

The solvable degree of a Schur algebra

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ABSTRACT. The solvable degree of a Schur algebra D is the greatest common divisor of all integers n such that $D^{n \times n}$ is a direct summand of the rational group algebra $\mathbb{Q}G$ for some finite solvable group G . The main result of this paper gives divisibility conditions on the solvable degree depending on the center and the rational primes that divide the discriminant of D .

KEYWORDS: Schur subgroup of Brauer group; direct summands of group algebras; Schur index; character degree

1 Introduction

An interesting number theoretic invariant of a finite group G is its rational group ring $\mathbb{Q}G$. This is a $|G|$ -dimensional semi-simple $\mathbb{Q}$ -algebra,

$$\mathbb{Q}G = \bigoplus_{i=1}^s D_i^{n_i \times n_i}$$

for rational division algebras D_i . Their centers, $K_i := Z(D_i)$, are exactly the character fields $\mathbb{Q}(\chi_i)$ for the (Galois orbits of) irreducible complex characters χ_i . In particular $K_i = \mathbb{Q}[\chi_i(g) : g \in G]$ is generated by sums of $|G|$ -th roots of unity and hence an abelian extension of $\mathbb{Q}$. Following Turull [17] we call the class $[D_i] := [\chi_i]$ the (inverse) Brauer element of χ_i in the Brauer group $\text{Br}(K_i)$ of central simple K_i -algebras.

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The Schur subgroup $S(K)$ of the Brauer group $\text{Br}(K)$ is the group of classes $[D]$ of all central simple K -algebras for which there is some $n \in \mathbb{N}$ and some finite group G such that $D^{n \times n}$ is a direct summand of the rational group algebra $\mathbb{Q}G$. The group $S(K)$ has been extensively studied since the middle of last century (see [21] and the more than 70 papers citing this book). We call a division algebra D a *Schur algebra* if $[D] \in S(K)$. The *degree* $\deg(D)$ (respectively the *solvable degree* $\deg_{\text{sol}}(D)$) of a Schur algebra D is the greatest common divisor of the values n , such that $D^{n \times n}$ is a direct summand of $\mathbb{Q}G$ for some finite group G (respectively some finite solvable group G). Clearly $\deg(D)$ divides $\deg_{\text{sol}}(D)$. There are Schur algebras D , where $\deg(D) \neq \deg_{\text{sol}}(D)$ (see Remark 4.4).

For solvable groups, there are divisibility relations for the character degrees depending on the character field (see Theorem 2.12). These provide our main tool to show the main result of this paper, Theorem 5.1, giving divisibility conditions on the solvable degree of a Schur algebra. We also give examples that prove that these are tight for certain infinite series of Schur algebras.

Section 2 of this paper summarizes the relevant results from the literature that are used in the proof of Theorem 5.1. Section 3 might be of independent interest; here we generalize the results from [20] to obtain conditions on the Schur index of an induced character. A more sophisticated theory based on Tignol's notion of generalised crossed product algebras is developed in [16]. Section 4 then introduces the degree and solvable degree of a Schur algebra and the last section is devoted to the statement and proof of our main result.

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2 Characters

2.1 The Brauer element

The set of rational primes is denoted by $\mathbb{P} := \{p \in \mathbb{N} \mid p \text{ is a prime}\}$.

Let K be a number field and D be a division algebra with center K . Then $\dim_K(D) = m^2$ where $m := \text{ind}(D)$ is the index of D . Note that the index, m , is also the order of $[D]$ in the Brauer group $\text{Br}(K)$. Division algebras over number fields satisfy the Hasse principle, i.e. they are isomorphic, if and only if all their completions are isomorphic:

A place P of K is either a prime ideal of the ring of integers in K (a finite place) or an embedding into the real or complex numbers (an infinite place). The corresponding completion $A_P = D \otimes_K K_P$ of D is a matrix ring over some K_P -division algebra $A_P = D_P^{n_P \times n_P}$. The division algebra D_P is uniquely determined by its *Hasse invariant*

$$\text{inv}_P(D) = \text{inv}(D_P) = \frac{a_P}{m_P} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \cong \text{Br}(K_P)$$

where $m_P = m/n_P$ is the index of D_P .

For a finite group G we denote by

$$\text{Irr}(G) := \{\chi : G \rightarrow \mathbb{C} \mid \chi \text{ is an irreducible character}\}$$

the set of irreducible complex characters of G . The characters $\chi \in \text{Irr}(G)$ correspond to the conjugacy classes of irreducible complex representations

$$\rho : G \rightarrow \text{GL}_{\chi(1)}(\mathbb{C}), \chi(g) = \text{trace}(\rho(g)) \text{ for all } g \in G.$$

We then also say that ρ affords the character χ . The character χ is called *faithful* if ρ is injective. The $\mathbb{Q}$ -algebra spanned by $\rho(G)$ is a simple algebra,

$$\langle \rho(g) \mid g \in G \rangle_{\mathbb{Q}} \cong D^{n \times n} \leq \mathbb{C}^{\chi(1) \times \chi(1)}$$

where $K := Z(D) = \mathbb{Q}(\chi)$ is the character field of χ .

Definition 2.1. *Let G be a finite group, $\chi \in \text{Irr}(G)$, $K = \mathbb{Q}(\chi)$, ρ and D as above.*

(a) (see [17]) *The inverse Brauer element of χ is the class*

$$[\chi] = [D] \in \text{Br}(K)$$

in the Brauer group of K .

(b) *The Schur index $m(\chi)$ is the order $m(\chi) = |\langle [\chi] \rangle|$ of the Brauer element of χ .*

(c) *For two fields $K \leq L$ we denote the natural morphism $\text{Br}(K) \rightarrow \text{Br}(L)$, $[D] \mapsto [D \otimes L] =: [D] \otimes L$.*

(d) *For a place P of K we define the P -local Schur index $m_P(\chi)$ to be the order of $[\chi] \otimes K_P \in \text{Br}(K_P)$ in the Brauer group of the P -adic completion of K .*

- (e) *The division algebras D with $[D] = [\chi]$ for some finite group G and some $\chi \in \text{Irr}(G)$ are called Schur algebras. The Schur subgroup $S(K)$ of the Brauer group $\text{Br}(K)$ is the subgroup consisting of all classes of Schur algebras with center K .*

Note that Turull defines the Brauer element of χ as the inverse of our inverse Brauer element. As the natural involution on $\mathbb{Q}G$, inverting the group elements, maps $[\chi]$ to $[\chi]^{-1} = [\bar{\chi}]$, the Brauer element of χ is just the inverse Brauer element of the complex conjugate character $\bar{\chi}$. So both have the same degree and also the same solvable degree.

Algorithms for computing Schur indices of characters are developed in [11] and [18]. They are based on the restriction to so called elementary subgroups and the following result from Brauer:

Theorem 2.2. *(see [2, Section (2A)]) Let $U \leq G$ be finite groups, $\chi \in \text{Irr}(G)$ and $\psi \in \text{Irr}(U)$, occurring in the restriction $\chi|_U$ with multiplicity $e := (\chi|_U, \psi)_U$. If $p \in \mathbb{P}$ does not divide $e[\mathbb{Q}(\psi, \chi) : \mathbb{Q}(\chi)]$ then the p -part of $m(\psi)$ equals to the p -part of $m(\chi)$.*

Theorem 2.2 also allows for computations of the inverse Brauer element, where we only need one special case:

Corollary 2.3. *Let $U \leq G$ be finite groups, $\chi \in \text{Irr}(G)$ and $\psi \in \text{Irr}(U)$. If ψ occurs in $\chi|_U$ with multiplicity 1 and $\mathbb{Q}(\psi) \leq \mathbb{Q}(\chi)$ then $[\chi] = [\psi] \otimes \mathbb{Q}(\chi)$.*

The two papers [3] and [4] list important properties of Schur algebras:

Theorem 2.4. *Let D be a Schur algebra, $K := Z(D)$ its center and $m = \text{ind}(D)$ the index of D .*

- (i) *K is an abelian number field containing primitive m -th roots of unity ([4, Theorem]), so*

$$\zeta_m := \exp\left(\frac{2\pi i}{m}\right) \in K.$$

- (ii) *Let $p \in \mathbb{P}$. Then there is a divisor m_p of m such that $m_P = m_p$ for all finite places P of K over p ([3, Theorem 1]).*

- (iii) *If $2 < p \in \mathbb{P}$ then m_p of (ii) divides $p - 1$ and $m_2 \in \{1, 2\}$ ([19, Satz 10, Satz 11]).*

(iv) Let P be a prime ideal of K and $\sigma \in \text{Gal}(K/\mathbb{Q})$, $b \in \{1, \dots, m-1\}$ be such that $\sigma(\zeta_m) = \zeta_m^b$. Then the Hasse invariants of D at P and $\sigma(P)$ satisfy $\text{inv}_P(D) = b \text{inv}_{\sigma(P)}(D)$.

Since the Galois group acts transitively on the prime ideals over p , Theorem 2.4 (iv) implies that the p -local invariants are fully determined by the Hasse invariant for one prime ideal over p . Conway polynomials provide a tool to single out a unique prime ideal $P := P(K, p)$ of K that contains p (see [10, Introduction, 3]). Then any Schur algebra D with $Z(D) = K$ is uniquely determined by the finite set

$$\text{Inv}(D) := \{(\text{inv}_{P(K,p)}, p) \in \mathbb{Q}/\mathbb{Z} \times \mathbb{P} \mid p \in \mathbb{P}, m_p(D) > 1\} \cup I$$

where $I = \emptyset$ if no real place of K ramifies in D and $I = \{\infty\}$ otherwise. We also put $\text{Inv}([\chi]) = \text{Inv}(D)$ for irreducible characters χ with $[\chi] = [D]$.

For many purposes the following coarser invariants suffice:

Definition 2.5. For a finite group G , $\chi \in \text{Irr}(G)$, $p \in \mathbb{P}$, and D a division algebra such that $[\chi] = [D]$ we put

$$m_p(D) = m_p([D]) = m_p([\chi]) = m_p(\chi)$$

to denote the p -local Schur index of χ . The set of ramified primes in $[\chi]$ is denoted by

$$\mathcal{P}([\chi]) := \mathcal{P}(\chi) := \{p \in \mathbb{P} \cup \{\infty\} \mid m_p(\chi) > 1\}$$

and $P(D) := P(\chi) := \prod \{p \in \mathcal{P}(\chi) \mid p \neq \infty\} = \prod \{p \in \mathbb{P} \mid m_p(\chi) > 1\}$.

2.2 Clifford Theory

Clifford Theory deals with the restriction and induction of characters for normal subgroups. Let G be a finite group and $N \trianglelefteq G$ be a normal subgroup of G . Then G acts on $\text{Irr}(N)$ by conjugation: For $\psi \in \text{Irr}(N)$ and $g \in G$ put $\psi^g : N \rightarrow \mathbb{C}$, $\psi^g(n) = \psi(gng^{-1})$ for all $n \in N$.

Theorem 2.6. (Clifford, see for instance [8, Theorem (6.5)]) The restriction of $\chi \in \text{Irr}(G)$ to the normal subgroup N is of the form

$$\chi|_N = e(\psi_1 + \dots + \psi_t)$$

where $\psi_i \in \text{Irr}(N)$ and $e \in \mathbb{N}$. Moreover $\{\psi_1, \dots, \psi_t\}$ is a full orbit under G . Put $\psi := \psi_1$. Then the stabiliser

$$T := I_G(\psi) := \{g \in G \mid \psi^g = \psi\}$$

is a subgroup of G of index $t = [G : T]$. There is a unique $\alpha \in \text{Irr}(T)$ such that

$$\alpha|_N = e\psi \text{ and } \chi = \alpha^G.$$

The character α is called the Clifford correspondent of χ over ψ . For the character fields we get

$$\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\alpha) \text{ and } \mathbb{Q}(\chi) \subseteq \mathbb{Q}(\alpha).$$

Theorem 2.7. (Gallagher, see for instance [8, Corollary (6.17)]) If $\chi \in \text{Irr}(G)$ is such that $\chi|_N =: \psi \in \text{Irr}(N)$, then the set of irreducible constituents of the induced character ψ^G is $\{\beta\chi \mid \beta \in \text{Irr}(G/N)\}$.

2.3 p' -special characters of solvable groups

In the proof of the main result we use p' -special characters. For a finite group G and a character $\chi \in \text{Irr}(G)$ we define the determinantal order $o(\chi)$ to be

$$|\langle \det(\rho(g)) \mid g \in G \rangle|$$

the order of the subgroup of the multiplicative group $\mathbb{Q}(\chi)^\times$ generated by the determinants of the images of the group elements under any complex representation ρ affording the character χ .

Definition 2.8. Let p be a prime and let G be a solvable group. An irreducible character $\chi \in \text{Irr}(G)$ is called p' -special if

- (i) p does not divide $\chi(1)$, and
- (ii) for every subnormal subgroup S of G and every irreducible constituent $\theta \in \text{Irr}(S)$ of $\chi|_S$, the determinantal order $o(\theta)$ is prime to p .

From [9, Lemma 3.3] we obtain

Lemma 2.9. Let G be solvable, $N \trianglelefteq G$, and let $\theta \in \text{Irr}(N)$ be p' -special.

- (i) If G/N is a p' -group, then every irreducible constituent of θ^G is p' -special.

- (ii) If G/N is a p -group and θ is G -invariant, then θ has a unique extension $\hat{\theta} \in \text{Irr}(G)$ whose determinantal order $o(\hat{\theta})$ is prime to p . Moreover this extension $\hat{\theta}$ is p' -special.

We use the following property of p' -special characters:

Lemma 2.10. ([7, Proposition 6.3 (b)]) Let $\chi \in \text{Irr}(G)$ be p' -special and let χ' denote the restriction of χ to the p -regular elements. Then χ' is an irreducible Brauer character of G .

Remark 2.11. As

$$m_p(\chi) \mid [\mathbb{Q}_p(\chi, \varphi) : \mathbb{Q}_p(\chi)],$$

for all p -modular constituents φ of χ , Lemma 2.10 implies that for any p' -special character χ the p -local Schur index $m_p(\chi) = 1$.

2.4 Character degrees for solvable groups

For an abelian number field K , the *conductor* $f(K)$ is the smallest natural number n such that K is contained in the n -th cyclotomic field, $K \subseteq \mathbb{Q}[\zeta_n]$. We also put $F(K) := \mathbb{Q}[\zeta_{f(K)}]$ to denote the conductor field.

Theorem 2.12. [5] Let χ be an irreducible character of some finite solvable group G . Then $[F(\mathbb{Q}(\chi)) : \mathbb{Q}(\chi)]$ divides $\chi(1)$.

3 Schur indices of induced representations

This section gives a generalization of [20, Theorem 2]. It is useful to apply the criterion prime per prime so we will work over local fields and consider the following situation:

G is a finite group, $N \trianglelefteq G$ a normal subgroup, $\alpha \in \text{Irr}(N)$ such that $\alpha^G = \chi \in \text{Irr}(G)$. Then $\chi(1) = [G : N]\alpha(1)$ and the restriction

$$\chi|_N = \sum_{gN \in G/N} \alpha^g.$$

Let p be a rational prime. Then the p -adic character fields, $E := \mathbb{Q}_p(\alpha)$ and $K := \mathbb{Q}_p(\chi)$, satisfy $K \leq E$.

Theorem 3.1. If E/K is unramified and $m_p(\alpha) = 1$ then $m_p(\chi)$ divides $\gcd(\alpha(1), [E : K])$.

Proof. (a) Similarly as in [20] we first construct a subgroup $H \leq G$ such that $H/N \cong \text{Gal}(E/K)$:

$$H := \{g \in G \mid \exists \sigma \in \text{Gal}(E/K) : \alpha(gng^{-1}) = \sigma(\alpha(n)) \text{ for all } n \in N\}.$$

(b) For the induced character $\psi = \alpha^H$ we then get $\mathbb{Q}_p(\psi) = \mathbb{Q}_p(\chi)$ and $\chi = \psi^G$ and hence $m_p(\psi) = m_p(\chi)$ by Corollary 2.3.

(c) The fact that $m_p(\alpha) = 1$ implies that E splits $[\psi]$ and $m_p(\psi)$ divides $[E : K]$.

(d) So it is enough to show that $m_p(\psi)$ divides $\alpha(1)$:

Put $d := \alpha(1)$ and $t := [E : K] = |H/N|$. Since E/K is unramified, its Galois group is cyclic,

$$\text{Gal}(E/K) = \langle \sigma \rangle.$$

Choose $h \in H$ with $\alpha(hnh^{-1}) = \sigma(\alpha(n))$ for all $n \in N$ and let $\rho : N \rightarrow \text{GL}_d(E)$ be a representation affording the character α . By abuse of notation, we apply elements of $\text{Gal}(E/K)$ also to matrices over E , by applying the Galois automorphisms to all entries of the matrix. The representations ρ^σ and ρ^h with

$$\rho^\sigma(n) = \sigma(\rho(n)) \text{ and } \rho^h(n) = \rho(hnh^{-1}) \text{ for all } n \in N$$

have the same character and hence are equivalent. So there is a matrix $T \in \text{GL}_d(E)$ such that

$$\sigma(\rho(n)) = T\rho(hnh^{-1})T^{-1} \text{ for all } n \in N.$$

For $1 \leq i \leq t$ put $X_j := \sigma^j(T) \cdots \sigma(T)T$. Inductively we get

$$\sigma^j(\rho(n)) = X_{j-1}\rho(h^jnh^{-j})X_{j-1}^{-1} \text{ for all } n \in N, 1 \leq j \leq t. \quad (1)$$

The induced representation $\rho^H : H \rightarrow \text{GL}_{dt}(E)$ can be defined by putting

$$\rho^H(n) := \text{diag}(\rho(n), \sigma(\rho(n)), \dots, \sigma^{t-1}(\rho(n))) \text{ for all } n \in N$$

and

$$\tilde{h} := \rho^H(h) = \begin{pmatrix} 0 & T^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \sigma(T)^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \sigma^{t-2}(T)^{-1} \\ X_{t-2}\rho(h^t) & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

As both α and ψ are irreducible characters, the K -algebra $B := \langle \rho^H(g) \mid g \in H \rangle_K$ is a central simple algebra containing the subalgebra $A := \langle \rho^H(n) \mid n \in N \rangle_K$. Since $m_p(\alpha) = 1$, we have $A \cong E^{d \times d}$, by our choice of basis

$$A = \{\text{diag}(M, \sigma(M), \dots, \sigma^{t-1}(M)) \mid M \in E^{d \times d}\}.$$

Conjugation with the matrix

$$\tilde{\sigma} := \text{diag}(T, \sigma(T), \dots, \sigma^{t-1}(T))\tilde{h} = \begin{pmatrix} 0 & I_d & 0 & \cdots & 0 \\ 0 & 0 & I_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I_d \\ X_{t-1}\rho(h^t) & 0 & \cdots & \cdots & 0 \end{pmatrix} \in B$$

induces the Galois automorphism σ on A . In particular $\tilde{\sigma}^t \in A$ commutes with all elements in A , so is a scalar matrix

$$\tilde{\sigma}^t = X_{t-1}\rho(h^t) \otimes I_t = \lambda I_{dt}$$

for some $\lambda \in E^\times$. Then $\lambda \in K^\times$ because $\sigma(\lambda) = \lambda$. As $A \cong E^{d \times d}$ this establishes an isomorphism between B and the matrix ring $(E/K, \sigma, \lambda)^{d \times d}$ of the cyclic algebra $(E/K, \sigma, \lambda)$ (see [15, Section 30]). In particular

$$[B] = [(E/K, \sigma, \lambda)] \in \text{Br}(K)$$

and hence

$$[B]^d = [(E/K, \sigma, \lambda^d)] \in \text{Br}(K).$$

Now

$$\lambda^d = \det(\lambda I_d) = \det(X_{t-1}\rho(h^t)) = N_{E/K}(\det(T)) \det(\rho(h^t)).$$

Let $\mathcal{O}_E$ resp. $\mathcal{O}_K$ denote the valuation rings of E resp. K . As E/K is unramified the norm is surjective on units, $N_{E/K}(\mathcal{O}_E^\times) = \mathcal{O}_K^\times$. The determinant of $\rho(h^t)$ is an element of finite order in $K^\times$, so $\det(\rho(h^t)) \in \mathcal{O}_K^\times$, which shows that $\lambda^d \in N_{E/K}(E^\times)$. Therefore $[B]^d = [(E/K, \sigma, \lambda^d)] = [K] \in \text{Br}(K)$, so the order $m_p(\psi)$ of $[B]$ in the Brauer group of K divides $d = \alpha(1)$. $\square$

4 The degree of a Schur algebra

Let $\mathcal{G}$ respectively $\mathcal{G}_{\text{sol}}$ denote the category of all finite groups, respectively the one of all finite solvable groups. Let K be a number field.

Definition 4.1. *For a central simple K -division algebra D we put*

$$\deg(D) := n \text{ respectively } \deg_{\text{sol}}(D) := n$$

if $n\mathbb{Z}$ is the ideal generated by

$$\{t \mid \exists G \in \mathcal{G} \text{ (resp. } \mathcal{G}_{\text{sol}}) \text{ such that } D^{t \times t} \text{ is a direct summand of } \mathbb{Q}G\}.$$

If $[D] \notin S(K)$ then we put $\deg(D) = \deg_{\text{sol}}(D) = \infty$.

Remark 4.2. Clearly the Galois group $\text{Gal}(K/\mathbb{Q})$ of K acts on the Schur subgroup $S(K)$. Schur algebras in the same Galois orbit have the same degree.

4.1 Some definite quaternion algebras of degree 1

If the center of a Schur algebra D is a totally real field, $Z(D) = K$, then either $D = K$ or D is a quaternion algebra. In the latter case, there are $a, b \in K$ such that

$$D = (a, b)_K = \langle 1, i, j, ij \mid i^2 = a, j^2 = b, ij = -ji \rangle.$$

Then D is called a *definite* quaternion algebra, if $\infty \in \text{Inv}(D)$, i.e. $D \otimes_K \mathbb{R}$ is a division algebra (for one and hence all real embeddings of K into $\mathbb{R}$).

The definite quaternion algebras D that are direct summands of $\mathbb{Q}G$ for some finite group G are classified in [13, Theorem 6.1]:

Theorem 4.3. *Assume that K is a real number field and let D be a definite quaternion algebra with center K such that D is a direct summand of $\mathbb{Q}G$ for some finite group G . Then one of the following holds:*

- *For $K = \mathbb{Q}$ there are two such quaternion division algebras: $D = (-1, -1)_{\mathbb{Q}}$ (with $\text{Inv}(D) = \{(\frac{1}{2}, 2), \infty\}$) and $D = (-1, -3)_{\mathbb{Q}}$ (with $\text{Inv}(D) = \{(\frac{1}{2}, 3), \infty\}$).*
- *For $[K : \mathbb{Q}] = 2$ there are three such quaternion division algebras, the ones where $K = \mathbb{Q}[\sqrt{p}]$ for $p = 2, 3, 5$ and $\text{Inv}(D) = \{\infty\}$.*

- If $[K : \mathbb{Q}] \geq 3$ then $F(K) = \mathbb{Q}[\zeta_m]$ for some even natural number m , K is the maximal real subfield of $F(K)$, i.e. $[F(K) : K] = 2$, and $G = C_m.C_2 = Q_{2m}$ is the quaternion group of order $2m$.

$$\text{Inv}(D) = \begin{cases} \{\infty\} & \text{if } [K : \mathbb{Q}] \text{ even} \\ \{(\frac{1}{2}, p), \infty\} & \text{if } m = 2p^a \text{ for a prime } p \equiv 3 \pmod{4}. \end{cases}$$

There are more examples of quaternion algebras D with $\deg(D) = 1$:

Remark 4.4. Take $D = (-1, -11)_{\mathbb{Q}}$ (i.e. $Z(D) = \mathbb{Q}$, $\text{Inv}(D) = \{(\frac{1}{2}, 11), \infty\}$). Then $D^{5 \times 5}$ is a direct summand of $\mathbb{Q}G$ for

$$G = C_{22}.C_{10} = \langle a, b \mid a^{22}, b^{10} = a^{11}, (a^2)^b = a^4 \rangle.$$

By [6] (see also [18] for some corrections) the group $H = 2.M_{22}.2i$ has a character χ of degree 252 with $[\chi] = [D]$, so $\mathbb{Q}H$ has a direct summand $D^{126 \times 126}$. So here we find $\deg(D) = 1$. However, Theorem 5.1 below shows that $\deg_{\text{sol}}(D) = 5$.

4.2 Schur algebras of even degree

Using the classification of finite simple groups, Navarro and Tiep have shown the following result on character fields of odd degree irreducible characters:

Theorem 4.5. ([12, Theorem A1]) *Let ψ be an absolutely irreducible ordinary character of some finite group G such that $\psi(1)$ is odd. If $\zeta_{2^a} \in F(\mathbb{Q}(\psi))$ then $\zeta_{2^a} \in \mathbb{Q}(\psi)$.*

This theorem implies that there are Schur algebras D with $\deg(D) \neq 1$:

Corollary 4.6. *Let D be a Schur algebra of odd index such that $K := Z(D)$ does not satisfy Theorem 4.5, i.e. there is $a \in \mathbb{N}$ such that $\zeta_{2^a} \in F(K) \setminus K$. Then $\deg(D)$ is even.*

It is easy to construct finite groups G and irreducible characters $\chi \in \text{Irr}(G)$ such that $[\chi] = [D]$ as in Corollary 4.6. Put $H := C_7 : C_9$ to be one of the groups from Corollary 5.6 below. Then there is a faithful character $\chi \in \text{Irr}(H)$ with $\mathbb{Q}(\chi) = \mathbb{Q}[\sqrt{-7}, \zeta_3] =: K$ such that $\text{Inv}([\chi]) = \{(\frac{1}{3}, 7)\}$. Now take $G := H \times D_{16}$ and $\psi := \chi \otimes \varphi$, where $\varphi \in \text{Irr}(D_{16})$ is faithful of degree 2 and $\mathbb{Q}(\varphi) = \mathbb{Q}[\sqrt{2}]$. Then $[\psi]$ is the class of a Schur algebra as in Corollary 4.6.

Indefinite quaternion division algebras D with rational center might also have even degree. Corollary 5.5 shows that for such quaternion algebras D the solvable degree is even and we are not aware of an irreducible rational character χ with Frobenius Schur indicator $+$ and Schur index 2 such that $\chi(1) \equiv 2 \pmod{4}$.

5 The solvable degree of a division algebra

5.1 The main theorem and some examples

Theorem 5.1. *Let G be a finite solvable group, $\chi \in \text{Irr}(G)$. Put $K := \mathbb{Q}(\chi)$ and recall that*

$$P(\chi) = \prod \{p \in \mathbb{P} \mid m_p(\chi) > 1\}.$$

Then the following two divisibility conditions hold:

(a) $[F(K[\zeta_{P(\chi)}]) : K]$ divides $\chi(1)$.

(b) $[K[\zeta_{2P(\chi)}] : K]$ divides $\chi(1)$.

Remark 5.2. Clearly (a) implies (b) if either $2 \notin \mathcal{P}(\chi)$ or the prime 2 is ramified in $K/\mathbb{Q}$.

Remark 5.3. The authors are aware of two papers, [14] and [22], that deal with variations of the degree of a Schur algebra. Pendergrass shows that for any finite group G and any $\chi \in \text{Irr}(G)$ we have $\chi(1) \geq [K[\zeta_{2P(\chi)}] : K]$ where $K := \mathbb{Q}(\chi)$. The paper [22] claims that Theorem 5.1 (b) holds for arbitrary groups G , which is known to be wrong (see for instance Remark 4.4).

For some Schur algebras D we can find solvable groups G where $\chi(1) = [K[\zeta_{2P(\chi)}] : K]$, as the well known examples in the following three corollaries show:

Corollary 5.4. *Let p be an odd prime and $K \leq \mathbb{Q}[\zeta_p]$ be such that $[K : \mathbb{Q}]$ is odd. Then K is a real number field and $t := [\mathbb{Q}[\zeta_p] : K]$ is even. Let D be the quaternion algebra with center K ramified at all infinite places of K and at the unique place dividing p , so $\mathcal{P}([D]) = \{\infty, p\}$. Then $\deg_{\text{sol}}(D) = t/2$.*

Proof. Let σ be a generator of $\text{Gal}(\mathbb{Q}[\zeta_p]/K)$. Then $\sigma(\zeta_p) = \zeta_p^a$ for some $1 \leq a \leq p-1$. Put

$$G := \langle x, y, z \mid x^p = 1, x^y = x^a, y^t = z, z^2 = 1, z \text{ central} \rangle \cong C_{2p}.C_t.$$

Up to the action of the outer automorphism group there is a unique faithful $\chi \in \text{Irr}(G)$. Then $\chi(1) = t$ and $[\chi] = [D]$. $\square$

Corollary 5.5. *The solvable degree of a rational quaternion division algebra D is*

$$\deg_{\text{sol}}(D) = \frac{[\mathbb{Q}[\zeta_{2P(D)}] : \mathbb{Q}]}{2}.$$

Proof. For the odd primes p dividing $P(D)$ we take $K = \mathbb{Q}$ in Corollary 5.4, define $G_p := C_{2p} \cdot C_{p-1}$ and $\chi_p \in \text{Irr}(G_p)$ to be the group and the irreducible character from the proof of Corollary 5.4. If 2 ramifies in D then we put $G_2 := Q_8 := C_4 \cdot C_2$ to be the quaternion group of order 8 and $\chi_2 \in \text{Irr}(G_2)$ with $\chi_2(1) = 2$. Let G denote the central product of all G_p with $m_p(D) = 2$ and $\chi := \prod_{p|P(D)} \chi_p \in \text{Irr}(G)$. Then $\chi(1) = [\mathbb{Q}[\zeta_{2P(D)}] : \mathbb{Q}]$ and $[\chi] = [D]$. $\square$

A series of examples with $m(\chi) > 2$ and $\deg_{\text{sol}}([\chi]) = 1$ can be constructed from [1]:

Corollary 5.6. *Let p, q be odd primes such that q divides $p - 1$. Let $E \leq \mathbb{Q}[\zeta_p]$ be the subfield of index $q = [\mathbb{Q}[\zeta_p] : E]$ and put $K = E[\zeta_q]$. For any $b \in \{1, \dots, q - 1\}$ there is a unique Schur algebra D with center K and $\text{Inv}(D) = \{(\frac{b}{q}, p)\}$. All these D have $\deg(D) = \deg_{\text{sol}}(D) = 1$.*

Proof. Let σ be a generator of $\text{Gal}(\mathbb{Q}[\zeta_p]/E)$. Then $\sigma(\zeta_p) = \zeta_p^a$ for some $1 \leq a \leq p - 1$ such that $a + p\mathbb{Z} \in (\mathbb{Z}/p\mathbb{Z})^\times$ is of order q . Put

$$G := \langle x, y, z \mid x^p = 1, x^y = x^a, y^q = z, z^q = 1, z \text{ central} \rangle \cong C_p : C_{q^2}.$$

The Galois group of $\mathbb{Q}[\zeta_q]/\mathbb{Q}$ acts regularly on the set of prime ideals of K that divide p and also on the set

$$X := \{\chi \in \text{Irr}(G) \mid \chi(1) = q\}$$

of faithful irreducible characters of G . For $\chi \in X$ we get $\text{Inv}([\chi]) = \{(\frac{b}{q}, p)\}$ for some $b \in \{1, \dots, q - 1\}$. All these $q - 1$ values occur. So one of these characters $\chi \in X$ satisfies $[\chi] = [D]$. An explicit formula for computing the Hasse invariants of these characters is given in [17, Theorem 5.1]. $\square$

5.2 Proof of Theorem 5.1

This section is devoted to a proof of Theorem 5.1. The proof is split into four lemmas, where we always keep the notation of Theorem 5.1 and the following

Assumption 5.7. *G is a finite solvable group and $\chi \in \text{Irr}(G)$ is a faithful irreducible character of G that is a counterexample to Theorem 5.1. We also assume that G is minimal, i.e. Theorem 5.1 holds for all proper subgroups $U < G$ and all $\psi \in \text{Irr}(U)$.*

Lemma 5.8. *Let $N \trianglelefteq G$ be a normal subgroup of G . Then $\chi|_N = e(\psi_1 + \dots + \psi_t)$ with $e \in \mathbb{N}$ and $\psi_i \in \text{Irr}(N)$ for $i = 1, \dots, t$. Put $\psi := \psi_1$ and $T := I_G(\psi)$ the inertia group of ψ . Then $T \trianglelefteq G$ is normal in G .*

Let $\alpha \in \text{Irr}(T)$ be the Clifford correspondent of χ lying above ψ , i.e. $\alpha|_N = e\psi$ and $\chi = \alpha^G$, and put $L := \mathbb{Q}(\alpha)$. Then $[L : K] = [G : T]$ and $G/T \cong \text{Gal}(L/K)$.

Proof. Put $H := N_G(T)$ and let $\beta := \alpha^H$. Then $T \leq H \leq G$ and $\beta^G = (\alpha^H)^G = \alpha^G = \chi$. In particular $\beta \in \text{Irr}(H)$ and $K \leq \mathbb{Q}(\beta) \leq L$.

We first show that $\mathbb{Q}(\beta) = K$. Let $\sigma \in \text{Gal}(L/K)$. Since $\chi^\sigma = \chi$, the character $\psi^\sigma \in \text{Irr}(N)$ is again an irreducible constituent of $\chi|_N$, so there is $x \in G$ with $\psi^\sigma = \psi^x$. On the other hand, $T^x = I_G(\psi^x) = I_G(\psi^\sigma) = I_G(\psi) = T$, since the Galois conjugation commutes with the conjugation action. So $x \in N_G(T) = H$ and hence $\beta^\sigma = \beta$.

Now $\mathbb{Q}(\beta) = \mathbb{Q}(\chi) = K$ so $[\beta] = [\chi] \in \text{Br}(K)$ by Corollary 2.3. The minimality of G hence implies that $H = G$ and hence $T \trianglelefteq G$.

As in the proof of Theorem 3.1 (a) we now put

$$H := \{g \in G \mid \exists \sigma \in \text{Gal}(L/K) : \alpha(gng^{-1}) = \sigma(\alpha(n)) \text{ for all } n \in T\}.$$

Then $\beta := \alpha^H \in \text{Irr}(H)$ satisfies $\mathbb{Q}(\beta) = K$ and $\chi = \beta^G$. This again implies that $[\beta] = [\chi]$ by Corollary 2.3. The minimality of G implies that $H = G$ and hence $G/T = H/T \cong \text{Gal}(L/K)$. $\square$

We now construct a chief series of G as follows: Let G_1 be a minimal normal subgroup of G and $\chi_1 \in \text{Irr}(G_1)$ be a constituent of $\chi|_{G_1}$. Let $T_1 := I_G(\chi_1)$ be the inertia group of χ_1 . Then

$$G_1 \leq T_1 \leq G$$

and also T_1 is normal in G . Let $G_2 \leq T_1$ be such that G_2/G_1 is a minimal normal subgroup of G/G_1 and put $T_2 := I_G(\chi_2)$ where $\chi_2 \in \text{Irr}(G_2)$ is a constituent of $\chi|_{G_2}$ such that $(\chi_2)_{|G_1}$ is a multiple of χ_1 .

As $I_G(\chi_2)$ also stabilises the restriction of χ_2 to the normal subgroup G_1 we have that $T_2 \leq T_1$. We continue like this to obtain a normal series

$$1 = G_0 < G_1 < G_2 < \dots < G_n < T_n =: G_{n+1} \trianglelefteq G$$

such that $Q_j := G_j/G_{j-1}$ are minimal normal subgroups of G/G_{j-1} for $j = 1, \dots, n+1$. In particular all these Q_j are elementary abelian.

Lemma 5.9. *Put $T := T_n$ and write $\chi|_T = \alpha_1 + \dots + \alpha_t$. Put $\alpha := \alpha_1$. Then $\zeta_{P(\chi)} \in \mathbb{Q}(\alpha)$.*

Proof. For $j = 1, \dots, n+1$ the restriction of χ_j to G_{j-1} is a multiple of χ_{j-1} , in particular

$$\mathbb{Q}(\chi_1) \leq \mathbb{Q}(\chi_2) \leq \dots \leq \mathbb{Q}(\chi_n) \leq \mathbb{Q}(\alpha).$$

Assume that $p \in \mathcal{P}(\chi)$ is odd and that $\mathbb{Q}[\zeta_p] \not\subseteq \mathbb{Q}(\alpha)$. Then $\mathbb{Q}[\zeta_p] \not\subseteq \mathbb{Q}(\chi_j)$ for all j . We then show inductively that every χ_j is p' -special:

Clearly $\chi_0 = 1$ is p' -special.

Let $j > 0$ and assume that χ_{j-1} is p' -special.

If Q_j is not a p -group, then by Lemma 2.9 also χ_j is p' -special.

Assume that Q_j is an elementary abelian p -group. As $G_j \leq I_G(\chi_{j-1})$ Lemma 2.9 says that the character χ_{j-1} has a unique p' -special extension $X := \hat{\chi}_{j-1} \in \text{Irr}(G_j)$. Then $X|_{G_{j-1}} = \chi_{j-1} \in \text{Irr}(G_{j-1})$. By Theorem 2.7 the induced character is

$$\chi_{j-1}^{G_j} = \sum_{\lambda \in \text{Irr}(Q_j)} \lambda X.$$

By Frobenius reciprocity also χ_j is a constituent of $\chi_{j-1}^{G_j}$, so there is some $\lambda \in \text{Irr}(Q_j)$ such that $\chi_j = \lambda X$. As all irreducible characters λ of the abelian group Q_j are linear, $\chi_j(1) = X(1) = \chi_{j-1}(1)$. Now χ_{j-1} is p' -special, hence $\chi_j(1)$ is not a multiple of p . By assumption $\mathbb{Q}[\zeta_p]$ is not a subfield of $\mathbb{Q}(\chi_j)$, then also the determinantal order of χ_j is not divisible by p and hence χ_j is p' -special.

In particular also α is p' -special which implies that the restriction α' of α to the p -regular elements of T is an irreducible Brauer character by Lemma 2.10. Moreover $\mathbb{Q}(\alpha)$ is generated by p' -power roots of unity, in particular

$\mathbb{Q}(\alpha)/\mathbb{Q}$ is unramified at p . So also $\mathbb{Q}(\alpha)/K$ is unramified at all prime ideals P of K that lie above p . Therefore the restrictions α'_i of the α_i to the p -regular elements of T yield t distinct irreducible p -Brauer characters. So the restriction χ' of χ to the p -regular elements of G is the induced character

$$\chi' = (\alpha')^G$$

and hence an irreducible Brauer character of G . By Remark 2.11 this implies that $m_p(\chi) = 1$ contradicting the assumption. $\square$

For G , χ , $K = \mathbb{Q}(\chi)$ as in Assumption 5.7 and T , $\alpha \in \text{Irr}(T)$, $E := \mathbb{Q}(\alpha)$ and $t = [G : T] = [E : K]$ as in Lemma 5.9 we hence have

- (a) $K \leq E$ and $[E : K] = t$ (by Lemma 5.8).
- (b) $\zeta_{P(\chi)} \in E$ (by Lemma 5.9).
- (c) $[F(E) : E] \mid \alpha(1)$ (by Theorem 2.12).
- (d) $\zeta_4 \in F(E)$ if 2 ramifies in E over $\mathbb{Q}$.

Corollary 5.10. $[F(K[\zeta_{P(\chi)}]) : K] \mid \chi(1)$, so χ satisfies Theorem 5.1 (a).

Proof. As $K[\zeta_{P(\chi)}] \leq E$ we have that

$$[F(K[\zeta_{P(\chi)}]) : K] \mid [F(E) : K] = [F(E) : E][E : K] \mid \alpha(1)t = \chi(1).$$

$\square$

In particular χ satisfies Theorem 5.1 if $2 \notin \mathcal{P}(\chi)$. So in the following we assume that $m_2(\chi) \neq 1$ and hence $m_2(\chi) = 2$.

Lemma 5.11. *If 2 is ramified in $E/\mathbb{Q}$, then $[F(K[\zeta_{2P(\chi)}]) : K] \mid \chi(1)$. In particular χ satisfies Theorem 5.1 (b).*

Proof. If 2 ramifies in E then it also ramifies in $F(E)$, so $\zeta_4 \in F(E)$. In particular $K[\zeta_{2P(\chi)}]$ is a subfield of $F(E)$ and hence also $F(K[\zeta_{2P(\chi)}]) \leq F(E)$ so again

$$[F(K[\zeta_{2P(\chi)}]) : K] \mid [F(E) : K] = [F(E) : E][E : K] \mid \alpha(1)t = \chi(1).$$

$\square$

Lemma 5.12. *If $\alpha(1)$ is even then $[K[\zeta_{2P(\chi)}] : K] \mid \chi(1)$, so χ satisfies Theorem 5.1 (b).*

Proof. We have that $K[\zeta_{P(\chi)}] \leq E$ and $[K[\zeta_{2P(\chi)}] : K[\zeta_{P(\chi)}] \mid 2$. So

$$[K[\zeta_{2P(\chi)}] : K] \mid 2[E : K] = 2t \mid \alpha(1)t = \chi(1).$$

□

By Lemma 5.11 and 5.12 we are left with the situation that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is not ramified at the prime 2 and $\alpha(1)$ is odd. Then $m_2(\alpha) = 1$, because the Schur index of α divides the character degree $\alpha(1)$. Therefore Theorem 3.1 tells us that $m_2(\chi) = 1$. This finishes the proof of Theorem 5.1.

References

- [1] S. A. Amitsur, Finite subgroups of division rings. Trans. Am. Math. Soc. **80** (1955) 361–386.
- [2] Richard Brauer, On the algebraic structure of group rings. J. Math. Soc. Japan **3** (1951) 237–251.
- [3] Mark Benard, The Schur subgroup I. J. Algebra **22** (1972) 374–377.
- [4] Mark Benard, Murray M. Schacher, The Schur Subgroup II. J. Algebra **22** (1972) 378–385.
- [5] G.-Martin Cram, On the field of character values of finite solvable groups. Arch. Math. **51** (1988) 294–296.
- [6] Walter Feit, The computation of some Schur indices. Israel J. Math. **46** (1986) 274–300.
- [7] Dilip Gajendragadkar, A characteristic class of characters of finite π -separable groups. J. Algebra **59** (1979) 237–259.
- [8] Martin Isaacs, *Character Theory of Finite Groups*. AMS Chelsea Publishing (1976).
- [9] Martin Isaacs, Induction and restriction of π -special characters. Can. J. Math. **38** (1986) 576–604.

- [10] Christoph Jansen, Klaus Lux, Richard A. Parker, Robert A. Wilson, *An atlas of Brauer characters*. London Mathematical Society Monographs. Oxford: Clarendon Press (1995)
- [11] Falko Lorenz, Über die Berechnung des Schurschen Indizes von Charakteren endlicher Gruppen. *J. Number Theory* **3** (1971) 60–103.
- [12] Gabriel Navarro, Pham Huu Tiep, The fields of values of characters of degree not divisible by p , *Forum Math. Pi* **9** (2021) Paper No. e2, 28p
- [13] Gabriele Nebe, Finite quaternionic matrix groups. *Represent. Theory* **2** (1998) 106–223.
- [14] J. W. Pendergrass, The size of a simple component of a group algebra. *Commun. Algebra* **4** (1976) 1071–1076.
- [15] Irving Reiner, *Maximal Orders*. Lond. Math. Soc. Monogr. 5 (1975).
- [16] David Schlang, PhD Thesis, RWTH Aachen University (in preparation)
- [17] Alexandre Turull, On the calculation of local invariants of irreducible characters. *J. Algebra* **557** (2020) 129–164.
- [18] Bill Unger, An algorithm for computing Schur indices of characters. *J. Symb. Comput.* **93** (2019) 148–160.
- [19] Ernst Witt, Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zahlkörper. *J. Reine Angew. Math.* **190** (1952) 231–245.
- [20] Toshihiko Yamada, On the Schur index of a monomial representation. *Proc. Japan Acad.* 45 (1969) 522–525.
- [21] Toshihiko Yamada, *The Schur subgroup of the Brauer group*, Lecture Notes in Math. **397** Springer (1974)
- [22] Toshihiko Yamada, Seiichi Ito, The degree of a Schur algebra. *TRU Math.* **22** (1986) 53–61.