

Unitary Discriminants of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$

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Abstract

We give a full list of the unitary discriminants of the even degree indicator 'o' ordinary irreducible characters of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$.

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1 Introduction

Let G be a finite group, K a number field, $n \in \mathbb{N}$ and let $\rho : G \rightarrow \mathrm{GL}_n(K)$ be a K -representation of G of degree n . Then ρ is uniquely determined by its character $\chi : G \rightarrow K, g \mapsto \mathrm{trace}(\rho(g))$. The ordinary character table for G lists the values of all irreducible characters on the conjugacy classes of G . Together with an additional number-theoretic invariant, the Brauer element of χ (see Definition 2.13), it contains all necessary information about the linear actions of G over number fields, i.e. the representations ρ as above.

As G is finite, the image $\rho(G) \subseteq \mathrm{GL}_n(K)$ is contained in either a symplectic, unitary, or orthogonal group. The papers [14] and [13] develop methods towards classifying the orthogonal and unitary groups that contain $\rho(G)$. These methods are then applied to compute the orthogonal discriminants of the even degree indicator + irreducible characters and the unitary discriminants of the even degree indicator 'o' characters of a large portion of the groups in the ATLAS of finite groups [3].

The ATLAS of finite groups contains the character tables of small finite simple groups, including all sporadic simple groups. Most of the non-abelian simple

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groups are finite groups of Lie Type. They fall in infinite series for which there are so called generic character tables parameterising the representations. For these infinite series, it is necessary to compute the corresponding generic orthogonal and unitary discriminants.

The first formula in the literature for such generic discriminants is the one by Jantzen and Schaper (see for instance [7]), giving the composition factors of the discriminant groups of the Specht modules for all symmetric groups. The first generic orthogonal character tables have been obtained in [2] for the groups $SL_2(q)$ and all prime powers q . Later the authors of this paper computed the generic orthogonal discriminants of the groups $SL_3(q)$ and $SU_3(q)$ [6], again for all prime powers q .

This paper continues this investigation by determining the unitary discriminants of the groups $SL_3(q)$ and $SU_3(q)$ for all prime powers q . The computation illustrates three different methods: For $SL_3(q)$ Harish-Chandra induction from the parabolic subgroup $GL_2(q) \times \mathbb{F}_q^2$ is enough to obtain all unitary discriminants (Theorem 3.4 and 3.5).

For the unitary groups we also distinguish between even and odd defining characteristic. Section 4 lists the important players and the relevant facts that apply to both situations. For 2-powers q the irreducible even degree indicator 'o' characters of $SU_3(q)$ have degree $q(q^2 - q + 1)$. These characters appear in a rank 2 monomial representation that can be analysed using similar methods as for rank 2 permutation representations (see Section 5). The last section deals with unitary group in odd defining characteristic. Here we need to use the full strength of the methods from [13] and a metabelian subgroup of the Borel subgroup to finally deduce the results for odd q .

This paper is the starting point of a long term project to determine the unitary discriminants of ordinary irreducible characters of finite groups of Lie Type. On the one hand, the explicit results obtained in this paper allow to obtain unitary discriminants for characters that are Harish-Chandra induced from parabolic subgroups with Levi factor of Type U_3 or L_3 . On the other hand, the methods applied here can be generalised to obtain (perhaps less explicit) results for finite groups of Lie Type of higher rank.

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2 Methods

2.1 Quadratic and Hermitian forms

Let L be a field of characteristic $\neq 2$ and $\sigma \in \text{Aut}(L)$ an automorphism of order 1 or 2. Put $K := \text{Fix}_\sigma(L)$ to denote the fixed field of σ . An L/K Hermitian space is a finite dimensional L -vector space V together with a non-degenerate L/K -Hermitian form $H : V \times V \rightarrow L$, i.e. a K -bilinear map such that $H(av, w) = aH(v, w)$ and $H(v, w) = \sigma(H(w, v))$ for all $v, w \in V$ and $a \in L$. Let $N := N_{L/K}(L^\times)$ denote the norm subgroup of K^\times . Then

$$(K^\times)^2 \leq N \leq K^\times$$

and $N = (K^\times)^2$ if $\sigma = \text{id}$. In the latter case, $L = K$ and H is usually called a symmetric bilinear form.

Definition 2.1. The *discriminant* of H is the class

$$\text{disc}(H) := (-1)^{\binom{n}{2}} \det(H_B)N \in K^\times/N,$$

where $n = \dim(V)$ and $H_B := (H(b_i, b_j))_{i,j=1}^n \in L^{n \times n}$ is the Gram matrix of H with respect to any basis $B = (b_1, \dots, b_n)$ of V .

For $L \neq K$ the class of the discriminant defines a unique quaternion algebra over K which we call the discriminant algebra of H :

Definition 2.2. a) For $a, b \in K^\times$ let $(a, b)_K$ denote the central simple K -algebra with K -basis $(1, i, j, ij)$ such that $i^2 = a$, $j^2 = b$, $ij = -ji$.

b) For a quadratic extension $L = K[\sqrt{\delta}]$ of K we put

$$(L, b)_K := (\delta, b)_K.$$

Then the class of $[(L, b)_K] \in \text{Br}(L, K)$ lies in the Brauer group of central simple K -algebras that are split by L . We denote by

$$\text{disc}_L([(L, b)_K]) := bN_{L/K}(L^\times)$$

the L -discriminant of $[(L, b)_K] \in \text{Br}(L, K)$.

c) For $\sigma \neq \text{id}$ the *discriminant algebra* of the Hermitian form H with discriminant dN is defined as the class $\Delta(H) := [(L, d)_K] \in \text{Br}(L, K)$.

Note that $\Delta(H)$ is the Clifford invariant of the quadratic form $Q_H : V_K \rightarrow K, v \mapsto H(v, v)$ (see for instance [15, Remark (10.1.4)]). This result also implies the well definedness of the discriminant algebra, which is also guaranteed by the following remark.

Remark 2.3. For a quadratic extension L/K and $a, b \in K^\times$ we have $(L, a)_K \cong (L, b)_K$ if and only if $aN_{L/K}(L^\times) = bN_{L/K}(L^\times)$.

Remark 2.4. Division algebras over number fields K have the local-global property: They are isomorphic if and only if they become isomorphic over all completions of K . In particular, we can identify a quaternion algebra \mathcal{Q} by listing all the places \wp_1, \dots, \wp_h of K , where the completion is still a division algebra. The primes \wp_1, \dots, \wp_h are the *ramified primes* in \mathcal{Q} and we put

$$\mathcal{Q} =: \mathcal{Q}_K(\wp_1, \dots, \wp_h).$$

The finite ramified places in \mathcal{Q} are exactly those that divide the discriminant ideal of a maximal \mathbb{Z}_K -order in \mathcal{Q} . If $b \in \mathbb{Z}_K \setminus \{0\}$ then $\mathbb{Z}_L \oplus \mathbb{Z}_L j$ is a \mathbb{Z}_K -order in $(L, b)_K$ of discriminant $\text{disc}_K(L)^2 b^2$. In this situation we hence have the following remark:

Remark 2.5. Let K be a number field and assume that $b \in \mathbb{Z}_K \setminus \{0\}$. Then any finite place \wp that ramifies in $(L, b)_K$ divides $\text{disc}_K(L)b$.

It is a general principle that odd degree extensions are not relevant for isometry of quadratic or Hermitian forms.

Lemma 2.6. (see [8, Corollary 6.16]) *Let F be an odd degree extension of K . Put $E := FL$ and extend σ to a field automorphism σ of E with fixed field F . Let (V, H) and (W, H') be two L/K Hermitian spaces. If the E/F Hermitian spaces $(V \otimes_L E, H_E)$ and $(W \otimes_L E, H'_E)$ are isometric, then also $(V, H) \cong (W, H')$.*

2.2 Unitary stable characters

For a finite group G let $\text{Irr}(G)$ denote the set of irreducible complex characters of G . The *Frobenius-Schur indicator* $\text{ind}(\chi)$ of $\chi \in \text{Irr}(G)$ takes values in $\{+, -, 'o'\}$. We have $\text{ind}(\chi) = 'o'$ if and only if the character field $\mathbb{Q}(\chi)$ is not real, $\text{ind}(\chi) = +$ if there is a real representation affording the character χ , and $\text{ind}(\chi) = -$ for real characters χ that are not afforded by a real representation. We denote by

$$\text{Irr}^+(G) := \{\chi \in \text{Irr}(G) \mid \text{ind}(\chi) = +, \chi(1) \text{ even}\}$$

the even degree indicator $+$ characters of G and by

$$\text{Irr}^o(G) := \{\chi \in \text{Irr}(G) \mid \text{ind}(\chi) = 'o', \chi(1) \text{ even}\}$$

the even degree indicator $'o'$ characters of G .

Definition 2.7. ([14, Definition 5.12]) A character χ of a finite group G is called *orthogonal* if there is a representation ρ affording the character χ and fixing a non-degenerate quadratic form. An orthogonal character is called *orthogonally stable* if and only if all its indicator + constituents have even degree.

Then [14, Theorem 5.15] shows that the orthogonally stable characters are exactly those orthogonal characters that have a well defined discriminant:

Definition 2.8. Let χ be an orthogonally stable character of a finite group G with character field $K := \mathbb{Q}(\chi)$. Then the *orthogonal discriminant* $\text{disc}(\chi) \in K^\times / (K^\times)^2$ is the unique square class of the character field such that for any orthogonal representation $\rho : G \rightarrow \text{GL}_n(L)$ and any non-degenerate $\rho(G)$ -invariant quadratic form Q we have that $\text{disc}(Q) = \text{disc}(\chi)(L^\times)^2$.

The paper [13] transfers the notion of orthogonal stability to the Hermitian case:

Definition 2.9. [13, Definition 5.10] An ordinary character χ of a finite group G is called *unitary stable* if all irreducible constituents of χ have even degree.

Note that a unitary stable orthogonal character is orthogonally stable, but the converse is not always the case. Similarly as in the orthogonal case we get that a unitary stable character has a well defined unitary discriminant:

Remark 2.10. ([13, Proposition 5.13]) Let χ be a unitary stable character of the finite group G and assume that the character field $L := \mathbb{Q}(\chi)$ is not real. Denote by K its real subfield. Then there is a unique class

$$[\Delta(\chi)] \in \text{Br}(L, K)$$

such that for all representations $\rho : G \rightarrow \text{GL}_n(M)$ and all $\rho(G)$ -invariant non-degenerate Hermitian forms H we have

$$\Delta(H) = [\Delta(\chi) \otimes_K M^+]$$

where M^+ is the maximal real subfield of M .

$[\Delta(\chi)]$ is called the *unitary discriminant algebra* of χ .

Whereas orthogonal discriminants of orthogonally stable characters are essentially independent of the chosen splitting field, for unitary discriminants this field matters.

Definition 2.11. Let χ be a unitary stable character of the finite group G and let L be a totally complex number field with real subfield $K \neq L$. Assume that there is an L -representation ρ of G affording the character χ . Then all $\rho(G)$ -invariant non-degenerate Hermitian forms H have the same discriminant $\text{disc}(H) =: dN_{L/K}(L^\times)$. Then $\text{disc}_L(\chi) := dN_{L/K}(L^\times)$ is called the *L-discriminant* of χ and $\Delta_L(\chi) := (L, d)_K$ the *L-discriminant algebra* of χ .

2.3 Induced representations

Many irreducible characters χ of finite groups G are imprimitive, i.e. induced from a character ψ of a proper subgroup U . Then a G -invariant form in the induced representation is just the orthogonal sum of $[G : U]$ copies of a U -invariant form. However, the character field of ψ might be larger than the one of χ , and we only get the discriminant over the character field of ψ (see [13, Remark 7.2]). In view of Lemma 2.6 it is helpful to know when $[\mathbb{Q}(\psi) : \mathbb{Q}(\chi)]$ is odd:

Lemma 2.12. *Let G be a finite group, $U \leq G$, $\psi \in \text{Irr}(U)$ such that $\chi := \psi^G \in \text{Irr}(G)$. Then $\mathbb{Q}(\chi) \leq \mathbb{Q}(\psi)$. Let $\Psi := \{\psi_1, \dots, \psi_h\}$ be the set of constituents of the restriction $\chi|_U$ of χ to U of degree $\psi_i(1) = \psi(1)$ and assume that the cardinality, h , of Ψ is odd. Then there is $i_0 \in \{1, \dots, h\}$ such that $[\mathbb{Q}(\psi_{i_0}) : \mathbb{Q}(\chi)]$ is odd.*

Proof. By Frobenius reciprocity $\Psi = \{\psi_i \in \text{Irr}(U) \mid \chi = \psi_i^G\}$. For $\psi_i \in \Psi$ a full regular orbit under the Galois group $\text{Gal}(\mathbb{Q}(\psi_i)/\mathbb{Q}(\chi))$ is contained in Ψ and Ψ is a disjoint union of such Galois orbits. As $|\Psi|$ is odd at least one of these orbits has odd length. \square

2.4 Schur indices

Schur indices play an important role in the computation of unitary discriminants of unitary stable characters. As described in Theorem 2.18 below we need all local Schur indices to compute these discriminants. These are encoded in the Brauer element, a notion coined in [18, Definition 2.1]:

Definition 2.13. Let χ be an irreducible ordinary character of some finite group G . Let K be some field containing the character field $\mathbb{Q}(\chi)$ and let ρ be a K -representation of G affording the character $m\chi$ for some positive integer m . Then m is minimal if and only if $\text{End}_{KG}(\rho) =: D$ is a central simple division algebra over K . In this case $m^2 = \dim_K(D)$ and $m_K(\chi) := m$ is the *Schur index* of χ over K . The class $[\chi]_K := [D] \in \text{Br}(K)$ of D in the Brauer group $\text{Br}(K)$ of K is called the *Brauer element* of χ over K .

If $K = \mathbb{Q}(\chi)$ is the character field of χ , then we sometimes omit the field, so $[\chi] := [\chi]_{\mathbb{Q}(\chi)}$.

The field K is called a *splitting field* of χ if the Brauer element $[\chi]_K = [K]$ is trivial.

To compute unitary discriminants we sometimes need to compute local Schur indices of certain characters. General computational methods are described in [19] and [9]. When other methods fail they all fall back to the following very useful result from [1]:

Theorem 2.14. [1, Theorem 8.1] Let χ be an ordinary irreducible character of a finite group lying in a p -block with cyclic defect group. Let φ be a p -modular constituent of χ . Then the p -adic Schur index of χ is

$$m_{\mathbb{Q}_p(\chi)}(\chi) = [\mathbb{Q}_p(\chi, \varphi) : \mathbb{Q}_p(\chi)].$$

Explicit computations of Schur indices for monomial characters can be obtained from [21, Theorem 2] which we recall for the reader's convenience.

Theorem 2.15. [21, Theorem 2] Let K be a field of characteristic 0. Let H be a normal subgroup of a finite group G and let ψ be a linear character of H such that the induced character $\chi := \psi^G$ is irreducible. Then the character field $K(\chi)$ is a subfield of $K(\psi)$. Put $\Gamma := \text{Gal}(K(\psi), K(\chi))$ and let

$$F := \{\mathbf{g} \in G \mid \psi^{\mathbf{g}} = \psi^{\gamma(\mathbf{g})} \text{ for some } \gamma(\mathbf{g}) \in \Gamma\}.$$

Then $\gamma : F/H \cong \Gamma$ is a group isomorphism. Choose coset representatives $\mathbf{g}_1, \dots, \mathbf{g}_n$ of F in H , put $\gamma_i := \gamma(\mathbf{g}_i H)$ and let $\mathbf{h}_{ij} \in H$ such that $\mathbf{g}_i \mathbf{g}_j = \mathbf{g}_k \mathbf{h}_{ij}$ for some $k \in \{1, \dots, n\}$. Then

$$\beta : \Gamma \times \Gamma \rightarrow K(\psi), \beta(\gamma_i, \gamma_j) := \psi(\mathbf{h}_{ij})$$

is a factor system and $[\chi]_{K(\chi)}$ is the inverse of the class of the crossed product algebra defined by β .

2.5 Unitary discriminants of real characters

Let χ be an irreducible real character of G of even degree $2m = \chi(1)$. Then its Frobenius-Schur indicator is either $-$ or $+$ and the Brauer element $[\chi]$ is represented by a quaternion algebra \mathcal{Q} over the totally real number field $K = \mathbb{Q}(\chi)$. The following results hold in more generality (see [13]), but we only use them here for splitting fields L of χ so for simplicity we assume that L is a totally complex quadratic extension of K that is a maximal subfield of \mathcal{Q} .

Proposition 2.16. a) ([13, Proposition 5.12]) If the indicator of χ is $-$, then

$$\text{disc}_L(\chi) = \text{disc}_L([\chi]_K)^m \text{ and } \Delta(\chi) = [\chi]_K^m.$$

If m is even then $\text{disc}_L(\chi)$ is trivial.

b) ([13, Proposition 5.12], [8, Proposition 10.33]) Assume that the indicator of χ is $+$. Then the following hold:

(i) The character χ is orthogonally stable and has a well defined orthogonal discriminant $\text{disc}(\chi)$.

(ii) The orthogonal discriminant $\text{disc}(\chi)$ is represented by $(-1)^m$ times the reduced norm of a skew symmetric unit in any representation affording χ (see [12, Proposition 2.2]).

(iii) The L -discriminant and the discriminant algebra of χ are obtained as

$$\text{disc}_L(\chi) = \text{disc}(\chi) \text{disc}_L([\chi]_K)^m \text{ and } \Delta(\chi) = [(L, \text{disc}(\chi))_K][\chi]_K^m.$$

(iv) If m is even or $[\chi]_K = [K]$, then $\text{disc}_L(\chi) = \text{disc}(\chi)$.

Combining Proposition 2.16 with [11, Theorem 4.7] yields an easy formula for the unitary discriminant of a unitary stable rational character of a 2-group.

Corollary 2.17. *Let G be a 2-group and $\chi \in \text{Irr}(G)$ be an irreducible rational character of degree $\chi(1)$ a multiple of 4. Then the unitary (or orthogonal) discriminant of χ is 1 over any splitting field of χ .*

2.6 Orthogonal subalgebras

The paper [13] transfers the computational methods from [14] to the Hermitian case. One important notion is the one of orthogonal subalgebras. For $\chi \in \text{Irr}^o(G)$ the following method is quite useful: Put $L := \mathbb{Q}(\chi)$ and assume that there is an L -representation $\rho : G \rightarrow \text{GL}_{2m}(L)$ affording the character χ . Let K denote the real subfield of L and let H be a non-degenerate $\rho(G)$ -invariant L/K -Hermitian form. Assume that there is $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$ such that $\chi \circ \alpha = \bar{\chi}$. Then $\chi + \bar{\chi}$ extends to an irreducible character $\mathfrak{X} = \text{Ind}_G^{\mathcal{G}}(\chi)$ of the semidirect product $\mathcal{G} := G \rtimes \langle \alpha \rangle$.

Let $A := \text{Fix}_\alpha(\rho) = \langle \rho(g) + \rho(\alpha(g)) \mid g \in G \rangle_K$ denote the α -fixed algebra. Then A is invariant under the adjoint involution of H . Denote the restriction of this involution to A by ι_A .

Theorem 2.18. ([13, Theorem 9.1])

- (a) $[A] = [\mathfrak{X}]_K \in \text{Br}(L, K)$.
- (b) Assume that the Frobenius-Schur indicator of \mathfrak{X} is $+$.
 - (i) Then ι_A is an orthogonal involution on A . Its discriminant $\text{disc}(\iota_A)$ is $(-1)^m$ times the square class in K of the reduced norm of any skew symmetric unit X in A (see [12, Proposition 2.2]).
 - (ii) The discriminant of H is $\text{disc}(H) = \text{disc}_L([A])^m \text{disc}(\iota_A)$.
- (c) Assume that the Frobenius-Schur indicator of \mathfrak{X} is $-$.
Then $\text{disc}(H) = \text{disc}_L([\mathfrak{X}]_K)^m$ and $\Delta(H) = [\mathfrak{X}]_K^m$.

2.7 A generalisation of the Q_8 -trick

Sometimes subgroups help us to predict all unitary discriminants of faithful characters. The following result is a generalisation of the Q_8 -trick [13, Section 8.1]. A variant is later used in Theorem 3.1.

Theorem 2.19. *Let $d \geq 2$ and $m := 2^d a$ for some $a \in \mathbb{N}$. Let p, ℓ be not necessarily distinct odd primes and let $q := p^\ell$ be some power of p . Let G be some subgroup of $\mathrm{GL}_m(q)$ containing $\mathrm{SO}_m^+(p)$ (e.g. $\mathrm{SL}_m(q) \leq G \leq \mathrm{GL}_m(q), \mathrm{SU}_m(q) \leq G \leq \mathrm{GU}_m(q), \mathrm{SO}_m^+(q) \leq G \leq \mathrm{GO}_m^+(q)$). Let χ be an irreducible faithful ordinary or ℓ -Brauer character of G . Then*

- (a) *The character degree $\chi(1)$ is a multiple of 2^d .*
- (b) *If the indicator of χ is $+$, then its orthogonal discriminant is 1.*
- (c) *If χ is an ordinary character of indicator 'o', then its unitary discriminant is 1.*

Proof. Consider the group $U := 2_+^{1+2d} \cong \otimes^d D_8$. Then U has a unique irreducible character ψ that restricts non-trivially to the center $Z(U) \cong C_2$. This character has degree $\psi(1) = 2^d$, indicator $+$, and is the character of an integral irreducible representation $\rho : U \rightarrow \mathrm{SL}_{2^d}(\mathbb{Z})$. Moreover, ψ is orthogonally stable of orthogonal discriminant 1. Reducing ρ mod p hence shows that $U \leq \mathrm{SO}_m^+(p) \leq \mathrm{SL}_m(p)$, so $U \leq G$. As χ is a faithful character of G , its restriction to U is a multiple of ψ . In particular, $2^d = \psi(1)$ divides $\chi(1)$. Also (b) and (c) follow from the fact that $\chi|_U = a\psi$ is orthogonal and unitary stable. \square

3 The special linear group

Let p be a prime and let q be a power of p . The group $\mathrm{SL}_3(q)$ is the group of 3×3 -matrices over \mathbb{F}_q of determinant 1. It contains a maximal parabolic subgroup

$$P := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in \mathrm{SL}_3(q) \right\}$$

of odd index $[\mathrm{SL}_3(q) : P] = q^2 + q + 1$. Note that P is the semidirect product $P \cong \mathrm{GL}_2(q) \ltimes \mathbb{F}_q^2$ where the action of $\mathrm{GL}_2(q)$ on \mathbb{F}_q^2 is given by

$$\mathbf{g} \cdot \mathbf{h} = \det(\mathbf{g})(\mathbf{g}\mathbf{h}) \text{ for } \mathbf{g} \in \mathrm{GL}_2(q), \mathbf{h} \in \mathbb{F}_q^2.$$

The center of P is the center Z of $\mathrm{SL}_3(q)$, hence isomorphic to C_3 if $q - 1$ is a multiple of 3 and trivial otherwise. Let

$$d := |Z| = \gcd(3, q - 1) \text{ and } \omega := \exp\left(\frac{2\pi i}{3}\right).$$

3.1 The characters of $\mathrm{GL}_2(q)$

As we use Harish-Chandra induction from the Levi factor $\mathrm{GL}_2(q)$ of the subgroup P , we first deal with this group.

Theorem 3.1. *Assume that q is a power of some odd prime p . Then all characters $\psi \in \mathrm{Irr}^\circ(\mathrm{GL}_2(q))$ have unitary discriminant $\mathrm{disc}(\psi) = (-1)^{\psi(1)/2}$.*

Proof. If ψ is not faithful, then the character field of ψ is real (see [17]). So ψ is a faithful irreducible character of $\mathrm{GL}_2(q)$. Then the restriction of ψ to the subgroup $D_8 \leq \mathrm{GL}_2(q)$ is a multiple of the unique character of D_8 that restricts non-trivially to the center of D_8 . This character has degree 2, indicator $+$, trivial Schur indices and orthogonal discriminant -1 . As in Theorem 2.19 we conclude that the unitary discriminant is $\mathrm{disc}(\psi) = (-1)^{\psi(1)/2}$. \square

Remark 3.2. Assume that q is a power of 2. Then $\mathrm{GL}_2(q)$ has $q - 1$ irreducible characters of degree q . All these characters restrict to the Steinberg character of degree q of $\mathrm{SL}_2(q)$ and hence have unitary/orthogonal discriminant $(-1)^{q/2}(q + 1)$ (see [2, Theorem 6.2]).

3.2 The characters of P

Now let $\chi \in \mathrm{Irr}(P)$ be an irreducible character of $P \cong \mathbb{F}_q^2 \rtimes \mathrm{GL}_2(q)$ restricting non-trivially to the abelian normal subgroup $A := \mathbb{F}_q^2$ of order q^2 . As P acts transitively on $A \setminus \{1\}$, the restriction of χ to A is a multiple of the sum of all non-trivial linear characters ψ of A . By Clifford theory, the character χ is induced up from a character of the inertia subgroup $T_\psi \cong H \rtimes A$ of any of these characters ψ , where

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-2} \end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\} \leq \mathrm{GL}_2(q)$$

is isomorphic to $(\mathbb{F}_q^\times \times \mathbb{F}_q)$. The center of T_ψ is Z and the index of T_ψ in P is $q^2 - 1$. The irreducible characters ϕ of H consist of $(q - 1)$ linear characters and d^2 characters of degree $(q - 1)/d$. Inducing the characters $\phi \otimes \psi$ with $\phi \in \mathrm{Irr}(H)$ from $H \rtimes A$ to P we obtain $(q - 1)$ irreducible characters of degree $(q^2 - 1)$ and d^2 characters of degree $(q - 1)^2(q + 1)/d$.

Remark 3.3. The irreducible characters $\chi \in \mathrm{Irr}(P)$ for which the restriction of χ to a Sylow p -subgroup of P does not contain the trivial character are exactly the d^2 characters of degree $\chi(1) = (q - 1)^2(q + 1)/d$. By Theorem 2.14 all these characters have trivial Schur index. The d characters χ that restrict trivially to the center are rational, the other $d^2 - d$ characters χ have character field $\mathbb{Q}(\omega)$. All these characters have trivial unitary, resp. orthogonal, discriminant.

Proof. Write $q - 1 = ab$ such that a is a power of 3 and b is not a multiple of 3 and put

$$U := \langle P', g^a \mid g \in P \rangle$$

to be the normal subgroup of index a in P (so $U = P$ if $q \not\equiv 1 \pmod{3}$). Let $\chi \in \text{Irr}(P)$ be one of the irreducible characters of degree $(q - 1)^2(q + 1)/d$ from Remark 3.3. By Clifford theory the restriction $\chi|_U$ of χ to U is a sum of a 3-power number of conjugate characters of the same degree, in particular the degrees of the constituents of $\chi|_U$ are even. Moreover these constituents are rational and hence $\chi|_U$ is an orthogonally stable rational character that restricts orthogonally stably to a Sylow p -subgroup of U . Now [11, Theorem 4.3 and Corollary 4.4] yields

$$\text{disc}(\chi|_U) = (-1)^{\chi(1)/2} p^{\chi(1)/(p-1)} (\mathbb{Q}^\times)^2 = 1$$

and hence also the unitary discriminant of χ is 1. □

3.3 The characters of $\text{SL}_3(q)$

Theorem 3.4. *If q is odd then all characters $\chi \in \text{Irr}^o(\text{SL}_3(q))$ have unitary discriminant $(-1)^{\chi(1)/2}$.*

Proof. We use the description in [17] of the characters of $\text{SL}_3(q)$. The characters χ of degree $(q + 1)(q^2 + q + 1)$ and $(q - 1)(q^2 + q + 1)$ are Harish-Chandra induced from the maximal parabolic subgroup P from a character ψ of degree $(q + 1)$ resp. $(q - 1)$ of the Levi complement $\text{GL}_2(q)$.

If $\chi(1) = (q - 1)(q^2 + q + 1)$, then there is a unique such character ψ with $\chi = \psi^G$. In particular $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$. If $\mathbb{Q}(\psi^G)$ is not real, then also $\psi \in \text{Irr}^o(\text{GL}_2(q))$ and Theorem 3.1 shows that the unitary discriminant of ψ and hence the one of $\chi = \psi^G$ is $(-1)^{(q-1)/2}$.

If $\chi(1) = (q + 1)(q^2 + q + 1)$, then there are three such characters ψ inducing to the same character χ . As the Galois group of $\mathbb{Q}(\psi)/\mathbb{Q}(\chi)$ acts on the constituents of degree $q + 1$ of $\chi|_P$ it has at least one orbit of odd length. So one of these characters ψ satisfies $[\mathbb{Q}(\psi) : \mathbb{Q}(\chi)]$ is odd. By Lemma 2.6, this again allows to conclude that the unitary discriminant of χ is the same as the one of ψ .

It remains to consider the characters χ of degree $(q - 1)^2(q + 1)$ and $(q - 1)^2(q + 1)/3$ (if $q \equiv 1 \pmod{3}$).

For both degrees, the character χ does not appear in the permutation character of G on the $(q - 1)^2(q + 1)$ cosets of a Sylow p -subgroup S of G : This is clear if $\chi(1) = (q - 1)^2(q + 1) = [G : S]$ because all constituents of 1_S^G have degree $\leq [G : S] - 1$. If $\chi(1) = (q - 1)^2(q + 1)/3$, we note that the center of G has order 3 and orbits of length 3 on the cosets of G/S . So if χ occurs in 1_S^G , then this permutation character has three distinct constituents of degree $(q - 1)^2(q + 1)/3$ leading to the same contradiction as before.

In particular, the restriction of χ to P is a sum of the characters from Remark 3.3, showing again that the unitary discriminant of χ is trivial.

From the tables in [16], we see that the characters of degree $q(q+1)$ and $(q+1)(q^2+q+1)/3$ are rational. Their orthogonal discriminant can be read off from [6, Theorem 4.7]. \square

Theorem 3.5. *If q is even then all characters $\chi \in \text{Irr}^o(\text{SL}_3(q))$ have degree $q(q^2+q+1)$ and unitary discriminant $(-1)^q(q+1)$.*

Proof. For 2-powers q , the even degree irreducible characters of $\text{SL}_3(q)$ are as follows:

- (i) one character of degree q^2+q and Frobenius-Schur indicator $+$,
- (ii) the Steinberg character of degree q^3 and Frobenius-Schur indicator $+$,
- (iii) $q-1$ characters of degree $q(q^2+q+1)$, one of which is rational and of indicator $+$, while the others have indicator 'o' (see [17], [16]).

From [17, Table VI] and the arguments given just before we conclude that the $q-1$ characters from (iii) are Harish-Chandra induced from the $q-1$ irreducible characters ψ of degree q of the Levi factor $\text{GL}_2(q)$ of P . It follows that they have the same character field $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$ and discriminant. From Remark 3.2 we get $\text{disc}(\psi) = (-1)^{q/2}(q+1)$ whenever the character field $\mathbb{Q}(\chi)$ is not real, i.e. $\chi \in \text{Irr}^o(\text{SL}_3(q))$. As the index of P in $\text{SL}_3(q)$ is odd, also the discriminant of χ is $(-1)^{q/2}(q+1)$. \square

4 The special unitary group, general results

4.1 The special unitary group

Let p be a prime and let q be a power of p . The special unitary group $\text{SU}_3(q)$ is the stabiliser in $\text{SL}_3(q^2)$ of a non-degenerate Hermitian form on $\mathbb{F}_{q^2}^3$. Up to isometry there is a unique such form. We put

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and denote by Φ the \mathbb{F}_q -linear map on $\mathbb{F}_{q^2}^{3 \times 3}$ that raises each matrix entry to the q -th power. Then

$$\text{SU}_3(q) := \{\mathbf{g} \in \text{SL}_3(q^2) \mid \Phi(\mathbf{g})^{tr} \cdot \Omega \cdot \mathbf{g} = \Omega\}.$$

Let

$$B := \left\{ \begin{pmatrix} d & a & b \\ 0 & e & c \\ 0 & 0 & f \end{pmatrix} \in \mathrm{SU}_3(q) \right\} \text{ and } U := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}_3(q) \right\}.$$

Then U is the unipotent radical of B and a Sylow p -subgroup of $\mathrm{SU}_3(q)$, and $B = N_{\mathrm{SU}_3(q)}(U) = U \rtimes T$ is a (standard) Borel subgroup, where $T := \{\mathrm{diag}(d, e, f) \in \mathrm{SU}_3(q)\}$ is a maximal torus.

We also put

$$\mathbf{w} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

to denote a generator of the Weyl group of $\mathrm{SU}_3(q)$. Then

$$\mathrm{SU}_3(q) = B \dot{\cup} B\mathbf{w}B \text{ and } B \cap \mathbf{w}B\mathbf{w} = T.$$

We fix an element $\mathbf{t} \in T$ such that $T = \langle \mathbf{t} \rangle \cong C_{q^2-1}$ and put

$$\mathbf{t}_0 := \mathbf{t}^{(q^2-1)/2} = \mathrm{diag}(-1, 1, -1)$$

to denote the element of order 2 in T .

The center $Z := Z(U) \cong (\mathbb{F}_q, +)$ of the unipotent radical is generated as a normal subgroup of B by

$$\mathbf{z} := \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z,$$

where $b \in \mathbb{F}_{q^2}^\times$ is an element such that $b + b^q = 0$. Then \mathbf{t}_0 commutes with the elements of Z and inverts the classes in U/Z , i.e.

$$\mathbf{t}_0 \mathbf{h} Z \mathbf{t}_0 = \mathbf{h}^{-1} Z$$

for all $\mathbf{h} \in U$. We denote by A_0 the subgroup

$$A_0 := \langle \mathbf{z}, \mathbf{t} \rangle \cong (\mathbb{F}_q, +) \rtimes (\mathbb{F}_{q^2}^\times, \cdot).$$

We also need two rational quaternion algebras:

$$\mathcal{Q}_p := \mathcal{Q}_{\mathbb{Q}}(\infty, p) \text{ and } \mathcal{Q}_2 := \mathcal{Q}_{\mathbb{Q}}(\infty, 2) = (-1, -1)_{\mathbb{Q}}$$

ramified at the infinite place and the prime p resp. 2. For $p = 2$ we have that $\mathcal{Q}_2 = \mathcal{Q}_p$ and for $p \equiv 3 \pmod{4}$ the algebra $\mathcal{Q}_p = (-1, -p)_{\mathbb{Q}}$.

4.2 Restrictions to the Borel subgroup B

The character table and the conjugacy classes of B have already been obtained in [4, Table 2.1]. That paper also contains the character table of $\mathrm{SU}_3(q)$ using the same notation as in [16]. The index of the character names always gives their degree. To give the characters of B in a unified way we denote by

$$d := \gcd(3, q + 1) \in \{1, 3\}$$

the order of the center of $\mathrm{SU}_3(q)$, which is also the center of B . We put

$$\delta := \exp(2\pi i/(q^2 - 1)) \text{ and } \omega := \exp(2\pi i/3)$$

to denote a primitive $(q^2 - 1)$ th, resp. third, complex root of unity.

Proposition 4.1. [4, Table 2.1 b)] *The irreducible characters of B are as follows:*

- a) *The $q^2 - 1$ linear characters $\vartheta_1^{(u)}$ for $0 \leq u \leq q^2 - 2$ with character field $\mathbb{Q}(\delta^u)$.*
- b) *The $q + 1$ characters $\vartheta_{q^2-q}^{(u)}$ for $0 \leq u \leq q$ with character field $\mathbb{Q}(\delta^{(q-1)u})$.
Then $[\vartheta_{q^2-q}^{(0)}] = [\mathcal{Q}_p]$ and all the other characters have trivial Brauer element.*
- c) *The d^2 characters $\vartheta_{(q^2-1)/d}^{(u,v)}$ for $0 \leq u, v \leq d - 1$ with character field $\mathbb{Q}(\omega^u)$.*

Note that for $d = 1$ the character $\vartheta_{(q^2-1)/d}^{(0,0)}$ is denoted by ϑ_{q^2-1} in [4].

We want to describe the restrictions of the irreducible characters of $\mathrm{SU}_3(q)$ to B . For that, we introduce two further families of characters of B .

Definition 4.2. We define

$$\varphi^{(v)} := \left(d \sum_{j=1}^{(q+1)/d} \vartheta_{q^2-q}^{(dj+v)} \right), \quad \tau^{(u)} = \sum_{j=0}^{d-1} \vartheta_{(q^2-1)/d}^{(u,j)}$$

for arbitrary integers v and u . Note that if $d = 1$, then $\tau^{(0)} = \vartheta_{q^2-1}^{(0,0)}$. Here and in the following we consider the upper index u of $\vartheta_1^{(u)}$ modulo $q^2 - 1$ and the one of $\vartheta_{q^2-q}^{(u)}$ modulo $q + 1$.

Proposition 4.3. *The restrictions to B of the characters of $\mathrm{SU}_3(q)$ are given by:*

- i) $\mathrm{Res}_B(\chi_1) = \vartheta_1^{(0)}$.
- ii) $\mathrm{Res}_B(\chi_{q(q-1)}) = \vartheta_{q^2-q}^{(0)}$.

- iii) $\text{Res}_B(\chi_{q^3}) = \vartheta_1^{(0)} + \tau^{(0)} + \varphi^{(0)} - \vartheta_{q^2-q}^{(0)}$.
- iv) $\text{Res}_B(\chi_{q^2-q+1}^{(u)}) = \vartheta_1^{((q-1)u)} + \vartheta_{q^2-q}^{(u)}$, for $1 \leq u \leq q$.
- v) $\text{Res}_B(\chi_{q(q^2-q+1)}^{(u)}) = \vartheta_1^{((q-1)u)} + \tau^{(u)} + \varphi^{(u)} - \vartheta_{q^2-q}^{(u)} - \vartheta_{q^2-q}^{(-2u)}$, for $1 \leq u \leq q$.
- vi) $\text{Res}_B(\chi_{(q-1)(q^2-q+1)}^{(u,v,w)}) = \tau^{(u+v+w)} + \varphi^{(u+v+w)} - \vartheta_{q^2-q}^{(u-2v-2w)} - \vartheta_{q^2-q}^{(v-2u-2w)} - \vartheta_{q^2-q}^{(w-2u-2v)}$,
for $1 \leq u < v \leq (q+1)/d, v < w \leq (q+1), u+v+w \equiv 0 \pmod{q+1}$.
- vii) $\text{Res}_B(\chi_{(q-1)(q^2-q+1)/3}^{(u)}) = \vartheta_{(q^2-1)/3}^{(0,u)} + 1/3\varphi^{(0)} - \vartheta_{q^2-q}^{(0)}$, for $0 \leq u \leq 2$.
- viii) $\text{Res}_B(\chi_{q^3+1}^{(u)}) = \vartheta_1^{(u)} + \vartheta_1^{(-qu)} + \tau^{(u)} + \varphi^{(u)} - \vartheta_{q^2-q}^{(u)}$,
for $1 \leq u \leq q^2 - 1, (q-1) \nmid u$. Note that $\chi_{q^3+1}^{(u)} = \chi_{q^3+1}^{(-uq)}$.
- ix) $\text{Res}_B(\chi_{(q+1)(q^2-1)}^{(u)}) = \tau^{(u)} + \varphi^{(u)}$, for $1 \leq u < q^2 - q + 1, (q^2 - q + 1)/d \nmid u$.
Note that $\chi_{(q+1)(q^2-1)}^{(u)} = \chi_{(q+1)(q^2-1)}^{(-uq)} = \chi_{(q+1)(q^2-1)}^{(uq^2)}$ where the upper indices are understood modulo $q^2 - q + 1$.
- x) $\text{Res}_B(\chi_{(q+1)(q^2-1)/3}^{(u)}) = \vartheta_{(q^2-1)/3}^{(1,u)} + 1/3\varphi^{(1)}$, for $0 \leq u \leq 2$.
- xi) $\text{Res}_B(\chi_{(q+1)(q^2-1)/3}^{(u)'}) = \vartheta_{(q^2-1)/3}^{(2,u)} + 1/3\varphi^{(2)}$, for $0 \leq u \leq 2$.

Note that the characters $\chi_{(q-1)(q^2-q+1)/3}^{(u)}$, $\chi_{(q+1)(q^2-1)/3}^{(u)}$ and $\chi_{(q+1)(q^2-1)/3}^{(u)'}$ only exist if $d = 3$.

5 The unitary discriminants of $\text{SU}_3(q)$ for q even

In this section we assume that $q \geq 4$ is a power of 2. As $\chi_{q(q-1)}$ and χ_{q^3} are rational characters we get that

$$\text{Irr}^o(\text{SU}_3(q)) = \{\chi_{q(q^2-q+1)}^{(u)} \mid 1 \leq u \leq q\}.$$

The aim of this section is the proof of the following theorem.

Theorem 5.1. *Let $q \geq 4$ be a power of 2. Then $\text{disc}(\chi) = q + 1$ for all $\chi \in \text{Irr}^o(\text{SU}_3(q))$.*

From Proposition 4.3 we get

Remark 5.2. For $1 \leq u \leq q$ the sum

$$M^{(u)} := \chi_{q(q^2-q+1)}^{(u)} + \chi_{(q^2-q+1)}^{(u)} = (\vartheta_1^{((q-1)u)})^{\text{SU}_3(q)}$$

is the monomial character induced from the character $\vartheta_1^{((q-1)u)}$ of B . The character fields of $M^{(u)}$, $\chi_{q(q^2-q+1)}^{(u)}$, $\chi_{(q^2-q+1)}^{(u)}$ and $\vartheta_1^{((q-1)u)}$ are identical and isomorphic to $K^{(u)} := \mathbb{Q}(\delta^{(q-1)u})$. Let $V^{(u)}$ denote the $K^{(u)} \text{SU}_3(q)$ module affording the character $M^{(u)}$. Then $V^{(u)}$ is an orthogonal sum of the two absolutely irreducible submodules $V_{q(q^2-q+1)}^{(u)}$ and $V_{(q^2-q+1)}^{(u)}$. As $\text{SU}_3(q)$ fixes the standard form I_{q^3+1} on $V^{(u)}$ the discriminant of $\chi_{q(q^2-q+1)}^{(u)}$ is the discriminant of the restriction of the standard form to the submodule $V_{(q^2-q+1)}^{(u)}$.

Remark 5.3. In the notation of Section 4.1, we have

$$\text{SU}_3(q) = B \cup B\mathbf{w}B = B \dot{\cup} \bigcup_{\mathbf{h} \in U} B\mathbf{wh}.$$

So a basis of $V^{(u)}$ is given by $\{B\} \cup \{B\mathbf{wh} \mid \mathbf{h} \in U\}$.

In this notation we obtain the following

Lemma 5.4. *The Schur basis of $\text{End}_{\text{SU}_3(q)}(V^{(u)})$ is (I_{q^3+1}, E) where*

$$E_{B,B} = 0, E_{B,B\mathbf{wh}} = 1, E_{B\mathbf{wh},B} = 1 \text{ for all } \mathbf{h} \in U.$$

Proof. By the well known formulas for the Schur basis elements (see for instance [10, Proposition (1.10)]) we have $E_{B,B} = 0$ and $E_{B,B\mathbf{wh}} = \vartheta_1^{((q-1)u)}(\mathbf{h}) = 1$. To compute $E_{B\mathbf{wh},B}$ we need to write $\mathbf{h}^{-1}\mathbf{w} = \mathbf{g}\mathbf{wh}'$ for $\mathbf{h}' \in U$, $\mathbf{g} \in B$. But then $\mathbf{g}^{-1}\mathbf{h}^{-1} = \mathbf{wh}'\mathbf{w}^{-1}$ is an element of 2-power order in B and hence also $\mathbf{g} \in U$, so $\vartheta_1^{((q-1)u)}(\mathbf{g}) = 1$, whence

$$E_{B\mathbf{wh},B} = \vartheta_1^{((q-1)u)}(\mathbf{g})\vartheta_1^{((q-1)u)}(\mathbf{h}') = 1.$$

□

Lemma 5.5. *The eigenvalues of E are ϵq and $-\epsilon q^2$ for some $\epsilon \in \{1, -1\}$ with multiplicities $q(q^2 - q + 1)$ and $(q^2 - q + 1)$.*

Proof. The 2-dimensional $K^{(u)}$ -space generated by (I_{q^3+1}, E) is a ring, in particular E^2 is a $K^{(u)}$ -linear combination of E and I_{q^3+1} and E has exactly 2 distinct eigenvalues. Also the multiplicity of the eigenvalues are given by the dimensions of the irreducible constituents of $M^{(u)}$. Let a denote the eigenvalue of E occurring with multiplicity $q^2 - q + 1$ and b the one with multiplicity $q(q^2 - q + 1)$. Then

$$a(q^2 - q + 1) + bq(q^2 - q + 1) = \text{trace}(E) = 0,$$

so $a = -bq$. As the diagonal entries of E are 0 and the first diagonal entry of E^2 is q^3 by Lemma 5.4, we obtain the constant coefficient of the minimal polynomial of E as

$$-q^3 = ab = -b^2q \text{ and hence } b = \epsilon q$$

for some $\epsilon = \pm 1$. □

Corollary 5.6. *The rows of $E + \epsilon q^2 I_{q^3+1}$ span the submodule $V_{q(q^2-q+1)}^{(u)}$ and the rows of $E - \epsilon q I_{q^3+1}$ the submodule $V_{(q^2-q+1)}^{(u)}$.*

As U is a normal subgroup of B , we obtain a basis of the 2-dimensional B -eigenspace as follows.

Remark 5.7. The B -eigenspace for the character $\vartheta_1^{((q-1)u)}$ in $V^{(u)}$ is $W^{(u)} := \langle B, \sum_{\mathbf{h} \in U} B\mathbf{wh} \rangle$. Put

$$v^{(u)} := -\epsilon q^2 B + \sum_{\mathbf{h} \in U} B\mathbf{wh} \text{ and } w^{(u)} := \epsilon q B + \sum_{\mathbf{h} \in U} B\mathbf{wh}.$$

where ϵ is as in Lemma 5.5. Then

$$\langle v^{(u)} \rangle = W^{(u)} \cap V_{q(q^2-q+1)}^{(u)} \text{ and } \langle w^{(u)} \rangle = W^{(u)} \cap V_{(q^2-q+1)}^{(u)}.$$

The next lemma completes the proof of Theorem 5.1.

Lemma 5.8. *The unitary discriminant of $\chi_{q(q^2-q+1)}^{(u)}$ is represented by $q + 1$.*

Proof. The restriction of $M^{(u)}$ to B is

$$\text{Res}_B(M^{(u)}) = \vartheta_1^{((q-1)u)} + \vartheta_{q^2-q}^{(u)} + \text{Res}_B(\chi_{q(q^2-q+1)}^{(u)}).$$

The restriction of the character $\vartheta_{q^2-q}^{(u)}$ to the Sylow 2-subgroup U of B is a sum of $q - 1$ rational characters of degree q . As q is a multiple of 4, Corollary 2.17 implies that this restriction is unitary stable of unitary discriminant 1. So the discriminant of the submodule $V_{(q^2-q+1)}^{(u)}$ of $V^{(u)}$ is the square length of $w^{(u)}$, which is $q^2 + q^3 = (q + 1)q^2$. As the product of the discriminants of the two submodules $V_{(q^2-q+1)}^{(u)}$ and $V_{q(q^2-q+1)}^{(u)}$ of $V^{(u)}$ is $\text{disc}(V^{(u)}) = 1$, we obtain

$$\text{disc}(V_{q(q^2-q+1)}^{(u)}) = q + 1 = \text{disc}(\chi_{q(q^2-q+1)}^{(u)}).$$

□

6 The unitary discriminants of $\mathrm{SU}_3(q)$ for q odd

6.1 The strategy for q odd

Let q be odd and let $\chi \in \mathrm{Irr}^o(\mathrm{SU}_3(q))$ be an even degree indicator 'o' irreducible complex character. The goal of this subsection is to outline the strategy which we will use in the sequel to calculate the unitary discriminant of χ .

The restriction of χ to B decomposes as

$$\mathrm{Res}_B(\chi) = \chi_T + \chi_U,$$

where χ_T is the U -fixed part of $\mathrm{Res}_B(\chi)$. The character χ_T is also known as the *Harish-Chandra restriction* of χ .

For orthogonal characters χ the restriction $R := \mathrm{Res}_U(\chi_U)$ to U is an orthogonally stable character of the p -group U , so the determinant of R and hence the orthogonal determinant of χ_U is given in [11, Theorem 4.3 and Corollary 4.4]. As q is odd, R is a sum of odd degree characters and hence never unitary stable. However, it turns out that χ_U is a unitary stable character of B and hence has a well defined unitary discriminant. To compute $\mathrm{disc}(\chi_U)$, we restrict further to the metabelian subgroup $A_0 \leq B$ from Section 4.1 (Section 6.2).

Remark 6.1. The complex irreducible characters χ of $\mathrm{SU}_3(q)$ for which χ_T is non-zero are monomial characters on the set of $\chi(1) = q^3 + 1$ isotropic points in the natural 3-dimensional unitary geometry. Here we use condensation techniques to find $\mathrm{disc}(\chi_T)$:

Put $J_U := \frac{1}{|U|} \sum_{\mathbf{h} \in U} \mathbf{h}$ to denote the projection onto the U -fixed space and let V_χ be a $\mathbb{Q}(\chi)$ $\mathrm{SU}_3(q)$ -module affording the character χ . Then the decomposition $\chi = \chi_T + \chi_U$ corresponds to the orthogonal decomposition

$$V_\chi = V_\chi J_U \perp V_\chi(1 - J_U)$$

and hence

$$\mathrm{disc}(\chi) = \mathrm{disc}(V_\chi J_U) \mathrm{disc}(\chi_U).$$

The discriminant of the 2-dimensional module $V_\chi J_U$ is computed by obtaining the action of $J_U \mathbf{t} J_U$ and $J_U \mathbf{w} J_U$ on this module.

6.2 The unitary discriminants of A_0

In this section we compute the unitary discriminants of the subgroup A_0 of B defined in Section 4.1, which are then used in Section 6.3 to obtain the unitary discriminants of the irreducible characters of degree $q^2 - q$ of B .

Recall that $A_0 = \langle \mathbf{z}, \mathbf{t} \rangle \cong (\mathbb{F}_q, +) \rtimes (\mathbb{F}_{q^2}^\times, \cdot)$. The subgroup $\langle Z(U), \mathbf{t}^{q-1} \rangle$ is an abelian normal subgroup of order $q(q+1)$ and index $(q-1)$ in A_0 . The center $Z(A_0) \cong C_{q+1}$ is generated by \mathbf{t}^{q-1} . Let $\lambda^{(u)}$ be the linear character of $Z(A_0)$ defined by

$$\lambda^{(u)}(\mathbf{t}^{q-1}) = \delta^{(q-1)u} \text{ for } u = 0, \dots, q.$$

The group A_0 has (q^2-1) linear characters, the ones that restrict trivially to $Z(U)$, and $q+1$ irreducible characters of degree $q-1$, $\mu_{q-1}^{(u)}$ for $u = 0, \dots, q$, that restrict to $Z(A_0)$ as $(q-1)\lambda^{(u)}$.

Theorem 6.2. *The non-linear irreducible characters of A_0 are the characters $\mu_{q-1}^{(u)}$, $0 \leq u \leq q$ of degree $q-1$ and character field $\mathbb{Q}(\mu_{q-1}^{(u)}) = \mathbb{Q}(\delta^{(q-1)u})$.*

- a) $\mu_{q-1}^{(0)}$ is the character of a rational representation. Its orthogonal discriminant is $(-1)^{(q-1)/2}q$.
- b) $\mu_{q-1}^{((q+1)/2)}$ is a rational character of Frobenius-Schur indicator $-$. Its Brauer element is $[\mu_{q-1}^{((q+1)/2)}] = [\mathcal{Q}_p]$.
- c) For $u \notin \{0, (q+1)/2\}$ the Frobenius-Schur indicator of $\mu_{q-1}^{(u)}$ is 'o' and the discriminant is

$$\text{disc}(\mu_{q-1}^{(u)}) = (-1)^{(q-1)/2}q^{u-1} = \begin{cases} -q & \text{if } (u, q) \equiv (0, 3) \pmod{(2, 4)} \\ -1 & \text{if } (u, q) \equiv (1, 3) \pmod{(2, 4)} \\ q & \text{if } (u, q) \equiv (0, 1) \pmod{(2, 4)} \\ 1 & \text{if } (u, q) \equiv (1, 1) \pmod{(2, 4)}. \end{cases}$$

To prove the theorem, we restrict further to a subgroup A of A_0 : Write $q+1 = eb$, where b is odd and e is a power of 2 and put $\mathbf{t}_1 := \mathbf{t}^b$ a generator of the subgroup of order $e(q-1)$ of the torus T . Put

$$A := \langle \mathbf{z}, \mathbf{t}_1 \rangle.$$

Then $A_0 = A \times \langle \mathbf{t}^{(q-1)e} \rangle$ and the irreducible characters of A_0 are obtained as tensor product $\mu \otimes \lambda$, where μ is an irreducible character of A and λ a linear character of the cyclic group $\langle \mathbf{t}^{(q-1)e} \rangle$ of order b .

We first compute the unitary/orthogonal discriminants and Schur indices of the even degree irreducible characters of A .

Lemma 6.3. *For $u = 0, \dots, q$ we put $\mu := \text{Res}_A(\mu_{q-1}^{(u)})$. Then μ only depends on $u \pmod{e}$.*

- a) If e divides u , then μ is the character of a rational representation. Its orthogonal discriminant is $(-1)^{(q-1)/2}q$.

- b) If $u \pmod{e} = e/2$, then μ is a rational character of Frobenius-Schur indicator $-$. Its Brauer element is $[\mu] = [\mathcal{Q}_p]$, the class of the rational quaternion algebra ramified only at p and ∞ .
- c) If $u \pmod{e} \notin \{0, e/2\}$, then $e \geq 4$ and hence $q \equiv 3 \pmod{4}$ is an odd power of a prime $p \equiv 3 \pmod{4}$. Then the character field of μ is $\mathbb{Q}(\mu) = \mathbb{Q}(\zeta_e^u)$ and contains a primitive fourth root of unity. The unitary discriminant of μ is

$$\text{disc}(\mu) = \begin{cases} -q & \text{if } u \text{ is even} \\ -1 & \text{if } u \text{ is odd.} \end{cases}$$

Proof. Put $Z := Z(U)$ and $\mathbf{t}_2 := \mathbf{t}^{(q-1)b} = \mathbf{t}_1^{q-1}$ to denote a generator of the center of A . Then the character μ is a monomial character induced from a linear character, λ , of the normal subgroup $Z \times \langle \mathbf{t}_2 \rangle$. Theorem 2.15 says that, for any field K , the Brauer element $[\mu]_{K(\mu)}$ of μ is the inverse of the class of the crossed product algebra $\mathcal{Q} := (K(\lambda), \Gamma)$ in the Brauer group of $K(\mu)$, where Γ is the Galois group of $K(\lambda)/K(\mu)$. In our case, $K(\lambda)$ is generated by $\lambda(Z) = \langle \exp(2\pi i/p) \rangle$ and $\lambda(\mathbf{t}_2) = \zeta_e^u$. Let $e_0 := e/\text{gcd}(e, u)$ denote the order of $\lambda(\mathbf{t}_2)$. For $K = \mathbb{Q}$ we have $\Gamma = C_{p-1} = \langle \sigma \rangle = \text{Gal}(\mathbb{Q}(\exp(2\pi i/p))/\mathbb{Q})$, where the cocycle is given by $\sigma^{p-1} = \zeta_e^u$.

Let $\langle a \rangle = (\mathbb{Z}/p\mathbb{Z})^\times$ and put

$$A' := \langle \mathbf{z}', \mathbf{t}'_2, \sigma \mid \mathbf{z}'^p = 1, \mathbf{t}'_2{}^{e_0} = 1, \mathbf{z}'^\sigma = \mathbf{z}'^a, \sigma^{p-1} = \mathbf{t}'_2 \rangle.$$

Then $|A'| = p(p-1)e_0$ and A' is a group all of whose Sylow subgroups are cyclic. Moreover, $[\mathcal{Q}] = [\chi]$ for an irreducible faithful character χ of degree $p-1$ of the group A' .

- a) If $e_0 = 1$ and $K = \mathbb{Q}$, then $\mathcal{Q} = \mathbb{Q}^{p-1 \times p-1}$, so we get the Schur indices in a). For the orthogonal determinant, note that the representation corresponding to μ fixes the root lattice A_{q-1} of determinant q .
- b) If $e_0 = 2$, so $\lambda(\mathbf{t}_2) = -1$, then again χ is rational. Now the Frobenius-Schur indicator of χ is $-$. To compute the local Schur indices of χ we use Theorem 2.14. Over the completions at primes dividing $p-1$ this Schur-index is 1, so it remains to compute $m_{\mathbb{Q}_p}(\chi)$. Any p -modular constituent of χ is a (faithful) representation of $A'/\langle \mathbf{z}' \rangle$, a cyclic group of order $(p-1)e_0 = 2(p-1)$ and hence its character field is the cyclotomic field of order $2(p-1)$ over \mathbb{Q}_p . This has degree 2 over \mathbb{Q}_p , and hence $m_{\mathbb{Q}_p}(\chi) = 2$. So $[\chi] = [\mathcal{Q}_p]$ is the class of the rational quaternion algebra ramified at p and ∞ .
- c) If $e_0 \geq 4$, then the indicator of χ is 'o', in particular the Schur indices of χ at the infinite places are 1. Moreover in this case q and hence p is $\equiv 3 \pmod{4}$

and \mathbb{Q}_p does not contain a e_0 -th root of unity, so $\mathbb{Q}_p(\chi)$ is the unramified extension of degree 2 of \mathbb{Q}_p . As in (b) this field is also the character field of all p -modular constituents of χ , so the p -adic Schur index of χ is 1 by Theorem 2.14. For odd prime divisors ℓ of $p - 1$, the character χ remains irreducible modulo ℓ and hence again all Schur indices are 1. So the only prime where χ can have a nontrivial local Schur index is the unique prime of $\mathbb{Q}(\zeta_e^u)$ that divides 2. As the sum of the Hasse invariants is trivial, all local Schur indices are 1.

To compute the unitary discriminants of the characters in c) we use the strategy from [13, Section 10] as described in Section 2.6. Here we have $q \equiv 3 \pmod{4}$ and $e_0 \geq 4$. The character field $L := \mathbb{Q}(\mu) = \mathbb{Q}(\zeta_e^u)$ is a complex cyclotomic field of 2-power order. Let K denote its maximal real subfield. Let $\rho : A \rightarrow \mathrm{GL}_{p-1}(L)$ denote the representation affording the character μ .

The group A admits an automorphism $\alpha \in \mathrm{Aut}(A)$ with

$$\alpha|_Z = \mathrm{id}|_Z, \alpha(\mathbf{t}_1) = \mathbf{t}_1^q.$$

The restriction of α to the center of A inverts all elements of $\langle \mathbf{t}_2 \rangle$ and hence α interchanges μ with its complex conjugate character. In particular, K is the fixed field of the restriction of α to L . Put $\tilde{A} := A \rtimes \langle \alpha \rangle$ to denote the semidirect product of A with the group $\langle \alpha \rangle$ of order 2. Let $\tilde{\rho} := \rho^{\tilde{A}}$ be the induced representation and

$$X := \langle \rho(A) \rangle_L^\alpha := \{x \in \langle \rho(A) \rangle_L \mid \alpha(x) = x\}$$

denote the fixed algebra of α in the enveloping algebra of $\rho(A)$.

Then by Theorem 2.18 the class of X is the class of the enveloping algebra of $\tilde{\rho}(\tilde{A})$ in the Brauer group of K . To determine $[X] \in \mathrm{Br}(K)$ we compute the local Schur indices of $\tilde{\rho}$. Now $\tilde{\rho}$ is induced from the same linear character of the abelian normal subgroup $Z \times \langle \mathbf{t}_2 \rangle$ as ρ .

Recall that we are in the case $q \equiv 3 \pmod{4}$ and put $c := \frac{q-1}{2}$. Then c is odd and $\mathbf{t}_1^c \alpha$ acts as complex conjugation on $\mathbb{Q}(\exp(2\pi i/p), \zeta_e^u)$ and satisfies

$$(\mathbf{t}_1^c \alpha)^2 = \mathbf{t}_1^c \mathbf{t}_1^{cq} \alpha^2 = \mathbf{t}_1^{c(q+1)} = \mathbf{t}_0.$$

This allows to conclude that the real Schur index of $\tilde{\rho}$ is 2 if and only if $\tilde{\rho}(\mathbf{t}_0) = -1$, so if and only if u is odd.

For the odd primes ℓ dividing $q - 1$ the character of $\tilde{\rho}$ is in an ℓ -block of defect 0, so all ℓ -local Schur indices are 1. As 2 is totally ramified in the character field of $\tilde{\rho}$, the 2-adic Schur index can be read off from the p -adic Schur indices. As above we can use [21] to pass to the group $A' \rtimes \langle \alpha \rangle$ whose Sylow p -subgroups are cyclic.

As \mathbb{Q}_p contains primitive $(p-1)$ st roots of unity, Theorem 2.14 yields that these p -adic Schur indices are all 1. So we get

$$[X]_K = \begin{cases} [K] & \text{if } u \text{ is even} \\ [(-1, -1)_K] & \text{if } u \text{ is odd.} \end{cases}$$

We are now in the position to apply Theorem 2.18. From the above we obtain that X is an orthogonal subalgebra if and only if u is even. Then orthogonal determinant of the induced involution on X can be obtained as the determinant of any skew symmetric unit in X (see [12, Proposition 2.2]). Now X contains $\rho(Z)$ and the skew element $\rho(\mathbf{z}) - \rho(\mathbf{z}^{-1})$ has determinant $p^{(q-1)/(p-1)} \in q(\mathbb{Q}^\times)^2$. By Theorem 2.18 we hence have $\text{disc}(\mu) = -q$ here.

The Frobenius-Schur indicator of the character of $\tilde{\rho}$ is -1, if and only if u is odd, so here the restriction of the involution to X is symplectic. By Theorem 2.18 (c) the discriminant of μ is the L -discriminant of $[X]_K$. As $L = \mathbb{Q}(\zeta_e)$ contains a primitive fourth root of unity we have $[X]_K = (-1, -1)_K = (L, -1)_K$, so the L -discriminant of $[X]_K$ is -1. \square

When computing unitary discriminants of $\chi \in \text{Irr}^o(G)$ for $G \in \{A_0, B, \text{SU}_3(q)\}$ we will face the situation that the restriction of χ to A contains a constituent from Lemma 6.3 (b). Then Proposition 2.16 (a) shows that the contribution of this character to the unitary discriminant is trivial, if $q \equiv 1 \pmod{4}$. However, in the case where $q \equiv 3 \pmod{4}$ we need to compute the discriminant of $[\mathcal{Q}_p]$ over the character field of χ . It turns out that in our situations $\mathbb{Q}(\chi)$ satisfies the assumption of the next lemma, showing that the $\mathbb{Q}(\chi)$ -discriminant of $[\mathcal{Q}_p]$ is $-p$.

Lemma 6.4. *Let p be a prime, $p \equiv 3 \pmod{4}$. Let L be an abelian non-real number field with conductor dividing $p^a + 1$ for some odd integer a . Then L splits \mathcal{Q}_p and $\text{disc}_L(\mathcal{Q}_p) = -p$.*

Proof. Let K denote the real subfield of L . We show that $(L, -p)_K \cong \mathcal{Q}_p \otimes K$.

Let ζ be a primitive $(p^a + 1)$ th root of unity. Then $\mathbb{Q}_p(\zeta)$ is the unramified extension of degree $2a$ of \mathbb{Q}_p . As the a th power of the Frobenius inverts ζ and hence is the complex conjugation on $\mathbb{Q}(\zeta)$, all p -adic completions of any non-real subfield of $\mathbb{Q}(\zeta)$ have even degree over \mathbb{Q}_p . Moreover the p -adic completions of the maximal real subfield $\mathbb{Q}(\zeta + \zeta^{-1})$ have odd degree, a , over \mathbb{Q}_p .

In particular, L splits \mathcal{Q}_p as it splits this algebra at the infinite places and at the unique ramified place, p . Similarly we get that $\mathcal{Q}_p \otimes K$ is the quaternion algebra over K that is exactly ramified at all infinite places of K and all the places of K that divide p . The same ramification behaviour holds for $(L, -p)_K$ for the infinite primes and the ones dividing p .

It remains to show that no other primes ramify in $(L, -p)_K$.

We first consider the dyadic primes λ of K . If $p \equiv -1 \pmod{8}$ then $\sqrt{-p} \in \mathbb{Q}_2$, so we can assume that p and hence p^a is congruent to 3 mod 8. In particular, $\mathbb{Q}_2(\sqrt{-p})$ is the unramified quadratic extension of the 2-adics. Moreover $p^a + 1 = 4 \cdot x$ for some odd number x and the completion $L_{(2)}$ of L at λ is a subfield of $\mathbb{Q}_2(i)\mathbb{Q}_2(\zeta_x)$, a field with inertia subfield $\mathbb{Q}_2(\zeta_x)$ and ramification degree 2. As $L_{(2)}/K_{(2)}$ is ramified of degree 2, the completion $K_{(2)}$ of K at λ is totally unramified of degree, say, $f := [K_{(2)} : \mathbb{Q}_2]$ over the 2-adics. If f is even, then the unramified quadratic extension $\mathbb{Q}_2(\sqrt{-p})$ is contained in $K_{(2)}$, so $(L, -p)_K$ is split at λ . So f is odd and the Galois group of $L_{(2)}/\mathbb{Q}_2$ is abelian of order $2f$. In particular,

$$L_{(2)} = K_{(2)}\mathbb{Q}_2(\sqrt{a})$$

is the compositum of $K_{(2)}$ with a ramified quadratic extension $\mathbb{Q}_2(\sqrt{a})$ of \mathbb{Q}_2 and

$$(L_{(2)}, -p)_{K_{(2)}} = (a, -p)_{\mathbb{Q}_2} \otimes K_{(2)}.$$

Now the conductor of $\mathbb{Q}_2(\sqrt{a})$ is not a multiple of 8, so a is a unit in \mathbb{Z}_2 and hence a norm in the unramified extension $\mathbb{Q}_2(\sqrt{-p})/\mathbb{Q}_2$. Therefore $(a, -p)_{\mathbb{Q}_2}$ is split and so is $(L_{(2)}, -p)_{K_{(2)}}$.

Now let $\ell \neq p$ be an odd rational prime contained in some prime λ of K that ramifies in $(L, -p)_K$. The only prime ramifying in $\mathbb{Q}(\sqrt{-p})/\mathbb{Q}$ is p . Therefore λ ramifies in L/K , and hence ℓ divides the conductor of L , so ℓ divides $p^a + 1$, i.e. $(-p)^a \equiv 1 \pmod{\ell}$. Recall that a is odd. So $-p$ is a square mod ℓ and hence also in \mathbb{Q}_ℓ . Therefore primes dividing ℓ cannot ramify in $(L, -p)_K$. \square

From Lemma 6.3 we now conclude Theorem 6.2:

Proof. (of Theorem 6.2) We have $A_0 = A \times \langle \mathbf{t}^{(q-1)e} \rangle$ and $\mu_{q-1}^{(u)} = \mu \otimes \lambda$, where $\mu = \text{Res}_A(\mu_{q-1}^{(u)})$ is unitary stable and λ a linear character of the cyclic group $\langle \mathbf{t}^{(q-1)e} \rangle$ of order b . If $u = 0$ resp. $u = (q+1)/2$, then $\lambda = 1$. So case (a) and (b) of Theorem 6.2 follow from case (a) and (b) of Lemma 6.3. In all other cases, the character field $L := \mathbb{Q}(\mu_{q-1}^{(u)}) = \mathbb{Q}(\delta^{(q-1)u})$ is a complex number field; in particular the Frobenius-Schur indicator of $\mu_{q-1}^{(u)}$ is ‘o’. If the Frobenius-Schur indicator of μ is ‘o’ or +, then the unitary discriminant of $\mu_{q-1}^{(u)}$ is represented by any representative of $\text{disc}(\mu)$, for short

$$\text{disc}(\mu_{q-1}^{(u)}) = \text{disc}(\mu \otimes \lambda) = \text{disc}(\mu).$$

So it remains to consider the case where $u \equiv e/2 \pmod{e}$, i.e. $[\mu] = [\mathcal{Q}_p]$, but $u \neq (q+1)/2$. For $q \equiv 3 \pmod{4}$ the character field L satisfies the assumption of Lemma 6.4. Using Proposition 2.16 (a) we get that

$$\text{disc}(\mu_{q-1}^{(u)}) = \begin{cases} 1 & q \equiv 1 \pmod{4} \\ -q & q \equiv 3 \pmod{4} \end{cases}.$$

\square

6.3 The unitary discriminants of B

Theorem 6.5. *The irreducible even degree characters of B are the characters $\vartheta_{q^2-q}^{(v)}$, $0 \leq v \leq q$ of degree $q(q-1)$ and the characters $\vartheta_{(q^2-1)/d}^{(u,v)}$, $0 \leq u, v \leq d-1$.*

- a) $\vartheta_{q^2-q}^{((q+1)/2)}$ is the character of a rational representation. Its orthogonal discriminant is $(-1)^{(q-1)/2}q$.
- b) $\vartheta_{q^2-q}^{(0)}$ is a rational character of Frobenius-Schur indicator $-$. Its Brauer element is $[\vartheta_{q^2-q}^{(0)}] = [\mathcal{Q}_p]$.
- c) For $v \notin \{0, (q+1)/2\}$ the Frobenius-Schur indicator of $\vartheta_{q^2-q}^{(v)}$ is 'o', the character field is $L = \mathbb{Q}(\vartheta_{q^2-q}^{(v)}) = \mathbb{Q}[\delta^{v(q-1)}]$ and the unitary discriminant is

$$\text{disc}(\vartheta_{q^2-q}^{(v)}) = (-1)^{(q-1)/2}q^{v-1} = \begin{cases} -q & \text{if } (v, q) \equiv (0, 3) \pmod{(2, 4)} \\ -1 & \text{if } (v, q) \equiv (1, 3) \pmod{(2, 4)} \\ q & \text{if } (v, q) \equiv (1, 1) \pmod{(2, 4)} \\ 1 & \text{if } (v, q) \equiv (0, 1) \pmod{(2, 4)}. \end{cases}$$

- (d) The characters $\vartheta_{(q^2-1)/d}^{(u,v)}$ have trivial unitary discriminant.

Proof. **(d)** The irreducible characters $\vartheta_{(q^2-1)/d}^{(u,v)}$ of B are trivial on $Z(U)$ and induced from a non-trivial linear character of $U/Z(U)$, an elementary abelian group of order q^2 . Let \mathbf{t}_0 be the element of order 2 in the torus T and consider the semidirect product $H := U \rtimes \langle \mathbf{t}_0 \rangle$, a normal subgroup of B . Conjugation by \mathbf{t}_0 inverts the elements of $U/Z(U)$, so the restriction $R := \text{Res}_H(\vartheta_{(q^2-1)/d}^{(u,v)})$ of $\vartheta_{(q^2-1)/d}^{(u,v)}$ to H is the sum of orthogonal irreducible characters of H of degree 2. The images of the corresponding degree 2 representations are dihedral groups of order $2p$, so these constituents are orthogonal and of Schur index 1. As the restriction of R to U is orthogonally stable, the orthogonal discriminant of the unitary stable character real character R is 1 by the formula in [11, Theorem 4.3 and Corollary 4.4]. Moreover, no constituent of R has a non-trivial Schur index, so we conclude that $[\vartheta_{(q^2-1)/d}^{(u,v)}] = 1$ and the unitary discriminant of $\vartheta_{(q^2-1)/d}^{(u,v)}$ is represented by 1.

(a), (b), (c) It remains to consider the characters $\vartheta_{q^2-q}^{(v)}$. These restrict non-trivially to $Z(U)$. As $Z(U)$ is a normal subgroup of B , the trivial character of $Z(U)$ does not occur in the restriction of $\vartheta_{q^2-q}^{(v)}$ to $Z(U)$, so the restriction of $\vartheta_{q^2-q}^{(v)}$ to the group A_0 is a sum of the characters $\mu_{q-1}^{(u)}$ from Theorem 6.2. From the character table of B in [4] we obtain the restriction of $\vartheta_{q^2-q}^{(v)}$ to the center of A_0 as

$$\text{Res}_{Z(A_0)}(\vartheta_{q^2-q}^{(v)}) = \sum_{u \neq v} (q-1)\lambda^{(u)}$$

and hence

$$\text{Res}_{A_0}(\vartheta_{q^2-q}^{(v)}) = \sum_{u \neq v} \mu_{q-1}^{(u)}.$$

(a) If $v = (q+1)/2$, then $\vartheta_{q^2-q}^{(v)}$ is a rational orthogonal character. Its discriminant is given in [6, Theorem 4.7].

(b) For $v = 0$, $\vartheta_{q^2-q}^{(v)}$ is again rational, but now its restriction to A_0 contains the indicator – character $\mu_{q-1}^{((q+1)/2)}$ with multiplicity 1 showing that the Brauer element $[\vartheta_{q^2-q}^{(0)}] = [\mu_{q-1}^{((q+1)/2)}] = [\mathcal{Q}_p]$.

(c) In the remaining cases, the character field L of $\vartheta_{q^2-q}^{(v)}$ satisfies the assumptions of Lemma 6.4 if $q \equiv 3 \pmod{4}$. Moreover $R = \text{Res}_{A_0}(\vartheta_{q^2-q}^{(v)})$ is unitary stable, so we obtain the unitary discriminant of $\vartheta_{q^2-q}^{(v)}$ from Theorem 6.2: If v is even, then R contains $(q+1)/2$ summands $\mu_{q-1}^{(u)}$ with odd u and $(q-1)/2$ summands $\mu_{q-1}^{(u)}$ with even u , so

$$\text{disc}(\vartheta_{q^2-q}^{(v)}) = \begin{cases} -q & q \equiv 3 \pmod{4} \\ 1 & q \equiv 1 \pmod{4}. \end{cases}$$

If v is odd, then R contains $(q-1)/2$ summands $\mu_{q-1}^{(u)}$ with odd u and $(q+1)/2$ summands $\mu_{q-1}^{(u)}$ with even u , so

$$\text{disc}(\vartheta_{q^2-q}^{(v)}) = \begin{cases} -1 & q \equiv 3 \pmod{4} \\ q & q \equiv 1 \pmod{4}. \end{cases}$$

□

For the characters $\varphi^{(v)}$ and $\tau^{(u)}$ from Definition 4.2 we find the following.

Corollary 6.6. *The unitary discriminant of $\tau^{(u)}$ is trivial. Let L be a complex number field so that L satisfies the assumption of Lemma 6.4 in the case where $q \equiv 3 \pmod{4}$. Then*

$$\text{disc}_L(\varphi^{(v)}) = \begin{cases} q & \text{if } q \equiv 1 \pmod{4} \\ 1 & \text{if } q \equiv 3 \pmod{4} \end{cases}$$

Proof. All summands of $\tau^{(u)}$ are of unitary discriminant 1 and hence $\text{disc}(\tau^{(u)}) = 1$. As L satisfies the assumptions of Lemma 6.4 we see from Proposition 2.16 that

$$\text{disc}_L(\vartheta_{q^2-q}^{(0)}) = \begin{cases} 1 & q \equiv 1 \pmod{4} \\ -q & q \equiv 3 \pmod{4} \end{cases}$$

and hence $\vartheta_{q^2-q}^{(0)}$ contributes in the same way as all the other $\vartheta_{q^2-q}^{(v)}$ with even v . The character $\varphi^{(u)}$ is a sum of $q+1$ irreducible characters $\vartheta_{q^2-q}^{(v)}$ of which $(q+1)/2$

have an odd upper index v and $(q+1)/2$ have an even upper index v . If $q \equiv 1 \pmod{4}$ then $(q+1)/2$ is odd and hence Theorem 6.3 yields $\text{disc}_L(\varphi^{(v)}) = q$. For $q \equiv 3 \pmod{4}$ the number $(q+1)/2$ is even, so Theorem 6.3 implies that $\text{disc}_L(\varphi^{(v)}) = 1$. \square

6.4 The unitary discriminants of $\text{SU}_3(q)$ for q odd

Theorem 6.7. *Let q be a power of an odd prime p and put*

$$f(u) := \begin{cases} -(\delta^{(q+1)u} - \delta^{-(q+1)u})^2 & \text{if } (q-1)/2 \text{ does not divide } u, \\ -(\delta^u + \delta^{qu})^2 & \text{if } (q-1)/2 \text{ does divide } u. \end{cases}$$

The following table gives the unitary discriminant $\text{disc}(\chi)$ for the characters $\chi \in \text{Irr}^o(\text{SU}_3(q))$.

χ	parameters	$\text{disc}(\chi)$	
		$q \equiv 1 \pmod{4}$	$q \equiv 3 \pmod{4}$
$\chi_{(q-1)(q^2-q+1)}^{(u,v,w)}$	$1 \leq u < v \leq (q+1)/d,$ $v < w \leq q+1$ $u+v+w \equiv 0 \pmod{q+1}$	q	$-q$
$\chi_{q^3+1}^{(u)}$	$1 \leq u < (q^2-1)$ $(q-1) \nmid u, (q+1) \nmid u$ $(u) = (-uq)$	$-q^{u+1}f(u)$	$q^{u+1}f(u)$
$\chi_{(q+1)(q^2-1)}^{(u)}$	$1 \leq u < q^2 - q + 1$ $(q^2 - q + 1)/d \nmid u$	q	1
$\chi_{(q+1)(q^2-1)/3}^{(u)}$	$0 \leq u \leq 2$	q	1
$\chi_{(q+1)(q^2-1)/3}^{(u)'}$	$0 \leq u \leq 2$	q	1

Proof. The unitary discriminants of the characters χ of $\text{SU}_3(q)$ are obtained by restriction to the Borel subgroup B . We have

$$\text{Res}_B(\chi) = \chi_T + \chi_U$$

where χ_U is unitary stable. Note that for $q \equiv 3 \pmod{4}$ and all characters $\chi \in \text{Irr}^o(\text{SU}_3(q))$, the character field $L = \mathbb{Q}(\chi)$ satisfies the assumption of Lemma 6.4. So we obtain $\text{disc}(\chi_U)$ from Theorem 6.5 using Proposition 4.3.

For all characters $\chi \in \text{Irr}^o(\text{SU}_3(q))$, except for the ones of degree $q^3 + 1$, the character χ_T is 0, so it remains to consider the characters $\chi^{(u)} := \chi_{q^3+1}^{(u)}$. Here $\chi_U^{(u)}(1) = q^3 - 1$ and

$$\text{disc}(\chi_U^{(u)}) = (-1)^{q-1} q^{1+u}.$$

To handle the discriminant of the U -fixed space we use the strategy and notation from Remark 6.1.

Let Φ denote the Frobenius automorphism as in Section 4.1. Then $\chi^{(u)} \circ \Phi = \chi^{(-u)}$ is the complex conjugate character and we are in the position to apply Theorem 2.18 to compute the discriminant of the 2-dimensional $J_U \mathbb{Q} \mathrm{SU}_3(q) J_U$ -module $W^{(u)} := \mathbb{Q}[\delta^u] V^{(u)} J_U$. As in Remark 5.7, a $\mathbb{Q}[\delta^u]$ -basis for $W^{(u)}$ is given by

$$(B, \sum_{\mathbf{h} \in U} B \mathbf{w} \mathbf{h}).$$

We put $L := \mathbb{Q}[\delta^u + \delta^{-qu}] = \mathbb{Q}(\chi^{(u)})$ to denote the character field of $\chi^{(u)}$ and let K be its maximal real subfield. Then K is also the fixed field of Φ in L . We denote the matrix representation of $J_U \mathbb{Q} \mathrm{SU}_3(q) J_U$ on $W^{(u)}$ by ρ and put

$$R := \langle \rho(J_U \mathbf{g} J_U) \mid \mathbf{g} \in \mathrm{SU}_3(q) \rangle_L.$$

Then R is the L -algebra generated by the two matrices

$$\rho(J_U \mathbf{t} J_U) = \mathrm{diag}(\delta^u, \delta^{-qu}) \text{ and } W := \rho(J_U \mathbf{w} J_U) = \begin{pmatrix} 0 & 1 \\ q^3 & 0 \end{pmatrix}.$$

The form of W can be obtained from explicit computations in the Yokonuma algebra. Alternatively, note that the second basis vector is the image of the first one and now the second row is obtained from the fact that W is self adjoint with respect to the invariant form $\mathrm{diag}(q^3, 1)$.

The Frobenius automorphism Φ commutes with J_U , fixes \mathbf{w} and maps \mathbf{t} to \mathbf{t}^q . The elements

$$x := \mathbf{t}^{(q+1)} - \mathbf{t}^{-(q+1)}, \quad y := \mathbf{t} + \mathbf{t}^q - \mathbf{t}^{-1} - \mathbf{t}^{-q}$$

give rise to skew-symmetric elements $X := \rho(J_U x J_U)$ and $Y := \rho(J_U y J_U)$ in the Φ fixed algebra R^Φ . If u is not a multiple of $(q-1)/2$, then

$$\det(X) = -(\delta^{(q+1)u} - \delta^{-(q+1)u})^2 =: f(u)$$

is non-zero. If u is a multiple of $(q-1)/2$, then $2u/(q-1)$ is odd, as u is not a multiple of $(q-1)$. As $\delta^{(q^2-1)/2} = -1$, we compute $\delta^{qu} = -\delta^{-u}$ and hence

$$\det(Y) = -(\delta^u + \delta^{qu} - \delta^{-u} - \delta^{-qu})^2 = -4(\delta^u + \delta^{qu})^2 =: 4f(u)$$

is non-zero.

To conclude that $\mathrm{disc}(\chi_T^{(u)}) = -f(u)$ using Theorem 2.18 it remains to show that $R^\Phi \cong K^{2 \times 2}$. This is clear if q is a square, since then the minimal polynomial of W is reducible over \mathbb{Q} . So assume that q is not a square.

If u is not a multiple of $(q-1)/2$, then R^Φ is spanned as a K -algebra by $X_0 := 2X - \text{trace}(X)$ and W , so

$$R^\Phi \cong (\gamma^2, q^3)_K$$

where $\gamma = (\delta^{u(q+1)} - \delta^{-u(q+1)}) \in \mathbb{Q}[\delta]$. Then $K[X_0] \cong K[\gamma] = \mathbb{Q}[\delta^{u(q+1)}]$ is a maximal subfield of R^Φ and conjugation by W induces the non-trivial Galois automorphism of $K[\gamma]/K$. By Remark 2.4, $R^\Phi \cong K^{2 \times 2}$ if and only if no place \wp of K ramifies in R^Φ . As $q^3 > 0$ all infinite places of K are unramified in R^Φ . By Remark 2.5, the finite places of K that can possibly ramify in R^Φ are those dividing $q \text{disc}(K[\gamma]/K)$.

The places \wp dividing q are split in $K[\gamma]/K$, so they do not ramify in R^Φ . Note that $K[\gamma] = \mathbb{Q}[\delta^{u(q+1)}]$ is a cyclotomic field and K is its maximal real subfield. By [20, Proposition 2.15], there are no finite ramified places in $K[\gamma]/K$ unless $(q-1)/\gcd(2u, q-1)$ is a power of some prime ℓ . In this case there is only one finite place \wp of K that is ramified in $K[\gamma]/K$. As the number of ramified places of R^Φ is even, also $R^\Phi \otimes K_\wp$ is split.

If $(q-1)/2$ divides u , then

$$R^\Phi = \langle Y/2, W \rangle_K \cong (\gamma^2, q^3)$$

where $\gamma = \delta^u + \delta^{qu} = \delta^u - \delta^{-u}$. So $K[Y] \cong K[\gamma] \cong K[\delta^u]$ is a maximal subfield of R^Φ and conjugation by W yields the non-trivial Galois automorphism of $K[\gamma]/K$. Similarly as before, we conclude that q is a norm in $K[\gamma]/K$ and there is at most one finite place of K that is ramified in $K[\gamma]/K$. As before this implies that R^Φ is split. \square

Note that the proof above also shows that the Brauer elements $[\chi_{q^3+1}^{(u)}]$ are trivial, a result that is also obtained for all $\chi \in \text{Irr}^o(\text{SU}_3(q))$ from Proposition 4.3. Note that the Schur indices for the irreducible characters of $\text{PSU}_3(q)$ have been obtained by Gow [5], who also shows that all Schur indices of $\text{SU}_3(q)$ divide 2.

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