{i \in T} l(I, i)/2 = 1/2 \sum_{i \in N - I} l(I, i) 2^{n-1} = f 2^{n-1}.$$ 

Hence $n(\beta_{i,T}^I)$ divides $2^{l(I,T)}$ by Lemma 3.6 and Lemma 2.12.

In particular if $T = \{i\}$ and $i, i-1 \notin I \subset N$ this crucial lemma yields $n(\beta_{i,i}^I) = 2$. But

$$\beta_{i,i} = \varphi_{i+\{i\},i} \varphi_{i,i} = \varphi_{i,i} \varphi_{i+\{i\},i} \in 2\text{End}_R(P_{i+\{i\}}),$$ 

by Theorem 3.1 since $(I + \{i\}|i,i| + \{i\}) \in X$. So Lemma 2.10 implies that $\beta_{i,i}^I/2 \in \text{End}_R(P_{i+\{i\}})^*$ is a unit. Hence one gets the following lemma.

Lemma 3.10 If $i-1, i \notin I$, then $\varphi_{i,i}$ is injective and $\beta_{i,i}^I \in 2(\text{End}_R(P_{i+\{i\}})^*)$. Therefore there is a unit $u_{i,i} \in \text{End}_R(P_{i+\{i\}})^*$, such that

$$u_{i,i} \beta_{i,i}^I = 2id_{P_{i+\{i\}}} \in \text{End}_R(P_{i}).$$

Definition 3.11 For $I \subset N$ let $pr_I := \sum_{j \in I} e_j \in E$ be the identity on $P_I$. Let $i \notin I \subset N$. If $i-1 \notin I$, define

$$\beta_{i,i} = 2pr_{i+\{i\}}.$$ 

Otherwise let $j := j(I, i)$ and $J := J(I, i)$. Then

$$\beta_{i,i} := 2^{[J]+1} pr_{J-i} pr_I.$$ 

For $T \subset N - I$ let

$$\beta_{i,T} := \prod_{i \in T} \beta_{i,i}$$

where the empty product

$$\beta_{i,\emptyset} := pr_I$$

is the unit element in $\text{End}_R(P_I)$. 

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Theorem 3.12 Let $H, I \subset N$.

(a) The $\beta_{i,T}$ with $T \subset N - I$ form an $R$-basis of the lattice End$_{RG}(P_I)$.

(b) If Hom$_{RG}(P_H, P_I) \neq \{0\}$ then
\[ \text{Hom}_{RG}(P_H, P_I) \cong (\text{pr}_H(\text{End}_{RG}(P_{H \cap I})))\text{pr}_I \]
as End$_{RG}(P_H) - \text{End}_{RG}(P_I)$-bimodule.

Proof. (a) Let $I \subset N$ and $i \in N - I$. If $i - 1 \notin I$, then $\beta_{i,i} = \beta_{i,i}u_{i,i} \in \text{End}_{RG}(P_I)$ by Lemma 3.10. If $i - 1 \in I$, then let $j := j(I,i)$ and $J := J(I,i)$ be as in Definition 3.2. For $l = j, \ldots, i - 1$ define $I_l := I - \{j, \ldots, l\}$ and let $u_l := u_{i,i+1}$ be the unit of Lemma 3.10. It easily follows from [5] that
\[ c_l \subset c_{l-1} \subset \ldots \subset c_{l-J}. \]
Therefore
\[ \beta_{l,i}^{j-1} \prod_{l=0}^{i-1} u_l = (\varphi_{l-J,j} \varphi_{l,j} u_j) (\varphi_{l-j,j+1} \varphi_{l-j+1,j+1} u_{j+1}) \ldots \]
\[ \ldots (\varphi_{l-j,i-1} \varphi_{l-j+1,i-1} u_{i-1}) (\varphi_{l-j+1,i} \varphi_{l-j,i} u_{i-1}) = \]
\[ (2pr_j)(2pr_{j-1}) \ldots (2pr_{j-i})(2pr_{j-i+1}) = 2^{j+1}pr_{j-i}. \]
Hence the elements $\beta_{i,T}$ with $T \subset N - I$ lie in the ring End$_{RG}(P_I)$.

For $T \subset N - I$ let $a(I,T)_j, a(I,T)_j \in R (j = 1, \ldots, s)$ be such that $\beta_{i,T} = \sum_{j=1}^s a(I,T)_j e_j$ and $\beta_{i,T} = \sum_{j=1}^s a(I,T)_j e_j$. Since $\beta_{i,T}$ is obtained from $\beta_{i,T}$ by multiplication with units in local rings, there is $k_{i,T} \in R^+$ with
\[ k_{i,T} a(I,T)_j \equiv a(I,T)_j \pmod{2n(\beta_{i,T})} \]
for all $1 \leq j \leq s$. Replacing $\beta_{i,T}$ by $k_{i,T} \beta_{i,T}$ we assume that $k_{i,T} = 1$. Lemma 3.9 yields that $n(\beta_{i,T}) = n(\beta_{i,T}) = 2^{l_{i,T}}$ and the product of the norms is
\[ \prod_{T \subset N - I} n(\beta_{i,T}) = 2^{n-1}f \]
the index of End$_{RG}(P_I)$ in $\oplus_{j \in c_I} R e_j$. Here $n := |N - I|$. Let $B' := (a(I,T)_j)_{T,j}$,
$B := (a(I,T)_j)_{T,j}$ and $D := \text{diag}(n(\beta_{i,T}) | T \subset N - I)$. Then $B' = DU$ for some $U \in GL_n(R)$ and $B = DV$ with $V \equiv U \pmod{2R}$. Therefore the determinant of $V$ is also a unit in $R$ and $(\beta_{i,T} | T \subset N - I)$ is an $R$-basis of End$_{RG}(P_I)$.

(b) By Lemma 3.5
\[ \text{Hom}_{RG}(P_H, P_I) = \beta_{H \subset H \cap I} \text{End}_{RG}(P_{H \cap I}) \beta_{H \cap I \subset I}. \]
Multiplication with $\beta_{H \subset H \cap I} \beta_{H \cap I \subset I} \in E$ induces the desired bimodule isomorphism. 

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Remark 3.13 This theorem shows that the endomorphism rings of the projective indecomposable RSL\(_2(2^f)\)-modules have a rational structure: Let \( I \subset N \) and \( c_i = \{t_1, \ldots, t_n\} \). Then there is a (symmetric) \( \mathbb{Z}_2 \)-order

\[
O_I = \bigoplus_{T \subset N-I} \mathbb{Z}_2 \beta_{T,I} \subset \bigoplus_{i=1}^n \mathbb{Z}_2 \epsilon_i
\]

with \( R \otimes_{\mathbb{Z}_2} O_I = \text{End}_{RG}(P_I) \).

In particular the order \( O_0 \) is isomorphic to the group ring \( \mathbb{Z}_2(C_2^f) \) of the Sylow 2-subgroup of \( G \):

**Proposition 3.14** Let \( O_0 = \bigoplus_{T \subset N} \mathbb{Z}_2 \beta_{T,I} \) be as in the remark above. Then

\[
O_0 \cong \mathbb{Z}_2(C_2^f).
\]

In particular the endomorphism ring of \( P_0 \)

\[
\text{End}_{RG}(P_0) \cong R(C_2^f)
\]

is isomorphic to the group ring over \( R \) of the Sylow 2-subgroup of \( G \).

**Proof.** The \( R \)-order \( \text{End}_{RG}(P_0) \subset E \) is generated by \( \beta_{0,0} = \text{id}_{P_0} = 1 \in E \) and \( x_i := \beta_{0,i} = 2 \sum_{e \in c(i)} \epsilon_i, \ i = 1, \ldots, f \). The Frobenius automorphism \( F \) acts on \( \text{End}_{RG}(P_0) \) mapping \( x_i \) to \( x_{i+1} \) (\( i = 1, \ldots, f \)) where as usual the indices are taken modulo \( f \). With the notation of [5], the ordinary irreducible characters in the principal block of \( RG \) are \( 1, \eta_j, \delta_i \) where \( j = 1, \ldots, 2^{f-1} - 1 \) and \( i = 1, \ldots, 2^{f-1} \).

The Frobenius automorphism \( F \) acts on the irreducible characters \( \eta_j \) and \( \delta_i \) by multiplying the indices with 2, where the indices of \( \eta \) are taken modulo \( 2^f - 1 \) and the ones of \( \delta \) modulo \( 2^f + 1 \) and in both cases identified with the negative index. In the rest of the proof these characters are used to index the primitive idempotents of \( E \). Then by [5]

\[
x_f = 2 \left( \sum_{j \text{ odd}}^{2^{f-1}-1} \epsilon_{\eta_j} + \epsilon_{\delta_j} \right).
\]

Choose a generating set \( \{b_0, \ldots, b_{f-1}\} \) of \( C_2^f \) and let \( 0 \leq i = \sum_{j=0}^{f-1} z_j 2^j < 2^f \) with \( z_j \in \{0,1\} \). Then define the linear character \( \chi_i \) of \( C_2^f \) via \( \chi_i(b_j) = (-1)^{\ z_j} \). Let \( h(i) := \sum_{j=0}^{f-2} z_j 2^j \) and define a bijection between the set of characters \( \{\chi_0, \ldots, \chi_{2^f-1}\} \) and the set of characters in the principal block of \( G \) via

\[
\chi_i \mapsto \begin{cases} 
1 & \text{if } i = 0 \\
\eta_{h(i)} & \text{if } i > 0 \text{ and } z_{f-1} = 0 \\
\delta_{h(i)} & \text{if } z_{f-1} = 1.
\end{cases}
\]
This bijection induces a bijection of the primitive idempotents of $E$ and $KC_2^f$ and hence an isomorphism $\varphi : E \to KC_2^f$. Clearly $\chi_i(b_0) = 1$ if and only if $z_0 = 0$. Therefore one finds $\varphi(x_f) = 1 - b_0 \in \mathbb{Z}_2C_2^f$.

The action of $F$ on $KC_2^f$ induced via $\varphi$ is given by $b_i \mapsto b_{i+1}$ if $i = 0, \ldots, f-3$, $b_{f-2} \mapsto \sum_{t=0}^{f-2} b_t$ and $b_{f-1} \mapsto b_f-1$. One has $\varphi(x_1) = 1 - b_1, \ldots, \varphi(x_{f-3}) = 1 - b_{f-2}, \varphi(x_{f-2}) = 1 - \sum_{t=0}^{f-2} b_t, \varphi(x_f) = 1 - b_{f-1} \in \mathbb{Z}_2C_2^f$. Together with $1 = \varphi(1)$ these elements span $\mathbb{Z}_2C_2^f$, therefore $\varphi|_{O_0} : O_0 \to \mathbb{Z}_2C_2^f$ is an isomorphism. □

Note that this proposition also shows that $\text{End}_{RG}(P_0)$ is also symmetric with respect to $Tr_v$ where $v := 2^{-f} \sum_{t=1}^{s} \epsilon_t$.

Since the decomposition numbers of $RG$ are $1$, the projections $\Lambda \epsilon_i$ of $\Lambda$ onto the simple components of $K \otimes_R \Lambda$ are graduated orders. From Theorem 3.12 one gets a simple description of the graduated hull $\oplus_{t=1}^{s} \epsilon_i \Lambda$ of $\Lambda$, which also yields the theorem stated in the introduction.

**Theorem 3.15** For $1 \leq t \leq s$ there is an exponent matrix $M^{(t)} = (m^{(t)}_{H,I})$ where $H, I \subset N$, $t \in c_H \cap c_I$ for $\Lambda \epsilon_t$ such that

$$m^{(t)}_{H,I} = |H - I| \quad (\text{for all } H, I \text{ such that } t \in c_H \cap c_I).$$

**Proof.** Let $M^{(t)}$ be an exponent matrix of $\Lambda \epsilon_t$ such that the row corresponding to $M_0$ consists of 0 only

$$m^{(t)}_{0,H} = 0 \text{ for all } H \subset N, t \in c_H.$$  

From Lemma 3.9 it follows that

$$n(\beta_{H \supseteq H \cap t}^{H \cap c_I}) = 2^{|H - I|}$$

and

$$n(\beta_{H \supseteq H \cap t}^{H \cap c_I}) = 2^{|t - H|}.$$  

Then Theorem 3.12 (b) implies that for $I, H \subset N$ such that $t \in c_H \cap c_I$

$$m^{(t)}_{H,I} + m^{(t)}_{I,H} = |H - I| + |I - H|.$$  

In particular

$$m^{(t)}_{H,0} = |H| \text{ for all } H \subset N, t \in c_H.$$  

Since the simple $kG$-modules are all self-dual, one gets from Lemma 2.3

$$m^{(t)}_{H,I} = m^{(t)}_{I,H} - m^{(t)}_{I,0} + m^{(t)}_{H,0} = m^{(t)}_{I,H} - |I| + |H|.$$  

Therefore the Theorem follows. □

Theorem 3.15 and Theorem 3.12 do not describe $\Lambda$ up to Morita equivalence, because they do not give a system of simultaneous embeddings $\varphi_t : P_t \hookrightarrow P_0$.
\((I \subset N)\) that determines the multiplication in \(\Lambda\). If the isomorphism type of \(\Lambda\) is uniquely determined by that of \(\Lambda/2\Lambda\), then one can show that \(\Lambda\) is as given in the following conjecture.

**Conjecture:** There are embeddings \(\varphi_i : P_I \hookrightarrow P_\emptyset, I \subset N\), such that for all \(I, H \subset N\)

\[
\text{Hom}_{RG}(\varphi_H(P_H), \varphi_I(P_I)) = 2^{[I - \{H \cap I\}]}(\text{pr}_H(\text{End}_{RG}(\varphi_{H \cap I}(P_{H \cap I})) \text{pr}_I)) \subset E.
\]

One easily shows using the combinatorical description of the sets \(c_I\) in [5] that the \(R\)-order \(\Lambda = \text{End}_{RG}(\oplus_{I \subset N} P_I)\) in the conjecture satisfies \(\Lambda/2\Lambda \cong kQ/X\) as in Theorem 3.1.

**Proposition 3.16** The conjecture is true for \(f \leq 6\).

**Proof.** For all proper subsets \(\emptyset \neq I \subset N\) choose \(i \in I\) with \(i - 1 \notin I\). Let \(T\) be the set of such pairs \((I, i)\). Via the monomorphism \(\varphi_{I - \{i\}} : P_I \to P_{I - \{i\}}\) in Lemma 3.10 one embeds \(P_I\) into \(P_{I - \{i\}}\) for all \((I, i) \in T\), such that recursively all projective indecomposable \(RG\)-modules are embedded into \(P_\emptyset\). For \(I \subset N\) let \(\varphi_I : P_I \hookrightarrow P_\emptyset\) denote this embedding. Then for all \((I, i) \in T\) it holds that

\[
\text{Hom}_{RG}(\varphi_I(P_I), \varphi_{I - \{i\}}(P_{I - \{i\}})) = \text{pr}_I(\text{End}_{RG}(\varphi_{I - \{i\}}(P_{I - \{i\}}))).
\]

One possibility for \(T\) in the case \(f = 5\) is given in figure 1, which shows the Ext-quiver for the principal block of \(\mathbb{P}_2 SL_2(2^5)\). The vertices of the quiver are indexed by the projective indecomposable modules \(P_I\), where only the elements of \(I\) are given and \(0\) stands for the empty set. The arrows \(P_I \to P_{I + \{i\}}\) and \(P_{I - \{i\}} \to P_I\) in \(Q\) are represented by one straight line. The mappings \(\varphi_I\) are constructed along the bold edges.

So to show the Proposition for the case \(f = 5\), it remains to show that the homomorphism \(\text{pr}_I\) lies in \(\text{Hom}_{RG}(\varphi_I(P_I), \varphi_{I - \{i\}}(P_{I - \{i\}}))\) for all thin edges \((P_I, P_{I - \{i\}})\) with \(i \in I \subset N\), \(i - 1 \notin I\) and \((I, i) \notin T\).

We prove this for the example \(I = \{2, 4, 5\}\) and \(i = 4\). Let

\[
h \in \text{Hom}_{RG}(\varphi_{\{2, 4, 5\}}(P_{\{2, 4, 5\}}), \varphi_{\{4, 5\}}(P_{\{4, 5\}}))
\]

be a lift of \(\Psi(\alpha_{4, 2, 5})\). Theorem 3.1 says that there is an \(x \in \text{End}_{RG}(\varphi_{\{2\}}(P_{\{2\}}))\) with

\[
h = h\text{pr}_{\{2, 5\}} = \text{pr}_{\{2, 4, 5\}}\text{pr}_{\{4, 5\}}(1 + 2x) = \text{pr}_{\{2, 4, 5\}}(1 + 2x).
\]

Now

\[
2\text{End}_{RG}(\varphi_{\{2\}}(P_{\{2\}})) \subset \text{End}_{RG}(\varphi_{\{2, 5\}}(P_{\{2, 5\}})),
\]

so \(1 + 2x\) is a unit in \(\text{End}_{RG}(\varphi_{\{2, 5\}}(P_{\{2, 5\}}))\). Therefore

\[
\text{pr}_{\{2, 4, 5\}} \in \text{Hom}_{RG}(\varphi_{\{2, 4, 5\}}(P_{\{2, 4, 5\}}), \varphi_{\{2, 5\}}(P_{\{2, 5\}})).
\]

The other cases are completely analogous. If \(f = 6\), one additionally needs the action of the Galois group \(\text{Gal}(K/Q_2) \cong C_f\) on the quiver \(Q\). \(\square\)
Remark 3.17 If the conjecture is true, then one does not need irrationalities to describe $RG$. More precisely if $KG \cong \bigoplus_{t=1}^{n} K^{n \times n}$ then there is a $\mathbb{Q}_2$-algebra $A \cong \bigoplus_{t=1}^{n} \mathbb{Q}_2^{n \times n}$ and a (symmetric) $\mathbb{Z}_2$-order $\Lambda_0 \subset A$ such that $R \otimes_{\mathbb{Z}_2} \Lambda_0 = RG$.

References


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