On Siegel modular forms of weight 12

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Abstract * We calculate the action of some explicit Hecke operators on the space generated by the 24 isometry classes of even unimodular lattices in dimension 24 and hence the 24 eigenforms for the Hecke algebra. We find the degrees of these eigenforms in most cases and so determine the dimensions of the spaces of Siegel modular cusp forms of weight 12 generated by theta series.

1 Introduction

Since the work of Witt [11] there was much interest in studying linear relations between Siegel theta series of even unimodular lattices of a given dimension n. For $n = 16$ the theta series of the two lattices are linearly dependent in degrees ≤ 3 and linearly independent in degree 4 (and give the Schottky cusp form of degree 4 and weight 8). The next interesting case of dimension 24 was initiated by Erokhin [6], where it was proved that the 24 theta series in question are linearly independent in degree 12. Recently this topic was reconsidered in the interesting paper [3], where the authors give a new construction for the resulting unique Siegel cusp form of degree 12 and weight 12 and find many interesting properties of this cusp form.

In this paper we study the whole filtration on the vector space V of formal linear combinations of the 24 isometry classes of Niemeier lattices, given by the theta series of different degrees.

We start from the explicit Hecke operator K on V, which is given by the adjacency matrix of the Kneser 2-neighbour graph (with the natural multiplicities), which was essentially found by R. Borcherds for the purpose of classifying odd unimodular lattices in dimension 24 (see [4], Chapter 17). The operator K fixes the filtration. Also it has a simple spectrum and the 24 eigenvectors give the 24 Siegel cusp forms, which are eigenfunctions for the Hecke algebra, of different degrees and weight 12.

It is a non trivial problem to find the exact degrees of these 24 eigenforms, which we have solved only partially: We consider a natural Hermitian scalar product on V (an algebraic analogue for the Petersson scalar product for Siegel cusp forms), and introduce a multiplication for which the dual filtration behaves well (Proposition 2.3). The resulting graduated algebra has nice properties that enable us to place most of the 24 eigenforms exactly in the filtration. In two cases we use a different construction with theta series with harmonic coefficients (Section 3.3). As a result we calculate the dimensions of the subspaces of Siegel cusp forms of weight 12, that are generated by theta series, in most of the degrees (see Theorem 3.7). There remains essentially one undecided case, where we could not decide whether an explicit cusp form of degree 10 is zero or not.

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2 The general situation

2.1 Notation

Let $\mathcal{G}$ be a genus of lattices in Euclidean space $(\mathbb{R}^m, (,))$. For $\Gamma \in \mathcal{G}$, we denote by $[\Gamma]$ its isometry class. Let $V = V(\mathcal{G})$ be the $\mathbb{C}$-vector space with basis $([\Gamma] \mid \Gamma \in \mathcal{G})$. If a sum runs over isometry classes $[\Gamma]$ we always mean, that it runs over all of the finitely many isometry classes of lattices in $\mathcal{G}$.

Since we only need the situation that $m \equiv 0 \pmod 8$ and $\mathcal{G}$ is the genus of even unimodular lattices of rank m, we will henceforth assume this.

Let $M_{n,k}$ be the vector space of Siegel modular forms of weight k and degree n with respect to the full symplectic group $\mathrm{Sp}(2n,\mathbb{Z})$. Then for each $n \geq 0$ and each $\Gamma \in \mathcal{G}$, the degree-n Siegel theta series

$$\Theta_\Gamma^{(n)} \in M_{n,m/2}.$$

Since we fix the dimension n, $M_{n,m/2}$ will be denoted by M_n. Extending $\Theta^{(n)}$ linearly on V we obtain a linear map

$$(1) \quad \Theta^{(n)} : V \to M_n : \Theta^{(n)}(\sum_{[\Gamma]} c_\Gamma[\Gamma]) := \sum_{[\Gamma]} c_\Gamma \Theta_\Gamma^{(n)}$$

The maps $\Theta^{(n)}$ commute with the Siegel Φ-operator, i.e. $\Phi \circ \Theta^{(n)} = \Theta^{(n-1)}$.

2.2 The basic filtration

Let $V_n := \ker(\Theta^{(n)})$ be the kernel of $\Theta^{(n)}$, i.e. those linear combinations of lattices which have trivial degree-n Siegel theta series.

By (1) we get the filtration

$$V =: V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \ldots \supseteq V_m = \{0\}.$$

$V_0 = \{v = \sum_{[\Gamma]} c_\Gamma[\Gamma] \mid \sum_{[\Gamma]} c_\Gamma = 0\}$ is of codimension 1 in V.

Let M'_n be the image of $\Theta^{(n)}$. Then M'_n is the subspace of M_n spanned by degree-n Siegel theta series. S. Böcherer [2] has proved that $M'_n = M_n$ for $n < m/4$ and it is an important open problem if $M'_n = M_n$ for all n.

If S_n denotes the space of cusp forms in M_n and $S'_n := M'_n \cap S_n$, then

$$V_{n-1}/V_n \cong S'_n.$$

2.3 The scalar product on V and the dual filtration

Let $\langle , \rangle$ denote the Hermitian scalar product on V defined by

$$\langle [\Gamma], [\Lambda] \rangle := (\#\mathrm{Aut}(\Gamma))\delta_{[\Gamma],[\Lambda]}.$$

Then $\langle , \rangle$ is non degenerate and positive, that is it gives on V a Hilbert space structure. Let W_n be the orthogonal complement of V_n. We then have the ascending filtration

$$0 = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq \ldots \subseteq W_m = V$$

For an even, symmetric, positive semi definite $n \times n$-matrix $N \in \mathrm{Sym}^{(n)}_{\geq 0}(\mathbb{Z})$ let

$$a_N([\Gamma]) := \#\{(\gamma_1, \ldots, \gamma_n) \in \Gamma^n \mid (\gamma_i, \gamma_j) = N_{ij}\}$$

be the N-the Fourier coefficient of $\Theta^{(n)}([\Gamma])$ and extend this definition linearly on V. Denote by

$$b_N := \sum_{[\Gamma]} \frac{1}{\#\mathrm{Aut}(\Gamma)} a_N([\Gamma])[\Gamma] \in V$$

Then clearly

$$a_N(v) = \langle v, b_N \rangle \text{ for all } v \in V.$$

(So b_N plays the role of the Poincaré series in the theory of modular forms.)

Proposition 2.1

$$W_n = \langle b_N \mid N \in \mathrm{Sym}^{(n)}_{\geq 0}(\mathbb{Z}) \rangle$$

Proof. The inclusion $\supseteq$ follows because $\langle v, b_N \rangle = a_N(v) = 0$ for all $v \in V_n, N \in \mathrm{Sym}^{(n)}_{\geq 0}(\mathbb{Z})$. On the other hand, the orthogonal complement of the right hand side in W_n is 0: Let $x \in W_n$ with $a_N(x) = 0$ for all $N \in \mathrm{Sym}^{(n)}_{\geq 0}(\mathbb{Z})$. Then $x \in V_n$ and therefore $x \in W_n \cap V_n = 0$. $\square$

Remark 2.2 *Let $Y_n := V_{n-1} \cap W_n$. Then $Y_n \cong V_{n-1}/V_n \cong S'_n$ is canonically isomorphic to S'_n $(n = 0, \ldots, m)$ and one obtains an orthogonal decomposition $V = \sum_{n=0}^{m} Y_n$.*

2.4 The multiplication on V.

The rule

$$[\Gamma] \circ [\Lambda] := \#(\mathrm{Aut}(\Gamma))\delta_{[\Gamma],[\Lambda]}[\Gamma]$$

defines a commutative and associative multiplication on V. The elements $(\#\mathrm{Aut}(\Gamma))^{-1}[\Gamma]$ form a set of pairwise commuting idempotents summing up to the unit element

$$e := \sum_{[\Gamma]} \frac{1}{\#\mathrm{Aut}(\Gamma)} [\Gamma].$$

With respect to this multiplication the Hermitian form $\langle , \rangle$ is associative, i.e.

$$\langle v_1 \circ v_2, v_3 \rangle = \langle v_1, v_2 \circ v_3 \rangle \text{ for all } v_1, v_2, v_3 \in V.$$

The second filtration by the W_n behaves well with respect to this multiplication:

Proposition 2.3

$$W_n \circ W_l \subseteq W_{n+l}$$

Proof. By Proposition 2.1 it is enough to show that

$$b_{N_n} \circ b_{N_l}$$

is a linear combination of some $b_{N_{n+l}}$ for any $N_n \in \mathrm{Sym}^{(n)}_{\geq 0}(\mathbb{Z})$, $N_l \in \mathrm{Sym}^{(l)}_{\geq 0}(\mathbb{Z})$. But

$$b_{N_n} \circ b_{N_l} = \sum_{[\Gamma]} \frac{1}{\#\mathrm{Aut}(\Gamma)} a_{N_n}(\Gamma) a_{N_l}(\Gamma)[\Gamma].$$

Now

$$a_{N_n}(\Gamma) a_{N_l}(\Gamma) = \sum_M a_M(\Gamma)$$

where the sum goes over all matrices $M \in \mathrm{Sym}^{n+l}_{\geq 0}(\mathbb{Z})$ of the form $M = \begin{pmatrix} N_n & \star \\ \star & N_l \end{pmatrix}$. This is a very well known property of the Fourier coefficients of the theta series (cf. [10], Theorem 1). □

From Proposition 2.3 one deduces the following property of the V_n.

Corollary 2.4 *For all $j > i$ one has*

$$W_i \circ V_j \subseteq V_{j-i}$$

2.5 Hecke operators.

There are some natural linear operators on V which preserve the V_n and the scalar product and which act on $V_{n-1}/V_n \cong S'_n$ as certain combinations of Hecke operators.

We define these operators in terms of correspondences on lattices. Fix a lattice Γ, a prime p and a natural number d. Then

$$K_{p,d}([\Gamma]) := \sum_{\Lambda} [\Lambda],$$

where the sum runs over all lattices Λ in the genus of Γ such that

$$\Gamma/(\Lambda \cap \Gamma) \cong \Lambda/(\Lambda \cap \Gamma) \cong \mathbb{F}_p^d \text{ and } (\Lambda \cap \Gamma) \text{ is of level } p.$$

Clearly the result depends only on the isometry class of Γ and not on the particular choice of $\Gamma \in [\Gamma]$.

We also define operators $T(p)$ for primes p, where

$$T(p)([\Gamma]) := \sum_{E \subset \Gamma/p\Gamma} [\sqrt{p}^{-1} \Gamma_E]$$

where the sum runs over all maximal isotropic subspaces E in $(\Gamma/p\Gamma, (,))$ and $\Gamma_E := \{\gamma \in \Gamma \mid \gamma + p\Gamma \in E\}$.

All these operators $K_{p,d}$ and $T(p)$ pairwise commute. They are also symmetric with respect to $\langle,\rangle$. So there is a basis of V consisting of pairwise orthogonal common eigenvectors $d_1, \ldots, d_s$ for all $K_{p,d}$ and $T(p)$ where $s = \dim(V)$ is the number of isometry classes of lattices in $\mathcal{G}$. Since the subspaces $Y_n = V_{n-1} \cap W_n$ are invariant under all the operators $K_{p,d}$ and $T(p)$, each eigenvector lies in some Y_n. If $d_i \in Y_n$, then n is called the degree of d_i. We order the d_i by increasing degree. Then $d_1 = e$ (up to a scalar) is the only element of degree 0, etc.

For each $1 \leq i \leq s$ we define $v(i) \in \{-1, \ldots, m-1\}$ by

$$d_i \in V_{v(i)}, \quad d_i \notin V_{v(i)+1}.$$

Analogously let $w(i) \in \{0, \ldots, m\}$ be defined by

$$d_i \in W_{w(i)}, \quad d_i \notin W_{w(i)-1}.$$

If the degree of d_i is n then $v(i) \leq n-1$ and $w(i) \geq n$. The next lemma shows that one has equality if the eigenspaces are 1-dimensional.

Lemma 2.5 *Let $1 \leq i \leq s$ and assume that d_i generates the full eigenspace to the corresponding character of the Hecke algebra. Then $w(i) = v(i) + 1$.*

Proof. Let $n := w(i)$. Then $d_i \notin W_{n-1}$. Since W_{n-1} is generated by some of the eigenvectors and the eigenvectors to different eigenvalues are orthogonal, it follows that $d_i \perp W_{n-1}$ and so $d_i \in V_{n-1} = W_{n-1}^{\perp}$. If $d_i \in V_n$, then $\langle d_i, d_i \rangle = 0$ which is a contradiction. Therefore $n - 1 = v(i)$. □

3 The special case $m = 24$.

Now let $m = 24$ and $\mathcal{G}$ be the genus of even unimodular 24-dimensional lattices. Then $\mathcal{G}$ contains 24 isometry classes of lattices, represented by the 24 Niemeier lattices $\Gamma_1, \ldots, \Gamma_{24}$ in the order of [4], Table 16.1. The root system of Γ_i is

i	1	2	3	4	5	6	7	8
	D_{24}	$D_{16}E_8$	E_8^3	A_{24}	D_{12}^2	$A_{17}E_7$	$D_{10}E_7^2$	$A_{15}D_9$
i	9	10	11	12	13	14	15	16
	D_8^3	A_{12}^2	$A_{11}D_7E_6$	E_6^4	$A_9^2D_6$	D_6^4	A_8^3	$D_5^2A_7^2$
i	17	18	19	20	21	22	23	24
	A_6^4	$D_4A_5^4$	D_4^6	A_4^6	A_3^8	A_2^{12}	A_1^{24}	$\emptyset$

The basis elements $[\Gamma_i]$ of V are abbreviated to e_i $(i = 1, \ldots, 24)$.

The filtration

$$V = V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \ldots \supseteq V_{12} = 0$$

ends with V_{12} (see [6], [3]). Dually we have $W_{12} = V$.

Let $K := K_{2,1}$ and $T(2)$ be the Hecke operators as defined in Section 2.5.

Theorem 3.1 *(a) The Hecke operator K with respect to the basis above is given by right multiplication with the matrix on page 11. K has 24 distinct integral eigenvalues $ev(1) > ev(2) > \ldots > ev(24)$ with*

$$\begin{array}{llll} ev(1) = 8390655, & ev(2) = 4192830, & ev(3) = 2098332, & ev(4) = 1049832, \\ ev(5) = 533160, & ev(6) = 519120, & ev(7) = 268560, & ev(8) = 244800, \\ ev(9) = 145152, & ev(10) = 126000, & ev(11) = 99792, & ev(12) = 91152, \\ ev(13) = 89640, & ev(14) = 69552, & ev(15) = 51552, & ev(16) = 45792, \\ ev(17) = 35640, & ev(18) = 21600, & ev(19) = 17280, & ev(20) = 5040, \\ ev(21) = -7920, & ev(22) = -16128, & ev(23) = -48528, & ev(24) = -98280. \end{array}$$

(b) The operator $T(2)$ is a polynomial in K, and can be calculated from the image $[\Gamma_1]T(2) = e_1T(2) = 561196350(1127e_1 + 159390e_2 + 75900e_3 + 3349444e_5 + 10200960e_7 + 507907620e_9 + 2271527104e_{14} + 20891566080e_{16} + 53441979360e_{19} + 284357427200e_{21} + 255926200320e_{23} + 9248440320e_{24})$. *The explicit matrix can be obtained from [9].*

Proof. To determine the operator K, we use the Kneser 2-neighbour graph calculated by R. Borcherds during the classification of odd unimodular lattices of dimension 24.

If the intersection of the two even unimodular lattices $L_1, L_2 \in \mathcal{G}$ is of index 2 in these lattices, then $L_1 \cap L_2$ is contained in a unique odd unimodular lattice O. If O has a vector of norm 1, then the reflection along this vector maps L_1 to L_2. Therefore the two even neighbours of O are isometric. If O does not contain vectors of norm 1, then O and the two even neighbours L_1 and L_2 are listed in [4], Table 17.1. (Note that there are misprints in the tables given in the first and second edition.) The lattice O yields $\frac{\#\mathrm{Aut}(L_1)}{\#\mathrm{Aut}(O)}$ neighbours of L_1 that are isometric to L_2. To calculate K_{ij} with $i \neq j$, one adds these numbers where $L_1 \cong \Gamma_i$ and $L_2 \cong \Gamma_j$. The diagonal entry K_{ii} then is $(2^{11}+1)(2^{12}-1) - \sum_{j\neq i} K_{ij}$.

One then easily determines the eigenvalues of the matrix K. Because the eigenvalues of K are distinct and $T(2)$ commutes with K, the operator $T(2)$ is a polynomial in K. To determine $T(2)$ it therefore suffices to calculate the image of the Leech lattice $[\Gamma_{24}]T(2)$ which is done in [5]. □

Definition 3.2 *Let $d_i \in V$ be an eigenvector of K to the eigenvalue $ev(i)$.*

The eigenvectors $d_1, \ldots, d_{24} \in V$ can be calculated explicitly as linear combinations of the e_i. All this data may be obtained from [9].

3.1 Inequalities for the degree of d_i.

For $i = 1, \ldots, 12$ let $w(i)$ and $v(i)$ be defined as in Section 2.5. Then by Lemma 2.5, the degree of d_i is $w(i) = v(i) - 1$. The Fourier coefficients of $\Theta^{(n)}(e_i)$ which correspond to the irreducible root systems

$0, A_1, A_2, A_3, A_4, D_4, A_5, D_5, A_6, D_6, E_6, A_7, D_7, E_7, A_8, D_8, E_8, A_9, D_9, A_{10}, D_{10}, A_{11}, D_{11}, D_{12}$

are given in [3]. From this table one easily calculates the corresponding Fourier coefficients of $\Theta^{(n)}(d_i)$, which shows the following inequalities for the $w(i)$:

Lemma 3.3

$$
\begin{array}{llllll}
w(1) \leq 0, & w(2) \leq 1, & w(3) \leq 2, & w(4) \leq 3, & w(5) \leq 4, & w(6) \leq 4, \\
w(7) \leq 5, & w(8) \leq 5, & w(9) \leq 6, & w(10) \leq 6, & w(11) \leq 6, & w(12) \leq 7, \\
w(13) \leq 8, & w(14) \leq 7, & w(15) \leq 8, & w(16) \leq 7, & w(17) \leq 8, & w(18) \leq 8, \\
w(19) \leq 9, & w(20) \leq 9, & w(21) \leq 10, & w(22) \leq 10, & w(23) \leq 11, & w(24) \leq 12.
\end{array}
$$

3.2 Equalities for the $w(i)$

Proposition 3.4 *We have*

$$
\begin{array}{llllll}
w(1) = 0, & w(2) = 1, & w(3) = 2, & w(4) = 3, & w(5) = 4, & w(6) = 4, \\
w(7) = 5, & w(8) = 5, & w(9) = 6, & w(10) = 6, & w(11) = 6, & w(12) = 6,7, \\
w(13) = 6,7,8, & w(14) = 7, & w(15) = 6,7,8, & w(16) = 7, & w(17) = 8, & w(18) = 8, \\
w(19) = 7,8,9, & w(20) = 9, & w(21) = 8,9,10, & w(22) = 10, & w(23) = 11, & w(24) = 12.
\end{array}
$$

Proof. Clearly $w(1) = 0$. Since V_0 is of dimension one and d_2 is linearly independent of d_1, also $w(2) = 1$ is clear. In [3], a nonzero cusp form of degree 12 is constructed, as linear combinations of Siegel theta series. By Lemma 3.3 all the other eigenforms d_i with $i < 24$ have degree < 12. Therefore the cusp form of [3] is a multiple of d_{24} and $w(24) = 12$. To prove the proposition, we calculate the multiplication $\circ$ on V with respect to the basis $(d_1, \ldots, d_{24})$ and use the property that $W_i \circ W_j \subseteq W_{i+j}$ by Proposition 2.3.

We see that

$$d_i \circ d_j = A_{ij} d_{24} + \sum_{l=1}^{23} b_{ij}^l d_l$$

with a nonzero coefficient A_{ij} for the following pairs (i, j):

$$(2,23),\ (3,22),\ (4,20),\ (5,17),\ (6,18),\ (7,14),\ (8,16),\ (9,9),\ (10,10),\ (11,11)$$

(The structure constants for the multiplication in the basis of the d_i can also be obtained from [9].) One concludes, that for all the 17 elements occurring in one of those pairs, the inequality in Lemma 3.3 is indeed an equality:

Let us illustrate the reasoning by treating the first pair $(2, 23)$. Seeking for a contradiction we assume that $w(2) < 1$ or $w(23) < 11$. Then the sum $w(2)+w(23) \leq 11$ and therefore $d_2 \circ d_{23} \in W_{11}$ by Proposition 2.3. Since $d_1, \ldots, d_{23} \in W_{11}$ the fact that $A_{2,23} \neq 0$ implies that $d_{24} \in W_{11}$ which contradicts $w(24) = 12$. Analogously one gets all the $w(i)$ except for $i = 12, 13, 15, 19$ and 21.

The fact that $A_{12,12}$, $A_{13,13}$ and $A_{15,15} \neq 0$ shows that d_{12}, d_{13}, and d_{15} do not lie in W_5. Therefore $w(12) = 6$ or 7, $w(13) = 6, 7$, or 8 and $w(15) = 6, 7$, or 8 using Lemma 3.3. Using $A_{7,19} \neq 0$ and $A_{5,21} \neq 0$, one obtains the remaining bounds $w(19) = 7, 8$, or 9 and $w(21) = 8, 9$, or 10. $\square$

3.3 $w(13) = w(15) = 8$

In this section we calculate $w(13)$ and $w(15)$. The idea is based on the following construction of cusp forms:

Let Γ be an even unimodular lattice of dimension m and $E \subset \Gamma \otimes \mathbb{C}$ an isotropic subspace of dimension n, $E = \langle e_1, \ldots, e_n \rangle$.

Then for even natural l and $Z \in \mathcal{H}_n$ in the Siegel upper half space we let

$$\Theta_{\Gamma,E,l}(Z) := \sum_{\gamma_1,\ldots,\gamma_n \in \Gamma} (\det(e_i, \gamma_j))^l \exp(\pi i \sum_{i,j=1}^{n} (\gamma_i, \gamma_j) Z_{ij})$$

Then $\Theta_{\Gamma,E,l}$ is a Siegel modular form of degree n and weight $l + \frac{m}{2}$ (cf. [7]).

Proposition 3.5 $w(13) = w(15) = 8$.

<u>Proof.</u> Let $X = P(W_8)$ be the projection of W_8 obtained by taking only the coefficients of the generating theta series that correspond to

$$E_8,\ A_8,\ D_8,\ E_7a,\ D_7c,\ D_7b,$$

where Gram matrices for the last three lattices are

$$E_7a := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 4 \end{pmatrix},\ D_7c := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 4 \end{pmatrix},$$

$$\text{and } D_7b := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 4 \end{pmatrix}$$

Then $\dim(X) \leq 4$ and X is spanned by $P(d_{17})$, $P(d_{18})$, and possibly $P(d_{13})$ and $P(d_{15})$, if $w(13) = w(15) = 8$. One easily checks that the four vectors $P(d_j)$ $(j = 13, 15, 17, 18)$ are linearly independent.

Let $\Gamma := E_8 \perp E_8$ be the decomposable even unimodular lattice of dimension 16. Let $x_1, \ldots, x_8 \in \Gamma$ be 8 pairwise orthogonal roots in the first copy of E_8 and $y_1, \ldots, y_8 \in \Gamma$ similar elements in the second copy. Define $e_j := x_j + iy_j$, $j = 1, \ldots, 8$, where $i = \sqrt{-1}$ and let $E := \langle e_1, \ldots, e_8 \rangle$. Then E is a totally isotropic subspace of $\Gamma \otimes \mathbb{C}$. Hence $\Theta := \Theta_{\Gamma,E,4}$ lies in W_8. Therefore $P(\Theta)$ can be written uniquely as a linear combination of $P(d_{13})$, $P(d_{15})$, $P(d_{17})$ and $P(d_{18})$. One checks that the coefficients of the first three basis vectors in this linear combination are not zero. Therefore d_{13} and d_{15} are in W_8 and the proposition follows.

The calculation of $P(\Theta)$:
The coefficients of Θ that correspond to E_8, A_8 and D_8 can be calculated easily, since all sublattices of Γ isometric to one of these three lattices already lie in one of the orthogonal summands E_8 of Γ. Since E_8 has exactly one sublattice isometric to E_8, 135 sublattices D_8 and 960 sublattices A_8, one calculates the first three coefficients of $P(\Theta)$ to be $2 \cdot 2^{16} = 131072$, $270 \cdot 2^4 \cdot 2^{16} = 283115520$, respectively $1920 \cdot 3^4 \cdot 2^{16} = 10192158720$. In the other 3 cases, either the sublattice lies in one of the two orthogonal summands E_8 or the basis vector of length 4 is a sum $v_1 + v_2$, where v_j is a root in the jth orthogonal summand E_8 of Γ $(j = 1, 2)$. Since the determinants of the Gram matrices are 5, 9, respectively 12, only the lattice D_7c is a sublattice of E_8 (giving a summand $2 \cdot 1080 \cdot 3^4 \cdot 2^{16}$ of the corresponding coefficient). To treat the other cases, one fixes 8 vectors in E_8 with inner product matrix $A - diag(0, 0, 0, 0, 0, 0, 0, 2)$, where A is one of the 3 matrices above. If the 8. vector corresponds to v_1, one has to go through all 240 possibilities for v_2 and sum up the fourth powers of the corresponding determinants (e.g. with maple). The result must be multiplied with the number of different sublattices obtained in this way, i.e. with $2 \cdot 240$ in the case E_7a, $2 \cdot 1080$ in the case D_7c respectively $2 \cdot 1080 \cdot 2$ in the case D_7b, where one has 2 possibilities for v_1. In total one obtains $P(\Theta) =$

$$131072 \quad 283115520 \quad 10192158720 \quad -10569646080 \quad -76865863680 \quad -380507258880\ .$$

$\square$

Since $d_2 \circ d_{12}$ is a linear combination of d_9, d_{12}, d_{13} and d_{15} with nonzero coefficients at d_{13} and d_{15} the fact that $w(13) = w(15) = 8$ and $w(2) = 1$ now implies that $w(12) = 7$.

Corollary 3.6 *$w(12) = 7$.*

Summarizing we get the following theorem.

Theorem 3.7 *The dimensions of the spaces of cusp forms S'_n generated by Siegel theta series of weight* 12 *are*

n	0	1	2	3	4	5	6	7	8	9	10	11	12
dim	1	1	1	1	2	2	3	$3-4$	$4-6$	$1-3$	$1-2$	1	1

From [2] follows that the space of Siegel cusp forms of degree 5 is generated by Siegel theta series. Therefore we have the following corollary:

Corollary 3.8 *The dimension of the space of cusp forms of degree* 5 *weight* 12 *is* $\dim(S_5) = 2$.

Since $d_2 \circ d_{19}$ involves d_{21} with a non zero coefficient, one can also show that $w(21) = 10$ implies that $w(19) = 9$.

Conjecture $w(21) = 10$.

Theorem 3.9 *If the conjecture is true then* $\dim(S'_n)$ *is as follows:*

n	0	1	2	3	4	5	6	7	8	9	10	11	12
dim	1	1	1	1	2	2	3	3	4	2	2	1	1

3.4 Open questions.

1) Prove the conjecture above.

2) Which of the eigenforms d_i satisfy the Ramanujan conjecture? Note that d_3, d_5, d_{11}, d_{13}, and d_{24} are Ikeda lifts from elliptic modular forms (see [8]) so they do not satisfy this conjecture.

21253	1470942	0	4194304	2704156	0	0	0	0	0	0	0
135	48960	12870	0	491400	1966080	1921920	2211840	1737450	0	0	0
0	54675	43605	0	0	0	5832000	0	2460375	0	0	0
25	0	0	65780	0	1081575	0	2042975	0	5200300	0	0
1	1980	0	0	68558	0	243936	901120	980100	2097152	3244032	0
0	153	0	1296	0	77022	43758	192780	0	616896	2951676	0
0	126	90	0	3969	36864	128808	51840	357210	0	2774016	395136
0	18	0	256	1820	20160	6435	80628	115830	465920	1035216	0
0	3	1	0	420	0	9408	24576	112959	0	688128	0
0	0	0	26	169	2574	0	18590	0	167388	267696	0
0	0	0	0	27	1272	1419	4266	13365	27648	280296	19404
0	0	0	0	0	0	4374	0	0	0	419904	86616
0	0	0	0	0	90	90	1060	2025	11680	60300	0
0	0	0	0	3	0	360	0	5400	0	73728	10368
0	0	0	0	0	27	0	243	729	6804	18144	0
0	0	0	0	0	0	1	20	240	512	10704	784
0	0	0	0	0	0	0	0	0	294	1764	0
0	0	0	0	0	0	0	0	0	0	180	12
0	0	0	0	0	0	0	0	45	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	853776	0	0	0	0	0	0	0	0	0	0
2756754	0	1750320	0	0	0	0	0	0	0	0	0
2322432	1666980	0	653184	0	0	0	0	0	0	0	0
3395392	0	1647360	1621620	0	0	0	0	0	0	0	0
1376256	658560	1048576	4128768	0	0	343000	0	0	0	0	0
1492777	0	1840410	1656369	2944656	0	0	0	0	0	0	0
795960	174636	506880	3576474	1824768	1164240	0	0	0	0	0	0
0	531441	0	5668704	0	1679616	0	0	0	0	0	0
328990	44100	446400	1655640	2188800	3016440	0	635040	0	0	0	0
245760	197620	0	2557440	1048576	3686400	405000	0	160000	0	0	0
210924	0	281556	918540	2286144	2667168	0	2000376	0	0	0	0
65408	18130	76800	891828	974848	3104640	122500	2634240	490000	0	0	0
27930	2401	61740	314874	884296	2355381	0	3241350	1500625	0	0	0
6156	1350	11520	160380	376704	1665828	50625	3110400	2767500	240000	0	0
0	6480	0	276480	0	2211840	172098	2359296	3317760	0	46656	0
375	0	2500	39375	150000	900000	15625	2532780	3750000	1000000	0	0
0	7	0	3024	28672	330624	9072	1548288	4043560	2322432	104976	0
0	0	0	0	0	40095	0	577368	3247695	3994056	531441	0
0	0	0	0	0	0	1771	0	1457280	5275648	1651860	4096
0	0	0	0	0	0	0	0	0	0	8292375	98280

The Hecke operator $K = K_{2,1}$.

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