The group ring of $SL_2(p^f)$ over p-adic integers.

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Situation and Results for $p = 2$.

Let $3 \leq f \in \mathbb{N}$, R be the ring of integers in the unramified extension K of degree f of $\mathbb{Q}_2$ and $k := R/2R \cong \mathbb{F}_{2^f}$ the residue class field. Let $G := SL_2(2^f)$ denote the group of 2×2-matrices over k of determinant 1. Then (K, R, k) is a 2-modular splitting system for G.

The simple kG-modules S_I are indexed with the subsets I of $N := \{1, \ldots, f\}$, such that $dim(S_I) = 2^{|I|}$.

Theorem. Let V be a simple KG-module of dimension n with corresponding representation Δ_V and $C_V := \{I_1, \ldots, I_r\}$ be the set of indices of the 2-modular constituents of V and put $n_j := 2^{|I_j|}$ $1 \leq j \leq r$. Then there is a basis of V such that

$$\Delta_V(RG) = \{(X_{ij})_{1 \leq i,j \leq r} \in R^{n \times n} \mid X_{ij} \in 2^{|I_i - I_j|} R^{n_i \times n_j}\}.$$

The endomorphism rings and homomorphism spaces of the projective indecomposable RG-lattices can be described explicitely. In particular one finds

Corollary. The endomorphism ring of the projective cover of the trivial RG-module is isomorphic to the group ring of the Sylow 2-subgroup of G.

All these results follow from a description of the group algebra kG in characteristic 2.

Similar results are obtained for $p > 2$.

The method.

Let $P_1, \ldots, P_h$ be the projective indecomposable RG-lattices and

$$\Lambda := End_{RG}(P_1 \oplus \ldots \oplus P_h) = \bigoplus_{1 \leq i,j \leq h} Hom_{RG}(P_i, P_j)$$

be the basic order Morita equivalent to RG.

Then the $P_i/2P_i =: \overline{P_i}$ are the projective indecomposable kG-modules and kG is Morita equivalent to $\overline{\Lambda} := End_{kG}(\oplus_{i=1}^h \overline{P_i}) = \Lambda/2\Lambda$.

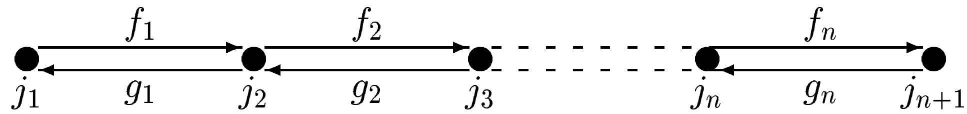
A good system of generators for the basic algebra $\overline{\Lambda}$ is given by the vertices and arrows in the Ext-quiver of kG. The products of these generators along certain paths in this Ext-quiver form a k-basis of $Hom_{kG}(\overline{P_i}, \overline{P_j})$ $(1 \leq i, j \leq h)$.

The generators lift to generators of Λ. The same products as above in these lifts form an R-basis of $Hom_{RG}(P_i, P_j)$ $(1 \leq i, j \leq h)$.

It is easier to calculate in Λ than in $\overline{\Lambda}$:

In our special situation $End_{RG}(P_i)$ is commutative since $K \otimes_R P_i$ is the direct sum of pairwise non isomorphic simple KG-modules. Embedding all the P_i into the direct sum W over representatives of the isomorphism classes of simple KG-modules one obtains simultaneous embeddings of the homomorphism spaces $Hom_{RG}(P_i, P_j)$ into the commutative K-algebra $E = End_{KG}(W) \cong Z(KG)$ such that products can be calculated in E. Note that the primitive idempotents of E form a canonical basis. Hence each element of $Hom_{RG}(P_i, P_j)$ can be written as a linear combination of these idempotents. If $i = j$ then this expression does not depend on the choice of the embedding $P_i \hookrightarrow W$.

For $G = SL_2(2^f)$ there are k-bases for $End_{kG}(P_i)$ obtained as products, that correspond to closed paths $f_1 f_2 \dots f_n g_n \dots g_2 g_1$ in the Ext-quiver of the following shape:



Since E is commutative, the product of the lifts $F_i, G_i \in \Lambda$ of the elements in $\overline{\Lambda}$ corresponding to the f_i, g_i in the picture above can be calculated as

$$(F_1 \dots F_n)(G_n \dots G_1) = (F_1G_1)(F_2G_2)\dots(F_nG_n).$$

The endomorphisms $F_iG_i \in End_{RG}(P_{j_i})$ factor through $P_{j_{i+1}}$ and therefore are not invertible. Hence their coefficients with respect to the canonical basis of E are divisible by 2. This gives lower bounds on the index of $End_{RG}(P_{j_1})$ in the maximal order. These bounds are sharp as it can be calculated from the fact that $End_{RG}(P_{j_1})$ is a symmetric order. In particular this implies that the minimum of the 2-adic valuations of the coefficients of F_iG_i is 1.
The relations for $\overline{\Lambda}$ say that $G_iF_i \in 2End_{RG}(P_{j_{i+1}})$. Therefore $G_iF_i = 2u_i$ for some unit in $End_{RG}(P_{j_{i+1}})$ and

$$F_iu_i^{-1}G_i = u_i^{-1}G_iF_i = 2id_{P_{j_{i+1}}} \in End_{RG}(P_{j_i}) \cap End_{RG}(P_{j_{i+1}}).$$

This calculates the endomorphism rings of the projective indecomposable RG-lattices explicitely and the $End_{RG}(P_i) - End_{RG}(P_j)$-bimodule $Hom_{RG}(P_i, P_j)$ up to isomorphism.

The Theorem follows by calculating the ideal generated by the coefficients of the elements of $Hom_{RG}(P_i, P_j)$ at the primitive idempotent that belongs to V.

The same methods, with some additional technical difficulties, give the result for odd primes p.