The group ring of $SL_2(p^f)$ over p-adic integers for p odd.

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ABSTRACT: Let $p > 2$ be a prime, $R = \mathbb{Z}_p[\zeta_{p^f-1}]$, $K = \mathbb{Q}_p[\zeta_{p^f-1}]$ and $G = SL_2(p^f)$. The group ring RG is calculated nearly up to Morita equivalence: The projections of RG into the simple components of KG are given explicitly and the endomorphism rings and homomorphism bimodules between the projective indecomposable RG-lattices are described.

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1 Introduction

This paper investigates the group ring over p-adic integers of the special linear groups $G := SL_2(p^f)$ of degree 2 for odd primes p and $f > 2$. For $f = 2$ the group ring $\mathbb{Z}_p SL_2(p^2)$ has been described up to Morita equivalence in [Neb 98]. The case $p = 2$ is slightly easier and has been treated in [Neb 00]. The main strategy is the same as for $p = 2$. There are additional technical difficulties due to the fact that for odd primes p, there are two pairs of ordinary characters of degree $\frac{p^f \pm 1}{2}$ that have the same p-modular constituents.

In the first part of the paper the general method, already used to describe RG for $p = 2$ in [Neb 00] is repeated and adopted to the slightly more complicated situation here, that we do not work with a splitting field. The second part is devoted to the special situation $G = SL_2(p^f)$, $K = \mathbb{Q}_p[\zeta_{p^f-1}]$, $R = \mathbb{Z}_p[\zeta_{p^f-1}]$, $k = R/pR$. Since details are needed, the technical description of kG given in [Kos 98] is repeated briefly in the first section. To describe RG it is useful to know some facts about the direct summands of $K \otimes_R P$ for the projective indecomposable RG-lattices P. The necessary information is derived in section 3.2 from the description of the decomposition numbers in [HSW 82]. After all these preparations the section 3.3 gives a description of RG. The endomorphism rings of the projective indecomposable RG-lattices are calculated and the homomorphism lattices between two such modules are determined up to bimodule isomorphisms. Finally the projections of RG into the simple components of KG are described explicitly. This allows to read off the inclusion patterns for the irreducible RG-lattices as it is illustrated for the example $G = SL_2(3^3)$.

This paper is intended to make one part of my habilitation thesis [Neb 99] available to a wider audience. I thank Dr. Alexander Zimmermann for giving me a copy of the preprint of [Kos 98].

2 General definitions.

Throughout the paper let p be a prime, K a finite extension of $\mathbb{Q}_p$, R its ring of integers with maximal ideal πR and residue class field $k := R/\pi R$. Let A be a finite dimensional semisimple K-algebra

$$A = \bigoplus_{t=1}^{s} A\epsilon_t \cong \bigoplus_{t=1}^{s} K_t^{n_t \times n_t}$$

where $\epsilon_1, \ldots, \epsilon_s$ are the central primitive idempotents of A and K_t are K-division algebras.

Let Δ be an R-order in A. This section develops a language to describe Δ up to Morita equivalence, if it fulfills additional conditions. It does not weaken the tools to calculate Δ, if one makes unramified extensions of K. Therefore it is convenient to assume that K is big enough for Δ (cf. [Jac, Definition 9.10]) in the sense that

- k is a splitting field for $k \otimes_R \Delta$ and
- the division algebras K_t are totally ramified field extensions of K.

So let K be big enough for Δ and let $P_1, \ldots, P_h$ represent the isomorphism classes of projective indecomposable Δ right modules.

Then Δ is Morita equivalent to

$$\Lambda := \mathrm{End}_\Delta(P_1 \oplus \ldots \oplus P_h) = \bigoplus_{i,j=1}^{h} \mathrm{Hom}_\Delta(P_i, P_j)$$

and Λ is a basic order in the sense that the simple Λ-modules are one dimensional vector spaces over k.

Let V_t be the simple $A\epsilon_t$-module and define the A-module

$$V := \bigoplus_{t=1}^{s} V_t.$$

The main assumption of this paper is that for $1 \leq i \leq h$ the endomorphism rings $\mathrm{End}_\Delta(P_i)$ are commutative. This is equivalent to say that the multiplicities of the simple A-modules in $K \otimes_R P_i$ are ≤ 1. Then V has a unique A-submodule isomorphic to $K \otimes_R P_i$. For all $1 \leq i \leq h$ choose an embedding

$$\varphi_i : P_i \hookrightarrow V.$$

Let Q_i be the unique A-invariant complement of $K \otimes_R \varphi_i(P_i)$ in V,

$$V = (K \otimes_R \varphi_i(P_i)) \oplus Q_i.$$

Then the Δ-homomorphisms $\varphi \in \mathrm{Hom}_\Delta(P_i, P_j)$ for $1 \leq i, j \leq h$ are considered as A-endomorphisms of V by letting $\varphi_{|Q_i} = 0$. So let

$$E := \mathrm{End}_A(V) \cong \bigoplus_{t=1}^{s} K_t \cong Z(A).$$

Definition 2.1 *For $i = 1, \ldots, h$ let $\varphi_i^{-1} : V \to K \otimes_R P_i$ be the right inverse of φ_i with $\varphi_i^{-1}(Q_i) = 0$. Then for $1 \le i, j \le h$ there are embeddings*

$$\mathrm{Hom}_\Delta(P_i, P_j) \hookrightarrow E, \varphi \mapsto \varphi_i^{-1}\varphi\varphi_j.$$

Via these embeddings $\mathrm{Hom}_\Delta(P_i, P_j)$ *is viewed as a subset*

$$\Lambda_{ij} := \varphi_i^{-1}\mathrm{Hom}_\Delta(P_i, P_j)\varphi_j \subset E.$$

Note that Λ_{ij} in general depends on the choice of φ_i and φ_j but Λ_{ii} does not.

The primitive idempotents of E form a canonical "basis" of E in the sense that the elements of E can be written uniquely as $\sum_{t=1}^s a_t\epsilon_t \in E$ with $a_t \in K_t$.

If $\varphi_{ij} = \sum_{t=1}^s a_t\epsilon_t \in \Lambda_{ij}$ and $\varphi_{ji} = \sum_{t=1}^s b_t\epsilon_t \in \Lambda_{ji}$ then

$$\varphi := \varphi_{ij}\varphi_{ji} == \sum_{t=1}^s a_t b_t \epsilon_t = \varphi_{ji}\varphi_{ij} \in \Lambda_{ii} \cap \Lambda_{jj}.$$

Since φ corresponds to an endomorphism of P_i that factors through P_j, it is not a unit in Λ_{ii}. Therefore one concludes that $a_t b_t \in \pi_t R_t$ lies in the maximal ideal $\pi_t R_t$ of the ring of integers R_t in K_t for all $t = 1, \ldots, s$.

To measure the difference of φ being a unit in Λ_{ii} we introduce a norm on E. To this purpose let L be the compositum of the fields K_t $(1 \le t \le s)$ with ring of integers O and maximal ideal $\wp$.

Definition 2.2 *Let $0 \ne \varphi = \sum_{t=1}^s a_t\epsilon_t \in E$ with $a_t \in K_t$. The fractional O-ideal $\sum_{t=1}^s a_t O$ is called the* norm *of φ,*

$$n(\varphi) := \sum_{i=t}^s a_t O \subseteq L.$$

It follows directly from the definition that

$$n(\varphi)n(\psi) \text{ divides } n(\varphi\psi)$$

for all $\varphi, \psi \in E - \{0\}$ and for $i \in \mathbb{N}$

$$n(\varphi^i) = n(\varphi)^i.$$

The unit groups Λ_{ii}^* of the local rings Λ_{ii} consist precisely of the norm O elements:

$$\Lambda_{ii}^* = \{\varphi \in \Lambda_{ii} \mid n(\varphi) = O\} \text{ for all } 1 \le i \le h.$$

As seen above this implies that

$$n(\varphi_{ij}\varphi_{ji}) \subseteq \wp \text{ for all } 1 \le i \ne j \le h,\ 0 \ne \varphi_{ij} \in \Lambda_{ij},\ 0 \ne \varphi_{ji} \in \Lambda_{ji}.$$

In the applications Δ will always be a ring direct summand of a group ring RG for some finite group G. roup rings have the distinguished property that they are symmetric orders: For the regular trace tr_{reg} of $A = KG$ one has $tr_{reg}(g) = 0$

for all $1 \neq g \in G$ and $tr_{reg}(1) = |G|$. Hence $\{g^{-1} \mid g \in G\}$ is the reciprocal basis of G with respect to the bilinear form $Tr_{reg} : A \times A \to K$, $Tr_{reg}(a,b) := \frac{1}{|G|} tr_{reg}(ab)$ and $RG = RG^{\#} := \{a \in KG \mid Tr_{reg}(a, RG) \subseteq R\}$. This observation allows for instance to calculate the index of RG in any maximal overorder. One easily sees that also the blocks of RG are symmetric with respect to the restriction of Tr_{reg} (cf. [Thé 95, Proposition (1.6.2)]). For calculations it is easier to work with the reduced trace of KG: Let $\chi_t(1)$ denote the dimension of an absolutely irreducible constituent of the simple $KG\epsilon_t$-module and let $u_t := \frac{1}{|G|}\chi_t(1)\epsilon_t \in Z(KG\epsilon_t)^*$, $t = 1, \ldots, s$. Then $\frac{1}{|G|} tr_{reg}(a) = \sum_{t=1}^{s} tr_{red}(u_t a)$ for all $a \in KG$.

Assume now that Δ is a symmetric order with respect to the associative bilinear form $Tr_u : (a,b) \mapsto \sum_{t=1}^{s} tr_{red}(a u_t \epsilon_t b)$ with $u = \sum_{t=1}^{s} u_t \epsilon_t$ and $u_t \in K_t$. Since there is an idempotent $e \in \Delta$, such that $\Lambda \cong e\Delta e$ [Thé 95, Proposition (1.6.2)] shows that Λ is also symmetric.

Some additional notation is needed:

If $L \subset M$ are two R-lattices with $M/L \cong \oplus_{i=1}^{t} R/\pi^{x_t}R$ the index of L in M is the ideal $[M : L] := \pi^{x_1+\ldots+x_t}R$.

For $1 \leq i \leq h$ let

$$c_i := \{1 \leq t \leq s \mid \epsilon_t P_i \neq 0\}$$

denote the constituents of the KG-module $K \otimes_R P_i$.

Let ν_π be the extension of the π-adic valuation of K to L ($\nu_\pi(\pi) = 1$) and for fractional O-ideals I in L let $\nu_\pi(I) := min\{\nu_\pi(x) \mid x \in I\}$. For $t = 1, \ldots, s$ let $\mathcal{D}(R_t)$ be the inverse different of R_t, i.e. the dual of R_t with respect to the trace bilinear form over R.

Lemma 2.3 *Let $1 \leq j \leq h$. Let*

$$D_j := \prod_{t \in c_j} [R_t : u_t^{-1}\mathcal{D}(R_t)]O.$$

If $(\psi_1, \ldots, \psi_l)$ is an R-basis of Λ_{jj}, then

$$2\sum_{i=1}^{l} \lfloor \nu_\pi(n(\psi_i)) \rfloor \leq \nu_\pi(D_j).$$

Proof. Let $M := \oplus_{t \in c_j} R_t \epsilon_t$ be the maximal R-order in $K\Lambda_{jj}$. Then the dual of M with respect to Tr_u is $M^{\#} = \oplus_{t \in c_j} u_t^{-1}\mathcal{D}(R_t)\epsilon_t$. Since $M^{\#} \subseteq \Lambda_{jj} = \Lambda_{jj}^{\#} \subseteq M$ with $[\Lambda_{jj} : M^{\#}] = [M : \Lambda_{jj}]$ one has

$$[M : \Lambda_{jj}]^2 = [M : M^{\#}] = D_j.$$

The Lemma follows, because $\pi^{\sum_{i=1}^{l} \lfloor \nu_\pi(n(\psi_i)) \rfloor}$ divides $[M : \Lambda_{jj}]$. □

To describe the action of Δ on the irreducible A-modules we need some additional notation as it is developed in [Ple 83]. The main idea is not to consider norms of individual elements of $\mathrm{Hom}_\Delta(P_i, P_j)$ but to consider the ideal generated by the coefficients a_t at ϵ_t ($t \in c_i \cap c_j$) for all these homomorphisms. If the center of $\Delta\epsilon_t$ is the ring of integers R_t in K_t, then this ideal is a certain power $\pi_t^{m_{ij}^{(t)}} R_t$ of the maximal ideal in R_t. Then the order $\oplus_{t=1}^s \Delta\epsilon_t$ is called a graduated order (see Definition 2.4) and determined by the matrices $M^{(t)} := (m_{ij}^{(t)})_{i,j | t \in c_i \cap c_j}$ up to Morita equivalence.

Definition 2.4 *An R-order Γ in the simple K-algebra $D^{n\times n}$ is called* graduated, *if there are $h, n_1, \ldots, n_h \in \mathbb{N}$ $(n = n_1 + \ldots + n_h)$ and $M = (m_{ij}) \in \mathbb{Z}_{\geq 0}^{h\times h}$ such that Γ is conjugated to*

$$\Lambda(\Omega, n_1, \ldots, n_h, M) :=$$
$$\{X = (x_{ij})_{i,j=1,\ldots,h} \in D^{n\times n} \mid x_{ij} \in (\mathcal{P}^{m_{ij}})^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq h\}.$$

where Ω the maximal R-order in the division algebra D with maximal ideal $\mathcal{P}$. If $\Gamma/J(\Gamma) \cong \oplus_{i=1}^h (\Omega/\mathcal{P})^{n_i\times n_i}$ then $m_{ii} = 0$, $m_{ij} + m_{ji} > 0$ and $m_{ij} + m_{jl} \geq m_{il}$ for all $1 \leq l, i \neq j \leq h$. In this case M is called an exponent matrix *of Γ.*
An R-order Γ in a semisimple K-algebra is called graduated, *if Γ is the direct sum of graduated orders in the simple components.*

The Γ-lattices in the simple $K \otimes_R \Gamma$-modules can be easily described by means of its exponent matrix M, see [Ple 83, Remark (II.4)]. One can always conjugate Γ such that $m_{1j} = 0$ for all $1 \leq j \leq h$.

From [Ple 83, Proposition (IV.1)] one finds

Lemma 2.5 *Let $M \in \mathbb{Z}^{h\times h}$ be an exponent matrix of a graduated R-order Γ in the simple K-algebra A. Assume that there is an involution $^\circ : \Gamma \to \Gamma$ (i.e. an R-order antiautomorphism of order ≤ 2) fixing the central primitive idempotents of $\Gamma/J(\Gamma)$. Then*

$$m_{ij} + m_{jl} + m_{li} = m_{ji} + m_{il} + m_{lj} \text{ for all } 1 \leq i, j, l \leq h.$$

If M is normalized such that the first row of M consists of 0 only then

$$m_{ij} + m_{j1} = m_{ji} + m_{i1} \text{ for all } 1 \leq i, j \leq h.$$

This lemma will be applied to epimorphic images of group rings RG. The natural involution $^\circ : RG \to RG$ is the R-linear map defined by $g \mapsto g^{-1}$ for all $g \in G$. If ϵ is a central primitive idempotent of KG with $\epsilon^\circ = \epsilon$ such that $\Gamma := RG\epsilon$ is a graduated order and all p-modular constituents of the character belonging to ϵ are self dual, then Γ satisfies the conditions of Lemma 2.5.

Remark 2.6 *Let $M = (m_{ij})_{1\leq i,j\leq h}$ be an exponent matrix of a graduated order as in Lemma 2.5 such that $m_{1,j} = 0$ for all $1 \leq j \leq h$. Let*

$$m(i,j) := m_{ij} + m_{ji} \text{ for all } 1 \leq i, j \leq h.$$

Then M is already determined by the $m(i,j)$ since $2m_{ij} = m(i,j) - m(j,1) + m(i,1)$. The $m(i,j)$ only depend on Γ and are called structure constants *of Γ.*

3 The group ring $\mathbb{Z}_p[\zeta_{p^f-1}]SL_2(p^f)$.

Throughout this chapter let p be an odd prime, $f \in \mathbb{N}$, and $R := \mathbb{Z}_p[\zeta_{p^f-1}]$ the ring of integers in the unramified extension K of $\mathbb{Q}_p$ of degree f with residue class field $ki := R/pR \cong \mathbb{F}_{p^f}$. We want to describe the group ring RG where $G = SL_2(k)$ is the group of 2×2 matrices over k with determinant 1. RG has three blocks, one of which is of defect 0 and the other two have the Sylow p-subgroups $\cong C_p^f$ of G as their defect groups.

The simple kG-modules can be obtained from Steinberg's tensor product theorem: Let F be the Frobenius automorphism $x \mapsto x^p$ of $k := \mathbb{F}_{p^s}$. For a vector space V let $V^{(i)}$ be the vector space obtained by twisting V i-times with F, hence $V^{(i)} = V$ with scalar multiplication $x \cdot v := x^{p^i} v$ $(x \in k, v \in V)$. The group $SL_2(k)$ acts as group of automorphisms on the algebra $k[X,Y]$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends X and Y to $dX - bY$, $-cX + aY$, respectively. For $0 \leq \lambda < p$ let $M_\lambda \subseteq k[X,Y]$ be the subspace of homogenous polynomials of degree λ and for $\lambda = (\lambda_0, \ldots, \lambda_{f-1})$ with $0 \leq \lambda_i < p$ let

$$M_\lambda := M_{\lambda_0} \otimes_k M_{\lambda_1}^{(1)} \otimes_k \ldots \otimes_k M_{\lambda_{f-1}}^{(f-1)}.$$

The modules $\bar{k}M_\lambda$ form a system of representatives of the isomorphism classes of simple $SL_2(p^f)$-modules over the algebraic closure $\bar{k}$ of k.

3.1 The group algebra kG.

In this section the description in [Kos 98] of the basic algebra belonging to the group algebra of G in characteristic p is repeated.

Let $P := \{0, \ldots, p-1\}^f - \{(p-1, \ldots, p-1)\}$. The elements of P are in bijection to the simple kG-modules in one of the two blocks of positive defect. To be consistent with the notation in [HSW 82] for the simple kG-modules the elements $\lambda \in P$ are indexed with the elements of $N := \{0, \ldots, f-1\}$. Then $\lambda = (\lambda_0, \ldots, \lambda_{f-1}) \in P$ corresponds to the simple kG-module M_λ.

The elements of N are considered modulo f.

[AJL 83] calculates the Ext-groups between the simple kG-modules: For $i \in N$ and $h = \pm 1$ let

$$P(i,h) := \{\lambda \in P \mid 0 \leq \lambda_{i-1} \leq p-2 \text{ und } 0 \leq \lambda_i + h \leq p-1\}$$

and $f(i,h) : P(i,h) \to P; \lambda \mapsto \gamma$, where $\gamma_j = \lambda_j$ for all $i, i-1 \neq j \in N$, $\gamma_i = \lambda_i + h$ and $\gamma_{i-1} = p - 2 - \lambda_{i-1}$.

Theorem 3.1 *([AJL 83, Corollary 4.5]) Let $\lambda, \gamma \in P$. Then* $\mathrm{Ext}^1_{kG}(M_\lambda, M_\gamma) \neq 0$ *if and only if $\gamma = f(i,h)(\lambda)$ for some $i \in N$, $h \in \pm 1$.*
If $f \geq 3$, then the non-trivial Ext*-groups are one dimensional.*

For $\lambda \in P$ let $\overline{P_\lambda}$ be the projective kG-module with head M_λ and

$$\overline{\Lambda} := \mathrm{End}_{kG}(\oplus_{\lambda \in P} \overline{P_\lambda})$$

the basic algebra Morita equivalent to the sum of the two blocks of defect f of kG. [Kos 98] calculates defining relations between the generators of $\overline{\Lambda}$ given by the Ext-quiver of kG:

For $\lambda \in P$ let

$$S(\lambda) := \{i \in N \mid \lambda_i = p-1\}.$$

Theorem 3.2 *([Kos 98, Theorem 2.2]) Let $f \geq 3$ and Q be the quiver whose vertices are indexed by the elements of P with arrows*

$$\alpha_{i,h,\lambda} : f(i,h)(\lambda) \to \lambda \textit{ for all } i \in N, h = \pm 1, \lambda \in P(i,h).$$

Let kQ be the path algebra of Q. Then for $\lambda, \gamma \in P$ the paths in Q from λ to γ can be written as

$$(\lambda|(i_1,h_1),\ldots,(i_l,h_l)|\gamma) = \alpha_{i_1,h_1,f(i_1,-h_1)(\lambda)}\alpha_{i_2,h_2,I}\ldots\alpha_{i_l,h_l,\gamma}.$$

$(\lambda||\lambda)$ is the idempotent in kQ, corresponding to the vertex λ. Then let X be the ideal of kQ, generated by

$$(f(j,r)(f(i,h)(\lambda))|(j,r),(i,h)|\lambda) - (f(j,r)(f(i,h)(\lambda))|(i,h)(j,r)|\lambda)$$
$$(i,j \in N; j \neq i-1,i,i+1; h,r = \pm 1; \lambda \in P(i,h) \cap P(j,r))$$

$$(\lambda|(i,h),(i,h)|f(i,h)(f(i,h)(\lambda))) \quad (i \in N; h = \pm 1; \lambda, f(i,h)(\lambda) \in P(i,h))$$

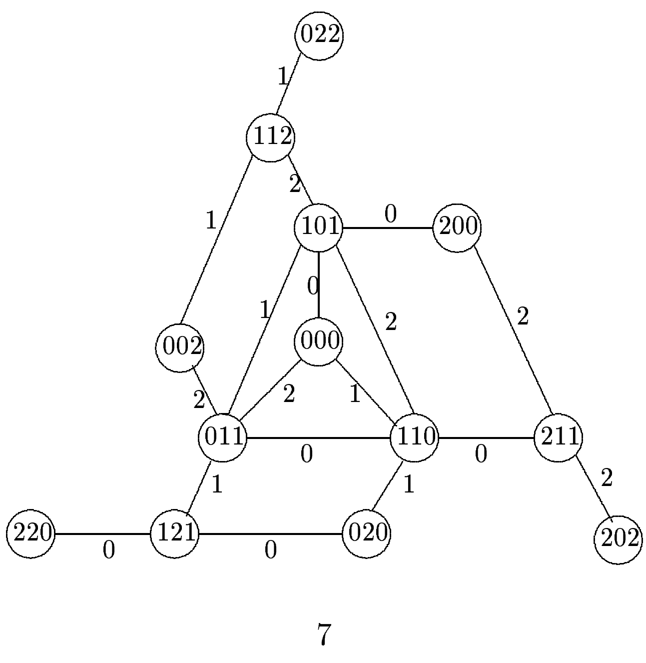
$$(\lambda|(i,1),(i,-1)|\lambda) \quad (i \in N; \lambda \in P(i,-1), i \in S(\lambda))$$

$$(\lambda|(i,1),(i,-1)|\lambda) - (\lambda|(i,-1),(i,1)|\lambda) \quad (i \in N; \lambda \in P(i,-1) \cap P(i,1))$$

$$(f(i+1,r)(f(i,h)(\lambda))|(i+1,r),(i,h)|\lambda) - (f(j,r)(f(i,h)(\lambda)|(i,-h)(i+1,r)|\lambda)$$
$$(i \in N; h,r = \pm 1; \lambda \in P(i,h); \lambda, f(i,h)(\lambda) \in P(i+1,r))$$

Then $\overline{\Lambda} \cong kQ/X$.

Example. The Ext-quiver of the principal block of $kSL_2(3^3)$. The arrows $\alpha_{i,h,\lambda}$ and $\alpha_{i,-h,f(i,h)(\lambda)}$ are indicated by a single edge labeled with i.



Definition 3.3 *For $i \in N$, $h = \pm 1$ and $\lambda \in P(i,h)$ the number i is called the* direction *of $\alpha_{i,h,\lambda} \in Q$.*

Let

$$\Psi : kQ \to \overline{\Lambda}$$

be the epimorphism of [Kos 98] with $Ker(\Psi) = X$. The images under Ψ of the paths in Q from the vertex λ to the vertex γ generate the vector space $\mathrm{Hom}_{kG}(\overline{P_\lambda}, \overline{P_\gamma})$ over k. [Kos 98, Proposition 9.2] constructs a k-basis for $\mathrm{Hom}_{kG}(\overline{P_\lambda}, \overline{P_\gamma})$. To describe this basis we need to resume the notation of [Kos 98]:

Definition 3.4 *(a) Let $\lambda, \gamma \in P$ with $S(\lambda) = S(\gamma)$ and $\emptyset \neq T = \{i_1, \ldots, i_l\} \subseteq N$ such that there is a path $(\lambda|(i_1,h_1), \ldots, (i_l,h_l)|\gamma) \in Q$ from λ to γ for suitable $h_1, \ldots, h_l \in \{\pm 1\}$. Then*

$$\langle \lambda \sim \gamma | T \rangle := \Psi((\lambda|(i_1,h_1), \ldots, (i_l,h_l)|\gamma))$$

denotes the image of such a path. $\langle \lambda \sim \lambda | \emptyset \rangle := \Psi((\lambda||\lambda))$.

(b) If $\lambda, \gamma \in P$ such that there is a path $\omega = (\lambda| \ldots |\gamma) \in Q$ of length $l(\omega) = |S(\lambda) - S(\gamma)|$ then $\langle \lambda \supset \gamma \rangle$ denotes the image of such a path under Ψ.

(c) If $\lambda, \gamma \in P$ such that there is a path $\omega = (\lambda| \ldots |\gamma) \in Q$ of length $l(\omega) = |S(\gamma) - S(\lambda)|$ then $\langle \lambda \subset \gamma \rangle$ denotes the image of such a path under Ψ.

(d) Assume that for $\lambda, \gamma \in P$ there exists some $x \in P$ with $S(x) = S(\lambda) \cap S(\gamma)$ such that $\langle \lambda \supset x \rangle$ exists. Then x is unique and denoted by

$$\lambda \ominus \gamma := x.$$

(e) Let $\lambda \in P$, $i \in N - S(\lambda)$. If $i-1 \in S(\lambda)$, then let $j-1 \notin S(\lambda)$ be such that $\{j, j+1, \ldots, i-1\} \subseteq S(\lambda)$. Then

$$[\lambda, i] := \Psi((\lambda|(j,1), \ldots, (i-1,1), (i,-1), (i,1), (i-1,-1), \ldots, (j,-1)|\lambda)).$$

If $i-1 \notin S(\lambda)$, then put

$$[\lambda, i] := \Psi((\lambda|(i,-1),(i,1)|\lambda)).$$

For $T \subset N - S(\lambda)$, the product $\prod_{i \in T}[\lambda, i]$ does not depend on the ordering of the factors and is denoted by $[\lambda, T]$.

Definition 3.5 *Let $\lambda, \gamma \in P$, $T_1 \subset N - (S(\lambda) \cup S(\gamma))$ be such that*

(1) $\lambda \ominus \gamma$ and $\gamma \ominus \lambda$ exist.

(2) $\langle (\lambda \ominus \gamma) \sim (\gamma \ominus \lambda), T_1 \rangle$ exists.

Then we define for $T_2 \subset N - (S(\lambda) \cup S(\gamma) \cup T_1)$

$$\langle\langle \lambda, T_1, T_2, \gamma \rangle\rangle := \langle \lambda \supset (\lambda \ominus \gamma) \rangle \langle (\lambda \ominus \gamma) \sim (\gamma \ominus \lambda), T_1 \rangle [\gamma \ominus \lambda, T_2] \langle (\gamma \ominus \lambda) \subset \gamma \rangle.$$

With this notation one has

Proposition 3.6 *([Kos 98, Proposition 9.2]) Let $\lambda, \gamma \in P$. The elements $\langle\langle\lambda, T_1, T_2, \gamma\rangle\rangle$, for which λ, γ, T_1 satisfy conditions (1) and (2) of Definition 3.5 and $T_2 \subset N - (S(\lambda) \cup S(\gamma) \cup T_1)$, form a basis of* $\mathrm{Hom}_{kG}(\overline{P_\lambda}, \overline{P_\gamma})$.

Remark 3.7 *Let λ, γ be as in Definition 3.5. The directions of the arrows in the path corresponding to $\langle\lambda \supset (\lambda \ominus \gamma)\rangle$ respectively $\langle(\gamma \ominus \lambda) \subset \gamma\rangle$ are precisely the elements of $S(\lambda) - S(\gamma)$ respectively $S(\gamma) - S(\lambda)$. Therefore the path belonging to $\langle\langle\lambda, T_1, T_2, \gamma\rangle\rangle$ is of the shape $\langle\lambda \sim \gamma, T\rangle$ for some subset $T \subset N$.*

Lemma 3.8 *Let $\lambda \in P$ and $\emptyset \neq T \subseteq N$ such that $\langle\lambda \sim \lambda, T\rangle$ exists. Then $T = N$ and there are signs $\nu_0, \ldots, \nu_{f-1} \in \{\pm 1\}$, such that $\lambda_i = \frac{p-2-\nu_i}{2}$ for $i = 0, \ldots, f-1$.*

Proof. The operators $f(i,h)$ and $f(i',h')$ $(i, i' \in N, h, h' = \pm 1)$ commute if one considers the entries of λ modulo 2. $f(i,h)$ changes precisely the parity of λ_i and λ_{i-1}. Hence $i \in T$ implies $i-1 \in T$ and therefore $T = N$. Moreover for all $i = 0, \ldots, f-1$ there is $\nu_i \in \{\pm 1\}$ such that

$$p - 2 - \lambda_i + \nu_i = \lambda_i \Leftrightarrow \lambda_i = \frac{p - 2 - \nu_i}{2}. \qquad \square$$

Definition 3.9 *Let $\tilde{P} := \{\lambda \in P \mid \lambda_i = \frac{p-2-\nu_i}{2}$ for $\nu_i = \pm 1$ for all $i \in N\}$. For $\lambda \in \tilde{P}$, the element $\langle\lambda \sim \lambda, N\rangle \in \overline{\Lambda}$ is called a* circle *through λ.*

Corollary 3.10 *(a) For $\lambda \in P - \tilde{P}$ the $2^{|N-S(a)|}$ endomorphisms $[\lambda, T]$ with $T \subseteq N - S(\lambda)$ form a k-basis of* $\mathrm{End}_{kG}(\overline{P_\lambda})$.

(b) If $\lambda \in \tilde{P}$ then there is a further basis element $\langle\lambda \sim \lambda, N\rangle$.

Definition 3.11 *Let $\lambda, \gamma \in P$ be such that* $\mathrm{Hom}_{kG}(\overline{P_\lambda}, \overline{P_\gamma}) \neq 0$. *If λ or γ does not lie in $\tilde{P}$ then there is a unique $T_1 \subset N - (S(\lambda) \cup S(\gamma))$, such that $\langle(\lambda \ominus \gamma) \sim (\gamma \ominus \lambda), T_1\rangle \in Q$. Then let*

$$d(\lambda, \gamma) := l(\langle\lambda \supset (\lambda \ominus \gamma)\rangle\langle(\lambda \ominus \gamma) \sim (\gamma \ominus \lambda), T_1\rangle\langle(\gamma \ominus \lambda) \subset \gamma\rangle)$$

denote the length of a shortest path from λ to γ in Q without repeated directions.

3.2 Decomposition numbers

To lift this description of kG to characteristic zero, we need some properties of the decomposition numbers of RG. In this section no central primitive idempotents are needed and the letter ϵ usually stands for ±1. A useful description of the decomposition numbers and the Cartan invariants of the group $SL_2(p^f)$ in characteristic p can be found in [HSW 82] (cf. also [Bur 76]).

In the notation of [HSW 82], the characters of the absolutely irreducible $\mathbb{C}SL_2(p^f)$-modules are 1, δ, δ', η, η', η_i, δ_j and St, where $1 \leq i \leq (p^f - 3)/2$ and $1 \leq j \leq (p^f - 1)/2$. The character degrees are $\eta(1) = \eta'(1) = (p^f + 1)/2$,

$\delta(1) = \delta'(1) = (p^f - 1)/2$, $\delta_i(1) = p^f - 1$, $\eta_i(1) = p^f + 1$ and $St(1) = p^f$. The action of F on the ordinary characters η_i and δ_j is given by multiplying the indices with p. $\delta_j^F = \delta_{pj}$ and $\eta_i^F = \eta_{pi}$ where the indices are considered modulo $p^f + 1$ respectively modulo $p^f - 1$ and are identified with their negatives. δ_j and η_i belong to the principal block of RG, if and only if j and i are even.

To describe the decomposition numbers define for $\lambda = (\lambda_0, \ldots, \lambda_{f-1})$ the set

$$W(\lambda) = \{0 \le \nu \le p^f - 1 \mid \nu = \sum_{i=0}^{f-1} \epsilon_i \tilde{\lambda}_i p^i \text{ for some } \epsilon_0, \ldots, \epsilon_{f-1} = \pm 1\}$$

where $\tilde{}$ is the bijection of the set $\{0, \ldots, p-1\}$ defined by $\tilde{x} = p - 1 - x$. Moreover let $V^\epsilon(\lambda) = W(\lambda) \cup (p^f - \epsilon - W(\lambda))$ for $\epsilon = \pm 1$.

For $j = 1, \ldots, (p^f - 3)/2$ let $d_{\lambda,j}^{(1)} := d_{\lambda,\eta_j}$ be the multiplicity of the Brauer character belonging to M_λ in the restriction of η_j to the p-regular classes of $SL_2(p^s)$. Analogously let $d_{\lambda,j}^{(-1)} := d_{\lambda,\delta_j}$ $(j = 1, \ldots, (p^f - 1)/2)$, $d_{\lambda,(p^f-1)/2}^{(1)} := d_{\lambda,\eta} = d_{\lambda,\eta'}$, and $d_{\lambda,(p^f+1)/2}^{(-1)} := d_{\lambda,\delta} = d_{\lambda,\delta'}$. Then one gets

Theorem 3.12 *([HSW 82, Theorem 2.7])*

a) $d_{\lambda,j}^{(\epsilon)} = 1$ *if* $j \in V^\epsilon(\lambda)$ *and* 0 *otherwise.*

b) $d_{\lambda,St} = 1$ *if* $\lambda = p^f - 1$ *and* 0 *otherwise.*
$d_{\lambda,1} = 1$ *if* $\lambda = 0$ *and* 0 *otherwise.*

The elements of $V^\epsilon(\lambda)$ are treated like the indices of η (if $\epsilon = 1$) respectively δ (if $\epsilon = -1$), they are considered modulo $p^f - \epsilon$ and identified with their negatives.

The main reason why the case $p > 2$ is much more complicated than the case $p = 2$ is the existence of the exceptional characters η, η', δ and δ'. If L is a projective indecomposable RG-lattice such that one of these four characters occurs as a constituent of the character of $K \otimes_R L$, then there is an additional generator, called a circle (cf. Definition 3.9) of the endomorphism ring $\mathrm{End}_{kG}(L/pL)$. By Brauer reciprocity, the set of such L is in bijection to the set of modular constituents of one of the exceptional characters. [HSW 82, Corollary 2.4] determines these modular characters:

Lemma 3.13 *η and η' respectively δ and δ' have the same p-modular constituents. The set of all these constituents is*

$$M := \{(\lambda_0, \ldots, \lambda_{f-1}) \mid \lambda_i = \frac{1}{2}(p - 2 + \epsilon_i) \text{ for } \epsilon_i \in \{\pm 1\}\}.$$

[HSW 82] even says that if $\lambda \in M$, then $\frac{p^f-\epsilon}{2}$ lies in $W(\lambda)$ for $\epsilon = \pm 1$ with $\lambda \equiv (p^f - \epsilon)/2 \pmod 2$. Therefore $\frac{p^f-\epsilon}{2} \in V^{-\epsilon}(\lambda)$ for all $\lambda \in M$ with $\lambda \equiv (p^f - \epsilon)/2$ (mod 2). Since the Frobenius automorphism F preserves the set of those modular characters λ one gets the following

Lemma 3.14 *Let $\lambda \in M$ be a p-modular constituent of δ and $\epsilon := -1$ or a p-modular constituent of η and $\epsilon := 1$. Then*

$$\chi_m^{(-\epsilon)} := \frac{1}{2}(p^f + \epsilon) - p^m \in V^{-\epsilon}(\lambda) \text{ for all } 0 \leq m \leq f-1.$$

Proof. The argumentation above shows that $\frac{1}{2}(p^f - \epsilon)p^m \in V^{-\epsilon}(\lambda)$ for $m = 0, \ldots, f-1$. One has $\frac{p^m+1}{2}(p^f + \epsilon) - \frac{p^{f+m} - \epsilon p^m}{2} = \frac{1}{2}(p^f + \epsilon) + \epsilon p^m$ and $(p^f + \epsilon) - (\frac{1}{2}(p^f + \epsilon) + \epsilon p^m) = \frac{1}{2}(p^f + \epsilon) - \epsilon p^m$. Therefore $\chi_m^{(-\epsilon)} \in V^{-\epsilon}(\lambda)$. □

Now the definition of a circle (Definition 3.9) is repeated in the language of Brauer characters.

Definition 3.15 *Let $\mathrm{Kr} := (\lambda^{(1)}, \ldots, \lambda^{(f)})$ be a sequence of p-Brauer characters of G such that*

$$\lambda^{(j)} = (\lambda_0^{(j)}, \ldots, \lambda_{f-1}^{(j)}) \text{ with } \lambda_i^{(j)} = \frac{1}{2}(p - 2 + \epsilon_i^{(j)})p^i \text{ for some } \epsilon_i^{(j)} = \pm 1$$

$(0 \leq i \leq f-1)$ $j = 1, \ldots, f$. Kr is called a circle, *if there is a bijection $\sigma : \{1, \ldots, f\} \to \{0, \ldots, f-1\}$ such that*

$$\epsilon_i^{(j)} = \begin{cases} \epsilon_i^{(j-1)} & i \neq \sigma(j), \sigma(j) - 1 \\ -\epsilon_i^{(j-1)} & i = \sigma(j) \text{ or } \sigma(j) - 1 \end{cases} \quad \text{for } j = 2, \ldots, f$$

where the indices $i \in \{0, \ldots, f-1\}$ are considered modulo f.

Lemma 3.16 *Let $\mathrm{Kr} := (\lambda^{(1)}, \ldots, \lambda^{(f)})$ be a circle and $\epsilon = \pm 1$. Assume that there is some $a \in \mathcal{X} := \bigcap_{j=1}^{f} V^{\epsilon}(\lambda^{(j)})$ with $0 \leq a < (p^f - \epsilon)/2$. Then there exists $0 \leq m \leq f-1$ such that $a = \chi_m^{(\epsilon)}$.*

Proof. By definition there are $\nu_0^{(j)}, \ldots, \nu_{f-1}^{(j)} \in \{\pm 1\}$ such that $\lambda_i^{(j)} = \frac{1}{2}(p-2+\nu_i^{(j)})$ $(j = 1, \ldots, f,\ i = 0, \ldots, f-1)$. Then $\tilde{\lambda}_i^{(j)} = \frac{p - \nu_i^{(j)}}{2}$. Now let $a \in \mathcal{X}$ with $0 \leq a \leq \frac{p^f - \epsilon}{2}$. Then there are $\epsilon_i^{(j)} \in \{\pm 1\}$ $(1 \leq j \leq f, 0 \leq i \leq f-1)$ with $\epsilon_{f-1}^{(j)} = 1$ for all $1 \leq j \leq f$, such that for all $1 \leq j \leq f$ either

$$a = \sum_{i=0}^{f-1} \epsilon_i^{(j)} \frac{p - \nu_i^{(j)}}{2} p^i = \frac{p^f - \epsilon_0^{(j)} \nu_0^{(j)}}{2} + \sum_{i=1}^{f-1} \frac{\epsilon_{i-1}^{(j)} - \epsilon_i^{(j)} \nu_i^{(j)}}{2} p^i$$

or

$$a = (p^f - \epsilon) - \sum_{i=0}^{f-1} \epsilon_i^{(j)} \frac{p - \nu_i^{(j)}}{2} p^i = \frac{p^f - 2\epsilon + \epsilon_0^{(j)} \nu_0^{(j)}}{2} - \sum_{i=1}^{f-1} \frac{\epsilon_{i-1}^{(j)} - \epsilon_i^{(j)} \nu_i^{(j)}}{2} p^i.$$

Let K_0 denote the set of $1 \leq j \leq f$ such that the first equation holds and K_1 the set of the other $1 \leq j \leq f$.

- We now show that if $K_1 = \emptyset$, then $a = \frac{p^f \pm \epsilon}{2}$.

Assume that $K_1 = \emptyset$. Then there is some $\nu \in \{\pm 1\}$ with $2a \equiv -\nu \pmod p$ and $\epsilon_0^{(j)}\nu_0^{(j)} = \nu$ for all $1 \le j \le f$. Therefore $a - \frac{p^f - \nu}{2} = \sum_{i=1}^{f-1} a_i p^i$ where

$$a_i := \frac{\epsilon_{i-1}^{(j)} - \epsilon_i^{(j)}\nu_i^{(j)}}{2} \in \{-1, 0, 1\} \text{ for all } 1 \le j \le f$$

and it remains to show that $a_i = 0$ for all $1 \le i \le f-1$. Seeking a contradiction we let $m \in \{1, \ldots, f-1\}$ be maximal such that $a_m \ne 0$. Then for $n = m+1, \ldots, f-1$ and all $1 \le j \le f$

$$\epsilon_{n-1}^{(j)} = \epsilon_n^{(j)}\nu_n^{(j)} = \prod_{i=n}^{f-1} \nu_i^{(j)}$$

and for $n = m$ one has

$$0 \ne a_m = \epsilon_{m-1}^{(j)} = -\epsilon_m^{(j)}\nu_m^{(j)} = -\prod_{i=m}^{f-1} \nu_i^{(j)} \text{ for all } 1 \le j \le f.$$

If $j := \sigma^{-1}(m)$ then $\sigma(j) - 1 = m - 1 \in \{0, \ldots, f-1\}$ and the condition that Kr is a circle yields $\prod_{i=m}^{f-1} \nu_i^{(j)} = -\prod_{i=m}^{f-1} \nu_i^{(j-1)}$, contradicting the equality for a_m above. Therefore all a_i are zero and a is as stated.

- Analogously one proves: If $K_0 = \emptyset$, then $a = \frac{p^f - \epsilon}{2}$ or $a = \frac{p^f - 3\epsilon}{2} = (p^f - \epsilon) - \frac{p^f + \epsilon}{2}$.
- Assume now that both sets $K_0, K_1 \ne \emptyset$. Then $a \equiv (p^f - \epsilon_0^{(j)}\nu_0^{(j)})/2 \pmod p$ and $a \equiv (p^f - 2\epsilon + \epsilon_0^{(j)}\nu_0^{(j)})/2 \pmod p$. Since $p > 2$ this implies that $\epsilon_0^{(j)}\nu_0^{(j)} = \epsilon$ for all $1 \le j \le f$. Let $j \in K_0$ and a_i be as above ($1 \le i \le f-1$). The uniqueness of the p-adic expansion of a gives

$$\frac{\epsilon_{i-1}^{(j)} - \epsilon_i^{(j)}\nu_i^{(j)}}{2} = a_i \text{ for all } j \in K_0$$

and

$$\frac{\epsilon_{i-1}^{(j)} - \epsilon_i^{(j)}\nu_i^{(j)}}{2} = -a_i \text{ for all } j \in K_1.$$

Again let m be maximal such that $a_m \ne 0$. Assume that there is another $1 \le l < m$ such that $a_l \ne 0$ and choose l maximal with this condition. Then for all $1 \le j \le f$ one has

$$\epsilon_{n-1}^{(j)} = \epsilon_n^{(j)}\nu_n^{(j)} = \prod_{i=n}^{f-1} \nu_i^{(j)} \text{ for all } n = m+1, \ldots, f-1$$

and

$$\epsilon_{n-1}^{(j)} = \prod_{i=n}^{m-1} \nu_i^{(j)} = -\prod_{i=n}^{f-1} \nu_i^{(j)} \text{ for all } n = l+1, \ldots, m-1.$$

For m and l one gets for all $j \in K_h,\ h = 0, 1$

$$0 \ne (-1)^h a_m = \epsilon_{m-1}^{(j)} = -\epsilon_m^{(j)}\nu_m^{(j)} = -\prod_{i=m}^{f-1} \nu_i^{(j)}$$

and

$$0 \neq (-1)^h a_l = \epsilon_{l-1}^{(j)} = -\epsilon_l^{(j)} \nu_l^{(j)} = \prod_{i=l}^{f-1} \nu_i^{(j)}.$$

In particular the quotient

$$a_l/a_m = -\prod_{i=l}^{m-1} \nu_i^{(j)}$$

is constant on $K_0 \cup K_1 = \{1, \ldots, f\}$. But the condition that Kr is a circle implies that $\prod_{i=l}^{m-1} \nu_i^{(j)} = -\prod_{i=l}^{m-1} \nu_i^{(j-1)}$ for $j := \sigma^{-1}(m)$. This is a contradiction, hence there is no such m with $1 \leq l < m$ and $a_l \neq 0$. Since $p^f - \epsilon - (\frac{p^f-\epsilon}{2} + p^m) = \frac{p^f-\epsilon}{2} - p^m$, $a = \chi_m^{(\epsilon)}$ as claimed in the Lemma. □

The next lemma determines the p-modular constituents of the ordinary character associated with $\chi_m^{(\epsilon)}$.

Lemma 3.17 *Let $\epsilon = \pm 1$ and $0 \leq m \leq f-1$. Let $\lambda = (\lambda_0, \ldots, \lambda_{f-1})$ denote a p-Brauer character of G with $\sum_{i=0}^{f-1} \lambda_i \equiv \chi_m^{(\epsilon)} \pmod 2$. Then $\chi_m^{(\epsilon)} \in V^\epsilon(\lambda)$ if and only if there are signs $\epsilon_i \in \{\pm 1\}$ such that*

$$\lambda_i = \frac{p-2+\epsilon_i}{2} \text{ for all } 0 \leq i \neq m \leq f-1$$

and

$$\lambda_m \in \{\frac{p-5}{2}, \frac{p-3}{2}, \frac{p-1}{2}, \frac{p+1}{2}\}.$$

Proof. Let λ be as in the Lemma. If $\lambda_m = \frac{p-3}{2}$ or $\lambda_m = \frac{p-1}{2}$, then $\frac{p^f-\epsilon}{2} + p^i \in V^\epsilon(\lambda)$ for all $0 \leq i \leq f-1$ by Lemma 3.14. Otherwise there is $\nu = \pm 1$, such that $\lambda_m + \nu = \frac{p-3}{2}$ or $\frac{p-1}{2}$. For $\lambda' = (\lambda_0, \ldots, \lambda_{m-1}, \lambda_m + \nu, \lambda_{m+1}, \ldots, \lambda_{f-1})$ one has by [HSW 82, Corollary 2.4] that $\frac{p^f-\epsilon}{2} \in W(\lambda')$. Therefore $\frac{p^f-\epsilon}{2} \pm p^m \in V^\epsilon(\lambda)$ and all sets $V^\epsilon(\lambda)$ in the Lemma contain $\chi_m^{(\epsilon)}$.

The sum of the degrees of these Brauer characters if $p^f + \epsilon$, which is the degree of the character belonging to $\chi_m^{(\epsilon)}$. □

The other important structure of the Ext-quiver of kG is given by the m-strings. Let $m \in N$ be fixed and $\lambda := \lambda^{(p-1)} \in P(m, -1)$ be such that $\lambda_m = p-1$. For $j = p-2, \ldots, 0$ define $\lambda^{(j)} := f(m,-1)(\lambda^{(j+1)})$. Then there is a sequence of relations $(\lambda|(m,1), (m,-1)|\lambda)$, $(\lambda^{(j)}|(m,1), (m,-1)|\lambda^{(j)}) - (\lambda^{(j)}|(m,-1), (m,1)|\lambda^{(j)})$ for $j = p-2, \ldots, 1$ in X.

Definition 3.18 *Let $\lambda \in P$, $m \in N$ with $\lambda_m = 0$, $\lambda_{m-1} < p-1$. Define $\lambda^{(0)} = \lambda$ and $\lambda^{(j+1)} = f(m,1)(\lambda^{(j)})$ for $j = 0, \ldots, p-2$. The sequence*

$$ST(\lambda, m) := (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p-1)})$$

*is called the m-*string with origin *λ or simply an m-*string.

Lemma 3.19 *Let* $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(p-1)})$ *be an* m*-string. Then* $W(\lambda^{(p-1)}) \subset W(\lambda^{(p-2)})$ *and for* $1 \leq j \leq p-2$ *one has* $W(\lambda^{(j)}) \subset W(\lambda^{(j-1)}) \cup W(\lambda^{(j+1)})$*. Moreover* $W(\lambda^{(j-1)}) - W(\lambda^{(j)}) \neq \emptyset$.

Proof. This follows from the equality

$$\sum_{i=0}^{f-1} \epsilon_i \tilde{\lambda}_i p^i = \sum_{i=0, i\neq j, j-1}^{f-1} \epsilon_i \tilde{\lambda}_i p^i - \epsilon_{j-1}(p - \tilde{\lambda}_{j-1}) p^{j-1} + \epsilon_j (\tilde{\lambda}_j - \epsilon_j \epsilon_{j-1}) p^j.$$

If $\lambda_j = p-1$, then $\tilde{\lambda}_j = 0$ and one may choose $\epsilon_j = -\epsilon_{j-1}$. □

We now consider the intersection $V^\epsilon(\lambda^{(j)}) \cap V^\epsilon(\lambda^{(j-2)})$ for m-strings.

Lemma 3.20 *Let* $l := \frac{p-1}{2}$ *and* $(\lambda^{(0)}, \dots, \lambda^{(p-1)})$ *be an* m*-string such that there is* $0 \leq i \leq p-3$ *and* $\nu = \pm 1$ *with* $V^\nu(\lambda^{(i)}) \cap V^\nu(\lambda^{(i+2)}) \neq \emptyset$*. Then* $i = l-2$ *or* $i = l-1$ *and*

$$\{\chi_m^{(\nu)}\} = V^\nu(\lambda^{(l-2)}) \cap V^\nu(\lambda^{(l-1)}) \cap V^\nu(\lambda^{(l)}) \cap V^\nu(\lambda^{(l+1)})$$

where one has to omit $\lambda^{(l-2)}$ *if* $p = 3$.

Proof. The lemma is only proved for $m = 0$. The other cases follow by applying the Frobenius automorphism F. Let $\lambda^{(i)} = (\lambda_0, \dots, \lambda_{f-1})$. Then $\lambda^{(i+2)} = (\lambda'_0, \dots, \lambda'_{f-1})$ with $\lambda'_i = \lambda_i$ for $i = 1, \dots, f-1$ and $\lambda'_0 := \lambda_0 + 2 \leq p-1$. Since $c_{\lambda,\lambda'} > 0$, there is $\epsilon = \pm 1$ such that

$$(W(\lambda) \cap W(\lambda')) \cup (W(\lambda) \cap (p^f - \epsilon - W(\lambda'))) \neq \emptyset.$$

Assume that there is some $\alpha \in W(\lambda) \cap W(\lambda')$. Then there are signs ϵ_i, ϵ'_i $(0 \leq i \leq f-1)$ with $\sum_{i=0}^{f-1} \epsilon_i \tilde{\lambda}_i p^i = \sum_{i=0}^{f-1} \epsilon'_i \tilde{\lambda}'_i p^i$. Hence

$$\epsilon_0 \tilde{\lambda}_0 - \epsilon'_0(\tilde{\lambda}_0 - 2) = p(\sum_{i=1}^{f-1} (\epsilon_i \tilde{\lambda}_i - \epsilon'_i \tilde{\lambda}_i) p^{i-1}).$$

The left hand side of the equation is an even number between $-2(p-1)$ and $2(p-1)$. The right hand side is divisible by p. Therefore $\epsilon_0 \tilde{\lambda}_0 - \epsilon'_0(\tilde{\lambda}_0 - 2) = 0$. This has no solution $\tilde{\lambda}_0 \geq 2$. Therefore there is some $\epsilon = \pm 1$ and $\alpha \in W(\lambda) \cap (p^f - \epsilon) - W(\lambda')$ with $0 \leq \alpha \leq p^f - \epsilon$. Then α can be written as

$$\alpha = \sum_{i=0}^{f-1} \nu_i \tilde{\lambda}_i p^i = p^f - \epsilon - \sum_{i=0}^{f-1} \nu'_i \tilde{\lambda}'_i p^i,$$

for suitable ν_i, $\nu'_i \in \{\pm 1\}$ with $\nu_{f-1} = \nu'_{f-1} = 1$. Hence

$$\nu_0 \tilde{\lambda}_0 + \nu'_0(\tilde{\lambda}_0 - 2) + \epsilon = p^f - p(\sum_{i=1}^{f-1} (\nu_i \tilde{\lambda}_i + \nu'_i \tilde{\lambda}_i) p^{i-1}).$$

The left hand side is an odd number between $-2p$ and $2p$ and the right had side is divisible by p. Therefore both sides equal $\nu_0 p$ and

$$\tilde{\lambda}_0 = \frac{1}{2}(p + 2 - \epsilon\nu_0)$$

and $i = l - 2$ or $i = l - 1$.

For the other λ_i the equality

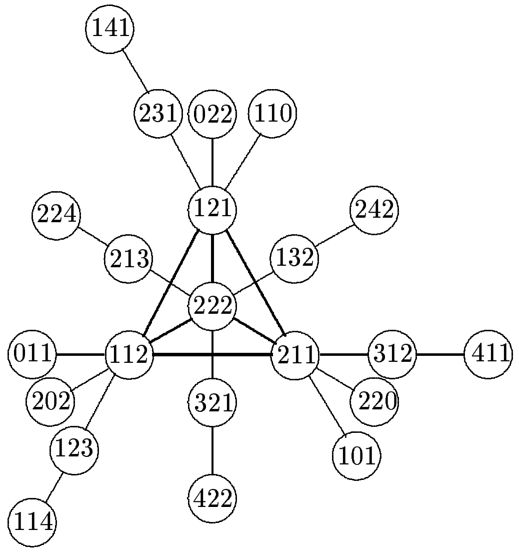
$$\sum_{i=1}^{f-1} \nu_i \tilde{\lambda}_i p^{i-1} = p^{f-1} - \nu_0 - \sum_{i=1}^{f-1} \nu_i' \tilde{\lambda}_i p^{i-1}$$

follows. Now considering $\overline{\lambda} := (\lambda_1, \ldots, \lambda_{f-1})$ as a Brauer character of $SL_2(p^{f-1})$ one obtains an element in $W(\overline{\lambda}) \cap p^{f-1} - \nu_0 - W(\overline{\lambda})$. By [HSW 82, Lemma 2.3] this intersection contains at most one element namely $\frac{1}{2}(p^{f-1} - \nu_0)$. Lemma 3.13 therefore implies that $\lambda_{f-1} = \lambda'_{f-1} < p - 1$. Moreover $\frac{1}{2}(p^{f-1} - \nu_0)$ is even if and only if $\sum_{i=1}^{f-1} \tilde{\lambda}_i$ is even. Therefore the numbers λ_i with $i \geq 1$ determine the sign ν_0 and the parity of λ_0 determines $\epsilon\nu_0$.

The element in $V^\epsilon(\lambda) \cap V^\epsilon(\lambda')$ is $\frac{1}{2}p(p^{f-1} - \nu_0) + \nu_0\tilde{\lambda}_0 = \frac{1}{2}(p^f - \nu) + \nu_0$. Therefore $\chi_0^{(\epsilon)} \in V^\epsilon(\lambda) \cap V^\epsilon(\lambda')$. With Lemma 3.16 and Lemma 3.17 the Lemma follows. □

We call an m-string as in the Lemma **tangent**, since the m-string satisfies the condition of the Lemma if and only if $\lambda^{(l)}$ and $\lambda^{(l-1)}$ lie on a circle. The arrows $(\lambda^{(l)}|(m, -1)|\lambda^{(l-1)})$ and $(\lambda^{(l-1)}|(m, 1)|\lambda^{(l)})$ in Q are called **exceptional arrows in direction** m.

Example. The tangent strings in the quiver of the principal 5-block of $SL_2(5^3)$. The circles (which are triangles here) and therefore also the exceptional arrows are the ones in the triangle (indicated by thick lines).



3.3 The group ring RG.

In this section we want to describe the basic order that is Morita equivalent to RG. The group ring RG has three blocks, one of which is of defect 0. The two other blocks are of defect f and will be treated separately. So let $\mathcal{B}$ denote either the principal block of G or the block containing the faithful irreducible characters and $y := 0$ or $y := 1$ in the respective cases. Let

$$P' := \{\lambda \in P \mid \sum_{i=0}^{f-1} \lambda_i \equiv y \pmod 2\}$$

be the set of indices of the simple $\mathcal{B}$-modules.

For $\lambda \in P'$ let P_λ be the projective indecomposable RG-module with head M_λ and let

$$\Lambda := \mathrm{End}_{RG}(\oplus_{\lambda \in P'} P_\lambda).$$

For $i \in N$, $h = \pm 1$ and $\lambda \in P(i,h) \cap P'$ let

$$\varphi_{i,h,\lambda} \in \mathrm{Hom}_{RG}(P_{f(i,h)(\lambda)}, P_\lambda)$$

be a lift of $\Psi(\alpha_{i,h,\lambda})$. If $T \subset N - S(\lambda)$ define the endomorphism $\beta'_{\lambda,T} \in \mathrm{End}_{RG}(P_\lambda)$ as the lift of $[\lambda, T]$ obtained by replacing the $\Psi(\alpha_{i,h,\gamma})$ in all definitions by $\varphi_{i,h,\gamma}$ ($\gamma \in P' \cap P(i,h)$) and let $\beta'_\lambda \in \mathrm{End}_{RG}(P_\lambda)$ be such a lift of $\langle \lambda \sim \lambda, N\rangle$ for all $\lambda \in \tilde{P} \cap P'$.

Corollary 3.21 *The lifts $\beta'_{\lambda,T}, \beta'_\lambda \in \mathrm{End}_{RG}(P_\lambda)$ of the basis vectors $[\lambda, T]$, $\langle \lambda \sim \lambda, N\rangle$ of $\mathrm{End}_{kG}(\overline{P_\lambda})$ form an R-basis of the lattice $\mathrm{End}_{RG}(P_\lambda)$.*

As in section 2 let V be the direct sum over a system of representatives of isomorphism classes of simple $K \otimes_R \mathcal{B}$-modules and $E := \mathrm{End}_{KG}(V)$. For $\lambda, \gamma \in P'$ the vector space $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma)$ is considered as embedded in E. Let $\epsilon_1, \ldots, \epsilon_s$ be the primitive idempotents of E.

If f is even, then K is a splitting field for KG. Then the ϵ_t are numbered in such a way that ϵ_1 corresponds to δ or η and ϵ_2 corresponds to δ' or η'. In this case put $O := R$ and $A := \{1, 2\}$.

If f is odd, let

$$\mu := (-1)^{(p-1)/2} \text{ and } O := R[\sqrt{\mu p}].$$

Then the KG-modules with character $\delta + \delta'$ and $\eta + \eta'$ are irreducible and O is isomorphic to the maximal order in the endomorphism ring of these modules. Then we order the idempotents such that $O \cong Z(\epsilon_1 \mathcal{B})$ and put $A := \{1\}$.

Let $\wp$ be the maximal ideal of O.

If δ belongs to $\mathcal{B}$, then let χ_m denote the index of $\eta_{\chi_m^{(1)}} \in \mathcal{B}$ and if η belongs to $\mathcal{B}$, then χ_m is the index of $\delta_{\chi_m^{(-1)}} \in \mathcal{B}$.

For $\lambda \in P'$ let

$$c_\lambda := \{1 \le t \le s \mid \epsilon_t P_\lambda \ne 0\}$$

be the set of the indices of the simple KG-modules isomorphic to a direct summand of $K \otimes_R P_\lambda$.

From Lemma 3.13 one gets

Remark 3.22 *Let $\lambda \in P'$. Then $\lambda \in \tilde{P}$ if and only if $1 \in c_\lambda$.*

Since $\mathrm{End}_{RG}(P_\lambda)$ is a symmetric order, one finds:

Remark 3.23 *Let $\lambda \in P'$ and put $n := |N - S(\lambda)|$.*

(a) If $\lambda \notin \tilde{P}$, then $|c_\lambda| = 2^n = dim_R(\mathrm{End}_{RG}(P_\lambda))$ and $\mathrm{End}_{RG}(P_\lambda)$ is a sublattice of $\oplus_{t\in c_\lambda} R\epsilon_t$ of index $p^{2^{n-1}f}R$.

(b) Let $\lambda \in \tilde{P}$.
If f is even then $|c_\lambda| = 2^n + 1 = dim_R(\mathrm{End}_{RG}(P_\lambda))$ and $\mathrm{End}_{RG}(P_\lambda)$ is a sublattice of $\oplus_{t\in c_\lambda} R\epsilon_t$ of index $p^{(2^n+1)f/2}R$.
If f is odd, then $|c_\lambda| + 1 = 2^n + 1 = dim_R(\mathrm{End}_{RG}(P_\lambda))$ and $\mathrm{End}_{RG}(P_\lambda)$ is a sublattice of $O\epsilon_1 \oplus \oplus_{t\in c_\lambda, t\neq 1} R\epsilon_t$ of index $p^{((2^n+1)f-1)/2}R$.

The next lemma is the crucial observation in the determination of Λ. In the moment, we can only prove it for endomorphisms of projective modules P_λ with $\lambda \notin \tilde{P}$.

Lemma 3.24 *Let $\lambda \in P' - \tilde{P}$ and $T \subseteq N - S(\lambda)$ and $\beta'_{\lambda,T}$ be as above. Then*

$$n(\beta'_{\lambda,T}) = p^{l(\lambda,T)}O \quad \textit{where } l(\lambda,T) := \sum_{i\in T} \frac{1}{2} l(\lambda,i).$$

Proof. Let $i \in N - S(\lambda)$ and $l := \frac{1}{2}l(\lambda,i)$. Then

$$\beta'_{\lambda,i} = f_1 \dots f_l g_l \dots g_1$$

is a product of $f_j \in \mathrm{Hom}_{RG}(P_{\lambda_j}, P_{\lambda_{j+1}})$ and $g_j \in \mathrm{Hom}_{RG}(P_{\lambda_{j+1}}, P_{\lambda_j})$ for certain pairwise distinct $\lambda_1, \dots, \lambda_{l+1} \in P$. In the commutative ring E this product can be evaluated as

$$\beta'_{\lambda,i} = (f_1 g_1)(f_2 g_2)\dots(f_l g_l).$$

Since the ϵ_1-coefficient of $\beta'_{\lambda,i}$ is 0, the properties of the norm (see section 2) imply that p^l divides $n(\beta'_{\lambda,i})$ and therefore $p^{l(\lambda,T)}$ divides $n(\beta'_{\lambda,T})$.

On the other hand

$$\sum_{T\subseteq N-S(\lambda)} l(\lambda,T) = \frac{1}{2}\sum_{T\subseteq N-S(\lambda)}\sum_{i\in T} l(\lambda,i) = \frac{1}{2}\sum_{i\in N-S(a)} l(\lambda,i)2^{n-1} = f2^{n-1}.$$

The Lemma now follows from Remark 3.23 and Lemma 2.3. □

In particular one finds

Corollary 3.25 *Let $\lambda \in P'$, $i \in S(\lambda)$, $i-1 \notin S(\lambda)$ and $\gamma := f(i,-1)(\lambda)$. Then $\varphi_{i,1,\gamma}\varphi_{i,-1,\lambda} \in p\mathrm{End}_{RG}(P_\lambda)^*$.*

Proof. Since $i-1, i \notin S(\gamma)$ one has $[\gamma, i] = \Psi((\gamma|(i,-1),(i,1)|\gamma))$. Therefore $\beta'_{\gamma,i} = \varphi_{i,-1,\lambda}\varphi_{i,1,\gamma}$ is an element of norm p according to Lemma 3.24. By Theorem 3.2 $\varphi_{i,1,\gamma}\varphi_{i,-1,\lambda} \in p\mathrm{End}_{RG}(P_\lambda)$. So there is a unit $u \in \mathrm{End}_{RG}(P_\lambda)$ with $\varphi_{i,1,\gamma}\varphi_{i,-1,\lambda} = pu$. $\square$

To calculate the other $\beta'_{\lambda,m}$ we need to consider m-strings. In the moment we only know the $\beta'_{\lambda,m}$ corresponding to the ends of the m-strings.

Lemma 3.26 *Let $(\lambda^{(0)}, \ldots, \lambda^{(p-1)})$ be an m-string with origin $\lambda^{(0)} \in P'$. For $j = 1, \ldots, p-1$ define*

$$f_j := \varphi_{\lambda^{(j)},m,-1}, \quad g_j := \varphi_{\lambda^{(j-1)},m,1} \in E.$$

(i) $g_{p-1}f_{p-1} \in p\mathrm{End}_{RG}(P_{\lambda^{(p-1)}})^*$.

(ii) If $1 \le j < p-1$ then

$$g_j f_j = \sum_{i=1}^{s} a_{ji}\epsilon_i$$

with $a_{ji} \in pO$ $(1 \le i \le s)$ and $a_{ji} \notin p\wp$ for all $i \in c_{\lambda^{(j)}} - c_{\lambda^{(j+1)}}$.

(iii) If $1 \le j < p-1$ then $g_j f_j - f_{j+1}g_{j+1} \in p\mathrm{End}_{RG}(P_{\lambda^{(j)}})^*$.

Proof. (i) is the statement of Corollary 3.25.
(ii) and (iii) is shown by induction.
Let $j = p-2$. Theorem 3.2 says that

$$g_j f_j - f_{j+1}g_{j+1} \in p\mathrm{End}_{RG}(P_{\lambda^{(j)}}).$$

Because of (i) the norm of this endomorphism is pO. Hence $\frac{1}{p}(g_j f_j - f_{j+1}g_{j+1}) \in \mathrm{End}_{RG}(P_{\lambda^{(j)}})^*$ which implies (iii) and (ii) for $j = p-2$.
Now assume that $0 \le j < p-2$ and (ii) holds for $j+1$. By Theorem 3.2 one has $g_j f_j - f_{j+1}g_{j+1} \in p\mathrm{End}_{RG}(P_{\lambda^{(j)}})$. Since (ii) holds for $j+1$ and $c_{\lambda^{(j+1)}} - c_{\lambda^{(j+2)}} \neq \emptyset$ the norm of this endomorphism is pO. Therefore (iii) and hence also (ii) holds for j. $\square$

Now we want to show an analogous statement to Lemma 3.24 also for $\lambda \in \tilde{P}$. To this purpose we recall what is already shown in section 3.2.

Remark 3.27 *Let $(\lambda^{(1)}, \ldots, \lambda^{(f)})$ be a circle in Q with $\lambda^{(i)} \in P'$. Then*

$$c_{\lambda^{(1)}} \cap \ldots \cap c_{\lambda^{(f)}} = A \cup \{\chi_0, \ldots, \chi_{f-1}\}.$$

Lemma 3.28 *Let $\lambda \in \tilde{P} \cap P'$, $T \subseteq N - S(\lambda)$ and $\beta'_{\lambda,T}$, β'_λ be as above. Then*

$$n(\beta'_{\lambda,T}) = p^{l(\lambda,T)}O \text{ and } n(\beta'_\lambda) = \begin{cases} p^{f/2}R & f \text{ even} \\ \wp^f & f \text{ odd} \end{cases}.$$

Proof. As in the proof of Lemma 3.24 one finds that $n(\beta'_{\lambda,T})$ is divisible by $p^{l(\lambda,T)}O$ if one uses Lemma 3.26 (ii). By Lemma 2.3 the norm of the last basis vector β'_λ divides $p^{f/2}R$ respectively $\wp^f = p^{\frac{f-1}{2}}\wp$ depending on whether f is even or odd. So it suffices to show that $p^{f/2}$ respectively $\wp^f$ divides the coefficients of β'_λ. Then the Lemma follows similarly as Lemma 3.24.

Let $(\lambda^{(1)}, \dots, \lambda^{(f)})$ be a circle through $\lambda =: \lambda^{(1)}$. Let $f_i \in \mathrm{Hom}_{RG}(P_{\lambda^{(i)}}, P_{\lambda^{(i+1)}})$ denote a lift of the image of $\Psi(\lambda^{(i)}, (m_i, \nu_i), \lambda^{(i+1)})$ $(1 \leq i \leq f)$ such that

$$\beta'_\lambda = \prod_{i=1}^{f} f_i = \sum_{t=1}^{s} b_t \epsilon_t$$

with $b_t \in O$. Remark 3.27 says that

$$\textbf{(1)} \quad b_t = 0, \text{ if } t \notin A \cup \{\chi_0, \dots, \chi_{f-1}\}.$$

Since $\lambda^{(i)}$ and $\lambda^{(i+1)}$ are neighbours on an m_i-string there is $\gamma^{(i)} \in P'$, such that $(\lambda^{(i+1)}|(m_i, \nu_i)|\gamma^{(i)}) \in Q$ or $p = 3$ and there is $\gamma^{(i)}$ with $(\gamma^{(i)}|(m_i, \nu_i)|\lambda^{(i)}) \in Q$.
We assume that for all $1 \leq i \leq f$ the condition (2) holds:

$$\textbf{(2)} \quad \text{There is } \gamma^{(i)} \in P', \text{ such that } (\lambda^{(i+1)}|(m_i, \nu_i)|\gamma^{(i)}) \in Q.$$

With analogous considerations one can also treat the other case. For $1 \leq i \leq f$ let h_i be a lift of $\Psi(\lambda^{(i+1)}|(m_i, \nu_i)|\gamma^{(i)})$ and let $g_i \in \mathrm{Hom}_{RG}(P_{\lambda^{(i+1)}}, P_{\lambda^{(i)}})$ be a lift of $\Psi(\lambda^{(i+1)}, (m_i, -\nu_i), \lambda^{(i)})$.
Theorem 3.2 states that $f_i h_i \in p\mathrm{Hom}_{RG}(P_{\lambda^{(i)}}, P_{\gamma^{(i)}})$. More precisely there are $e_i \in \mathrm{End}_{RG}(P_{\lambda^{(i+1)}})$ such that

$$\textbf{(3)} \quad f_i h_i = p \prod_{j=1, j\neq i}^{f} g_j e_i h_i \quad (1 \leq i \leq f)$$

by Proposition 3.6.
Lemma 3.26 yields that the coefficient of h_i at $\epsilon_{\chi_{m_i}}$ is not zero.
For $j = 1, \dots, f$ let x_j and y_j be the coefficients of f_j, and g_j and e the coefficient of e_i at $\epsilon_{\chi_{m_i}}$. Because of **(3)**

$$x_i = p \prod_{j=1, j\neq i}^{f} y_j e.$$

The coefficient of β'_λ at $\epsilon_{\chi_{m_i}}$ is

$$b_{\chi_{m_i}} = \prod_{j=1}^{f} x_j = p \prod_{j=1, j\neq i}^{f} x_j y_j e.$$

By Lemma 3.26 (ii) $x_j y_j R = pR$ therefore

$$(*) \; p^f \text{ divides } b_{\chi_{m_i}} \text{ for all } 1 \leq i \leq f.$$

Let $\beta''_\lambda = \prod_{j=f}^{1} g_j \in \mathrm{End}_{RG}(P_\lambda)$ be defined analogously to β'_λ going the other way around the circle. Then $n(\beta'_\lambda\beta''_\lambda)$ is divisible by p^f by Lemma 3.26 (iii). With the same Lemma (ii) one finds

(4) If $t \in A$, then the coefficients x'_j and y'_j of f_j and g_j at ϵ_t are not 0.

Let $b_t = \prod_{j=1}^{f} x'_j$ and $b'_t := \prod_{j=1}^{f} y'_j$. Then $p^f \mid b_t b'_t$. If f is odd, then $\wp^f \mid b_1 O$ or $\wp^f \mid b'_1 O$. Since one may take β''_λ instead of β'_λ as basis vector of $\mathrm{End}_{RG}(P_\lambda)$ it follows that $b_1 O = b'_1 O = \wp^f = n(\beta'_\lambda)$.
Assume now that f is even. Let $\Phi = Tr_u$ be the symmetrizing form of $\mathrm{End}_{RG}(P_\lambda)$. Then $\Phi(\beta'_\lambda, id_{|P_\lambda}) \in R$. But $p^f\Phi(\beta'_\lambda, id_{|P_\lambda}) \equiv b_1 + b_2 \pmod{p^f R}$. Therefore $b_1 \equiv -b_2 \pmod{p^f R}$. As in the case f odd one concludes that both b_1 and b_2 have p-adic valuation $\frac{f}{2}$ and $n(\beta'_\lambda) = p^{f/2}R$. □

The statement **(*)** of the proof above allows to conclude the following:

Corollary 3.29 *There is a unit $x \in R^*$, such that $\beta'_\lambda \equiv x\beta_\lambda \pmod{p^f \oplus_{i\in c_\lambda} R\epsilon_i}$, where*

$$\beta_\lambda := \begin{cases} p^{f/2}(\epsilon_1 - \epsilon_2) & f \text{ even} \\ p^{(f-1)/2}\sqrt{\mu p}\,\epsilon_1 & f \text{ odd}. \end{cases}$$

We now define generators $\beta_{\lambda,T}$ of $\mathrm{End}_{RG}(P_\lambda)$ that will replace the old $\beta'_{\lambda,T}$.

Definition 3.30 *(a) Let $j \in \{0, \dots, p-2\}$. If the m-string $(\lambda^{(0)}, \dots, \lambda^{(p-1)})$ is not tangent or $j \neq \frac{p-3}{2}$, then define*

$$\mathrm{pr}_{\lambda^{(j)},m} := \sum_{t\in c_{\lambda^{(j)}}\cap c_{\lambda^{(j+1)}}} \epsilon_t \in E.$$

Otherwise let χ_m be as in Lemma 3.20 and define

$$\mathrm{pr}_{\lambda^{(j)},m} := \sum_{t\in c_{\lambda^{(j)}}\cap c_{\lambda^{(j+1)}} - \{\chi_m\}} \epsilon_t \in E.$$

(b) For $\lambda \in P'$ let $\mathrm{pr}_\lambda := \sum_{t\in c_\lambda} \epsilon_t \in E$ be the unit element of $\mathrm{End}_{RG}(P_\lambda) \subset E$.

(c) Let $\lambda \in P'$, $m \in N - S(\lambda)$. If $m - 1 \notin S(\lambda)$, then λ lies on an m-string. Then let

$$\beta_{\lambda,m} := p \cdot \mathrm{pr}_{\lambda,m}.$$

If $m - 1 \in S(\lambda)$, then let $j - 1 \notin S(\lambda)$ be the unique element such that $J := \{j, j+1, \dots, m-1\} \subseteq S(\lambda)$. Then define $\gamma := f(m-1,1)(f(m-2,1)(\dots f(j+1,1)(f(j,1)(\lambda))\dots))$ and

$$\beta_{\lambda,m} := p^{|J|+1}\mathrm{pr}_{\gamma,m}\mathrm{pr}_\lambda.$$

(d) For $T \subset N - S(\lambda)$ let $\beta_{\lambda,T} := \prod_{m\in T} \beta_{\lambda,m}$, where $\beta_{\lambda,\emptyset} := \mathrm{pr}_\lambda$.

In the same spirit as Lemma 3.26 and using the notation there one shows

Lemma 3.31 *For* $j = 0, \ldots, p-2$

$$p \cdot \mathrm{pr}_{\lambda^{(j)},m} \in \mathrm{End}_{RG}(P_{\lambda^{(j)}}) \cap \mathrm{End}_{RG}(P_{\lambda^{(j+1)}}).$$

More precisely there are units $u_j, v_j \in \mathrm{End}_{RG}(P_{\lambda^{(j)}})^*$ *such that*

$$p \cdot \mathrm{pr}_{\lambda^{(j)},m} \equiv u_j f_{j+1} g_{j+1} \equiv g_{j+1} f_{j+1} v_{j+1} \pmod{\oplus_{l=0}^{f-1} p^{f+1} R\epsilon_{\chi_l}}.$$

With this lemma one now can give a precise description of the endomorphism rings of the projective indecomposable RG-lattices:

Theorem 3.32 *Let* $\lambda \in P'$.
If $\lambda \notin \tilde{P}$, *then*

$$\mathrm{End}_{RG}(P_\lambda) = \langle \beta_{\lambda,T} \mid T \subset N - S(\lambda) \rangle_R.$$

If $\lambda \in \tilde{P}$, *then*

$$\mathrm{End}_{RG}(P_\lambda) = \langle \beta_{\lambda,T}, \beta_\lambda \mid T \subset N - S(\lambda) \rangle_R.$$

Proof. From Lemma 3.31 and Lemma 3.26 it follows that $\beta_{\lambda,m} \in \mathrm{End}_{RG}(P_\lambda)$. If $\lambda \in \tilde{P}$, then Corollary 3.29 states that $\beta_\lambda \in \mathrm{End}_{RG}(P_\lambda)$. Write $\beta_{\lambda,T} = \sum_{t \in c_\lambda} a(\lambda,T)_t \epsilon_t$ and $\beta'_{\lambda,T} = \sum_{t \in c_\lambda} a(\lambda,T)'_t \epsilon_t$. Since the $\beta_{\lambda,m}$ are obtained from $\beta'_{\lambda,m}$ by multiplication with units in local rings, there are $k_T \in R^*$ with $k_T a(\lambda,T)_t \equiv a(\lambda,T)'_t \pmod{pn(\beta_{\lambda,T})}$ for all $t \in c_\lambda$. A relation between the $\beta_{\lambda,T}$ (modulo $\wp$) increases the index of the order generated by the $\beta'_{\lambda,T}$. Therefore the $\beta_{\lambda,T}$ generate the same R-order as the $\beta'_{\lambda,T}$ and the Theorem follows from Corollary 3.21. □

The remainder of the paper is devoted to the description of the homomorphism spaces between different projective indecomposable RG-lattices and the determination of the order $\Gamma := \bigoplus_{t=1}^s \Lambda \epsilon_t$. The order Γ is a graduated order. Since all p-modular characters of RG are self dual, one may calculate exponent matrices for Γ from the structure constants $m^{(t)}(\lambda,\gamma)$ $(\lambda,\gamma \in P', t \in c_\lambda \cap c_\gamma)$ by Remark 2.6.

Proposition 3.33 *Let* $\lambda, \gamma \in P'$ *such that* λ *or* γ *does not lie in* $\tilde{P}$ *and* $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma) \neq \{0\}$. *Let* $\mathrm{pr}_{\lambda,\gamma} := \sum_{t \in c_\lambda \cap c_\gamma} \epsilon_t$. *Then*

$$\mathrm{Hom}_{RG}(P_\lambda, P_\gamma) \cong (\mathrm{pr}_{\lambda,\gamma}(\mathrm{End}_{RG}(P_{\gamma \ominus \lambda}))$$

as $\mathrm{End}_{RG}(P_\lambda) - \mathrm{End}_{RG}(P_\gamma)$*-bimodule. The non trivial structure constants (see Remark 2.6) are*

$$m^{(t)}(\lambda,\gamma) = d(\lambda,\gamma),$$

where $d(\lambda,\gamma)$ *is as in Definition 3.11.*

Proof. Let $T_1 \subset N - (S(\lambda) \cup S(\gamma))$ be as in Definition 3.11. Proposition 3.6 says that $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma)$ is generated as $\mathrm{End}_{RG}(P_{\gamma \ominus a})$-module of any lift of

$$w_{\lambda,\gamma} := \langle \lambda \supset (\lambda \ominus \gamma) \rangle \langle (\lambda \ominus \gamma) \sim (\gamma \ominus \lambda), T_1 \rangle \langle (\gamma \ominus \lambda) \subset \gamma \rangle$$

whence $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma) \cong (\mathrm{pr}_{\lambda,\gamma}(\mathrm{End}_{RG}(P_{\gamma \ominus \lambda}))$.

Let $\beta_{\lambda,\gamma}$ be the product of the lifts of the $\varphi_{i,h,\lambda'}$ that occur in $w_{\lambda,\gamma}$. Analogously define $\beta_{\gamma,\lambda}$.
Since one of λ and γ, say λ, does not lie in $\tilde{P}$ one has $A \cap c_\lambda = \emptyset$.
Let $m \in \{0, \dots, f-1\}$ such that $\chi_m \in c_\lambda \cap c_\gamma$. Then by Lemma 3.17 $\lambda_m \in \{l-2, l+1\}$ where $l := \frac{p-1}{2}$

Assume that the path to $w_{\lambda,\gamma}$ contains an exceptional arrow $(\lambda'|(m, \pm 1)|\gamma')$ in direction m. Then $\lambda'_m = l-1$ and $\gamma'_m = l$ or $\lambda'_m = l$ and $\gamma'_m = l-1$. Since $w_{\lambda,\gamma}$ is of the shape $\langle \lambda \sim \gamma, T\rangle$ for some $T \subset N$, the entry λ_m is changed at most once again to $p-2-\lambda_m$. Since $p-2-(l-2) = l+1$ and $p-2-(l+1) = l-2 \neq l, l-1$, this is a contradiction.

Therefore $\beta_{\lambda,\gamma}\beta_{\gamma,\lambda} = \sum_{t \in c_\lambda \cap c_\gamma} x_t \epsilon_t$, with $x_t R = p^{d(\lambda,\gamma)} R$ by Lemma 3.31. The proposition follows. □

If both elements λ and γ lie in $\tilde{P}$, then the description of $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma)$ is more complicated, since this module is not a cyclic $\mathrm{End}_{RG}(P_\gamma)$-module.

Proposition 3.34 *Let $\lambda, \gamma \in \tilde{P} \cap P'$. Then there are $T_1, T_2 \subset N$ with $N = T_1 \dot\cup T_2$ such that $\langle \lambda \sim \gamma | T_i\rangle \in Q$ for $i = 1, 2$.
Let $d_i := |T_i|$ denote the length of these paths, where we assume w.l.o.g. that $d_1 \leq d_2$.
For $i = 1, 2$ let C_i be the set of indices of the simple KG-modules that lie in c_x, for all x in the path $\langle \lambda \sim \gamma | T_i\rangle$ and $R_i := \{\chi_m \mid m \in T_i\} \subset C_i$.
Let $t \in c_\lambda \cap c_\gamma$. Then*

$$m^{(t)}(\lambda, \gamma) = \begin{cases} d_1 & t \in C_1 - C_2 \\ d_2 & t \in C_2 - C_1 \\ f - d_i & t \in R_i \\ md_1 & t \in A \end{cases}$$

*where $m := 1$, if f is even and $m := 2$, if f is odd.
To get the isomorphism type of $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma)$ define*

$$\beta_1 := \sum_{t \in C_1 - R_1} \epsilon_t \quad \text{and } \beta_2 := \sum_{t \in C_2 - R_2 - A} \epsilon_t + b_2$$

with

$$b_2 := \begin{cases} p^{\frac{f}{2} - d_1}(\epsilon_1 - \epsilon_2), & \text{if } f \text{ is even} \\ p^{\frac{f-1}{2} - d_1}\sqrt{\mu p}\,\epsilon_1, & \text{if } f \text{ is odd} \end{cases}$$

Then

$$\mathrm{Hom}_{RG}(P_\lambda, P_\gamma) \cong \beta_1 \mathrm{End}_{RG}(P_\gamma) + \beta_2 \mathrm{End}_{RG}(P_\gamma) + \sum_{t \in R_1} p^{d_2} R\epsilon_t + \sum_{t \in R_2} p^{d_1} R\epsilon_t$$

as $\mathrm{End}_{RG}(P_\lambda) - \mathrm{End}_{RG}(P_\gamma)$-bimodule.

Proof. By Proposition 3.6 and Lemma 3.27

$$c_\lambda \cap c_\gamma = C_1 \cup C_2 \text{ and } C_1 \cap C_2 = A \cup R_1 \cup R_2 = A \cup \{\chi_0, \dots, \chi_{f-1}\}.$$

For $j = 1, 2$ let β_j be the product of the lifts of the $\varphi_{i,h,\lambda'}$ that occur in $\langle \lambda \sim \gamma | T_j \rangle$. Then β_1 and β_2 generate the $\mathrm{End}_{RG}(P_\gamma)$-module $\mathrm{Hom}_{RG}(P_\lambda, P_\gamma)$ by Proposition 3.6.

$$\text{For } j = 1, 2 \text{ write } \beta_j = \sum_{t \in C_j} x_t^{(j)} \epsilon_t \text{ with } x_t^{(j)} \in O.$$

Analogously define β_1', β_2' to the paths $\langle \gamma \sim \lambda | T_1 \rangle$ and $\langle \gamma \sim \lambda | T_2 \rangle$. Let $\,^\circ$ be the involution on Λ induced by the R-linear mapping on RG defined by $g \mapsto g^{-1}$ for all $g \in G$. Then $\,^\circ$ maps $\mathrm{End}_{RG}(P_\lambda)\beta_i\mathrm{End}_{RG}(P_\gamma)$ onto $\mathrm{End}_{RG}(P_\gamma)\beta_i'\mathrm{End}_{RG}(P_\lambda)$. Hence there are $a_\lambda^{(i)} \in \mathrm{End}_{RG}(P_\lambda), a_\gamma^{(i)} \in \mathrm{End}_{RG}(P_\gamma)$ and $f_t \in O$ $(t \in C_i)$ with

$$\beta_i^\circ = \sum_{t \in C_i} f_t (x_t^{(i)})^\circ \epsilon_t = a_\gamma^{(i)} \beta_i' a_\lambda^{(i)}.$$

Since β_i° also generates the bimodule $\mathrm{End}_{RG}(P_\gamma)\beta_i'\mathrm{End}_{RG}(P_\lambda)$ the elements $a_\gamma^{(i)}$ and $a_\lambda^{(i)}$ are units in the local rings $\mathrm{End}_{RG}(P_\gamma)$ and $\mathrm{End}_{RG}(P_\lambda) \subset E$. In particular the coefficients of β_i° and β_i' have the same p-adic valuation.

Now the structure constants are determined. To this purpose let $t \in c_\lambda \cap c_\gamma$. First assume that $t \notin C_1 \cap C_2$ and let $t \in C_i$ ($i = 1$ or 2). Then by Lemma 3.31

$$\epsilon_t \mathrm{Hom}_{RG}(P_\lambda, P_\gamma)\mathrm{Hom}_{RG}(P_\gamma, P_\lambda) = p^{d_i} R \epsilon_t$$

whence $m^{(t)}(\lambda, \gamma) = d_i$.
Now let $t = \chi_m \in R_i$ for some $0 \le m \le f - 1$. As in the proof of Lemma 3.28 **(3)** one sees that $f_t (x_t^{(i)})^2 \in p^f R$. Therefore the p-adic valuation $\nu_p(x_t^{(i)}) \ge \frac{1}{2}(f - \nu_p(f_t))$. For $\{i, l\} = \{1, 2\}$ one finds for $x_t^{(l)}$

$$\textbf{(1)} \quad f_t (x_t^{(l)})^2 R = p^{d_l} R.$$

Therefore $m^{(t)}(\lambda, \gamma) = d_l = f - d_i$.
Now assume $t \in A$. Then $\nu_p(f_t(x_t^{(i)})(x_t^{(i)})^\circ) = d_i$. Therefore $m^{(t)}(\lambda, \gamma) = m \cdot min(d_1, d_2) = m \cdot d_1$.
The coefficients of β_1 and β_2 can be determined as follows: Applying an isomorphism one may assume that $x_t^{(i)} = 1$ for all $t \in C_i - R_i - A$ and $i = 1, 2$ and that $x_t^{(1)} = 1$ for all $t \in A$. Let $i = 1$ or $i = 2$.
First it is shown:

$$\textbf{(2)} \quad \text{If } t \in R_i, \text{ then } p^{d_i} \text{ divides } x_t^{(i)}.$$

By Lemma 3.31 there are units $u_1, \ldots, u_{d_i}$ in the endomorphism rings of the projective indecomposable RG-lattices occurring in the path $\langle \lambda \sim \gamma | T_i \rangle$ such that $\beta_i \beta_i^\circ u = \sum_{t \in C_i - R_i} p^{d_i} \epsilon_t + p^{f + d_i} \sum_{t \in R_i} a_t \epsilon_t$, where $a_t \in O$ and $u := \prod_{l=1}^{d_i} u_l = \sum_{t \in C_i} v_t \epsilon_t$ with $v_t \in O^*$. Let $\{i, l\} = \{1, 2\}$. Then

$$f_t x_t^{(i)} (x_t^{(i)})^\circ v_t = \begin{cases} p^{f + d_i} a_t & t \in R_i \\ p^{d_i} & t \in C_i - R_i. \end{cases}$$

If $t \in R_i$ then $\nu_p(f_t) = d_l$ by **(1)** because $x_t^{(l)} = 1$. Since the p-adic valuations of $x^{(i)}$ and $(x^{(i)})^\circ$ are the same, $\nu_p(x_t^{(i)}) \ge \frac{1}{2}(f + d_i - \nu_p(f_t)) = \frac{1}{2}(f + d_i - d_l) = d_i$

for all $t \in R_i$. Hence (**2**) is proven.
If $t \in R_i$ then $x_t^{(l)}$ is chosen to be 1 and $p^{f-d_l}\epsilon_t\beta_l \in \mathrm{Hom}_{RG}(P_\lambda, P_\gamma)$. Therefore one can assume that $x_t^{(i)} = 0$ for all $t \in R_i$.
It suffices to determine the coefficients $x_t^{(2)}$ for $t \in A$. To this purpose consider $\beta_1^\circ\beta_2 = \sum_{t\in A} f_t x_t^{(2)}\epsilon_t \in \mathrm{End}_{RG}(P_\lambda) \cap \mathrm{End}_{RG}(P_\gamma)$. By Lemma 3.31 there is $u \in R^*$, such that $f_t = up^{d_1}$ for all $t \in A$. Since the norm of $\beta_1^\circ\beta_2$ is $p^{f/2}R$ (if f is even) respectively $\wp^f$ (if f is odd) by Lemma 3.28, one gets with Corollary 3.29, that $\beta_1^\circ\beta_2$ is an R^*-multiple of $p^{d_1}b_2$. Replacing β_2 by an R^*-multiple and adjusting the coefficients of β_2 at ϵ_t with $t \in C_2 - C_1$ one gets the proposition. □

3.4 The principal block of $SL_2(3^3)$

Let $G := SL_2(3^3)$ and $R := \mathbb{Z}_3[\zeta_{26}]$. The decomposition matrix of the principal block of RG:

		000	200	020	002	011	101	110	022	202	220	211	121	112
		1	3	3	3	4	4	4	9	9	9	12	12	12
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
δ	13	1	0	0	0	1	1	1	0	0	0	0	0	0
δ'	13	1	0	0	0	1	1	1	0	0	0	0	0	0
δ_4	26	0	1	0	1	1	1	0	0	0	0	0	0	1
δ_{12}	26	0	1	1	0	0	1	1	0	0	0	1	0	0
δ_8	26	0	0	1	1	1	0	1	0	0	0	0	1	0
δ_2	26	1	0	0	0	0	1	0	1	0	0	0	0	1
δ_6	26	1	0	0	0	0	0	1	0	1	0	1	0	0
δ_{10}	26	1	0	0	0	1	0	0	0	0	1	0	1	0
η_4	28	1	0	0	1	1	1	1	0	0	0	0	0	1
η_{12}	28	1	1	0	0	1	1	1	0	0	0	1	0	0
η_{10}	28	1	0	1	0	1	1	1	0	0	0	0	1	0
η_2	28	0	1	0	0	0	1	0	1	0	0	0	0	1
η_6	28	0	0	1	0	0	0	1	0	1	0	1	0	0
η_8	28	0	0	0	1	1	0	0	0	0	1	0	1	0

Corollary 3.35 *The action of RG on the simple KG-module with character χ is given by the graduated order Λ_χ, where, in contrary to Definition 2.4, not the dimensions but the names of the modular constituents of χ are given.*

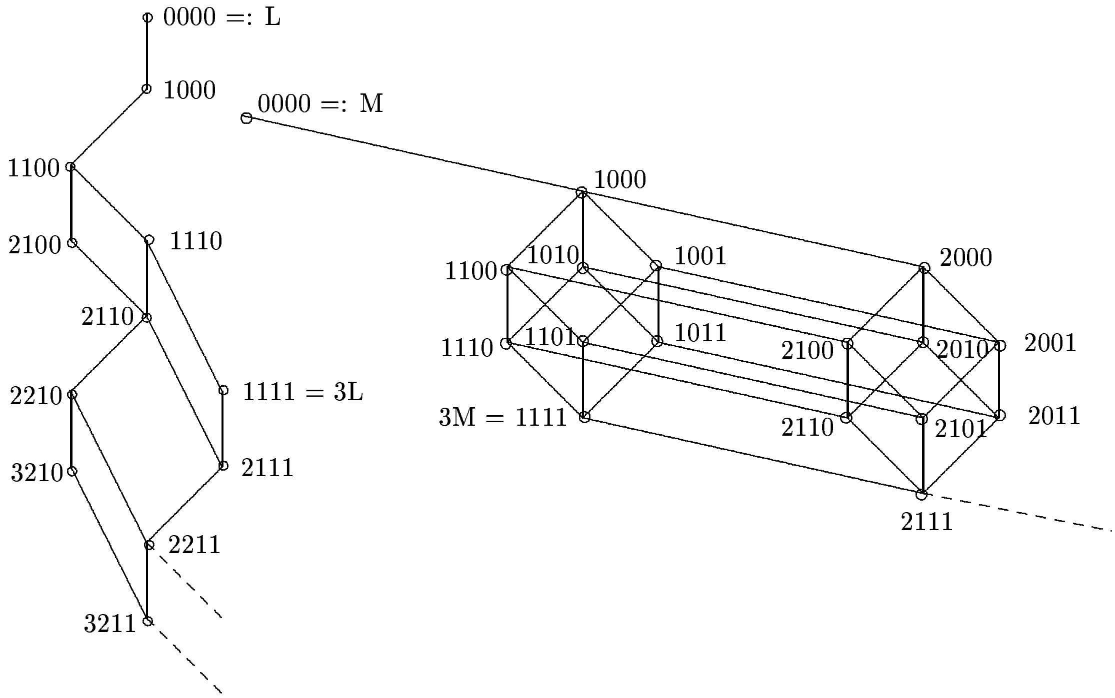
$\Lambda_1 = \Lambda(R, 000, (0))$, $\Lambda_{\delta+\delta'} = \Lambda(R[\sqrt{-3}], 000, 011, 101, 110, V)$,
$\Lambda_{\delta_4} = \Lambda(R, 200, 101, 112, 011, 002, N)$, $\Lambda_{\delta_2} = \Lambda(R, 000, 101, 112, 022, U)$,
$\Lambda_{\eta_2} = \Lambda(R, 200, 101, 112, 022, U)$, $\Lambda_{\eta_4} = \Lambda(R, 000, 101, 110, 011, 112, 002, M)$,
where

$$V := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}), \quad N := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \end{pmatrix}), \quad U := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$$\text{and } M := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 & 0 \end{pmatrix}).$$

The other projections of RG to the simple components of KG in the principal block are calculated by applying the Galois automorphism F.

These exponent matrices allow to read off the inclusion patterns of the irreducible RG-lattices as explained in [Ple 83, Remark (II.4)], from which we also use the notation. The inclusions of the lattices corresponding to δ_2, δ_6, δ_{10}, η_2, η_6, and η_8 (left picture) respectively to $\delta + \delta'$ (right picture) are given as follows:



Proof. To get a nice exponent matrix for Λ_χ one looks at the subgraph of the Ext-quiver on those vertices that correspond to the modular constituents of the character χ. If χ is Galois conjugate to δ_2 or η_2, then this subgraph is a straight line. If one orders the constituents along this line, one always finds an exponent matrix $U = U_{ij}$ with $U_{ij} = max(0, i - j)$ (cf. [Neb 99, Satz 5.6.1]). We only show how to get the exponent matrix in the most complicated case η_4. To order the modular constituents we start at one extremal point λ of the subgraph, say $\lambda = 000$. Note that the vertex 002 has distance 3 from 000, since only paths without repeated directions are allowed. We then list the vertices according to their distance to λ. Let M be the exponent matrix of Λ_{η_4} corresponding to the ordering of the modular constituents as above such that $m_{1,j} = 0$ for all j. Now η_4 corresponds to the character with number χ_2. The minimum of the length of the paths from 000 to 110, 101, or 011 is 1, but the edge $(000, 011)$ has direction 2. So by Proposition 3.34 the first 4 entries in the first column of M are $0, 1, 1, 2$. Since

112, 002 $\notin \tilde{P}$, Proposition 3.33 gives the remaining entries in the first column of M. The distance between 101 and 110 in Q is 1. But the edge joining the two vertices has direction 2. So by Proposition 3.34, the sum of the two entries $m_{23}+m_{32}$ is 2. Now Lemma 2.5 yields $m_{23}=m_{32}=1$. Analogously one calculates the remaining entries of M. □

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