Unitary Discriminants of SL_3(q)

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Abstract

We give a full list of the unitary discriminants of the even degree indicator 'o' ordinary irreducible characters of $\mathrm{SL}_3(q)$.

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1 Introduction

Let G be a finite group, K a number field, $n \in \mathbb{N}$ and let $\rho : G \to \mathrm{GL}_n(K)$ be a K-representation of G of degree n. Then ρ is uniquely determined by its character $\chi : G \to K, g \mapsto \mathrm{trace}(\rho(g))$. The ordinary character table for G lists the values of all irreducible characters on the conjugacy classes of G. Together with additional number-theoretic invariants, the local Schur indices, it contains all necessary information about the linear actions of G over number fields, i.e. the representations ρ as above.

As G is finite, the image $\rho(G) \subseteq \mathrm{GL}_n(K)$ is contained in either a symplectic, unitary, or orthogonal group. The papers [9] and [8] develop methods towards classifying the orthogonal and unitary groups that contain $\rho(G)$. These methods are then applied to compute the orthogonal discriminants of the even degree indicator + irreducible characters and the unitary discriminants of the even degree indicator 'o' characters of a large portion of the groups in the ATLAS of finite groups [3].

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The ATLAS of finite groups contains the character tables of small finite simple groups, including all sporadic simple groups. Most of the non-abelian simple groups are finite groups of Lie Type. They fall in infinite series for which there are so called generic character tables parameterising the representations. For these infinite series, it is necessary to compute the corresponding generic orthogonal and unitary discriminants.

The first formula in the literature for such generic discriminants is the one by Jantzen and Schaper (see for instance [5]) giving the module structure of the discriminant groups of the Specht modules for all symmetric groups. The first generic orthogonal character tables have been obtained in [2] for the groups $\mathrm{SL}_2(q)$ and all prime powers q. Later the authors of this paper computed the generic orthogonal discriminants of the groups $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ [4], again for all prime powers q.

This paper continues this investigation by determining the unitary discriminants of the groups $\mathrm{SL}_3(q)$ for all prime powers q. In this case Harish-Chandra induction from the parabolic subgroup $\mathrm{GL}_2(q) \ltimes \mathbb{F}_q^2$ is enough to obtain all unitary discriminants (Theorem 3.4 and 3.5).

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2 Methods

2.1 Quadratic and Hermitian forms

Let L be a field and $\sigma \in \mathrm{Aut}(L)$ an automorphism of order 1 or 2. Put $K := \mathrm{Fix}_\sigma(L)$ to denote the fixed field of σ. An L/K Hermitian space is a finite dimensional L-vector space V together with a non-degenerate L/K-Hermitian form $H : V \times V \to L$, i.e. a K-bilinear map such that $H(av, w) = aH(v, w)$ and $H(v, w) = \sigma(H(w, v))$ for all $v, w \in V$ and $a \in L$. Let $N := N_{L/K}(L^\times)$ denote the norm subgroup of $K^\times$. Then

$$(K^\times)^2 \leq N \leq K^\times$$

and $N = (K^\times)^2$ if $\sigma = \mathrm{id}$. In the latter case, $L = K$ and H is usually called a symmetric bilinear form.

Definition 2.1. The *discriminant* of H is the class

$$\mathrm{disc}(H) := (-1)^{\binom{n}{2}} \det(H_B) N \in K^\times/N,$$

where $n = \dim(V)$ and $H_B := (H(b_i, b_j))_{i,j=1}^n \in L^{n\times n}$ is the Gram matrix of H with respect to any basis $B = (b_1, \dots, b_n)$ of V.

It is a general principle that odd degree extensions are not relevant for isometry of quadratic or Hermitian forms.

Lemma 2.2. *(see [6, Corollary 6.16]) Let F be an odd degree extension of K. Put $E := FL$ and extend σ to a field automorphism σ of E with fixed field F. Let (V, H) and (W, H') be two L/K Hermitian spaces. If the E/F Hermitian spaces $(V \otimes_L E, H_E)$ and $(W \otimes_L E, H'_E)$ are isometric, then also $(V, H) \cong (W, H')$.*

2.2 Unitary stable characters

For a finite group G let $\mathrm{Irr}(G)$ denote the absolutely irreducible complex characters of G. The *Frobenius-Schur indicator* $\mathrm{ind}(\chi)$ of $\chi \in \mathrm{Irr}(G)$ takes values in $\{+, -, \text{'o'}\}$. We have $\mathrm{ind}(\chi) = \text{'o'}$ if and only if the character field $\mathbb{Q}(\chi)$ is not real, $\mathrm{ind}(\chi) = +$ if there is a real representation affording the character χ, and $\mathrm{ind}(\chi) = -$ for real characters χ that are not afforded by a real representation. We denote by

$$\mathrm{Irr}^+(G) := \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = +,\ \chi(1) \text{ even }\}$$

the even degree indicator + characters of G and by

$$\mathrm{Irr}^o(G) := \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = \text{'o'},\ \chi(1) \text{ even }\}$$

the even degree indicator 'o' characters of G.

Definition 2.3. A character χ of a finite group G is called *orthogonal* if there is a representation ρ affording the character χ and fixing a non-degenerate quadratic form. An orthogonal character is called *orthogonally stable* if and only if all its indicator + constituents have even degree.

Then [9, Theorem 5.15] shows that the orthogonally stable characters are exactly those orthogonal characters that have a well defined discriminant:

Definition 2.4. Let χ be an orthogonally stable character of a finite group G with character field $K := \mathbb{Q}(\chi)$. Then the *orthogonal discriminant* $\mathrm{disc}(\chi) \in K^\times/(K^\times)^2$ is the unique square class of the character field such that for any orthogonal representation $\rho : G \to \mathrm{GL}_n(L)$ and any non-degenerate $\rho(G)$-invariant quadratic form Q we have that $\mathrm{disc}(Q) = \mathrm{disc}(\chi)(L^\times)^2$.

The paper [8] transfers the notion of orthogonal stability to the Hermitian case:

Definition 2.5. An ordinary character χ of a finite group G is called *unitary stable* if all absolutely irreducible constituents of χ have even degree.

Note that a unitary stable orthogonal character is orthogonally stable, but the converse is not always the case. Similarly as in the orthogonal case we get that a unitary stable character has a well defined unitary discriminant:

Definition 2.6. Let χ be a unitary stable character of the finite group G and let L be a totally complex number field with real subfield L^+. Assume that there is an L-representation ρ of G affording the character χ. Then all $\rho(G)$-invariant non-degenerate Hermitian forms H have the same discriminant $\mathrm{disc}(H) =: dN_{L/L^+}(L^\times)$. Then $\mathrm{disc}_L(\chi) := dN_{L/L^+}(L^\times)$ is called the *L-discriminant* of χ. If $L = \mathbb{Q}(\chi)$ is the character field of χ then we usually omit the index L and denote by $\mathrm{disc}(\chi) := \mathrm{disc}_{\mathbb{Q}(\chi)}(\chi)$ the *unitary discriminant* of χ.

2.3 Induced representations

Many irreducible characters χ of finite groups G are imprimitive, i.e. induced from a character ψ of a proper subgroup U. Then a G-invariant form in the induced representation is just the orthogonal sum of $[G : U]$ copies of a U-invariant form. However, the character field of ψ might be larger than the one of χ, and we only get the discriminant over the character field of ψ (see [8, Remark 5.11]). In view of Lemma 2.2 it is helpful to know when $[\mathbb{Q}(\psi) : \mathbb{Q}(\chi)]$ is odd:

Lemma 2.7. *Let G be a finite group, $U \leq G$, $\psi \in \mathrm{Irr}(U)$ such that $\chi := \psi^G \in \mathrm{Irr}(G)$. Then $\mathbb{Q}(\chi) \leq \mathbb{Q}(\psi)$. Let $\Psi := \{\psi_1, \ldots, \psi_h\}$ be the set of constituents of the restriction $\chi_{|U}$ of χ to U of degree $\psi_i(1) = \psi(1)$ and assume that the cardinality, h, of Ψ is odd. Then there is $i_0 \in \{1, \ldots, h\}$ such that $[\mathbb{Q}(\psi_{i_0}) : \mathbb{Q}(\chi)]$ is odd.*

Proof. By Frobenius reciprocity $\Psi = \{\psi_i \in \mathrm{Irr}(U) \mid \chi = \psi_i^G\}$. For $\psi_i \in \Psi$ a full regular orbit under the Galois group $\mathrm{Gal}(\mathbb{Q}(\psi_i)/\mathbb{Q}(\chi))$ is contained in Ψ and Ψ is a disjoint union of such Galois orbits. As $|\Psi|$ is odd at least one of these orbits has odd length. $\square$

2.4 A generalisation of the Q_8-trick

Sometimes subgroups help us to predict all unitary discriminants of faithful characters. The following result is a generalisation of the Q_8-trick [8, Section 6.1]. A variant is later used in Theorem 3.1.

Theorem 2.8. *Let $d \geq 2$ and $m := 2^d a$ for some $a \in \mathbb{N}$. Let p, ℓ be not necessarily distinct odd primes and let $q := p^f$ be some power of p. Let G be some subgroup of $\mathrm{GL}_m(q)$ containing $\mathrm{SO}_m^+(p)$ (e.g. $\mathrm{SL}_m(q) \leq G \leq \mathrm{GL}_m(q), \mathrm{SU}_m(q) \leq G \leq \mathrm{GU}_m(q), \mathrm{SO}_m^+(q) \leq G \leq \mathrm{GO}_m^+(q)$). Let χ be an absolutely irreducible faithful ordinary or ℓ-Brauer character of G. Then*

(a) The character degree $\chi(1)$ is a multiple of 2^d.

(b) If the indicator of χ is $+$, then its orthogonal discriminant is 1.

(c) If χ is an ordinary character of indicator 'o', then its unitary discriminant is 1.

Proof. Consider the group $U := 2_+^{1+2d} \cong \otimes^d D_8$. Then U has a unique absolutely irreducible character ψ that restricts non-trivially to the center $Z(U) \cong C_2$. This character has degree $\psi(1) = 2^d$, indicator $+$, and is the character of an integral absolutely irreducible representation $\Delta : U \to \mathrm{SL}_{2^d}(\mathbb{Z})$. Moreover, ψ is orthogonally stable of orthogonal discriminant 1. Reducing Δ mod p hence shows that $U \leq \mathrm{SO}_m^+(p) \leq \mathrm{SL}_m(p)$, so $U \leq G$.

As χ is a faithful character of G, its restriction to U is a multiple of ψ. $\square$

3 The special linear group

Let p be a prime and let q be a power of p. Put $\mathbb{F}_q^{n \times n}$ to denote the ring of $n \times n$ matrices over the finite field $\mathbb{F}_q$. The group $\mathrm{SL}_3(q)$ is

$$\mathrm{SL}_3(q) = \{\mathbf{g} \in \mathbb{F}_q^{3\times 3} | \det(\mathbf{g}) = 1\}.$$

It contains a maximal parabolic subgroup

$$P := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in \mathrm{SL}_3(q) \right\}.$$

Note that P is the semidirect product $P \cong \mathrm{GL}_2(q) \ltimes \mathbb{F}_q^2$ where the action of $\mathrm{GL}_2(q)$ on $\mathbb{F}_q^2$ is given by

$$\mathbf{g} \cdot \mathbf{h} = \det(\mathbf{g})(\mathbf{gh}) \text{ for } \mathbf{g} \in \mathrm{GL}_2(q), \mathbf{h} \in \mathbb{F}_q^2.$$

The center of P is the center Z of $\mathrm{SL}_3(q)$, hence isomorphic to C_3 if $q-1$ is a multiple of 3 and trivial otherwise. Let

$$d := |Z| = \gcd(3, q-1) \text{ and } \omega := \exp\left(\frac{2\pi i}{3}\right).$$

3.1 The characters of $\mathrm{GL}_2(q)$

As we use Harish-Chandra induction from the Levi factor $\mathrm{GL}_2(q)$ of the subgroup P, we first deal with this group.

Theorem 3.1. *Assume that q is a power of some odd prime p. Then all characters $\psi \in \mathrm{Irr}^o(\mathrm{GL}_2(q))$ have unitary discriminant* $\mathrm{disc}(\psi) = (-1)^{\psi(1)/2}$.

Proof. If ψ is not faithful, then the character field of ψ is real (see [11]). So ψ is a faithful irreducible character of $\mathrm{GL}_2(q)$. Then the restriction of ψ to the subgroup $D_8 \leq \mathrm{GL}_2(q)$ is a multiple of the unique character of D_8 that restricts non-trivially to the center of D_8. This character has degree 2, indicator $+$, trivial Schur indices and orthogonal discriminant -1. As in Theorem 2.8 we conclude that the unitary discriminant is $\mathrm{disc}(\psi) = (-1)^{\psi(1)/2}$. □

Remark 3.2. Assume that q is a power of 2. Then $\mathrm{GL}_2(q)$ has $q-1$ irreducible characters of degree q. All these characters restrict to the Steinberg character of degree q of $\mathrm{SL}_2(q)$ and hence have unitary/orthogonal discriminant $(-1)^{q/2}(q+1)$ (see [2, Theorem 6.2]).

3.2 The characters of P

Now let $\chi \in \mathrm{Irr}(P)$ be an irreducible character of $P \cong \mathbb{F}_q^2 \rtimes \mathrm{GL}_2(q)$ restricting non-trivially to the abelian normal subgroup $A := \mathbb{F}_q^2$ of order q^2. As P acts transitively on $A \setminus \{1\}$, the restriction of χ to A is a multiple of the sum of all non-trivial linear characters ψ of A. By Clifford theory, the character χ is induced up from a character of the inertia subgroup $T_\psi \cong H \ltimes A$ of any of these characters ψ, where

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-2} \end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\} \cong (\mathbb{F}_q^\times \ltimes \mathbb{F}_q)$$

with center Z. The index of T_ψ in P is q^2-1. The irreducible characters ϕ of H consist of $(q-1)$ linear characters and d^2 characters of degree $(q-1)/d$. Inducing the characters $\phi \otimes \psi$ with $\phi \in \mathrm{Irr}(H)$ from $H \ltimes A$ to P we obtain $(q-1)$ irreducible characters of degree (q^2-1) and d^2 characters of degree $(q-1)^2(q+1)/d$.

Remark 3.3. The irreducible characters $\chi \in \mathrm{Irr}(P)$ for which the restriction of χ to a Sylow p-subgroup of P does not contain the trivial character are exactly the d^2 characters of degree $\chi(1) = (q-1)^2(q+1)/d$. By Benard's theorem [1] all these characters have trivial Schur index. The d characters χ that restrict trivially to the center are rational, the other d^2-d characters χ have character field $\mathbb{Q}(\omega)$. All these characters have trivial unitary, resp. orthogonal, discriminant.

Proof. Write $q-1 = ab$ such that a is a power of 3 and b is not a multiple of 3 and put

$$U := \langle P', g^a \mid g \in P \rangle$$

to be the normal subgroup of index a in P (so $U = P$ if $q \not\equiv 1 \pmod 3$). Let $\chi \in \mathrm{Irr}(P)$ be one of the irreducible characters of degree $(q-1)^2(q+1)/d$ from Remark 3.3. By Clifford theory the restriction $\chi_{|U}$ of χ to U is a sum of a 3-power number of conjugate characters of the same degree, in particular the degrees of the constituents of $\chi_{|U}$ are even. Moreover these constituents are rational and hence $\chi_{|U}$ is an orthogonally stable rational character that restricts orthogonally stably to a Sylow p-subgroup of U. Now [7, Theorem 4.3 and Corollary 4.4] yields

$$\mathrm{disc}(\chi_{|U}) = (-1)^{\chi(1)/2} p^{\chi(1)/(p-1)} (\mathbb{Q}^\times)^2 = 1$$

and hence also the unitary discriminant of χ is 1. □

3.3 The characters of $\mathrm{SL}_3(q)$

Theorem 3.4. *If q is odd then all characters $\chi \in \mathrm{Irr}^o(\mathrm{SL}_3(q))$ have unitary discriminant $(-1)^{\chi(1)/2}$.*

Proof. We use the description in [11] of the characters of $\mathrm{SL}_3(q)$. The characters χ of degree $(q+1)(q^2+q+1)$ and $(q-1)(q^2+q+1)$ are Harish-Chandra induced from the maximal parabolic subgroup P from a a character ψ of degree $(q+1)$ resp. $(q-1)$ of the Levi complement $\mathrm{GL}_2(q)$.

If $\chi(1) = (q-1)(q^2+q+1)$, then there is a unique such character ψ with $\chi = \psi^G$. In particular $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$. So if $\mathbb{Q}(\psi^G)$ is not real, then also $\psi \in \mathrm{Irr}^o(\mathrm{GL}_2(q))$ and Theorem 3.1 shows that the unitary discriminant of ψ and hence the one of $\chi = \psi^G$ is $(-1)^{(q-1)/2}$.

If $\chi(1) = (q+1)(q^2+q+1)$, then there are three such characters ψ inducing to the same character χ. As the Galois group of $\mathbb{Q}(\psi)/\mathbb{Q}(\chi)$ acts on the constituents of degree $q+1$ of $\chi_{|P}$ it has at least one orbit of odd length. So one of these characters ψ satisfies $[\mathbb{Q}(\psi) : \mathbb{Q}(\chi)]$ is odd. By Lemma 2.2, this again allows to conclude that the unitary discriminant of χ is the same as the one of ψ.

It remains to consider the characters χ of degree $(q-1)^2(q+1)$ and $(q-1)^2(q+1)/3$ (if $q \equiv 1 \pmod 3$).

For both degrees, the character χ does not appear in the permutation character of G on the $(q-1)^2(q+1)$ cosets of a Sylow p-subgroup S of G: This is clear if $\chi(1) = (q-1)^2(q+1) = [G:S]$ because all constituents of 1_S^G have degree $\leq [G:S]-1$. If $\chi(1) = (q-1)^2(q+1)/3$, we note that the center $Z = \langle z \rangle$ of G has order 3 and orbits of length 3 on the cosets of G/S. So if χ occurs in 1_S^G, then this permutation character has three distinct constituents of degree $(q-1)^2(q+1)/3$ leading to the same contradiction as before.

In particular, the restriction of χ to P is a sum of the characters from Remark 3.3, showing again that the unitary discriminant of χ is trivial.

From the tables in [10], we see that the characters of degree $q(q+1)$ and $(q+1)(q^2+q+1)/3$ are rational. Their orthogonal discriminant can be read off from [4, Theorem 4.7]. □

Theorem 3.5. *If q is even then all characters $\chi \in \mathrm{Irr}^o(\mathrm{SL}_3(q))$ have degree $q(q^2+q+1)$ and unitary discriminant $(-1)^{q/2}(q+1)$.*

Proof. Now assume that q is a power of 2. Then the even degree irreducible characters of $\mathrm{SL}_3(q)$ are as follows:

(i) one character of degree q^2+q and Frobenius-Schur indicator $+$,

(ii) the Steinberg character of degree q^3 and Frobenius-Schur indicator $+$,

(iii) $q-1$ characters of degree $q(q^2+q+1)$, one of which is rational and of indicator $+$, while the others have indicator 'o' (see [11], [10]).

From [11, Table VI] and the arguments given just before we conclude that the $q-1$ characters $\chi \in \mathrm{Irr}(\mathrm{SL}_3(q))$ of degree $\chi(1) = q(q^2+q+1)$ are

Harish-Chandra induced from the $q-1$ irreducible characters ψ of degree q of the Levi factor $\mathrm{GL}_2(q)$ of P. It follows that they have the same character field $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$ and discriminant. From Remark 3.2 we get $\mathrm{disc}(\psi) = (-1)^{q/2}(q+1)$. As the index of P in $\mathrm{SL}_3(q)$ is odd, also the discriminant of χ is $(-1)^{q/2}(q+1)$. □

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