Low dimensional strongly perfect lattices. I: The 12-dimensional case.

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Abstract

It is shown that the Coxeter-Todd lattice is the unique strongly perfect lattice in dimension 12.

1 Introduction.

The notion of perfect lattices came up about 100 years ago in papers by Korkine and Zolotarev and especially Voronoi [18] during the study of dense lattice sphere packings. If the centers of the spheres in a packing form a lattice Λ in Euclidean space $(\mathbb{R}^n, (,))$ then the density of the sphere packing is proportional to the Hermite function of the lattice

$$\gamma(\Lambda) := \frac{\min(\Lambda)}{\det(\Lambda)^{1/n}}$$

where $\min(\Lambda) := \min\{(\lambda, \lambda) \mid 0 \neq \lambda \in \Lambda\}$ denotes the square of the minimal distance between distinct lattice points and $\det(\Lambda)$ is the square of the covolume of Λ in $\mathbb{R}^n$. In any dimension n the Hermite function γ has a global maximum on the set of n-dimensional lattices

$$\gamma_n := \max\{\gamma(\Lambda) \mid \Lambda \subset \mathbb{R}^n \text{ lattice } \}$$

the so called Hermite constant γ_n. Exact values for γ_n are only known for dimensions $n \leq 8$ and, due to a recent work by Cohn and Kumar [5], in dimension 24. New upper bounds for γ_n are given in the article [4] by Cohn and Elkies (see Theorem 3.2 for $n = 12$). The strategy due to Voronoi to find γ_n and the densest lattices in $\mathbb{R}^n$ is to determine all finitely many local maxima of γ on the set of similarity classes of n-dimensional lattices. To be a local maximum for γ, the lattice Λ has to be perfect and eutactic, two conditions on the geometry of the minimal vectors of Λ. For more information the reader is referred to Martinet's book [12]. Voronoi has developed a remarkable algorithm which permits in principle to enumerate all (finitely many similarity classes of) perfect lattices in a given dimension. But since the number of perfect lattices increases quite rapidely with the dimension, this seems to be unpracticable in dimensions ≥ 8.

In [17], the second author introduced the notion of strongly perfect lattices, which are those lattices, for which the minimal vectors form a 4-design (see Definition 2.2). They are perfect and eutactic and hence local maxima of the Hermite function. Up to rescaling they are rational, that is the mutual scalar products of lattice vectors lie in $\mathbb{Q}$ (this property is shared by all perfect lattices). Also most of the famous lattices such as the E_8-lattice, the Leech lattice and the Barnes-Wall lattices are strongly perfect.

There are two general approaches to study and construct strongly perfect lattices: by modular forms and by invariant theory of finite groups. The relation with modular forms arises, because the condition that the minimal vectors of Λ form a 4-design means the annulation of certain coefficients in a theta-series of Λ with harmonic coefficients. In this way one can prove the strong perfectness of many extremal lattices of small level (see [16] for the even unimodular and [2] for the modular case). By the way this is the only known way to prove the strong perfectness of an even unimodular 32-dimensional lattice without roots. By [9], there are more than 10^6 ot them, an explicit classification is not known.

If a rational lattice Λ has a big automorphism group $G := \mathrm{Aut}(\Lambda)$ which has no invariant harmonic polynomials of degree 2 and 4, a condition easily expressed in terms of the character of $G \leq O(n)$, then Λ is strongly perfect. There are many interesting lattices such as the Barnes-Wall lattices, the 248-dimensional Thompson-Smith lattice and others which are strongly perfect by this reason (see for example [10]).

The point of view of lattices is also useful for the study of general 4-designs. For example [3] treats an infinite number of new cases for the classification of tight spherical designs by considering the lattice generated by a given design.

The aim of this project, which is a continuation of [17] and [13] is to classify all strongly perfect lattices in a given small dimension. For $n \leq 24$, one only expects few n-dimensional strongly perfect lattices (see [17, Table 19.1, 19.2, pp. 82,83]) which possibly allow a classification. For $n \leq 11$ this was done in [17] and [13], here we deal with dimension 12. Our main theorem is

Theorem. (see Theorem 3.1) *The strongly perfect lattices in dimension* 12 *are similar to the Coxeter-Todd lattice* CT.

We prove this theorem by eliminating rather easily all possibilities for the Kissing number $2s = |\Lambda_{\min}|$ of a strongly perfect lattice in dimension 12, except for $s = 252$ and $s = 378$. To eliminate $s = 252$ is more difficult and involves the construction of higher-dimensional lattices up to dimension 18. In the last case, $s = 378$, the lattice Λ has the same Kissing number as the Coxeter-Todd lattice. One special property of the Coxeter-Todd lattice is that it is 3-modular and has a natural complex structure as a unimodular hermitian lattice over $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Its shortest vectors form the root system of the complex reflection group number 34 ($\cong 6.U_4(3).2$) in the Shephard-Todd classification [15]. During the identification of the putative strongly perfect lattice Λ with $s = 378$ we construct step by step parts of this rich structure which finally allows us to show that Λ is similar to the Coxeter-Todd lattice.

2 Some general equations.

2.1 General notation.

For a lattice Λ in n-dimensional Euclidean space we denote by Λ^* its dual lattice and by

$$\Lambda_a := \{\lambda \in \Lambda \mid (\lambda, \lambda) = a\}$$

the vectors of square length a. In particular, $\Lambda_{\min}$ is the set of minimal vectors in Λ, its cardinality is known as the Kissing number of the lattice Λ. There are general bounds on the cardinality of an antipodal spherical code and hence on the Kissing number of an n-dimensional lattice. For $n = 12$ the bound is $|\Lambda_{\min}| \leq 2 \cdot 614$ (see [1, Table 1]).

2.2 Designs and strongly perfect lattices

Let $(\mathbb{R}^n, (,))$ be the Euclidean space of dimension n. For $m \in \mathbb{R}$, $m > 0$ denote by

$$S^{n-1}(m) := \{y \in \mathbb{R}^n \mid (y, y) = m\}$$

the $(n-1)$-dimensional sphere of radius $\sqrt{m}$.

Definition 2.1 *A finite nonempty set $X \subset S^{n-1}(m)$ is called a* spherical t-design, *if*

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \int_{S^{n-1}(m)} f(x) d\mu(x)$$

for all polynomials of $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $\leq t$, where μ is the $O(n)$-invariant measure on the sphere, normalised such that $\int_{S^{n-1}(m)} 1 d\mu(x) = 1$.

Since the condition is trivially satisfied for constant polynomials f, and the harmonic polynomials generate the orthogonal complement $\langle 1 \rangle^{\perp}$ with respect to the $O(n)$-invariant scalar product $\langle f, g \rangle := \int_{S^{n-1}(m)} f(x) g(x) d\mu(x)$ on $\mathbb{R}[x_1, \ldots, x_n]$, it is equivalent to ask that

$$\sum_{x \in X} f(x) = 0$$

for all harmonic polynomials f of degree $\leq t$.

Definition 2.2 *A lattice $\Lambda \subset \mathbb{R}^n$ is called* strongly perfect, *if its minimal vectors $\Lambda_{\min}$ form a spherical 4-design.*

The most important property of strongly perfect lattices is that they provide interesting examples for local maxima of the Hermite function (see [17]).

Let Λ be a strongly perfect lattice of dimension n, $m := \min(\Lambda)$ and choose $X \subset \Lambda_m$ such that $X \cup -X = \Lambda_m$ and $X \cap -X = \emptyset$. Put $s := |X|$.

By [17] the condition that $\sum_{x \in X} f(x) = 0$ for all harmonic polynomials of degree 2 and 4 may be reformulated to the condition that for all $\alpha \in \mathbb{R}^n$

$$(D4)(\alpha) : \quad \sum_{x \in X} (x, \alpha)^4 = \frac{3sm^2}{n(n+2)} (\alpha, \alpha)^2.$$

Applying the Laplace operator $\sum_{i=1}^{n} \frac{\partial^2}{\partial \alpha_i^2}$ to $(D4)(\alpha)$ one obtains

$$(D2)(\alpha): \quad \sum_{x\in X}(x,\alpha)^2 = \frac{sm}{n}(\alpha,\alpha).$$

Note that it is enough to assume that there are constants c_t $(t=1,2)$ such that

$$\sum_{x\in X}(x,\alpha)^{2t} = c_t(\alpha,\alpha)^t \text{ for all } \alpha\in\mathbb{R}^n.$$

These constants c_t are then uniquely determined as one sees applying t times the Laplace operator with respect to α. For later use, we also remark that the condition $(D2)$ is equivalent to the 2-design property of $X\cup -X$.

Substituting $\alpha := \xi_1\alpha_1+\xi_2\alpha_2$ in $(D2)$ and comparing coefficients, one finds

$$(D11)(\alpha_1,\alpha_2) \quad \sum_{x\in X}(x,\alpha_1)(x,\alpha_2) = \frac{sm}{n}(\alpha_1,\alpha_2) \text{ for all } \alpha_1,\alpha_2\in\mathbb{R}^n.$$

Writing α as a linear combination of 4 vectors, $(D4)$ implies that for all $\alpha_1,\ldots,\alpha_4\in\mathbb{R}^n$

$$(D1111) \ \sum_{x\in X}(x,\alpha_1)(x,\alpha_2)(x,\alpha_3)(x,\alpha_4) =$$

$$= \frac{sm^2}{n(n+2)}((\alpha_1,\alpha_2)(\alpha_3,\alpha_4)+(\alpha_1,\alpha_3)(\alpha_2,\alpha_4)+(\alpha_1,\alpha_4)(\alpha_2,\alpha_3))$$

In particular

$$(D13)(\alpha_1,\alpha_2): \quad \sum_{x\in X}(x,\alpha_1)(x,\alpha_2)^3 = \frac{3sm^2}{n(n+2)}(\alpha_1,\alpha_2)(\alpha_2,\alpha_2)$$

$$(D22)(\alpha_1,\alpha_2): \quad \sum_{x\in X}(x,\alpha_1)^2(x,\alpha_2)^2 = \frac{sm^2}{n(n+2)}(2(\alpha_1,\alpha_2)^2+(\alpha_1,\alpha_1)(\alpha_2,\alpha_2))$$

$(D13)-(D11)(\gamma,\alpha)$ reads as

$$\sum_{x\in X}(x,\gamma)(x,\alpha)((x,\alpha)^2-1) = (\alpha,\gamma)\frac{sm}{n}(\frac{3m}{n+2}(\alpha,\alpha)-1) \text{ for all } \alpha,\gamma\in\mathbb{R}^n\ .$$

Note that for $\alpha\in\Lambda^*$ and $x\in\Lambda$, the product $(x,\alpha)((x,\alpha)^2-1)$ is divisible by 6 and hence $\frac{1}{6}((D13)-(D11))(\gamma,\alpha)\in\mathbb{Z}$ yields that

$$\frac{1}{6}\sum_{x\in X}(x,\gamma)(x,\alpha)((x,\alpha)^2-1) = (\alpha,\gamma)\frac{sm}{6n}(\frac{3m}{n+2}(\alpha,\alpha)-1)\in\mathbb{Z} \ \text{ for all } \alpha,\gamma\in\Lambda^*.$$

Putting $\alpha=\gamma$ in $(D13)-(D11)$ we obtain $\frac{1}{12}((D4)-(D2))(\alpha)\in\mathbb{Z}$ hence

$$\frac{1}{12}\sum_{x\in X}(x,\alpha)^2((x,\alpha)^2-1) = \frac{sm}{12n}(\alpha,\alpha)(\frac{3m}{n+2}(\alpha,\alpha)-1)\in\mathbb{Z} \ \text{ for all } \alpha\in\Lambda^*$$

since $(x,\alpha)^2((x,\alpha)^2-1)$ is divisible by 12 if $(x,\alpha)\in\mathbb{Z}$.

Notation: For a set X as above (usually clear from the context) and $\alpha\in\mathbb{R}^n$ we let

$$N_i(\alpha)=\{x\in X \mid (\alpha,x)=\pm i\} \text{ and } n_i(\alpha):=|N_i(\alpha)|.$$

Lemma 2.3 *(see [13, Lemma 2.1]) Let $X\subset\mathbb{R}^n$ be a 4-design and let $\alpha\in\mathbb{R}^n$ be such that $(x,\alpha)\in\{0,\pm1,\pm2\}$ for all $x\in X$. Put*

$$c:=\frac{sm}{6n}(\frac{3m}{n+2}(\alpha,\alpha)-1).$$

Then $n_2(\alpha)=c(\alpha,\alpha)/2$ and

$$\sum_{x\in N_2(\alpha)} x = c\alpha.$$

Lemma 2.3 will be often applied to $\alpha\in\Lambda^*$. Rescale Λ such that $\min(\Lambda)=m=1$ and let $r:=\min(\Lambda^*)$. Since $\gamma(\Lambda)\gamma(\Lambda^*)=\min(\Lambda)\min(\Lambda^*)\le\gamma_n^2$, we get $r\le\gamma_n^2$ and for $\alpha\in\Lambda_r^*$ we have $(\alpha,x)^2\le r$ for all $x\in\Lambda_1$. Hence if $r<9$ then $(\alpha,x)\in\{0,\pm1,\pm2\}$ for all $x\in X$ and Lemma 2.3 may be applied.

The next lemma yields good bounds on $n_2(\alpha)$.

Lemma 2.4 *Let Λ be a strongly perfect lattice. Let $m:=\min(\Lambda)$ and choose $\alpha\in\Lambda_r^*$. If $r\cdot m<8$, then*

$$n_2(\alpha)\le\frac{rm}{8-rm}.$$

Proof. Since $(x,x)(\alpha,\alpha)<8$ for all $x\in\Lambda_m$, the scalar product $|(x,\alpha)|<3$. Hence α satisfies the conditions of Lemma 2.3. Let $N_2(\alpha)=\{x_1,\ldots,x_k\}$ and $c=\frac{2k}{r}$ be the constant from Lemma 2.3. Then $(x_i,x_i)=m$ and $(x_i,x_j)\le\frac{m}{2}$ because the x_i are minimal vectors in Λ. Hence

$$\frac{4k}{r}=(x_1,c\alpha)=(x_1,x_1)+\sum_{i=2}^{k}(x_1,x_i)\le m+\frac{m(k-1)}{2}=\frac{m(k+1)}{2}$$

which yields that $k=|N_2(\alpha)|\le\frac{rm}{8-rm}$. □

Lemma 2.5 *([17, Théorème 10.4]) Let L be a strongly perfect lattice of dimension n. Then*

$$\gamma(L)\gamma(L^*)=\min(L)\min(L^*)\ge\frac{n+2}{3}.$$

A strongly perfect lattice L where equality holds is called of minimal type.

As an application this lemma allows to show that $|N_2(\alpha)|\ne1$.

Lemma 2.6 *Let Λ be a strongly perfect lattice and choose $\alpha\in\Lambda_r^*$ that satisfies the conditions of Lemma 2.3. If $n\ge11$ then $n_2(\alpha)\ne1$.*

Proof. Let $m := \min(\Lambda)$ and assume that $N_2(\alpha) = \{x\}$. Then $c\alpha = x$ for some constant c. Taking scalar products with x yields $c = \frac{m}{2}$ and hence $(\alpha, \alpha) = \frac{4}{m}$. Therefore

$$\frac{n+2}{3} \leq \min(\Lambda)\min(\Lambda^*) \leq 4$$

which implies that $n \leq 10$. □

The last lemma is quite useful in the investigation of lattices of minimal type:

Lemma 2.7 *Let $X \,\dot\cup\, -X \subset S^{n-1}(m)$ be a spherical 4-design and let $\alpha \in \mathbb{R}^n$ be such that $(x, \alpha) \in \{0, \pm 1\}$ for all $x \in X$. Let $M := N_1(\alpha)$ and let $\pi : \mathbb{R}^n \to \langle \alpha \rangle^\perp$ be the orthogonal projection onto the orthogonal complement of α. Then $\pi(M) \,\dot\cup\, -\pi(M) \subset S^{n-2}(m')$ is a spherical 2-design.*

Proof. Let $v \in \langle \alpha \rangle^\perp$, i.e. $(v, \alpha) = 0$. Then by $(D22)(\alpha, v)$ we find that

$$\sum_{x \in M} (x, v)^2 = \sum_{x \in \pi(M)} (x, v)^2 = c(v, v)$$

for some constant c not depending on v. Therefore $\pi(M) \cup -\pi(M)$ is a 2-design. □

2.3 Some general facts on lattices.

The next two lemmas about indices of sublattices are used quite often in the argumentation below. Since we are dealing with norms modulo some prime number p, we may pass to the localization $\mathbb{Z}_p := (\mathbb{Z} - p\mathbb{Z})^{-1}\mathbb{Z} \subset \mathbb{Q}$ of $\mathbb{Z}$ at p.

Lemma 2.8 *Let Γ be a $\mathbb{Z}_2$-lattice such that $(\gamma, \gamma) \in \mathbb{Z}_2$ for all $\gamma \in \Gamma$. Let $\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 2\mathbb{Z}_2\}$. If $\Gamma^{(e)}$ is a sublattice of Γ, then $[\Gamma : \Gamma^{(e)}] \in \{1, 2, 4\}$.*

Proof. Clearly $2\Gamma \subset \Gamma^{(e)}$ hence $\Gamma/\Gamma^{(e)}$ is a vector space over $\mathbb{F}_2$. Moreover $(\alpha, \beta) \in \frac{1}{2}\mathbb{Z}_2$ for all $\alpha, \beta \in \Gamma$ since the norms in Γ are integers. Let $\alpha, \beta, \gamma \in \Gamma - \Gamma^{(e)}$. Then $(\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + (\beta, \beta) + 2(\alpha, \beta) \in 2(\alpha, \beta) + 2\mathbb{Z}_2$ since both (α, α) and (β, β) are odd. In particular, if $(\alpha, \beta) \in \mathbb{Z}_2$, then $\alpha + \beta \in \Gamma^{(e)}$. Since one of (α, γ), (β, γ) or $(\alpha + \beta, \gamma)$ is integral, the three classes $\alpha + \Gamma^{(e)}, \beta + \Gamma^{(e)}, \gamma + \Gamma^{(e)}$ are $\mathbb{F}_2$-linearly dependent and $\dim_{\mathbb{F}_2}(\Gamma/\Gamma^{(e)}) \leq 2$. □

Lemma 2.9 *Let Γ be a $\mathbb{Z}_3$-lattice such that $(\gamma, \gamma) \in \mathbb{Z}_3$ for all $\gamma \in \Gamma$. Let $\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 3\mathbb{Z}_3\}$. Assume that*

$$(\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta) \in 3\mathbb{Z}_3 \textit{ for all } \alpha, \beta \in \Gamma.$$

Then $\Gamma^{(t)}$ is a sublattice of Γ and $[\Gamma : \Gamma^{(t)}] \in \{1, 3\}$.

Proof. By our assumption $(\alpha, \beta) \in 3\mathbb{Z}_3$ for all $\alpha, \beta \in \Gamma$ with $\alpha \in \Gamma^{(t)}$. Therefore $\Gamma^{(t)}$ is a sublattice of Γ. Clearly $3\Gamma \subset \Gamma^{(t)}$ hence $\Gamma/\Gamma^{(t)}$ is a vector space over $\mathbb{F}_3$. Let $\alpha, \beta \in \Gamma - \Gamma^{(t)}$. Then $(\alpha,\alpha)(\beta,\beta) \equiv (\alpha,\beta)^2 \pmod 3$ implies that $(\alpha,\alpha) \equiv (\beta,\beta) \pmod 3$ and one of $\alpha \pm \beta \in \Gamma^{(t)}$. Hence $[\Gamma : \Gamma^{(t)}] \leq 3$. □

We will also meet families of vectors $E := \{v_1, \ldots, v_k\}$ of equal norm, say $(v_i, v_i) = 1$ and non positive scalar products, i.e. $(v_i, v_j) \leq 0$ for all $i \neq j$ with $\sum_{i=1}^k v_i = 0$. Such a system is called decomposable, if $E = E' \dot\cup E''$ such that

$$\sum_{v \in E'} v = \sum_{v \in E''} v = 0.$$

It is classical (and easy) that for any indecomposable system E, the relation $\sum_{v \in E} v = 0$ is the only relation between the vectors in E. In particular

$$k = |E| = \dim\langle E \rangle + 1 \text{ for any indecomposable } E.$$

An arbitrary system E can be written as an orthogonal union $E = \dot\bigcup_{i=1}^t E_i$ with $E_i \perp E_j$, E_i indecomposable.

Lemma 2.10 *If t is the number of indecomposable components of E, then $|E| = \dim\langle E \rangle + t$*

Proof. For $t = 1$ this is clear by the argumentation above. For general E, all three functions $|E|$, $\dim\langle E \rangle$, $t(E)$ are additive for orthogonal union. □

Components of dimension 1 are just pairs $\{x, -x\}$. Components of dimension 2 are of the form $\{x_1, x_2, x_3\}$ with $x_1 + x_2 + x_3 = 0$ and $(x_i, x_i) = 1$, $(x_i, x_j) = -\frac{1}{2}$ for all $i \neq j$. They generate a root system A_2. For more information see [7].

3 Dimension 12

In this section we prove our main theorem:

Theorem 3.1 *Let Λ be a strongly perfect lattice of dimension 12. Then $\Lambda \cong CT$ is similar to the Coxeter-Todd lattice.*

The lattice CT, usually denoted by K_{12}, is an extremal 3-modular lattice in the sense of [14], which means that CT is similar to its dual lattice. Rescaled to minimum 4, the determinant of CT is 3^6. The Kissing number is $|CT_4| = 2 \cdot 378 = 756$. The Coxeter-Todd lattice is the densest known lattice in dimension 12.

The article [4] gives a good upper bound on the Hermite constants. In particular they yield

Theorem 3.2 *([4]) $\gamma_{12} \leq 2.522$.*

Remark 3.3 *No proper overlattice of CT has minimum 4.*

Proof. Let $\Gamma > CT$ be a proper overlattice of CT with $\min(\Gamma) = 4$. Then $\det(\Gamma) = \det(CT) \cdot [\Gamma : CT]^{-2} \leq 3^6 2^{-2}$. Hence the Hermite function

$$\gamma(\Gamma) = \frac{4}{\sqrt[12]{\det(\Gamma)}} \geq 2.59 > \gamma_{12}$$

by Theorem 3.2 which is a contradiction. □

Therefore it is enough to show that any strongly perfect lattice of dimension 12, that is generated by its minimal vectors, is similar to CT.

3.1 Kissing numbers.

Let Λ be a strongly perfect lattice in dimension 12, rescaled such that $\min(\Lambda) = 1$. Then by Lemma 2.5 we find

$$\frac{14}{3} \leq r := \min(\Lambda^*) \leq \gamma_{12}^2 \leq 6.37.$$

Hence $\alpha \in \Lambda_r^*$ satisfies the hypothesis of Lemma 2.3 and Lemma 2.4 yields

$$n_2(\alpha) \leq \frac{r}{8-r} \leq \frac{\gamma_{12}^2}{8-\gamma_{12}^2} \leq 3.88 < 4.$$

If $d := \det(\Lambda^*) = \frac{1}{\det(\Lambda)}$ denotes the determinant of Λ^*, then $d \leq \gamma_{12}^{12} \leq 66212.7$.

Let $s := |X|$ where $X \dot{\cup} -X = \Lambda_1$ be half the Kissing number of Λ. Then

$$\frac{12 \cdot 13}{2} = 78 \leq s \leq 614$$

where the lower bound follows from the fact that X is a 4-design and hence $\{x^{tr}x \mid x \in X\}$ spans the space of all symmetric matrices and the upper bound is the bound on the Kissing number of a 12-dimensional lattice as given in [1]. Moreover r is a rational solution of

$$n_2(\alpha) = \frac{sr}{12 \cdot 12 \cdot 14}(3r - 14)$$

by Lemma 2.3. Going through all possibilities by a computer we find:

Proposition 3.4 *With the notation above, one of the following holds:*

(a) $s = 168 = 2^3 \cdot 3 \cdot 7$ *and* $r = 6$.

(b) $s = 252 = 2^2 \cdot 3^2 \cdot 7$ *and* $r = 6$.

(c) $s = 378 = 2 \cdot 3^3 \cdot 7$ *and* $r = \frac{16}{3}$.

(d) $r = \frac{14}{3}$ *and* Λ *is of minimal type.*

3.2 The case $s = 168$, $r = 6$.

Proposition 3.5 *There is no strongly perfect lattice satisfying Proposition 3.4 (a).*

Proof. Let Λ be such a strongly perfect lattice with $\min(\Lambda^*) = r = 6$ and $s(\Lambda) = s = 168$. Then for all $\alpha \in \Lambda^*$

$$\begin{array}{lll} \sum_{x\in X}(x,\alpha)^2 & = 14(\alpha,\alpha) & \in \mathbb{Z} \\ \sum_{x\in X}(x,\alpha)^4 & = 3(\alpha,\alpha)^2 & \in \mathbb{Z} \end{array}$$

hence all norms in Λ^* are integers. The equalities $\frac{1}{6}((D13)-(D11))$ and $\frac{1}{12}((D2)-(D4))$ yield that

$$\begin{array}{ll} \frac{1}{6}(\alpha_1,\alpha_2)(3(\alpha_2,\alpha_2)-14) & \in \mathbb{Z} \\ \frac{1}{12}(\alpha_1,\alpha_1)(3(\alpha_1,\alpha_1)-14) & \in \mathbb{Z} \end{array}$$

for all $\alpha_1, \alpha_2 \in \Lambda^*$. Therefore Λ^* is an even lattice with $3 \mid (\alpha_1,\alpha_2)$ for all $\alpha_1,\alpha_2 \in \Lambda^*$ and hence $\Gamma := \frac{1}{\sqrt{3}}\Lambda^*$ is an even lattice with $\min(\Gamma) = 2$ and $\min(\Gamma^*) = 3$ which is a contradiction. □

3.3 Lattices of minimal type.

In this section we prove the following

Theorem 3.6 *Let Λ be a strongly perfect lattice of dimension 12 and of minimal type, i.e. $\min(\Lambda)\min(\Lambda^*) = \frac{n+2}{3} = \frac{14}{3}$ and let $s := \frac{1}{2}|\Lambda_{\min}|$. Then $s = 378$ or $s = 252$.*

Rescale Λ, such that $\min(\Lambda) = 1$. Then $\min(\Lambda^*) = \frac{14}{3}$. Applying $(D2)$ to $\alpha \in (\Lambda^*)_{14/3}$ yields that

$$\frac{14s}{12\cdot 3} = \frac{7s}{18} \in \mathbb{Z}.$$

The bound given in [1] yields $s \leq 614$. Hence

$$s = 18s_1 \leq 614 \text{ for some } s_1 \in \mathbb{N}, s_1 \leq 34.$$

Lemma 3.7 *7 divides s_1.*

Proof. Assume that 7 does not divide s_1 and choose $\alpha \in \Lambda^*$. The $(\alpha,\alpha) = \frac{p}{q}$ for some coprime integers p, q. Since $(D4)(\alpha) = 3^2 \cdot 2^{-2} \cdot 7^{-1} s_1 \frac{p^2}{q^2}$ is integral, this implies that $p = 7p_1$ is divisible by 7 and q^2 divides $3^2 s_1$. Moreover $((D4)-(D2))(\alpha)$ is divisible by 12 which yields that

$$(\star) \quad \frac{s_1}{2^4 q^2} 7p_1(3p_1 - 2q) \in \mathbb{Z} \ .$$

If q is even then p_1 is odd and $2^6 = 64 \mid 2^4 q^2 \mid s_1$ which contradicts the fact that $s_1 \leq 34$. Since q^2 divides $3^2 s_1$, the only possibilities for q are

$$q = 1, 3, 5, 3^2, 3\cdot 5.$$

•First assume that there is $\alpha \in \Lambda^*$ with $(\alpha,\alpha) = \frac{7a}{5}$ for some $a \in \mathbb{N}$ not divisible by 5.

Then $s_1 = 5^2$ is odd and hence by $(\star)$ the norms of the elements in Λ^* lie in $\frac{14}{15}\mathbb{Z}$. Let

$$\Gamma := \sqrt{\frac{15}{14}}\Lambda^*.$$

Then all norms in Γ are integers. For the scalar products we apply $(D22)$ to $\alpha, \beta \in \Gamma$ to find that

$$(\star\star) \quad \frac{7}{3}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z}.$$

Therefore Γ is an integral lattice. Let

$$\begin{array}{rcl} \Gamma^{(e)} & := & \{\gamma \in \Gamma \mid (\gamma,\gamma) \in 2\mathbb{Z}\} \\ \Gamma^{(t)} & := & \{\gamma \in \Gamma \mid (\gamma,\gamma) \in 3\mathbb{Z}\} \\ \Gamma' := \Gamma^{(e)} \cap \Gamma^{(t)} & := & \{\gamma \in \Gamma \mid (\gamma,\gamma) \in 6\mathbb{Z}\} \end{array}$$

Then $\Gamma^{(e)}$ is the even sublattice of Γ of index 2. Moreover $((D4)-(D2))(\alpha)$ yields that $\frac{7}{12}(\alpha,\alpha)((\alpha,\alpha)-5) \in \mathbb{Z}$ for all $\alpha \in \Gamma$. In particular the norms in $\Gamma^{(e)}$ are divisible by 4 and hence $\frac{1}{\sqrt{2}}\Gamma^{(e)}$ is still even. By $(\star\star)$ and Lemma 2.9 $\Gamma^{(t)}$ is a sublattice of index ≤ 3 in Γ. In particular $[\Gamma : \Gamma'] \leq 6$ and

$$\Delta := \frac{1}{\sqrt{6}}\Gamma'$$

is even. Since $\min(\Gamma^*)\det(\Gamma)^{1/12} \leq \gamma_{12} \leq 2.522$ we get

$$\det(\Gamma) \leq (2.522\frac{15}{14})^{12} < 151530.4$$

and therefore

$$\det(\Delta) \leq \frac{1}{6^{12}}6^2\det(\Gamma) \leq \frac{151530}{6^{10}} \leq 0.026$$

which is a contradiction since Δ is integral.

•Now assume that there is $\alpha \in \Lambda^*$ with $(\alpha,\alpha) = \frac{7p_1}{9}$

Then $s_1 = 3^2c$ for some $c \in \{1,2,3\}$ and hence by $(\star)$ the norms of the elements in Λ^* lie in $\frac{14}{9}\mathbb{Z}$. Let

$$\Gamma := \sqrt{\frac{9}{14}}\Lambda^*.$$

Then $\min(\Gamma) = 3$, $\min(\Gamma^*) = \frac{14}{9}$ and all norms in Γ are integers. For the scalar products we apply $(D22)$ to $\alpha, \beta \in \Gamma$ to find that

$$(D22)(\alpha,\beta): \quad \frac{7}{3}c(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z} \text{ for all } \alpha,\beta \in \Gamma.$$

In the new scaling the equation $((D4)-(D2))(\alpha)$ yields that

$$\frac{7}{12}c(\alpha,\alpha)((\alpha,\alpha)-3) \in \mathbb{Z} \text{ for all } \alpha \in \Gamma.$$

Assume first that c is odd. Then Γ is an integral lattice. Let

$$\Gamma^{(e)} := \{\gamma \in \Gamma \mid (\gamma, \gamma) \in 2\mathbb{Z}\}$$

be the even sublattice of Γ. Then the norms in $\Gamma^{(e)}$ are divisible by 4 and hence $\frac{1}{\sqrt{2}}\Gamma^{(e)}$ is an integral lattice. Therefore

$$1024 = 2^{10} \leq \frac{\det(\Gamma^{(e)})}{4} = \det(\Gamma) \leq (2.522\frac{9}{14})^{12} \leq 330$$

which is a contradiction.

Now assume that $c = 2$. Then $(D13) - (D11)(\alpha, \beta)$ yields that

$$\frac{7}{3}(\alpha, \beta)((\alpha, \alpha) - 3) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Gamma.$$

In particular $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Gamma$ with $(\alpha, \alpha) \in 2\mathbb{Z}$ and

$$\Gamma^{(e)} := \{\gamma \in \Gamma \mid (\gamma, \gamma) \in 2\mathbb{Z}\}$$

is a sublattice of Γ of index 2 or 4 in Γ by Lemma 2.8. Moreover $(D22)(\alpha, \beta)$ yields that

$$\Gamma^{(t)} := \{\gamma \in \Gamma \mid (\gamma, \gamma) \in 3\mathbb{Z}\}$$

is a sublattice of Γ of index 1 or 3 by Lemma 2.9. Put

$$\Gamma' := \Gamma^{(t)} \cap \Gamma^{(e)} .$$

Then Γ' is an even lattice of index ≤ 12 in Λ for which $\frac{1}{\sqrt{3}}\Gamma'$ is still even. Hence

$$3690 \leq \frac{3^{10}}{4^2} \leq \frac{\det(\Gamma')}{12^2} = \det(\Gamma) \leq (2.522\frac{9}{14})^{12} \leq 330$$

which is a contradiction.

- The remaining case is that all norms of elements in Λ^* lie in $\frac{7}{3}\mathbb{Z}$.

Let $\Gamma := \sqrt{\frac{3}{7}}\Lambda^*$. Then $\min(\Gamma) = 2$, $\min(\Gamma^*) = \frac{7}{3} > 2$ and all norms in Γ are integers. But Γ is not an integral lattice since $\min(\Gamma) < \min(\Gamma^*)$. In particular, there is $\alpha \in \Gamma$ with $(\alpha, \alpha) \in 1 + 2\mathbb{Z}$.

In the new scaling the equation $\frac{1}{12}((D4) - (D2))(\alpha)$ and $\frac{1}{6}((D13) - (D11))(\alpha, \beta)$ yield that

$$2^{-4}3^{-1}7s_1(\alpha, \alpha)((\alpha, \alpha) - 2) \in \mathbb{Z} \text{ for all } \alpha \in \Gamma$$
$$2^{-3}3^{-1}7s_1(\alpha, \beta)((\alpha, \alpha) - 2) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Gamma.$$

In particular

$$2^4 \mid s_1 \in \{16, 32\} \text{ and 3 divides } (\alpha, \beta)((\alpha, \alpha) - 2) \text{ for all } \alpha, \beta \in \Gamma.$$

As above we find that

$$\Gamma^{(t)} := \{\gamma \in \Gamma \mid (\gamma, \gamma) \in 3\mathbb{Z}\}$$

is a sublattice of Γ of index 3 such that $\sqrt{\frac{2}{3}}\Gamma^{(t)}$ is even. Therefore

$$14 \leq \frac{3^{10}}{2^{12}} \leq \frac{\det(\Gamma^{(t)})}{9} = \det(\Gamma) \leq (2.522\frac{3}{7})^{12} \leq 2.6$$

which is a contradiction. □

We therefore have $s = 2 \cdot 3^2 \cdot 7 \cdot s_2$ with $s_2 \in \{1, 2, 3, 4\}$. To obtain the theorem it remains to show that $s_2 = 3$ or $s_2 = 2$. We keep the scaling such that $\min(\Lambda) = 1$.

Lemma 3.8 *Assume that* $s_2 \neq 3$. *Then*

$$\Gamma^{(t)} := \{\gamma \in \Lambda^* \mid (\gamma, \gamma) \in \mathbb{Z}\}$$

is an even sublattice of Λ^* *of index* $[\Lambda^* : \Gamma^{(t)}] = 3$.

Proof. For $\alpha \in \Lambda^*$ we write $(\alpha, \alpha) = \frac{p}{q}$ with coprime integers p, q. Then $\frac{1}{12}((D4) - (D2))(\alpha) \in \mathbb{Z}$ reads as

$$(\star\star) \quad \frac{s_2}{2^4 q^2} p(3p - 14q) \in \mathbb{Z}.$$

Since 2^4 does not divide s_2, we find that p is even and $q \in \{1, 3\}$.

Moreover $\frac{1}{6}((D13) - (D11))(\alpha, \beta)$ yields that

$$(\star_2) \quad \frac{1}{2^3} s_2(\alpha, \beta)(3(\alpha, \alpha) - 14) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Lambda^*.$$

Hence if $s_2 \neq 3$, then $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha \in \Lambda^*$ for which $(\alpha, \alpha) \in \mathbb{Z}$, therefore

$$\Gamma^{(t)} := \{\gamma \in \Lambda^* \mid (\gamma, \gamma) \in \mathbb{Z}\}$$

is an even sublattice of Λ^*. The index of $\Gamma^{(t)}$ in Λ^* is 3, since for $\alpha, \beta \in \Lambda^* - \Gamma^{(t)}$, $(D22)(\alpha, \beta)$ implies that

$$\frac{3s_2}{4}(2(\alpha, \beta)^2 + (\alpha, \alpha)(\beta, \beta)) \in \mathbb{Z}.$$

Rescaling the bilinear form with 3, this allows to apply Lemma 2.9 to see that $[\Lambda^* : \Gamma^{(t)}] = 3$. □

Lemma 3.9 $s_2 \neq 1$.

Proof. Assume that $s_2 = 1$. The norms of the elements of the even lattice $\Gamma^{(t)}$ in Lemma 3.8 satisfy $(\star\star)$ with $q = 1$ and $s_2 = 1$. Moreover $\min(\Gamma^{(t)}) \geq \frac{14}{3}$. Since $p = 6$ does not satisfy $(\star\star)$ we find $\min(\Gamma^{(t)}) \geq 8$. Since $d := \det(\Lambda^*) \leq \gamma_{12}^{12} \leq 66212.7$ we get

$$\gamma(\Gamma^{(t)}) \geq \frac{8}{(9d)^{1/12}} \geq \frac{8}{(9 \cdot 66212.7)^{1/12}} \geq 2.64$$

contradicting that $\gamma_{12} \leq 2.522$. □

Lemma 3.10 $s_2 \neq 4$.

Proof. Assume that $s_2 = 4$. Then $\Gamma^{(t)}$ is an even lattice of minimum $\geq \frac{14}{3}$. Hence $\min(\Gamma^{(t)}) \geq 6$. If we show that $\Gamma^{(t)}$ does not contain vectors of square length 6, then we obtain a contradiction as in Lemma 3.9. So let $\gamma \in \Gamma^{(t)}$ be an element of norm $(\gamma, \gamma) = 6$. Then for all $x \in X$, $(x, \gamma) \in \{0, \pm 1, \pm 2\}$ and hence by Lemma 2.3 $n_2(\gamma) = 6 > \frac{6}{8-6} = 3$ contradicting the bound of Lemma 2.4. □

3.4 Résumé

Let us summarise what we have shown:

Theorem 3.11 *Let Λ be a strongly perfect lattice of dimension 12 and $s := \frac{1}{2}|\Lambda_{\min}|$ be half the kissing number of Λ. Then either $s = 252$ and $\min(\Lambda)\min(\Lambda^*) \in \{6, \frac{14}{3}\}$. or $s = 378$ and $\min(\Lambda)\min(\Lambda^*) \in \{\frac{16}{3}, \frac{14}{3}\}$.*

The next two sections treat the two possibilities $s = 252$ and $s = 378$ separately.

3.5 The case $s = 252$.

In this section we show

Theorem 3.12 *There is no strongly perfect lattice in dimension 12 with $s = 252$.*

Let Λ be such a strongly perfect lattice with $s(\Lambda) = s = 252$ scaled such that $\min(\Lambda) = 2$ and put $\Gamma := \Lambda^*$. Then $\min(\Gamma)$ is one of 3 or $\frac{7}{3}$ and for all $\alpha \in \Gamma$ equation $(D4)$ yields that $\sum_{x \in X}(x, \alpha)^4 = 18(\alpha, \alpha)^2$ Hence $(\alpha, \alpha) \in \frac{1}{3}\mathbb{Z}$. The equalities $\frac{1}{6}((D13) - (D11))$, $\frac{1}{12}((D4) - (D2))$, and $(D22)$ yield that

$$\begin{array}{ll} (\alpha,\beta)(3(\alpha,\alpha)-7) & \in \mathbb{Z} \\ \frac{1}{2}(\alpha,\alpha)(3(\alpha,\alpha)-7) & \in \mathbb{Z} \quad \text{for all } \alpha,\beta \in \Gamma. \\ 6(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) & \in \mathbb{Z} \end{array}$$

In particular, if $(\alpha, \alpha) \in 2\mathbb{Z}$, then $(\alpha, \beta) \in \mathbb{Z}$ for all $\beta \in \Gamma^*$.

Put

$$\Delta := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 2\mathbb{Z}\}.$$

Then Δ is an even sublattice of index $\iota := [\Gamma : \Delta]$ in Γ.

Lemma 3.13 *$\iota := |\Gamma/\Delta|$ is a divisor of 12. If $\iota \neq 2, 6$ then the quadratic group $(\Gamma/\Delta, -(,))$ is isometric to R^*/R for the following root lattice R:*

ι	3	4	12
R	A_2	D_4	$A_2 \perp D_4$

If $\iota = 2$ or $\iota = 6$ then

$$\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in \mathbb{Z}\}$$

is an odd integral sublattice of Γ of index $\iota/2$ such that Δ is the even sublattice of $\Gamma^{(t)}$. For $\iota = 6$ the quadratic group $\Gamma/\Gamma^{(t)}$ is isometric to $(A_2^*/A_2, -(,))$.

Proof. By Lemma 2.8 and 2.9 the index of Δ in Γ is a divisor of 12. Let us treat the primes 2 and 3 separately. Assume that 3 divides ι. Then there is $\alpha \in \Gamma$ with $(\alpha,\alpha) \in \frac{1}{3}\mathbb{Z} - \mathbb{Z}$. Equation $\frac{1}{12}((D4) - (D2))$ yields that $3(\alpha,\alpha) - 7 \in 3\mathbb{Z}$ hence $(\alpha,\alpha) \in \frac{1}{3} + \mathbb{Z}$. Hence the 3-Sylow subgroup of the quadratic group Γ/Δ is isometric to $(A_2^*/A_2, -(,))$. For the 2-Sylow subgroup note that the norms in Γ are 2-integral. Moreover $\alpha + \Delta = \beta + \Delta \neq \Delta$ if and only if $(\alpha,\alpha) \equiv (\beta,\beta) \equiv 1 \pmod{2\mathbb{Z}_2}$ and $(\alpha,\beta) \in \mathbb{Z}_2$. Hence if $|\Gamma/\Delta| = 4$ or 12, then the 2-Sylow subgroup of the quadratic group Γ/Δ is isometric to D_4^*/D_4. Also if $|\Gamma/\Delta| = 2$ or 6, then Γ is already 2-integral. □

Since $\min(\Gamma^*) = 2$ we find the bound

$$\det(\Gamma) \leq 16.17, \quad \det(\Delta) \leq 16.17\iota^2.$$

Since $\iota \leq 12$ this implies that $\min(\Delta) = 4$.

The strategy is to construct an even overlattice $\tilde{\Delta}$ of Δ, such that $\Delta = \Gamma \cap \tilde{\Delta}$ and hence

$$\Gamma/\Delta \cong (\Gamma + \tilde{\Delta})/\tilde{\Delta} \cong -R^*/R.$$

Denote the last isomorphism by φ. Then the subdirect product

$$M := (\Gamma + \tilde{\Delta}) \star R^* = \{(v_1, v_2) \in (\Gamma + \tilde{\Delta}) \perp R^* \mid v_2 = \varphi(v_1)\}$$

is an even lattice of dimension $12+\dim(R)$ and determinant $\det(M) = \det(\tilde{\Delta})/\det(R)$.

Lemma 3.14 *Let $\gamma \in \Delta_4$. Then $N_2(\gamma) = \{x_1, \ldots, x_5, y_1, \ldots, y_5\}$ with $\gamma = x_i + y_i$, $(x_i, y_i) = 0$ and $(x_i, y_j) = (x_i, x_j) = 1$ for all $i \neq j \in \{1, \ldots, 5\}$. The lattice generated by $N_2(\gamma)$ is isometric to the root lattice D_6.*

Proof. By Lemma 2.3 $|N_2(\gamma)| = 10$. Moreover if $x \in N_2(\gamma)$, then also $\gamma - x \in N_2(\gamma)$. This gives us the partition $N_2(\gamma) = \{x_1, \ldots, x_5, y_1, \ldots, y_5\}$ with $x_i + y_i = \gamma$. Taking scalar products with x_1 we get

$$2 = (x_1, \gamma) = (x_1, x_j) + (x_1, y_j).$$

Since $(x_1, x) \leq 1$ for all $x \in N_2(\gamma)$, $x \neq x_1$, we get $(x_1, x_j) = (x_1, y_j) = 1$ for all $j \geq 2$ as claimed. □

Proposition 3.15 *Fix some $\gamma \in \Delta_4$ and put*

$$\tilde{\Delta} := \langle \Delta, N_2(\gamma) \rangle.$$

Then $[\tilde{\Delta} : \Delta] \geq 8$. If $\iota = 3$ or $\iota = 12$ then $[\tilde{\Delta} : \Delta] = 8$ and $\tilde{\Delta}/\Delta$ is not cyclic.

Proof. First we note that $\tilde{\Delta}$ is an even lattice, $\Gamma \cap \tilde{\Delta} = \Delta$ and $\Gamma + \tilde{\Delta} \leq \tilde{\Delta}^*$. Therefore ι divides the determinant of $\tilde{\Delta}$.

We keep the notation of Lemma 3.14 and denote by $x \mapsto \overline{x}$ the natural projection $\tilde{\Delta} \to \tilde{\Delta}/\Delta$. Let $J := \{i \in \{1, \ldots, 5\} \mid 2\overline{x_i} \neq 0\}$ and $I := \{1, \ldots, 5\} - J$. Since

$\min(\Delta) = 4 > 2$, we have $\overline{x_i} \neq 0 \neq \overline{x_i - x_j}$ for $i \neq j$. Hence $\{0, \overline{x_1}, \ldots, \overline{x_5}\}$ are 6 distinct elements of $\tilde{\Delta}/\Delta$. Moreover, if $\overline{x_i} = -\overline{x_j}$ then $x_i + x_j - \gamma = x_j - y_i \in \Delta$ is an element of norm 2 for $j \neq i$. Therefore $\overline{x_i} = -\overline{x_j}$ if and only if $i = j \in I$. This implies that $|\tilde{\Delta}/\Delta| \geq 6 + |J|$.

If $|\tilde{\Delta}/\Delta| = 6$ then $J = \emptyset$ and $\tilde{\Delta}/\Delta$ is an elementary abelian 2-group. Therefore $|\tilde{\Delta}/\Delta| \geq 8$. If $|\tilde{\Delta}/\Delta| = 7$ or 9, then $\tilde{\Delta}/\Delta$ has no elements of order 2, hence $I = \emptyset$ and therefore $|\tilde{\Delta}/\Delta| \geq 11$. Therefore

$$|\tilde{\Delta}/\Delta| = 8 \text{ or } |\tilde{\Delta}/\Delta| \geq 10.$$

Now assume that $\iota \in \{3, 12\}$ and let R be the root lattice from Lemma 3.13. If $|\tilde{\Delta}/\Delta| \geq 10$ then $\det(\tilde{\Delta}) \leq \frac{\det(\Delta)}{100} \leq 23.3\iota/12$. Since $\det(\tilde{\Delta})$ is divisible by ι, we get $\det(\tilde{\Delta}) = \iota$ and

$$\tilde{\Delta}^* = \Delta + \Gamma.$$

Taking the subdirect product $M := (\Gamma + \tilde{\Delta}) \star R^*$ as described above, constructs an even unimodular lattice in dimension 14 or 18, which is a contradiction. □

Proposition 3.16 *$\iota = [\Gamma : \Delta] \neq 2$ or 6.*

Proof. Fix some $\gamma \in \Delta_4$ and put

$$\tilde{\Gamma} := \langle \Gamma^{(t)}, N_2(\gamma) \rangle.$$

As in the proof of Proposition 3.15 we see that $[\tilde{\Gamma} : \Gamma^{(t)}] \geq 8$.

Assume that $\iota = 2$. Then $\Gamma = \Gamma^{(t)}$ is an integral lattice of determinant ≤ 16.17. Hence $\det(\tilde{\Gamma}) \leq \frac{1}{8^2} 16.17 < 1$ which is a contradiction.

If $\iota = 6$, then $\Gamma/\Gamma^{(t)} \cong -A_2^*/A_2$. Taking the subdirect product $\Gamma \star A_2^*$ we obtain a 14-dimensional integral lattice of determinant $\leq \frac{3}{8^2} 16.17 < 1$ which is a contradiction. □

Definition 3.17 *Let*

$$M := (\Gamma + \tilde{\Delta}) \star R^* = \{(v_1, v_2) \in (\Gamma + \tilde{\Delta}) \perp R^* \mid v_2 = \varphi(v_1)\}.$$

Remark 3.18 *M is an even lattice of dimension $12 + \dim(R)$ and determinant*

$$\det(M) = \frac{\det(\tilde{\Delta})}{\det(R)} \leq \frac{\det(\Delta)}{8^2 \det(R)} = \iota \frac{\det(\Gamma)}{64} \leq 0.253\iota.$$

In particular $\iota \geq 4$ and hence $\iota \neq 3$.
Moreover $R^\perp = \{m \in M \mid (m, r) = 0$ for all $r \in R\} = \tilde{\Delta}$ contains a sublattice $R' = \langle N_2(\gamma) \rangle \cong D_6$.

Lemma 3.19 *The lattice R' is either a component of the root sublattice $\langle \tilde{\Delta}_2 \rangle$ or $R' \subseteq E_7 \subseteq \langle \tilde{\Delta}_2 \rangle$ is contained in a component isometric to E_7.*

Proof. Let $\tilde{R}$ be the component of $\langle\tilde{\Delta}_2\rangle$ containing R'. We have

$$\gamma = x_1 + y_1 \in R' \subset \tilde{R} \subset \tilde{\Delta} \subset \Lambda.$$

Since $N_2(\gamma) \subset R'$, there are no $x \in \tilde{R}_2 - R'$ satisfying $(\gamma, x) = 2$. Going through all possible root lattices, we see that only the two possibilities $\tilde{R} = R' \cong D_6$ or $\tilde{R} \cong E_7$ arise. □

We treat the two remaining cases $\iota = 4$ and $\iota = 12$ separately.

Lemma 3.20 $\iota \neq 4$.

Proof. If $\iota = 4$ then M is a 16-dimensional even unimodular lattice, hence $M \cong E_8 \perp E_8$ or $M \cong D_{16}^+$ and $\tilde{\Delta} = D_4^\perp$ is the orthogonal complement of a root system D_4 in M and hence has root system $E_8 \perp D_4$ respectively D_{12} contradicting Lemma 3.19. □

Lemma 3.21 $\iota \neq 12$.

Proof. If $\iota = 12$, then M is an even 18-dimensional lattice of determinant ≤ 3. Since there are no even unimodular 18-dimensional lattices and also no even lattices of determinant 2 and dimension 18, we have $\det(M) = 3$, $\det(\tilde{\Delta}) = 36$ and $\det(\Delta) = 8^2 \cdot 3 \cdot 12 = 2304$. There is one genus of 18-dimensional even lattices of determinant 3, it contains 6 isometry-classes, representatives $M_1, \ldots, M_6$ of which have root sublattices

$$E_7 \perp D_{10},\ E_8 \perp E_8 \perp A_2,\ A_2 \perp D_{16},\ A_{17},\ E_6 \perp E_6 \perp E_6,\ \text{and } D_7 \perp A_{11}.$$

By Lemma 3.19, the only possibility is $M = M_1$ with root system $E_7 \perp D_{10}$. Since all reflections along norm 2 vectors are automorphisms of M, there are up to isometries 3 embeddings of $R = A_2 \perp D_4$ into M:

$$(1) A_2 \perp D_4 \subset D_{10},\quad (2) A_2 \subset D_{10},\ D_4 \subset E_7,\quad (3) D_4 \subset D_{10},\ A_2 \subset E_7$$

(see for instance [9, Table 4]). The root systems of the orthogonal complements of R are

$$(1) A_3 \perp E_7,\quad (2) A_1^3 \perp D_7,\quad (3) A_5 \perp D_6.$$

By Lemma 3.19 possibility (2) is excluded. For the other two lattices we calculate all sublattices of index 8, such that the factor group is not cyclic to get a list of candidates for Δ. None of these lattices has minimum 4. □

This concludes the proof of Theorem 3.12.

3.6 The case $s = 378$

This section completes the proof of Theorem 3.1 by showing the following

Theorem 3.22 *If Λ is a strongly perfect lattice of dimension 12 with Kissing number $2 \cdot 378$ then Λ is similar to the Coxeter Todd lattice CT.*

Let $s = 2 \cdot 3^3 \cdot 7 = 378$ and rescale the strongly perfect 12-dimensional lattice Λ such that $\min(\Lambda) = \frac{4}{3}$. Because of Remark 3.3 we may and will assume that Λ is generated by its minimal vectors. Let $\Gamma := \Lambda^*$. Then $\min(\Gamma) = 4$ or $\min(\Gamma) = \frac{7}{2}$ and for all $\alpha, \beta \in \Gamma$

$$\begin{array}{lrl} (D2) & 2 \cdot 3 \cdot 7(\alpha,\alpha) & \in \mathbb{Z} \\ (D4) & 2^2 \cdot 3(\alpha,\alpha)^2 & \in \mathbb{Z} \\ 12^{-1}((D4)-(D2)) & \frac{1}{2}(\alpha,\alpha)(2(\alpha,\alpha)-7) & \in \mathbb{Z} \\ 6^{-1}((D13)-(D11)) & (\alpha,\beta)(2(\alpha,\alpha)-7) & \in \mathbb{Z} \end{array}$$

From $(D2)$ and $(D4)$ we find that $(\alpha,\alpha) \in \frac{1}{2}\mathbb{Z}$ for all $\alpha \in \Gamma$. If $(\alpha,\alpha) \in \mathbb{Z}$, then $\frac{1}{6}((D13) - (D11))$ implies that $(\alpha,\beta) \in \mathbb{Z}$ for all $\beta \in \Gamma$. In particular $(2\alpha,\beta) = 2(\alpha,\beta) \in \mathbb{Z}$ for all $\alpha,\beta \in \Gamma$. Let

$$\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha,\alpha) \in \mathbb{Z}\}.$$

Then $\Gamma^{(e)}$ is an integral sublattice of Γ and $\iota := [\Gamma : \Gamma^{(e)}] \leq 2$ since $(\alpha,\beta) \in \frac{1}{2}\mathbb{Z}$ for all $\alpha,\beta \in \Gamma$ and therefore $\Gamma^{(e)}$ is the kernel of the linear mapping $\Gamma \to \mathbb{Z}/2\mathbb{Z}, \alpha \mapsto 2(\alpha,\alpha)$.

Equality $\frac{1}{12}((D4) - (D2))$ yields that (α,α) is even whenever $\alpha \in \Gamma^{(e)}$, hence $\Gamma^{(e)}$ is an even lattice. Moreover the norms of the elements $\alpha \in \Gamma - \Gamma^{(e)}$ satisfy $2(\alpha,\alpha) - 7 \in 4\mathbb{Z}$, hence

$$(\alpha,\alpha) \in \frac{3}{2} + 2\mathbb{Z} \text{ for all } \alpha \in \Gamma - \Gamma^{(e)}.$$

Moreover

$$(\Gamma,\Gamma^{(e)}) \subseteq \mathbb{Z},\ \ (\Gamma - \Gamma^{(e)}, \Gamma - \Gamma^{(e)}) \in \frac{1}{2} + \mathbb{Z}$$

where the latter follows since $\alpha + \beta \in \Gamma^{(e)}$ for all $\alpha,\beta \in \Gamma - \Gamma^{(e)}$. Let $d := \det(\Gamma)$ and $d_0 := \det(\Gamma^{(e)}) = \iota^2 d$ for $\iota = [\Gamma : \Gamma^{(e)}] \in \{1,2\}$. Then

$$\min(\Lambda) \sqrt[12]{d} = \frac{4}{3} \sqrt[12]{d_0/\iota} \leq \gamma_{12} \leq 2.522$$

which yields $d_0 \leq 2097\iota^2$. On the other hand $\min(\Gamma^{(e)}) \geq 4$ yields that $d_0 \geq 254$. If $\min(\Gamma^{(e)}) \geq 6$, then $d_0 \geq 32875$ which is a contradiction. Therefore there is some $\gamma \in \Gamma^{(e)}$ with $(\gamma,\gamma) = 4$.

Hence we have shown that

Lemma 3.23 $\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha,\alpha) \in \mathbb{Z}\}$ *is an even sublattice of* Γ *of index* $\iota \leq 2$ *and minimum* 4. *We have* $\det(\Gamma) \leq 2097$ *and* $\det(\Gamma^{(e)}) \leq 2097\iota^2$.

The strategy of the proof of Theorem 3.22 is to construct an integral overlattice M of $\Gamma^{(e)}$ (respectively glue Γ with A_1^* to obtain an even overlattice of $\Gamma^{(e)} \oplus A_1$ in dimension 13 and then to find some overlattice M) of small determinant. Then we go through all possibilites for M and calculate $\Gamma^{(e)}$ as a sublattice of M. To prove the existence of such an integral overlattice M, we find vectors of even norm and integral scalar products as linear combinations of the vectors in X by analysing the possibilities for $N_2(\gamma)$ for $\gamma \in \Gamma$ satisfying the conditions of Lemma 2.3. The design properties of X allow to obtain very precise information about the sets $N_2(\gamma)$, if γ has norm 4 (see Propositions 3.26 and 3.28). Therefore we want to show that Γ_4 spans a space of dimension at least 8. This allows to construct an overlattice M (see Corollary 3.35). To use the theorem by Minkowski on the successive minima of Γ we need to bound the number of vectors of norm $\frac{7}{2}$ in Γ. Proposition 3.32 shows that there are at most two such vectors and Proposition 3.24 shows that no vector in Γ has norm $\frac{11}{2}$.

Proposition 3.24 *There is no $\gamma \in \Gamma$ with $(\gamma, \gamma) = \frac{11}{2}$.*

Proof. Let $\gamma \in \Gamma$ with $(\gamma, \gamma) = \frac{11}{2}$. Then by Lemma 2.3 $N_2(\gamma) = \{x_1, \ldots, x_{11}\}$ with $\sum_{i=1}^{11} x_i = 4\gamma$. Since $(x_i, \sum_{j=1}^{11} x_j) = 4(x_i, \gamma) = 8$ and $(x_i, x_j) \leq \frac{2}{3}$ for all $i \neq j$ we get that $(x_i, x_j) = \frac{1}{10}(8 - \frac{4}{3}) = \frac{2}{3}$ for all $i \neq j$ and $L := \langle x_1, \ldots, x_{11} \rangle \cong \sqrt{\frac{2}{3}} A_{11}$. We now enlarge $\Gamma^{(e)}$ to an integral overlattice

$$\Gamma^{(e)} \subset \tilde{\Gamma} \subset \langle \Gamma^{(e)}, N_2(\gamma) \rangle$$

by joining preimages of a maximal isotropic subspace of $L/(L^* \cap L) \otimes \mathbb{F}_3$. We find such a subspace of dimension 5 which allows us to construct an integral overlattice of $\Gamma^{(e)}$ of index 3^5 which contradicts the fact that $\det(\Gamma^{(e)}) \leq 4 \cdot 2097 < (3^5)^2$. In detail let

$$\begin{aligned} y_1 &:= x_3 + x_4 + x_5 \\ y_2 &:= x_6 + x_7 + x_8 \\ y_3 &:= x_9 + x_{10} + x_{11} \\ y_4 &:= x_3 - x_4 + x_7 - x_8 + x_9 - x_{11} \\ y_5 &:= x_1 + x_7 - x_8 + x_{10} - x_{11} \end{aligned}$$

Then the subspace $\langle y_1, \ldots, y_5 \rangle \leq L$ is an integral sublattice of L and the linear functionals $x \mapsto (x, y_i) \in \frac{1}{3}L^*$ $(i = 1, \ldots, 5)$ are linearly independent modulo L^*. Therefore

$$\tilde{\Gamma} := \langle \Gamma^{(e)}, y_1, \ldots, y_5 \rangle$$

is an integral overlattice of $\Gamma^{(e)}$ of index divisible by 3^5. On the other hand

$$\det(\tilde{\Gamma}) \leq \frac{1}{3^{10}} \det(\Gamma^{(e)}) \leq \frac{4 \cdot 2097}{3^{10}} \leq 0.15 < 1$$

yields a contradiction. □

The next aim is to investigate the vectors of norm 4 in Γ. From Lemma 2.3 and the equalities $(D2)$ and $(D4)$ we find

Lemma 3.25 *Let $\gamma \in \Gamma^{(e)}$ with $(\gamma, \gamma) = 4$. Then $N_2(\gamma) = \{x_1, x_2\}$ with $(x_1, x_2) = \frac{2}{3}$ and $\gamma = x_1 + x_2$. Moreover $|N_1(\gamma)| = 160$ and $|N_0(\gamma)| = 216$.*

Proposition 3.26 *Let $\gamma \in \Gamma^{(e)}$ with $(\gamma,\gamma) = 4$ and fix $x_1 \in N_2(\gamma) = \{x_1, x_2\}$. Then*

$$(x_1, X) \subset \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}\}$$

with

$$|N_0(x_1)| = 135,\ |N_{1/3}(x_1)| = 160,\ |N_{2/3}(x_1)| = 82,\ |N_{4/3}(x_1)| = 1.$$

Choose X such that $(x,\gamma) \geq 0$ for all $x \in X$ and $(x, x_1) \geq 0$ for all $x \in N_0(\gamma)$ and define

$$M_{ij}(\gamma, x_1) := \{x \in X \mid (x,\gamma) = i, (x, x_1) = j/3\}.$$

Then $M_{0,2}(\gamma, x_1) = \{x_1 - x_2\}$, $M_{2,2}(\gamma, x_1) = \{x_2\}$, $M_{2,4}(\gamma, x_1) = \{x_1\}$ and the $m_{ij} := |M_{ij}(\gamma, x_1)|$ are given in the following table:

m_{ij}	$j=0$	$j=1$	$j=2$	$j=4$
$i=0$	135	80	1	0
$i=1$	0	80	80	0
$i=2$	0	0	1	1

Proof. Choose X such that $(x,\gamma) \geq 0$ for all $x \in X$. Then

$$X = X_0 \cup X_1 \cup X_2$$

where $X_i = \{x \in X \mid (x,\gamma) = i\}$ and $X_2 = \{x_1, x_2\}$ by Lemma 3.25. We also choose the elements in X_0, such that $(x, x_1) \geq 0$ for all $x \in X_0$. Then the equation $(D11)$ with $\alpha_1 = \gamma$ yields that for all $\alpha \in \mathbb{R}^{12}$

$$\sum_{x \in X_1} (x,\alpha) + 2(x_1,\alpha) + 2(x_2,\alpha) = 2 \cdot 3 \cdot 7(\gamma,\alpha).$$

Since $\gamma = x_1 + x_2$ this gives

$$\sum_{x \in X_1} (x,\alpha) = 2^3 \cdot 5(\gamma,\alpha).$$

Similarly equality $(D22)$ yields

$$\sum_{x \in X_1} (x,\alpha)^2 = 2^3(\gamma,\alpha)^2 - 2^2(x_1,\alpha)^2 - 2^2(x_2,\alpha)^2 + 2^4(\alpha,\alpha)$$

and $(D13)$ gives

$$\sum_{x \in X_1} (x,\alpha)^3 = 2^2 3(\gamma,\alpha)(\alpha,\alpha) - 2(x_1,\alpha)^3 - 2(x_2,\alpha)^3$$

Let $x \in X_1$. Then $1 = (x,\gamma) = (x,x_1) + (x,x_2)$. Since $(x,x_i) \leq \frac{2}{3}$, this implies that $(x,x_1) \geq \frac{1}{3}$. For $i \in \mathbb{R}$ let $m_i := |\{x \in X_1 \mid (x,x_1) = i\}|$. Then the equalities above yield

$$\begin{aligned} \textstyle\sum_i m_i &= |X_1| = 2^5 5 \\ \textstyle\sum_i i m_i &= 2^4 5 \\ \textstyle\sum_i i^2 m_i &= 2^4 3^{-2} 5^2 \\ \textstyle\sum_i i^3 m_i &= 2^4 3^{-1} 5 \end{aligned}$$

Hence

$$\sum_i (i - \frac{1}{3})(\frac{2}{3} - i)m_i = \sum_i (-i^2 + i - \frac{2}{9})m_i = 0.$$

Since $\frac{1}{3} \le i \le \frac{2}{3}$ this yields that $(x, x_1) \in \{\frac{1}{3}, \frac{2}{3}\}$ for all $x \in X_1$. Moreover we find

$$m_{1,1} = 80, \quad m_{1,2} = 80.$$

Now equalities $(D2)$ and $(D4)$ yield the following equations for $n_i := \{x \in X_0 \mid (x_1, x) = i\}$:

$$\begin{array}{ll} \sum_i n_i & = |X_0| = 2^3 3^3 \\ \sum_i i^2 n_i & = 2^2 3^{-1} 7 \\ \sum_i i^4 n_i & = 2^5 3^{-3} \end{array}$$

By our assumption $(x, x_1) \ge 0$ for all $x \in X_0$. Moreover for $x \in X_0$ we have $(x, \gamma) = (x, x_1 + x_2) = 0$ hence $(x, x_2) = -(x, x_1)$. Since $(x_1, x_2) = 2/3$ we get $(x_1 - x_2, x_1 - x_2) = 4/3$ and hence $(x, x_1 - x_2) = 2(x, x_1) \le 2/3$ for all $x \in X_0$ with $x \ne x_1 - x_2$. Therefore $n_{2/3} = 1$ and $n_i \ne 0$ only for $i \in [0, \frac{1}{3}] \cup \{2/3\}$ and we get

$$\begin{array}{ll} \sum_{i \in [0, \frac{1}{3}]} n_i & = |X_0| - 1 = 2^3 3^3 - 1 \\ \sum_{i \in [0, \frac{1}{3}]} i^2 n_i & = 2^4 3^{-2} 5 \\ \sum_{i \in [0, \frac{1}{3}]} i^4 n_i & = 2^4 3^{-4} 5 \end{array}$$

from which we get

$$\sum_{i \in [0, \frac{1}{3}]} i^2 (i^2 - \frac{1}{9}) n_i = 0.$$

Therefore $n_i = 0$ for $i \notin \{0, \pm\frac{1}{3}, \pm\frac{2}{3}\}$ and we may use the 3 equalities above to calculate

$$n_0 = 135, \; n_{1/3} = 80, \; n_{2/3} = 1$$

from which the proposition follows. □

Corollary 3.27 *In the situation of Proposition 3.26 we have* $3x_1 \in \Gamma$ *and* $3x_2 \in \Gamma$. *Hence also* $\gamma' := x_1 - 2x_2 = \gamma - 3x_2 \in \Gamma$ *with* $(\gamma', \gamma') = 4$ *and* $(\gamma, \gamma') = -2$.

This allows us to define an equivalence relation on the set of norm 4 vectors $C := \Gamma_4$ in Γ. Note that C consists of the minimal vectors in the integral sublattice $\Gamma^{(e)}$ of Γ. In particular C is not empty by Lemma 3.23 and $|(\gamma_1, \gamma_2)| \le 2$ for distinct elements $\gamma_i \in C$. We call $\gamma_1, \gamma_2 \in C$ equivalent, if $\gamma_1 - \gamma_2 \in 3\Lambda$. Let $\mathcal{K}$ denote the set of equivalence classes. Then $\mathcal{K}$ forms a root system over $\mathbb{Z}[\zeta_3]$. More precisely we have:

Proposition 3.28 *(i) For all* $K \in \mathcal{K}$ *we have* $|K| = 3$.

(ii) If $K = \{u, v, w\}$ *then* $u + v + w = 0$ *and* $(u, v) = (u, w) = (v, w) = -2$. *Moreover* $N_2(u) = \{\frac{1}{3}(u - v), \frac{1}{3}(u - w) = \frac{1}{3}(2u + v)\}$.

(iii) If $K_1 \neq \pm K_2 \in \mathcal{K}$ then either $(k_1, k_2) = 0$ for all $k_i \in K_i$ (in this case the classes are called orthogonal*, $(K_1, K_2) := 0$) or there is $\epsilon \in \{\pm 1\}$ and a mapping $\varphi : K_1 \to K_2$ with $(k_1, \varphi(k_1)) = 2\epsilon$ and $(k_1, k_2) = -\epsilon$ for all $k_1 \in K_1, k_2 \in K_2 - \{\varphi(k_1)\}$. In this situation we will say that $(K_1, K_2) := \epsilon$.*

(iv) If $(K_1, K_2) = -1$, then $K_3 := K_1 + K_2 := \{k_1 + \varphi(k_1) \mid k_1 \in K_1\} \in \mathcal{K}$.

Proof. (i +ii) Let $u \in K \in \mathcal{K}$. Then $N_2(u) = \{x_u, y_u\}$ and $\{u, u - 3x_u, u - 3y_u\} \subset K$. On the other hand let $u \neq v \in K$. Then $x := \frac{1}{3}(u - v)$ is a non-zero vector in Λ and hence has square length $\geq \frac{4}{3}$. This implies that $(u, v) = -2$ and $x \in N_2(\gamma)$.
(iii) Since the differences of the elements in K_i lie in $3\Lambda = 3\Gamma^*$, the scalar products $(k_1, k_2) \equiv (k_1', k_2') \pmod 3$ are congruent modulo 3 for all $k_i, k_i' \in K_i, i = 1, 2$. Now these scalar products are integers of absolute value ≤ 2 with $\sum_{k_2 \in K_2} (k_1, k_2) = 0$ for all $k_1 \in K_1$ which only leaves the possibilities described in the proposition.
(iv) Clearly K_3 is again an equivalence class of norm 4 vectors in Γ. □

A closer analysis of the proof of Proposition 3.26 allows to define a normal subgroup of the automorphism group of Λ. For $\Lambda = CT$ this is the representation of a subgroup of index 2 in $\mathrm{Aut}(CT)$ as complex reflection group.

Proposition 3.29 *For all $K \in \mathcal{K}$ define an orthogonal mapping s_K by $(s_K)_{|L} = -\mathrm{id}_{|L}$ and $(s_K)_{|L^\perp} = \mathrm{id}_{|L^\perp}$, where $L = \langle K \rangle_{\mathbb{R}}$ is the vector space generated by K. If Λ is generated by its minimal vectors, then $s_K(\Lambda) = \Lambda$ hence $s_K \in \mathrm{Aut}(\Lambda)$. Moreover*

(i) $s_K^2 = \mathrm{id}$.

(ii) $(s_{K_1} s_{K_2})^2 = \mathrm{id}$ for all $K_1, K_2 \in \mathcal{K}$, $(K_1, K_2) = 0$.

(iii) $(s_{K_1} s_{K_2})^3 = \mathrm{id}$ for all $K_1, K_2 \in \mathcal{K}$, $(K_1, K_2) \neq 0$.

(iv) The subgroup $\langle s_K \mid K \in \mathcal{K} \rangle \leq \mathrm{Aut}(\Lambda)$ is a normal subgroup of the automorphism group of Λ.

Proof. Let $Y := X \cup -X = \Lambda_{\min}$. We only need to show that $s_K(Y) = Y$. The remaining properties of the s_K follow from direct calculations. In particular for all $g \in \mathrm{Aut}(\Lambda)$ the conjugate $s_K^g = s_{g(K)}$. Since $\mathrm{Aut}(\Lambda) = \mathrm{Aut}(\Gamma)$ permutes the elements of K, the s_K generate a normal subgroup of $\mathrm{Aut}(\Lambda)$. Let $\gamma \in K \in \mathcal{K}$ and let $N_2(\gamma) = \{x_1, x_2\}$. Then

$$\langle K \rangle_{\mathbb{R}} = \langle x_1, x_2 \rangle_{\mathbb{R}} = \langle \gamma, x_1 \rangle_{\mathbb{R}}.$$

First let $y \in Y$ with $(y, \gamma) = 0$. If $(y, x_1) = 0$, then $y \in K^\perp$ and $s_K(y) = y \in Y$. If $(y, x_1) = \frac{1}{3}$, then $(y, x_2) = -\frac{1}{3}$, $y - \frac{1}{2}(x_1 - x_2) \in K^\perp$, and $s_K(y) = y - (x_1 - x_2) \in Y$. Similarly, if $(y, x_1) = -\frac{1}{3}$, then $s_K(y) = y + (x_1 - x_2) \in Y$. If $(y, x_1) = \frac{2}{3}$, then $y = x_1 - x_2 \in K$ by Proposition 3.26. Therefore $s_K(y) = -y \in Y$.
If $(y, \gamma) = \pm 2$, then $y \in \pm N_2(\gamma) \subset \langle K \rangle$ and hence $s_K(y) = -y \in Y$.
It remains to consider the case that $(y, \gamma) = \pm 1$. Without loss of generality let $(y, \gamma) = 1$. Then by Proposition 3.26 (y, x_1) is one of $\frac{1}{3}$ or $\frac{2}{3}$. In the first case, the projection of y onto $\langle K \rangle_{\mathbb{R}}$ is $\frac{1}{2}x_2$ and hence $s_K(y) = y - x_2 \in Y$. In the second case, the projection of y onto $\langle K \rangle_{\mathbb{R}}$ is $\frac{1}{2}x_1$ and hence $s_K(y) = y - x_1 \in Y$. □

Proposition 3.30 *Let $\mathcal{K}$ denote the set of equivalence classes introduced in Proposition 3.28. If there are two classes $K_1, K_2 \in \mathcal{K}$ with $(K_1, K_2) = -1$ then $\Gamma \cong CT$ is the Coxeter-Todd lattice.*

Proof. Let $K_i := \{u_i, v_i, -(u_i+v_i)\}$ $(i = 1, 2)$ such that the Gram matrix of (u_1, v_1, u_2, v_2) is

$$A := \begin{pmatrix} 4 & -2 & -2 & 1 \\ -2 & 4 & 1 & -2 \\ -2 & 1 & 4 & -2 \\ 1 & -2 & -2 & 4 \end{pmatrix}.$$

Then $N_2(u_i) = \{x_i := \frac{1}{3}(u_i - v_i), y_i := \frac{1}{3}(2u_i + v_i)\}$ $(i = 1, 2)$ and $t := u_1 - v_2$ is a vector of norm 6 in Γ. We now want to investigate $N_2(t)$: The elements $y_1, x_2 - y_2$, and $x_1 + x_2$ have scalar product 2 with t and satisfy $y_1 + x_2 - y_2 + x_1 + x_2 = t$. With Lemma 2.3 we find $N_2(t) = \{z_1, \ldots, z_{12}, z_{13} := y_1, z_{14} := x_2 - y_2, z_{15} := x_1 + x_2\}$ with $\sum_{i=1}^{15} z_i = 5t$. Since $t = u_1 - v_2$ we get

$$2 = (z_i, t) = (z_i, u_1) - (z_i, v_2) = 1 - (-1) \text{ for all } i \in \{1, \ldots, 12\}$$

Hence $\{z_1, \ldots, z_{12}\} \subseteq N_1(u_1) \cap N_{-1}(v_2)$. Since $|(z_i, z_j)| \leq \frac{2}{3}$ for all $i \neq j$ and $(z_{13}, z_i) + (z_{14}, z_i) + (z_{15}, z_i) = (t, z_i) = 2$ for all $i \leq 12$, we find that $(z_j, z_i) = \frac{2}{3}$ for all $i \in \{1, \ldots, 12\}$, $j \in \{13, 14, 15\}$. For $i = 1, \ldots, 12$ let $\overline{z}_i := z_i - \frac{1}{3}t$. Then

$$(\overline{z}_i, u_1) = (\overline{z}_i, v_2) = (\overline{z}_i, z_j) = 0 \text{ for all } 1 \leq i \leq 12, j = 13, 14, 15.$$

Hence $\{z_1, \ldots, z_{12}\} \in \langle u_1, v_1, u_2, v_2 \rangle^{\perp}$ lie in an 8-dimensional space. Moreover

$$(\overline{z}_i, \overline{z}_j) = (z_i, z_j) - \frac{2}{3} \begin{cases} = \frac{2}{3} & i = j \\ \leq 0 & i \neq j \end{cases}.$$

By Lemma 2.10 there is a partition $\{1, \ldots, 12\} = I_1 \cup \ldots \cup I_k$ into disjoint sets I_j such that $\dim(\langle I_j \rangle) \geq |I_j| - 1$ for all $j = 1, \ldots, k$. Clearly $|I_j| > 1$. If $|I_j| = 2$, then $\frac{2}{3}t = z_i + z_l$ for some $i \neq l$ and hence $2t = 3(z_i + z_l) \in 3\Lambda$. Since $t \in \Lambda$ this implies that $t = 3t - 2t \in 3\Lambda$, hence $\frac{t}{3} \in \Lambda$ is a vector of norm $\frac{2}{3}$ contradicting the fact that $\min(\Lambda) = \frac{4}{3}$. Therefore $|I_j| \geq 3$ for all $j = 1, \ldots, k$. Since $k \geq 4$ and the I_j are disjoint, this implies that $k = 4$ and $|I_j| = 3$ for all j. Rearranging the z_i, we may assume that $I_j = \{2j - 1, 2j, 8 + j\}$ for $j = 1, \ldots, 4$. Put

$$\begin{aligned} a_j &:= (\overline{z}_{2j-1}, \overline{z}_{2j}) \\ b_j &:= (\overline{z}_{2j-1}, \overline{z}_{8+j}) \\ c_j &:= (\overline{z}_{2j}, \overline{z}_{8+j}). \end{aligned}$$

Then $a_j + b_j = a_j + c_j = b_j + c_j = \frac{-2}{3}$ which implies that $a_j = b_j = c_j = \frac{-1}{3}$ for all j. Hence Λ contains a sublattice $L := \langle u_1, t, x_2 - y_2, x_1 + x_2, z_1, \ldots, z_8 \rangle$ where the Gram matrix of L is

$$F := \frac{1}{3}\begin{pmatrix} 12 & 9 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 9 & 18 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 6 & 4 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 6 & 1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 6 & 2 & 2 & 4 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 6 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 6 & 2 & 2 & 2 & 2 & 4 & 1 & 2 & 2 & 2 & 2 \\ 3 & 6 & 2 & 2 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 \\ 3 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 1 & 2 & 2 \\ 3 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 4 & 2 & 2 \\ 3 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 1 \\ 3 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 4 \end{pmatrix}$$

Therefore Γ is a sublattice of the dual lattice $M := L^*$ with $\min(\Gamma) = 4$. In particular $\Gamma = \Gamma^{(e)}$, since M is already an even lattice. The determinant of M is $\det(M) = 81$, hence

$$[M : \Gamma] \leq \sqrt{\frac{2097}{81}} \leq 5.1.$$

Computations with MAGMA yield that $\mathrm{Aut}(M)$ has 6 orbits on the sublattices $M^{(2)}$ of index 2 in M, none of which satisfies $\min(M^{(2)*}) \geq \frac{4}{3}$. $\mathrm{Aut}(M)$ has 32 orbits on the sublattices $M^{(3)}$ of index 3 in M. only one of which satisfies $\min(M^{(3)*}) \geq \frac{4}{3}$. This lattice $M^{(3)}$ is isometric to the Coxeter Todd lattice CT.

$\mathrm{Aut}(M)$ has 253 orbits on the sublattices $M^{(5)}$ of index 5 in M. For none of these sublattices $M^{(5)}$ the dual $M^{(5)*}$ has minimum $\geq \frac{4}{3}$. □

The strategy is now to give a lower bound on the rank of the sublattice of Γ spanned by the norm 4-vectors using Minkowski's theorem on the successive minima of lattices. To this aim, we want to bound the number of vectors of norm $\frac{7}{2}$ in Γ.

Let $A := \{\alpha \in \Gamma \mid (\alpha, \alpha) = \frac{7}{2}\} = \Gamma_{7/2}$ and $C := \Gamma_4$ as above.

Lemma 3.31 *Let $\gamma \in C$ and denote $N_2(\gamma) := \{x_\gamma, y_\gamma\}$.*

(i) For all $\alpha \in A$ we have $(\alpha, \gamma) = (\alpha, x_\gamma) = (\alpha, y_\gamma) = 0$.

(ii) For $\alpha_1 \neq \pm\alpha_2 \in A$ we have $(\alpha_1, \alpha_2) = \pm\frac{1}{2}$.

Proof. (i) Since the scalar products $(\Gamma, \Gamma^{(e)})$ are integral, we have the possibilities $(\alpha, \gamma) \in \{0, \pm 1, \pm 2\}$. If $(\alpha, \gamma) = 1$, then $\alpha - \gamma \in \Gamma_{11/2}$ is a vector of norm $\frac{11}{2}$ in Γ which is impossible by Proposition 3.24. If $(\alpha, \gamma) = 2$, then $(\alpha, x_\gamma) = (\alpha, y_\gamma) = 1$ (since both scalar products are ≤ 1) and $\gamma - 3x_\gamma \in C$ has scalar product -1 with α contradicting Proposition 3.24. Therefore $(\alpha, \gamma) = 0$ for all $\alpha \in A$, $\gamma \in C$. Since $\gamma = x_\gamma + y_\gamma$ either both $(\alpha, x_\gamma) = (\alpha, y_\gamma) = 0$ or one of them is 1. As above the latter allows to construct a vector of norm $\frac{11}{2}$ in Γ.
(ii) The possible scalar products are $(\alpha_1, \alpha_2) \in \{\pm\frac{1}{2}, \pm\frac{3}{2}\}$. If $(\alpha_1, \alpha_2) = \frac{3}{2}$, then $\gamma := \alpha_1 - \alpha_2 \in C$ satisfies $(\gamma, \alpha_1) = 2$ contradicting (i). □

Proposition 3.32 $\frac{1}{2}|\Gamma_{7/2}| \leq 1$.

Proof. Assume that there are $\alpha_1 \neq \pm\alpha_2 \in \Gamma_{7/2}$. Then by Lemma 3.31 we have $(\alpha_1,\alpha_2) = \pm\frac{1}{2}$. Assume that $(\alpha_1,\alpha_2) = -\frac{1}{2}$ and put $t := \alpha_1+\alpha_2 \in \Gamma_6$. With Lemma 2.3 we find $N_2(t) = \{y_1,\ldots,y_{15}\}$ with $\sum_{i=1}^{15} y_i = 5t$. For $j=1,2$ we calculate $15 = 5(\alpha_j,t) = \sum_{i=1}^{15}(\alpha_j,y_i)$ hence $(\alpha_j,y_i) = 1$ for all i. For $i=1,\ldots,15$ let $\overline{y}_i := y_i - \frac{1}{3}t$. Then $(\alpha_j,\overline{y}_i) = 0$ for all $i \in \{1,\ldots,15\}, j = 1,2$ and the $\overline{y}_i$ lie in the 10-dimensional space $\langle\alpha_1,\alpha_2\rangle^\perp$. Moreover $(\overline{y}_i,\overline{y}_j) = (y_i,y_j) - \frac{2}{3}\left\{\begin{array}{ll} = \frac{2}{3} & i=j \\ \leq 0 & i\neq j \end{array}\right.$. By Lemma 2.10 there is a partition $\{1,\ldots,15\} = I_1\cup\ldots\cup I_k$ into disjoint sets I_j such that $\dim(\langle I_j\rangle) \geq |I_j|-1$ for all $j=1,\ldots,k$. As in the proof of Proposition 3.30 we find $k=5$ and $|I_j| = 3$ for all j. Rearranging the y_i, we may assume that $I_j = \{2j-1, 2j, 10+j\}$ for $j=1,\ldots,5$. Put

$$\begin{array}{ll} a_j & := (\overline{y}_{2j-1},\overline{y}_{2j}) \\ b_j & := (\overline{y}_{2j-1},\overline{y}_{10+j}) \\ c_j & := (\overline{y}_{2j},\overline{y}_{10+j}). \end{array}$$

Then $a_j+b_j = a_j+c_j = b_j+c_j = \frac{-2}{3}$ which implies that $a_j = b_j = c_j = \frac{-1}{3}$ for all j. Hence Λ is an overlattice of the lattice $L := \langle y_1,\ldots,y_{10},t,2\alpha_1\rangle$ where the Gram matrix of L is

$$F := \frac{1}{3}\begin{pmatrix} 4&1&2&2&2&2&2&2&2&2&6&6\\ 1&4&2&2&2&2&2&2&2&2&6&6\\ 2&2&4&1&2&2&2&2&2&2&6&6\\ 2&2&1&4&2&2&2&2&2&2&6&6\\ 2&2&2&2&4&1&2&2&2&2&6&6\\ 2&2&2&2&1&4&2&2&2&2&6&6\\ 2&2&2&2&2&2&4&1&2&2&6&6\\ 2&2&2&2&2&2&1&4&2&2&6&6\\ 2&2&2&2&2&2&2&2&4&1&6&6\\ 2&2&2&2&2&2&2&2&1&4&6&6\\ 6&6&6&6&6&6&6&6&6&6&18&18\\ 6&6&6&6&6&6&6&6&6&6&18&42 \end{pmatrix}$$

satifying $\min(\Lambda) = \frac{4}{3}$ and $\min(\Lambda^*) = \frac{7}{2}$. In particular $\Gamma^{(e)}$ is an even integral sublattice of L^* of minimum 4 and hence contained in the unique maximal integral sublattice M_0 of L^*. If B^* denotes the dual basis of the basis of L above, then M_0 is generated by $(b_1^*,\ldots,b_{10}^*, 2b_{11}^*+2b_{12}^*, b_{11}^*+3b_{12}^*)$ and has index 4 in L^* and determinant 3^4. M_0 is an overlattice of index 3 of the root lattice A_2^6. The lattice Γ is a sublattice of $M := M_0 + \mathbb{Z}\alpha_1$ of index $\leq \sqrt{4\cdot 2097/81} \leq 10.2$.

Computations with MAGMA yield that $\mathrm{Aut}(M)$ has 23 orbits on the sublattices $M^{(2)}$ of index 2 in M, only 4 of which satisfy $\min(M^{(2)*}) \geq \frac{4}{3}$. These sublattices have minimum $2,2,2,$ and $3/2$, hence are no candidates for Γ. All sublattices $M^{(4)}$ of these 4 sublattices $M^{(2)}$ of index 2 satisfy $\min(M^{(4)*}) < \frac{4}{3}$. Hence $[M:\Gamma] = [M_0:\Gamma_0]$ is not a power of 2.

Assume that 3 divides $[M:\Gamma]$. $\mathrm{Aut}(M)$ has 109 orbits on the sublattices $M^{(3)}$ of index 3 in M. only one of which satisfies $\min(M^{(3)*}) \geq \frac{4}{3}$. Since $\min(M^{(3)}) = \frac{3}{2}$, the lattice Γ is a proper sublattice of $M^{(3)}$. There are no sublattices $M^{(9)}$ of index 3 of $M^{(3)}$ such that $M^{(9)*}$ has minimum $\geq \frac{4}{3}$. A unique sublattice $M^{(6)}$ of index 2 in $M^{(3)}$ satisfies $M^{(6)*} = \frac{4}{3}$. This lattice is Coxeter-Todd lattice and has $\min(M_9) = 4 > \frac{7}{2}$.

Assume that 5 divides $[M : \Gamma]$. $\mathrm{Aut}(M)$ has 1771 orbits on the sublattices $M^{(5)}$ of index 5 in M. For none of these sublattices $M^{(5)}$ the dual $M^{(5)^*}$ has minimum $\geq \frac{4}{3}$.

It remains the case that $[M : \Gamma] = 7$. Here the orbit computations are too big to be performed with MAGMA. We therefore calculate all sublattices of M of index 7 by going through the 11-dimensional subspaces of $M/7M \cong \mathbb{F}_7^{12}$. These are parametrised by matrices in $\mathbb{F}_7^{11\times 12}$ that are in Hermite normal form. Building these matrices row by row, we only continue if the sublattice generated by $7M$ and the first rows has minimum $\geq \frac{7}{2}$. There are 31104 sublattices $M^{(7)}$ of M of index 7 that have minimum $\geq \frac{7}{2}$ but none of these lattices satisfies $\min(M^{(7)*}) \geq \frac{4}{3}$. □

Lemma 3.33 *The rank of the sublattice of* Γ *generated by the norm* 4 *vectors in* Γ *is at least* 7.

Proof. We use Minkowski's theorem on the successive minima

$$m_i := \min\{\lambda \in \mathbb{R} \mid L \text{ has } i \text{ linearly independent vectors of norm } \leq \lambda\}$$

of an n-dimensional lattice L stating that

$$m_1 \cdot m_2 \cdot \ldots \cdot m_r \leq \gamma_n^r \det(L)^{r/n} \quad \text{for all } 1 \leq r \leq n$$

(see for instance [12, Théorème II.6.8]).
Assume first that Γ does not contain vectors of norm $\frac{7}{2}$. By Proposition 3.24 there are no vectors of norm $\frac{11}{2}$ in Γ. Since

$$4^6 6^6 > 2.522^{12} 2097 \geq \gamma_{12}^{12} \det(\Gamma)$$

we get that the rank of the sublattice of Γ spanned by Γ_4 is at least 7.
Now assume that $\Gamma_{7/2} \neq \emptyset$. Then by Proposition 3.32 the set $\Gamma_{7/2} = \{\alpha, -\alpha\}$ has only 2 elements. Let π denote the orthogonal projection onto $\langle\alpha\rangle^\perp$ and let $M := \pi(N_1(\alpha))$. Then by Lemma 2.7, the set $\pi(M) \ \dot\cup \ -\pi(M)$ is a 2-design in $\langle\alpha\rangle^\perp$. In particular $\langle\pi(M)\rangle = \langle\alpha\rangle^\perp$ and hence M spans a subspace of dimension ≥ 11 of $\mathbb{R}^{12}$.
Choose $x \in M$ and let $G(x) := \{\gamma \in \Gamma \mid (\gamma, x) = 0\}$. Then $G(x)$ is an 11-dimensional lattice of determinant

$$\det(G(x)) = \frac{4}{3}\det(\Gamma) \leq 2796.$$

Since $(\alpha, x) = \pm 1$, the minimum of $G(x)$ is ≥ 4 and $G(x)$ contains no vectors of norm $\frac{11}{2}$ by Proposition 3.24. Now $\gamma_{11} \leq 2.39$ by [4] and

$$4^5 6^6 > 2.39^{11} 2796$$

which implies that the rank of the sublattice spanned by the norm 4-vectors in $G(x)$ is at least 6. This holds for any $x \in M$. Since M spans a space of dimension ≥ 11, not all norm 4-vectors of Γ can be orthogonal to all $x \in M$. Therefore there is some

$x \in M$, for which the rank of the sublattice spanned by the norm 4 vectors in Γ is strictly bigger than the one of the sublattice spanned by the norm 4 vectors in $G(x)$, hence $\dim\langle\Gamma_4\rangle \geq 7$. □

Assumption: In view of Proposition 3.30 we will assume in the following that all classes in $\mathcal{K}$ are pairwise orthogonal.

Then Lemma 3.33 and Proposition 3.28 directly imply

Corollary 3.34 *Γ has a sublattice $L \cong (A_1 \otimes A_2)^4$. The lattice $\Lambda \cap \mathbb{R}L$ contains a sublattice L' which is isometric to $(A_1 \otimes A_2^*)^4$ (under the same isometry).*

Corollary 3.35 *There is an even overlattice $\tilde{\Gamma}$ containing $\Gamma^{(e)}$ of index 9.*

Proof. Let $L' = \langle x_i, y_i | 1 \leq i \leq 4\rangle$ with $(x_i, y_i) = 2/3$, $(x_i, x_i) = (y_i, y_i) = 4/3$ for all i and all other scalar products are 0 be the lattice of Corollary 3.34. Then L' contains an even sublattice $\langle v_1 := x_1 + x_2 + x_3, v_2 := x_2 - x_3 + x_4\rangle$ such that v_1 and v_2 are linearly independent modulo $(L')^*$. Therefore $\tilde{\Gamma} := \langle \Gamma^{(e)}, v_1, v_2\rangle$ is the desired lattice. □

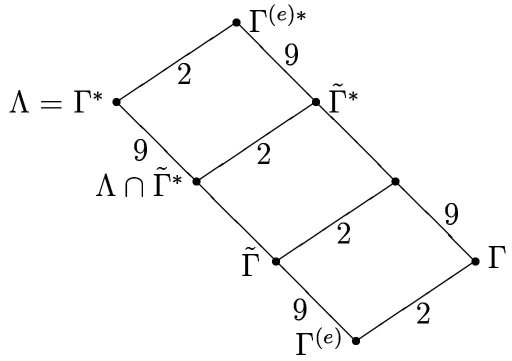
Note that $\det(\tilde{\Gamma}) \leq 4 \cdot 2097/81 \leq 103.6$ and if $\Gamma = \Gamma^{(e)}$, then $\tilde{\Gamma}$ is of determinant ≤ 25. If $\Gamma \neq \Gamma^{(e)}$ then we will take the subdirect product with A_1^* to obtain an even lattice of even determinant ≤ 50 in dimension 13. We therefore treat the two cases separately.

Proposition 3.36 *Assume that $\Gamma = \Gamma^{(e)}$. Then $\Gamma \cong CT$.*

Proof. We may assume that Λ is generated by its minimal vectors. Since these have norm $\frac{4}{3}$, the Sylow p-subgroup of the discriminant group Λ/Γ is generated by isotropic classes for all primes $p \neq 3$. In particular, if some prime $p \neq 3$ divides $\det(\Gamma)$, then $\tilde{\Gamma}$ has an integral overlattice of index p. In particular p^2 divides $\det(\tilde{\Gamma})$. By [6, Table 15.4, p. 387] there are no even 12-dimensional lattices of determinant 1, 2, 3, or 6. Therefore $\det(\tilde{\Gamma}) \neq p^2, 2p^2, 3p^2$ or $6p^2$ for all primes $p \neq 3$. Since $\det(\tilde{\Gamma}) \leq 25$ one gets $\det(\tilde{\Gamma}) = 9$ or 16. In the first case $\tilde{\Gamma}$ is in the genus of $A_2 \perp A_2 \perp E_8$. The genus contains 3 isometry classes, for only one the dual lattice has minimum $\geq 4/3$. This class is $E_6 \perp E_6$. The automorphism group of $E_6 \perp E_6$ has 20 orbits on the sublattices of index 3, only one of which consists of lattices of which the dual has minimum $\geq 4/3$. Continuing with this lattice, one finds 32 orbits of sublattices of index 3 under the automorphism group for only one of which the dual has minimum 4/3. This lattice is the Coxeter-Todd lattice CT.

In the second case, $\tilde{\Gamma}$ has an even overlattice $\tilde{\tilde{\Gamma}}$ which is in the genus of $E_8 \perp D_4$. The genus of $E_8 \perp D_4$ consists of 2 classes, the other is D_{12}. Both lattices have vectors of norm 1 in their duals, so this case is impossible. □

Now assume that $[\Gamma : \Gamma^{(e)}] = 2$. We then have the following situation:



Let $\varphi : \Gamma/\Gamma^{(e)} \to A_1^*/A_1$ be the unique isomorphism. Then we define

$$M := \Gamma \star A_1^* := \{x + y \in \Gamma \perp A_1^* \mid \varphi(x) = y\}.$$

The overlattice $\tilde{M} := \tilde{\Gamma} + M$ contains M of index 9 and is an even lattice in dimension 13 of even determinant ≤ 50. Moreover M contains a unique pair $\pm v$ of vectors of norm 2, The orthogonal complement $v^\perp := \{\gamma \in M \mid (v, \gamma) = 0\}$ is the lattice $\Gamma^{(e)}$. The orthogonal complement of v in $\tilde{M}$ is a sublattice of Λ and hence has minimum $\geq \frac{4}{3}$.

We will show that this situation is impossible:

Proposition 3.37 $\Gamma = \Gamma^{(e)}$.

Proof. Assume that $\Gamma \neq \Gamma^{(e)}$ and construct the lattice $\tilde{M}$ as above. Then $\det(\tilde{M}) = \frac{1}{2}\det(\tilde{\Gamma})$ is an even integer ≤ 50.

Moreover the condition (MIN) is satisfied:
(MIN): There is a vector $v \in \tilde{M}_2$ such that $L_v := \{\gamma \in M^* \mid (\gamma, v) = 0\}$ is a lattice of minimum $\geq \frac{4}{3}$.

As in the proof of Proposition 3.36 we may assume that Λ is generated by its minimal vectors. Since these have norm $\frac{4}{3}$, the orthogonal complement O of $\Gamma + A_1/\Gamma^{(e)} + A_1$ in M^*/M is generated by classes of norm $\frac{4}{3}$. In particular for all primes $p \neq 3$, the Sylow p-subgroup of O is generated by isotropic classes. Hence if some prime $p \neq 3$ divides $\frac{1}{2}\det(\tilde{M})$, then $\tilde{M}$ has an integral overlattice of index p. In particular p^2 divides $\frac{1}{2}\det(\tilde{M})$. By [6, Table 15.4, p. 387] there are no even 13-dimensional lattices of determinant 2. Therefore $\det(\tilde{M}) \neq 2p^2$ for all primes $p \neq 3$. Since $\det(\tilde{M}) \leq 50$ one gets $\det(\tilde{M}) = 6$, 16, 18, 24, 32 or 48.

det=6: In the first case $\tilde{M}$ is in the genus of $A_5 \perp E_8$. The genus contains 3 isometry classes two of which, say L_1 and L_2 (namely all but $A_5 \perp E_8$) satisfy condition (MIN). No sublattice of index 9 of L_1 satisfies (MIN) and a unique sublattice of index 9, say L', satisfies the condition (MIN). L' has root system A_1^3, hence $L' \neq M$.

det=24: If $\det(\tilde{M}) = 24$, then M is a sublattice of index 2 of the lattice L' constructed in the case that $\det(M) = 6$. But no such sublattice satisfies condition (MIN).

det=16: Assume now that $\det(\tilde{M}) = 16$. Then $\tilde{M}$ has an even overlattice $\tilde{\tilde{M}}$ containing M of index 2. The determinant of $\tilde{\tilde{M}}$ is 4. By [6, Table 15.4] there is a unique genus of even 13-dimensional lattices of deterimant 4 the genus of D_{13}. It contains 3 classes, none of which satisfies the condition (MIN).

det=18: Now assume that $\det \tilde{M} = 18$. Then $\tilde{M}$ is a maximal integral lattice (since there is no even lattice of determinant 2 in dimension 13). In particular the discriminant group of $\tilde{M}$ is not cyclic, and the 3-Sylow subgroup of $\tilde{M}^*/\tilde{M}$ is isometric to the unique anisotropic quadratic space of dimension 2 over $\mathbb{F}_3$. There is a unique genus of such lattices, namely the one of $E_6 \perp E_6 \perp A_1$. This genus has class number 7. Only the lattice $E_6 \perp E_6 \perp A_1$ satisfies condition (MIN). Only one sublattice of index 9 of this lattice satisfies condition (MIN) and this lattice is isometric to $A_1 \perp CT$.

det=32: If $\det \tilde{M} = 32$, then $\tilde{M}$ has an even overlattice $\tilde{\tilde{M}}$ of determinant 8. By [6, Table 15.4] there is a unique genus of even lattices of determinant 8 in dimension 13, the one of $A_1 \perp D_4 \perp E_8$. This genus contains 4 classes, none of which satisfies condition (MIN).

det=48: Assume finally that $\det(\tilde{M}) = 48$. Then there is an even overlattice $\tilde{\tilde{M}}$ of $\tilde{M}$ of determinant $\det(\tilde{\tilde{M}}) = 12$. There is one genus of such lattices of determinant 12, namely the one of $E_6 \perp D_7$. Its class number is 7 and none of the lattices satisfies condition (MIN). □

Together with Proposition 3.36 this concludes the proof of Theorem 3.22.

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