Symmetrizations of quadratic and hermitian forms

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Abstract.
The paper develops elementary linear algebra methods to compute the determinants of the tensor symmetrizations of quadratic and hermitian forms over fields of good characteristic. Explicit results are given for the partitions (n), (1^n), $(2,1^{n-2})$ and $(3,1^{n-3})$ as well as for all partitions of $n \leq 7$. For orthogonal groups these symmetrizations are not irreducible and we continue to find the determinants of their irreducible constituents, the refined symmetrizations, over fields of characteristic 0.
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1 Introduction

Let K be a field and V be a vector space over K of finite dimension, say, N. For $n \in \mathbb{N}$ the symmetric group S_n acts linearly on the nth tensor power $\otimes^n V$ by permuting the tensor factors thus turning $\otimes^n V$ into a KS_n-module. To avoid trivialities we assume that $n \geq 2$ and to avoid problems we assume that the characteristic of K is either 0 or $> n$. It is well known that the simple KS_n-modules are labelled by the partitions λ of n. A formula for a primitive idempotent $\tilde{e}_\lambda \in KS_n$ is given in [5, Theorem 3.1.10]. The direct summand

$$\mathrm{Sym}_\lambda(V) := \tilde{e}_\lambda(\otimes^n V)$$

is known as the λ-symmetrization of V. Its dimension $d(\lambda, N) := \dim(\mathrm{Sym}_\lambda(V))$ is the number of semi-standard λ-tableaux with content contained in $\{1, \ldots, N\}$ (see [5, Theorem 5.2.14]).

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Now let $B : V \times V \to K$ be a symmetric bilinear or Hermitian sesquilinear form on V. Then B defines a respective form $\otimes^n B$ on $\otimes^n V$ by

$$\otimes^n B(v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n) := \prod_{i=1}^{n} B(v_i, w_i).$$

We put $\mathrm{Sym}_\lambda(B)$ to denote the restriction of $\otimes^n B$ to $\mathrm{Sym}_\lambda(V)$.

One important invariant of B is its determinant $\det(B)$ which is defined to be the class of the determinant of a Gram matrix of B in $K/(K^\times)^2$ for symmetric bilinear forms and in $F/N_{K/F}(K^\times)$ for K/F Hermitian forms.

The main result of this note is a formula for the determinant of $\mathrm{Sym}_\lambda(B)$.

Theorem 1.1. $\det(\mathrm{Sym}_\lambda(B)) = c(\lambda, N) \det(B)^{d(\lambda,N)n/N}$, *where* $c(\lambda, N)$ *can be computed using combinatorial algorithms in* S_n.

Explicit formulas for $c(\lambda, N)$ have been obtained for symmetric and exterior powers of symmetric bilinear forms (where $\lambda = (n)$ respectively $\lambda = 1^n$) in [7] and [6]. In this paper we additionally derive such formulas for the two hooks $\lambda = (2, 1^{n-2})$ and $\lambda = (3, 1^{n-3})$. For small n more results are obtained by computer (see Table 5 for $n \leq 7$). These are helpful to compute determinants of even degree unitary characters as illustrated in Section 6, where the symmetrizations of the 12-dimensional unitary representation of $6.Suz$ are used to obtain most of the determinants of the faithful, simple $\mathbb{Q}[\sqrt{-3}](6.Suz)$-modules.

For symmetric bilinear forms B, the symmetrizations are usually not irreducible modules for the orthogonal group $O(B)$. The last section investigates certain $O(B)$-invariant submodules of $\mathrm{Sym}_\lambda(B)$, the refined symmetrizations. For $n \leq 6$ a table that can be used to compute their determinants is given in Section 7.4. As an application we obtain some orthogonal determinants for the sporadic simple Conway group Co_1, that were not contained in the database described in [1, BBBNP23] yet.

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2 Notation

We use the standard notation as given in the textbook [5]. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of a natural number n is a sequence of integers λ_i with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. It is visualised by its Young diagram with k rows and λ_i boxes in row i. A Young tableau t^λ is obtained by labelling these n boxes of the Young diagram by the n numbers in $\{1, \ldots, n\}$. Such a Young tableau defines two partitions of $\{1, \ldots, n\}$, the horizontal partition $\dot\bigcup_{j=1}^{k} P_j = \{1, \ldots, n\}$ where

P_j consists of the λ_j labels in row number j and a corresponding vertical partition given by the labels in the λ_1 columns of the Young tableau. The horizontal group H^λ of t^λ is the stabiliser in S_n of all sets in the horizontal partition and the vertical group G^λ the stabiliser of all sets in the vertical partition.

With this notation the formula for a primitive idempotent $\tilde{e}_\lambda$ from [5, Theorem 3.1.10] is $\tilde{e}_\lambda = \frac{d_\lambda}{n!} e_\lambda$ with

$$e_\lambda = \sum_{\sigma \in G^\lambda} \sum_{\rho \in H^\lambda} \mathrm{sign}(\sigma)\sigma\rho.$$

An N-semi-standard Young tableau t of shape λ is given by filling elements in $\{1, \ldots, N\}$ into the boxes of the Young diagram of λ such that they are non-decreasing along the rows and strictly increasing along the columns. The content $\mathrm{cont}(t)$ of t is the multi-set of its entries in $\{1, \ldots, N\}$. Put

$$T(\lambda, N) := \{t \mid t \text{ in an } N\text{-semi-standard Young tableau of shape } \lambda \}.$$

The following result is well known:

Remark 2.1. Let $d(\lambda, N) := |T(\lambda, N)|$ denote the number of N-semi-standard Young tableaux of shape λ. Then each element of $\{1, \ldots, N\}$ occurs with the same multiplicity in the union of the contents of the elements of $T(\lambda, N)$ and hence

$$\bigcup_{t \in T(\lambda,N)} \mathrm{cont}(t) = \{1^{d(\lambda,N)n/N}, \ldots, N^{d(\lambda,N)n/N}\}.$$

3 Proof of the main theorem

Let $(v_1, \ldots, v_N)$ be an orthogonal basis of (V, B) and put $a_i := B(v_i, v_i)$. Then $\det(B)$ is represented by $a_1 \cdots a_n$.

Remark 3.1. The pure tensors $v_{i_1} \otimes \ldots \otimes v_{i_n}$ form an orthogonal basis of $(\otimes^n V, \otimes^n B)$ with

$$\otimes^n B(v_{i_1} \otimes \ldots \otimes v_{i_n}, v_{i_1} \otimes \ldots \otimes v_{i_n}) = \prod_{j=1}^{n} a_{i_j}.$$

Fix a partition λ of n and fix a Young tableau t^λ to label the positions in $\otimes^n V$. Then any $t \in T(\lambda, N)$ defines a pure tensor

$$v(t) := w_1 \otimes \ldots \otimes w_n \in \otimes^n V$$

where $w_i = v_j$, if the box that is labelled by i in t^λ has content j in t.

Remark 3.2. Let $t \in T(\lambda, N)$ and $C := \mathrm{cont}(t)$ be its content. Then

$$\otimes^n B(v(t), v(t)) = \prod_{i \in C} a_i =: q(C)$$

only depends on C.

Lemma 3.3. *([4, Section 4.3], [5, Section 5.2]) A basis of* $\mathrm{Sym}_\lambda(V)$ *is given by*

$$\{e_\lambda(v(t)) \mid t \in T(\lambda, N)\}.$$

As the summands in $e_\lambda(v(t))$ are (up to a sign) pure tensors of the basis elements v_i with the same multi-set of indices, the value

$$\otimes^n B(e_\lambda(v(t)), e_\lambda(v(t))) = c(t)q(C)$$

is a constant multiple of $q(C)$ where $C = \mathrm{cont}(t)$.

The following trivial remark is fundamental for the computations:

Remark 3.4. For $t, s \in T(\lambda, N)$ we get

$$\otimes^n B(e_\lambda(v(t)), e_\lambda(v(s))) = 0$$

unless $\mathrm{cont}(t) = \mathrm{cont}(s)$.

Ordering the basis from Lemma 3.3 according to the content C of t we hence obtain a Gram matrix of $\mathrm{Sym}_\lambda(B)$ as a block diagonal matrix

$$\mathrm{diag}(q(C)X_C : C \in \mathrm{cont}(T(\lambda, N)))$$

and hence Theorem 1.1 follows with

$$c(\lambda, N) = \prod_{C \in \mathrm{cont}(T(\lambda,N))} \det(X_C).$$

For computing the Gram matrices X_C I wrote a small ad hoc program. Up to $n = 7$, the results are given in Table 5.

4 Small examples

4.1 The partition $(2, 1)$

This partition is the smallest case that it not yet available in the literature. To illustrate the algorithm used to compile Table 5 I explain these computations in detail. The formula in [5, Theorem 5.2.14] yields

$$\dim(\mathrm{Sym}_{(2,1)}(V)) = \frac{1}{3}N(N-1)(N+1), \text{ where } N = \dim(V).$$

So

$$\det(\mathrm{Sym}_{(2,1)}(B)) = \det(B)^{(N-1)(N+1)} c((2,1),N)$$

for some $c((2,1),N)$.

Proposition 4.1. *$c((2,1),N) = 3^{\binom{N}{3}} z^2$ for some $z \in \mathbb{N}$ with prime divisors in $\{2,3\}$.*

Proof. We give the N-semi-standard Young tableaux of shape $(2,1)$ by listing the entries in positions $((1,1),(1,2),(2,1))$ in this ordering.

Choosing $a < b < c$, $a,b,c \in \{1,\dots,N\}$ there are 4 sorts of N-semi-standard $(2,1)$-tableaux:

$$\{(a,a,b),(a,b,b),(a,b,c),(a,c,b)\}$$

Then

$$\begin{array}{rcl} e_{(2,1)}(a,a,b) &=& 2((a,a,b)-(b,a,a)) \\ e_{(2,1)}(a,b,b) &=& (a,b,b)-(b,b,a) \\ e_{(2,1)}(a,b,c) &=& (a,b,c)-(c,b,a)+(b,a,c)-(c,a,b) \\ e_{(2,1)}(a,c,b) &=& (a,c,b)-(b,c,a)+(c,a,b)-(b,a,c) \end{array}$$

giving rise to

$$\begin{array}{ll} X_{a^2b} = (8) & \text{with multiplicity } \binom{N}{2} \\ X_{ab^2} = (2) & \text{with multiplicity } \binom{N}{2} \\ X_{abc} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} & \text{with multiplicity } \binom{N}{3} \end{array}$$

yielding

$$\det(\mathrm{Sym}_{(2,1)}(B)) = 16^{\binom{N}{2}} 12^{\binom{N}{3}} \det(B)^{(N-1)(N+1)}.$$

□

4.2 Two partitions of 4

Proposition 4.2.

$$\dim(\mathrm{Sym}_{(3,1)}(V)) = \frac{1}{8}(N+2)(N+1)N(N-1) = 3\binom{N+2}{4}$$

and, up to squares,

$$\det(\mathrm{Sym}_{(3,1)}(B)) = 2^{\binom{N}{3}} \det(B)^{(N+2)(N+1)(N-1)/2}.$$

Proof. For $\lambda = (3,1)$ we compute the relevant X_C, $C \in \text{cont}(T(\lambda, N)$ as in the previous section and obtain:

$$X_{a^3b} = (72) \qquad \text{with multiplicity} \quad \binom{N}{2}$$
$$X_{a^2b^2} = (16) \qquad \text{with multiplicity} \quad \binom{N}{2}$$
$$X_{ab^3} = (8) \qquad \text{with multiplicity} \quad \binom{N}{2}$$
$$X_{a^2bc} = \begin{pmatrix} 24 & -8 \\ -8 & 24 \end{pmatrix} \qquad \text{with multiplicity} \quad \binom{N}{3}$$
$$X_{ab^2c} = \begin{pmatrix} 24 & -8 \\ -8 & 8 \end{pmatrix} \qquad \text{with multiplicity} \quad \binom{N}{3}$$
$$X_{abc^2} = \begin{pmatrix} 24 & -8 \\ -8 & 8 \end{pmatrix} \qquad \text{with multiplicity} \quad \binom{N}{3}$$
$$X_{abcd} = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{pmatrix} \qquad \text{with multiplicity} \quad \binom{N}{4}$$

Up to squares $c(\lambda, N) = \det(X_{a^2bc})^{\binom{N}{3}}$. □

Proposition 4.3.

$$\dim(\text{Sym}_{(2,2)}(V)) = \frac{1}{12}(N^4 - N^2)$$

and, up to squares,

$$\det(\text{Sym}_{(2,2)}(B)) = 3^{\binom{N}{4}} 2^{\binom{N}{3}} \det(B)^{N(N+1)(N-1)/3}$$

Proof. For $\lambda = (2,2)$ we obtain

$$X_{a^2b^2} = (64) \qquad \text{with multiplicity} \quad \binom{N}{2}$$
$$X_{a^2bc} = (32) \qquad \text{with multiplicity} \quad \binom{N}{3}$$
$$X_{ab^2c} = (8) \qquad \text{with multiplicity} \quad \binom{N}{3}$$
$$X_{abc^2} = (32) \qquad \text{with multiplicity} \quad \binom{N}{3}$$
$$X_{abcd} = \begin{pmatrix} 16 & -8 \\ -8 & 16 \end{pmatrix} \qquad \text{with multiplicity} \quad \binom{N}{4}$$

Up to squares $c(\lambda, N) = \det(X_{ab^2c})^{\binom{N}{3}} \det(X_{abcd})^{\binom{N}{4}}$. $\square$

4.3 The partitions (n) and (1^n)

The module $\mathrm{Sym}_{(1^n)}(V) = \Lambda^n(V)$ is better known as the n-th exterior power of V, whereas $\mathrm{Sym}_{(n)}(V)$ is the n-th symmetric power of V.

Proposition 4.4. *(see also [6])* $\dim(\mathrm{Sym}_{(1^n)}(V)) = \binom{N}{n}$ *and*

$$\det(\mathrm{Sym}_{(1^n)}(B)) = (n!)^{\binom{N}{n}} \det(B)^{\binom{N-1}{n-1}}.$$

Proof. Let $\lambda := (1^n)$. Then the horizontal group $H^\lambda = \{1\}$ and the vertical group $G^\lambda = S_n$. Moreover the N-semi-standard tableaux of shape λ are in bijection with the n-element subsets of $\{1, \ldots, N\}$, in particular

$$\dim(\mathrm{Sym}_\lambda(V)) = |T(\lambda, N)| = \binom{N}{n}$$

and any element $t \in T(\lambda, N)$ is uniquely determined by its content $C = \mathrm{cont}(t)$. So X_C is a scalar and equals to the length of the G^λ-orbit of t, which shows that

$$\otimes^n B(e_\lambda(v(t)), e_\lambda(v(t))) = n! q(C).$$

So

$$\det(\mathrm{Sym}_{(1^n)}(B)) = (n!)^{\binom{N}{n}} \prod_{1 \le i_1 < \ldots < i_n \le N} a_{i_1} \cdots a_{i_n}.$$

As each element $i \in \{1, \ldots, N\}$ occurs $\binom{N-1}{n-1}$ times in an n-element subset of $\{1, \ldots, N\}$ the latter product is just $\prod_{i=1}^N a_i^{\binom{N-1}{n-1}} = \det(B)^{\binom{N-1}{n-1}}$. $\square$

To state the result for $\lambda = (n)$ we introduce the set

$$\mathrm{Comp}(n, k) := \{(x_1, \ldots, x_k) \in \mathbb{Z}_{>0}^k \mid x_1 + \ldots + x_k = n\}$$

of all compositions of n.

Proposition 4.5. *(see also [7, Proposition 3.9])* $\dim(\mathrm{Sym}_{(n)}(V)) = \binom{N+n-1}{n}$ *and*

$$\det(\mathrm{Sym}_{(n)}(B)) = \prod_{k=1}^{n} \prod_{(x_1,\ldots,x_k) \in \mathrm{Comp}(n,k)} \left(\frac{n!}{(x_1!) \cdots (x_k!)}\right)^{\binom{N}{k}} \det(B)^{\binom{N+n-1}{n-1}}.$$

Proof. For $\lambda = (n)$ the vertical group of λ is trivial and the horizontal group $H^\lambda = S_n$. Now the N-semi-standard tableaux of shape λ are in bijection with the n-element multi-subsets of $\{1, \ldots, N\}$, in particular

$$\dim(\mathrm{Sym}_\lambda(V)) = |T(\lambda, N)| = \binom{N+n-1}{n}.$$

Again any element $t \in T(\lambda, N)$ is uniquely determined by its content $C = \mathrm{cont}(t)$ and X_C is a scalar. Instead of working with $e_\lambda v(t)$ we divide this vector by the order of the stabiliser of t and hence just work with the orbit sum

$$\sum\{v(\sigma(t)) \mid \sigma \in S_n\}.$$

This results in multiplying X_C be an integral square. If $C = \{n_1^{x_1}, \ldots, n_k^{x_k}\}$ with $n_1 \leq \ldots, \leq n_k$ then $X_C = \frac{n!}{(x_1!)\cdots(x_k!)}$. Therefore

$$c(\lambda, N) = \prod_{k=1}^{n} \prod_{(x_1,\ldots,x_k)\in\mathrm{Comp}(n,k)} (\frac{n!}{(x_1!)\cdots(x_k!)})^{\binom{N}{k}}.$$

□

4.4 A result for hooks

Another situation, where X_C can be computed for general λ and N, is if C is a subset of $\{1, \ldots, N\}$ or cardinality n, i.e. every element of the multi-set C occurs in C with multiplicity 1.

Lemma 4.6. *Assume that $\lambda = (\ell, 1^{n-\ell})$ is a hook. If C is multiplicity free then $\langle e_\lambda v(t) \mid t \in T(\lambda, N), \mathrm{cont}(t) = C\rangle \cong S^{\lambda'}$ is the irreducible representation of S_n isomorphic to the Specht module associated with the transposed partition $\lambda' = (n-\ell+1, 1^{\ell-1})$ and X_C is a Gram matrix of an S_n-invariant symmetric bilinear form on $S^{\lambda'}$ with*

$$\det(X_C) = ((\ell-1)!(n-\ell+1)!)^{\binom{n-\ell}{\ell-1}} n^{\binom{n-1}{\ell-1}}.$$

Proof. Let $C \subseteq \{1, \ldots, N\}$ with $|C| = n$ and let a be the minimal element of C. Then the semi-standard λ-tableaux with content C are the $\binom{n-\ell}{\ell-1}$ elements of

$$\{t_S \mid S \subseteq C \setminus \{a\}, |S| = \ell - 1\}$$

where the elements of the first row of t_I form the set $S \cup \{a\}$. As all summands in $e_\lambda v(t_S)$ are distinct we have $\otimes^n B(e_\lambda v(t_S), e_\lambda v(t_S)) = \ell!(n-\ell+1)!$. For $R \neq S \subseteq C \setminus \{a\}$ the common summands of $e_\lambda v(t_R)$ and $e_\lambda v(t_S)$ are those $v(t)$, where the set of elements in the first column of t are identical, i.e. the last $\ell-1$ elements of the first row are $\{a\}\cup(R\cap S)$, i.e. we only obtain non-zero inner product if $|R\cap S|=\ell-2$. So

$$\otimes^n B(e_\lambda v(t_S), e_\lambda v(t_R)) = \left\{ \begin{array}{ll} \ell!(n-\ell+1)! & \text{if } S=R \\ \pm(\ell-1)!(n-\ell+1)! & \text{if } |R\cap S| = \ell-2 \\ 0 & \text{if } |R\cap S| < \ell-2 \end{array} \right)$$

where the sign depends on the sign of the permutation $\pi_{R,S}$ mapping $[x,b,c,d,\ldots]$ to $[y,b,c,d,\ldots]$ where $S\setminus R=\{x\}$ and $R\setminus S=\{y\}$.

To compute the $(\ell-1)$st exterior power of the root lattice A_{n-1} we choose a basis

$$A_{n-1} = \langle b_2 := e_1 - e_2, \ldots, e_1 - e_n =: b_n \rangle$$

such that the Gram matrix of $(b_2,\ldots,b_n)$ is $I_{n-1}+J_{n-1}$ of determinant n. To an $\ell-1$ element subset $R=\{i_1,\ldots,i_{\ell-1}\}$ of $C\setminus\{1\}=\{2,\ldots,n\}$ we associate the basis vector $b_R = b_{i_1}\wedge\ldots\wedge b_{i_{\ell-1}}$. So in $\Lambda^{\ell-1}(A_{n-1})$ we obtain the inner product of the subsets R and S as the $\ell-1\times\ell-1$-minor

$$\det(I_{n-1}+J_{n-1})_{R\times S} = \pm\det(\begin{pmatrix} 0 & 0 \\ 0 & I_{R\cap S} \end{pmatrix} + J_{\ell-1}) = \left\{ \begin{array}{ll} \ell & \text{if } R=S \\ \mathrm{sign}(\pi_{R,S}) & \text{if } |R\cap S| = \ell-2 \\ 0 & \text{if } |R\cap S| < \ell-2. \end{array} \right.$$

So the lattice

$$(\mathbb{Z}^{\binom{n-\ell}{\ell-1}}, ((\ell-1)!(n-\ell+1)!)^{-1}X_C) \cong \Lambda^{\ell-1}(A_{n-1})$$

and hence

$$\det(X_C) = ((\ell-1)!(n-\ell+1)!)^{\binom{n-\ell}{\ell-1}} n^{\binom{n-1}{\ell-1}}.$$

□

4.5 The partitions $(2,1^{n-2})$

Proposition 4.7. *Let* $\lambda := (2,1^{n-2})$. *Then*

$$\dim(\mathrm{Sym}_\lambda(V)) = (n-1)\binom{N}{n} + (n-1)\binom{N}{n-1} = (n-1)\binom{N+1}{n}$$

and

$$c(\lambda,N) = n^{\binom{N}{n}}((n-1)!)^{(n-1)\binom{N+1}{n}}.$$

Proof. For $t\in T(\lambda,N)$ the content C of t has either $n-1$ or n distinct elements. If one element occurs twice in C, then this is either the minimal element of C and $t=(a,a,b,c,d,\ldots)$ such that

$$e_\lambda v(t) = 2\sum_{g\in S_{n-1}} \mathrm{sign}(g) v(gt)$$

or it is not the minimal element and the tableau is $s = (a, x, b, c, d, \ldots)$ where $x \in \{b, c, d, \ldots\}$ and then

$$e_\lambda v(s) = \sum_{g \in S_{n-1}} \mathrm{sign}(g) v(gs).$$

In both cases the tableau is uniquely determined by its content. We compute

$$B(e_\lambda v(s), e_\lambda v(s)) = B(1/2 e_\lambda v(t), 1/2 e_\lambda v(t)) = (n-1)! q(C).$$

This gives a contribution of $(n-1)\binom{N}{n-1}$ to $\dim(\mathrm{Sym}_\lambda(V))$ and of

$$(n-1)!^{(n-1)\binom{N}{n-1}} \text{ to } c(\lambda, N).$$

If every element in $C = \{a, b, c, d, \ldots\}$ occurs with multiplicity 1, then Lemma 4.6 shows that $X_C \cong (n-1)! A_{n-1}$. In total these C contribute $(n-1)\binom{N}{n}$ to the dimension and

$$(n(n-1)!^{(n-1)})^{\binom{N}{n}} \text{ to } c(\lambda, N).$$

□

4.6 The partitions $(3, 1^{n-3})$

Proposition 4.8. *Let* $\lambda := (3, 1^{n-3})$. *Then*

$$\dim(\mathrm{Sym}_\lambda(V)) = \binom{n-1}{2}(\binom{N}{n} + 2\binom{N}{n-1} + \binom{N}{n-2})$$

and

$$c(\lambda, N) = (n-2)!^x 2^y n^z$$

where

$$\begin{aligned} x &= (\tbinom{N}{n-2} + \tbinom{N}{n})\tbinom{n-1}{2}(n-2)\tbinom{N}{n-1} \\ y &= \tbinom{N}{n-2}\tbinom{n-2}{2} + (n-3)\tbinom{N}{n-1} + \tbinom{N}{n})\tbinom{n-1}{2} \\ z &= \tbinom{N}{n-1} + (n-2)\tbinom{N}{n}. \end{aligned}$$

Proof. For $t \in T(\lambda, N)$ the content C of t has either $n-2$, $n-1$ or n distinct elements.

If one element occurs three times in C, then this is either the minimal element of C and $t = (a, a, a, b, c, d, \ldots)$ such that

$$e_\lambda v(t) = 6 \sum_{g \in S_{n-2}} \mathrm{sign}(g) v(gt)$$

or it is not the minimal element and the tableau is $s = (a, x, x, b, c, d, \ldots)$ where $x \in \{b, c, d, \ldots\}$ and then

$$e_\lambda v(s) = 2 \sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gs).$$

In both cases the tableau is uniquely determined by its content. We compute

$$B(1/2 e_\lambda v(s), 1/2 e_\lambda v(s)) = B(1/6 e_\lambda v(t), 1/6 e_\lambda v(t)) = (n-2)! q(C)$$

So we obtain a contribution of $\binom{N}{n-2}(n-2)$ to the dimension and of

$$(n-2)!^{\binom{N}{n-2}(n-2)} \text{ to } c(\lambda, N).$$

If two elements occur twice in C, then either one of them is the minimal element of C and $t = (a, a, x, b, c, d, \ldots)$ where $x \in \{b, c, d, \ldots\}$. Then

$$e_\lambda v(t) = 2\Big(\sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gt) + \sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gt')\Big)$$

where $t' = (a, x, a, b, c, d, \ldots)$. Or $s = (a, x, y, b, c, d, \ldots)$ where $x < y \in C \setminus \{a\}$ and

$$e_\lambda v(s) = \sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gs) + \sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gs')$$

where $s' = (a, y, x, b, c, d, \ldots)$. We compute

$$B(1/2 e_\lambda v(t), 1/2 e_\lambda v(t)) = B(e_\lambda v(s), e_\lambda v(s)) = 2(n-2)! q(C)$$

So we obtain a contribution of $\binom{N}{n-2}\binom{n-2}{2}$ to the dimension and of

$$(2(n-2)!)^{\binom{N}{n-2}\binom{n-2}{2}} \text{ to } c(\lambda, N).$$

If C consists of $n-1$ distinct elements, then one element of C occurs twice. If this is the minimal element of C, then $t = (a, a, x, b, c, d, \ldots)$ where $x \notin \{a, b, c, d, \ldots\}$. Then

$$e_\lambda v(t) = 2\Big(\sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gt) + \sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gt') + \sum_{g \in S_{n-2}} \operatorname{sign}(g) v(gt'')\Big)$$

where $t' = (a, x, a, b, c, d, \ldots)$ and $t'' = (x, a, a, b, c, d, \ldots)$. We compute

$$B(1/2 e_\lambda v(t), 1/2 e_\lambda v(t)) = 3(n-2)!.$$

There are in total $n-2 = |\{x, b, c, d, \ldots\}|$ semi-standard λ tableaux with the same content. The inner product of distinct such tableaux is $\pm(n-2)!$ depending on the product of the signs of the permutations sorting the respective $[x, b, c, d, \ldots]$. So the determinant of X_C is

$$(n-2)!^{n-2}(\det(2I_{n-2} + J_{n-2})) = (n-2)!^{n-2}2^{n-3}n$$

adding $(n-2)\binom{N}{n-1}$ to the dimension and a factor

$$((n-2)!^{n-2}2^{n-3}n)^{\binom{N}{n-1}} \text{ to the determinant.}$$

If a non-minimal element occurs twice in C, then the tableau is either

$$t = (a, x, x, b, c, d, \ldots) \text{ or } s = (a, \min(x, y), \max(x, y), b, \ldots, x, \ldots)$$

We get

$$B(1/2e_\lambda v(t), 1/2e_\lambda v(t)) = 3(n-2)!, \ B(e_\lambda v(s), e_\lambda v(s)) = 4(n-2)!,$$

In total there are $n-2$ such semi-standard tableaux with the same content C, one tableau t and $(n-3)$ possibilities for tableaux of type s depending on the choice of $y \in \{b, c, d, \ldots\}$. For such s, s' we compute

$$B(1/2e_\lambda v(t), e_\lambda v(s)) = \pm 2(n-2)! = B(e_\lambda v(s), e_\lambda v(s')).$$

where the sign is the signum of the permutation mapping $[x, b, c, d, \ldots]$ to $[y, b, c, d, \ldots]$ respectively $[y, b, c, d, \ldots]$ to $[y', b, c, d, \ldots]$. So after multiplying those basis vectors $e_\lambda v(s)$ by -1 that have $B(1/2e_\lambda v(t), e_\lambda v(s)) = -2(n-2)!$ we obtain

$$X_C = (n-2)! \begin{pmatrix} 3 & 2 & \ldots & 2 \\ 2 & 4 & \ldots & 2 \\ \vdots & \ddots & \ddots & \vdots \\ 2 & \ldots & 2 & 4 \end{pmatrix}$$

of determinant $(n-2)!^{n-2}2^{n-3}n$. Such C contribute in total $(n-2)\binom{N}{n-1}$ to the dimension and

$$((n-2)!^{n-2}2^{n-3}n)^{\binom{N}{n-1}} \text{ to } c(\lambda, N).$$

If every element in $C = \{a, b, c, d, \ldots\}$ occurs with multiplicity 1, then Lemma 4.6 yields that $X_C \cong 2(n-2)!\Lambda^2(A_{n-1})$ has determinant

$$(2(n-2)!)^{\binom{n-1}{2}}n^{n-2}.$$

So we obtain an additive contribution of $\binom{n-1}{2}\binom{N}{n}$ to the dimension and a multiplicative contribution of

$$((2(n-2)!)^{\binom{n-1}{2}}n^{n-2})^{\binom{N}{n}} \text{ to the determinant.}$$

□

5 Determinants of symmetrizations

This section gives tables of dimensions and determinants of $(\mathrm{Sym}_\lambda(V), \mathrm{Sym}_\lambda(B))$ for partitions λ of n and all $n \le 7$. Here (V, B) is either a non-degenerate symmetric bilinear or Hermitian space over a field of characteristic not dividing $n!$. For fixed partition λ of n the dimension $d(\lambda) := \dim(\mathrm{Sym}_\lambda(V))$ is a polynomial in $N := \dim(V)$. We also give $c(\lambda)$ such that $\det(\mathrm{Sym}_\lambda(B)) = c(\lambda)\det(B)^{d(\lambda)n/N}$, up to squares.

n	λ	$d(\lambda)$	$c(\lambda)$
2	(2)	$N(N+1)/2$	$2^{\binom{N}{2}}$
	(1^2)	$N(N-1)/2$	$2^{\binom{N}{2}}$
3	(3)	$N(N+1)(N+2)/6$	$6^{\binom{N}{3}}$
	$(2,1)$	$N(N-1)(N+1)/3$	$3^{\binom{N}{3}}$
	(1^3)	$N(N-1)(N-2)/6$	$6^{\binom{N}{3}}$
4	(4)	$N(N+1)(N+2)(N+3)/24$	$2^{\binom{N}{2}+\binom{N}{4}}3^{\binom{N}{2}+\binom{N}{3}+\binom{N}{4}}$
	$(3,1)$	$N(N-1)(N+1)(N+2)/8$	$2^{\binom{N}{3}}$
	(2^2)	$N^2(N-1)(N+1)/12$	$2^{\binom{N}{3}}3^{\binom{N}{4}}$
	$(2,1^2)$	$N(N-1)(N-2)(N+1)/8$	$6^{\binom{N}{3}+\binom{N}{4}}$
	(1^4)	$N(N-1)(N-2)(N-3)/24$	$6^{\binom{N}{4}}$
5	(5)	$N(N+1)(N+2)(N+3)(N+4)/120$	$6^{\binom{N}{3}+\binom{N}{5}}5^{\binom{N}{5}}$
	$(4,1)$	$N(N-1)(N+1)(N+2)(N+3)/30$	$3^{\binom{N}{3}}5^{\binom{N}{5}}$
	$(3,2)$	$N^2(N-1)(N+1)(N+2)/24$	$3^{\binom{N}{5}}$
	$(3,1^2)$	$N(N-1)(N-2)(N+1)(N+2)/20$	$2^{\binom{N}{3}}5^{\binom{N}{5}}$
	$(2^2,1)$	$N^2(N-1)(N-2)(N+1)/24$	$3^{\binom{N}{3}}6^{\binom{N}{5}}$
	$(2,1^3)$	$N(N-1)(N-2)(N-3)(N+1)/30$	$5^{\binom{N}{5}}$
	(1^5)	$N(N-1)(N-2)(N-3)(N-4)/120$	$30^{\binom{N}{5}}$

λ	$d(\lambda)$	$c(\lambda)$
(6)	$N(N+1)(N+2)(N+3)(N+4)(N+5)/720$	$2^{\binom{N}{5}}3^{\binom{N}{3}}5^{\binom{N}{2}+\binom{N}{5}+\binom{N}{6}}$
$(5,1)$	$N(N-1)(N+1)(N+2)(N+3)(N+4)/144$	$2^{\binom{N}{3}}3^{\binom{N}{2}+\binom{N}{5}+\binom{N}{6}}$
$(4,2)$	$N^2(N-1)(N+1)(N+2)(N+3)/80$	$2^{\binom{N}{2}+\binom{N}{6}}5^{\binom{N}{5}}$
$(4,1^2)$	$N(N-1)(N-2)(N+1)(N+2)(N+3)/72$	$6^{\binom{N}{3}}3^{\binom{N}{5}}$
(3^2)	$N^2(N-1)(N+1)^2(N+2)/144$	$2^{\binom{N}{2}+\binom{N}{6}}3^{\binom{N}{3}+\binom{N}{5}}$
$(3,2,1)$	$N^2(N-1)(N-2)(N+1)(N+2)/45$	$3^{\binom{N}{3}}5^{\binom{N}{5}}$
$(3,1^3)$	$N(N-1)(N-2)(N-3)(N+1)(N+2)/72$	$3^{\binom{N}{5}}$
(2^3)	$N^2(N-1)^2(N-2)(N+1)/144$	$3^{\binom{N}{5}}$
$(2^2,1^2)$	$N^2(N-1)(N-2)(N-3)(N+1)/80$	$3^{\binom{N}{5}}30^{\binom{N}{6}}$
$(2,1^4)$	$N(N-1)(N-2)(N-3)(N-4)(N+1)/144$	$5^{\binom{N}{6}}30^{\binom{N}{5}}$
(1^6)	$N(N-1)(N-2)(N-3)(N-4)(N-5)/720$	$5^{\binom{N}{6}}$

λ	$d(\lambda)$	$c(\lambda)$
(7)	$\binom{N+6}{7}$	$6^{\binom{N}{5}}5^{\binom{N}{5}+\binom{N}{7}}7^{\binom{N}{3}+\binom{N}{5}+\binom{N}{7}}$
$(6,1)$	$N(N-1)(N+1)(N+2)(N+3)(N+4)(N+5)/840$	$3^{\binom{N}{3}+\binom{N}{5}}5^{\binom{N}{3}}7^{\binom{N}{3}+\binom{N}{5}+\binom{N}{7}}$
$(5,2)$	$N^2(N-1)(N+1)(N+2)(N+3)(N+4)/360$	$2^{\binom{N}{3}}3^{\binom{N}{3}+\binom{N}{5}+\binom{N}{7}}5^{\binom{N}{3}}$
$(5,1^2)$	$N(N-1)(N-2)(N+1)(N+2)(N+3)(N+4)/336$	$3^{\binom{N}{3}+\binom{N}{5}}7^{\binom{N}{5}}$
$(4,3)$	$N^2(N-1)(N+1)^2(N+2)(N+3)/360$	$3^{\binom{N}{3}+\binom{N}{7}}5^{\binom{N}{5}+\binom{N}{7}}$
$(4,2,1)$	$N^2(N-1)(N-2)(N+1)(N+2)(N+3)/144$	$2^{\binom{N}{7}}3^{\binom{N}{3}+\binom{N}{5}+\binom{N}{7}}$
$(4,1^3)$	$N(N-1)(N-2)(N-3)(N+1)(N+2)(N+3)/252$	$21^{\binom{N}{5}}$
$(3^2,1)$	$N^2(N-1)(N-2)(N+1)^2(N+2)/240$	$2^{\binom{N}{3}+\binom{N}{7}}3^{\binom{N}{5}+\binom{N}{7}}$
$(3,2^2)$	$N^2(N-1)^2(N-2)(N+1)(N+2)/240$	$3^{\binom{N}{5}+\binom{N}{7}}5^{\binom{N}{7}}$
$(3,2,1^2)$	$N^2(N-1)(N-2)(N-3)(N+1)(N+2)/144$	$2^{\binom{N}{7}}3^{\binom{N}{5}+\binom{N}{7}}$
$(3,1^4)$	$N(N-1)(N-2)(N-3)(N-4)(N+1)(N+2)/336$	$30^{\binom{N}{5}}105^{\binom{N}{7}}$
$(2^3,1)$	$N^2(N-1)^2(N-2)(N-3)(N+1)/360$	$15^{\binom{N}{7}}$
$(2^2,1^3)$	$N^2(N-1)(N-2)(N-3)(N-4)(N+1)/360$	$3^{\binom{N}{7}}$
$(2,1^5)$	$N(N-1)(N-2)(N-3)(N-4)(N-5)(N+1)/840$	$7^{\binom{N}{7}}$
(1^7)	$\binom{N}{7}$	$35^{\binom{N}{7}}$

6 Unitary determinants of the Suzuki group

To illustrate the use of the determinants of the symmetrizations we give a small example. The covering group $6.Suz$ of the Suzuki group has a unitary representation of degree 12 over the field $K = \mathbb{Q}[\sqrt{-3}]$. This representation fixes a Hermitian form B of determinant 1. The symmetrizations of this 12-dimensional Hermitian form up to degree 7 allow to find the determinants of quite a few irreducible unitary characters of covers of the Suzuki group. Note that $\binom{12}{k}$ is even for $k = 1, 2, 3, 5, 6, 7, 9, 10, 11$ and odd for $k = 4, 8$.

partition	χ	$\chi(1)$	det
$(1,1)$	78	66	1
(2)	80	78	1
$(2,1)$	48	572	1
(3)	46	364	1
(2^2)	85	1716	3
(1^5)	$154 + 156$	$12 + 780$	1×1
$(2,1^3)$	$156 + 162$	$780 + 4368$	1×1
$(2^2,1)$	164	8580	1
$(3,1^2)$	174	12012	1
$(3,2)$	170	12012	1
$(4,1)$	172	12012	1
(5)	160	4368	1
$(4,1^2)$	21	50050	1
(1^7)	$153 + 155$	$12 + 780$	1×1
$(2,1^5)$	$153 + 155 + 157 + 163$	$12 + 780 + 924 + 8580$	1
$(2^2,1^3)$	$155 + 161 + 163 + 179$	$780 + 4368 + 8580 + 27456$	1
$(2^3,1)$	$155 + 161 + 163 + 185$	$780 + 4368 + 8580 + 42900$	1
$(3,1^4)$	$157 + 175 + 179$	$924 + 23100 + 27456$	1
$(3,2,1^2)$	$163 + 179 + 199$	$8580 + 27456 + 144144$	1
$(3,2^2)$	$173 + 185 + 191$	$12012 + 42900 + 77220$	1
$(3^2,1)$	$169 + 199$	$12012 + 144144$	1
$(4,1^3)$	$175 + 193$	$23100 + 105600$	1
$(4,2,1)$	207	300300	1
$(4,3)$	$169 + 201$	$12012 + 144144$	1
$(5,1^2)$	203	171600	1
$(5,2)$	$171 + 205$	$12012 + 180180$	1
$(6,1)$	$159 + 195$	$4368 + 112320$	1
(7)	$159 + 177$	$4368 + 27456$	1

Just considering the faithful characters (with numbers 153 to 210) of $6.Suz$ this allows us to conclude that all the characters that occur in one of the symmetrizations above of one of the two complex conjugate characters 153 or 154 have unitary determinant 1. The missing ones are $165-168, 181-184, 187-190$ which have character fields of degree 4 over the rationals, and $197, 198$ and $209, 210$. The latter four characters occur in $\mathrm{Sym}_\lambda(\chi_{153})$ resp. $\mathrm{Sym}_\lambda(\chi_{154})$ for $\lambda = (3, 2^4)$ resp. $\lambda = (4, 2, 1^5)$ which are sums of even degree absolutely irreducible characters with character field $\mathbb{Q}[\sqrt{-3}]$.

The complex conjugate characters χ_{21} and χ_{22} are the unique absolutely irreducible characters of the simple Suzuki group Suz of even degree and indicator o. They have degree 50050 and the computation above shows that their determinant is 1. Similary we can conclude that all (4 pairs of) even degree indicator o characters of $2.Suz$ have determinant 1, where we need to consider

$$\lambda = (4, 1^5) \text{ of dim. } 1/6480 \prod_{i=-5}^{3} (N+i) \text{ and det } 3^{\binom{N}{7}}$$

and

$$\lambda = (3^3) \text{ of dim. } 1/8640(N-2)(N-1)^2N^3(N+1)^2(N+2) \text{ and det } 6^{\binom{N}{3}}10^{\binom{N}{7}}15^{\binom{N}{9}}.$$

For the faithful characters of $3.Suz$, these symmetrizations show the ones of degree $66, 78$, and 1716 have determinant 1.

7 Refined symmetrizations for orthogonal groups

In this section we assume that $\mathrm{char}(K) = 0$ and that $B : V \times V \to K$ is a symmetric bilinear form. Then there are $\binom{n}{2}$ linearly independent $O(B)$-invariant epimorphisms π_{ij} for all $1 \le i < j \le n$ by evaluating B in positions i, j of the tensors:

$$\pi_{ij} : \otimes^n V \to \otimes^{n-2} V, \pi_{ij}(w_1 \otimes \ldots \otimes w_n) = B(w_i, w_j) w_1 \otimes \ldots \otimes w_n$$

where in the last tensor product the vectors w_i and w_j are omitted. There are $O(B)$-invariant monomorphism $\varphi_{ij} : \otimes^{n-2} V \to \otimes^n V$ with $\pi_{ij} \circ \varphi_{ij} = N \, \mathrm{id}_{\otimes^{n-2} V}$: Choose a basis $(v_1, \ldots, v_N)$ of V and define $(v_1', \ldots, v_N')$ to denote its dual basis, i.e. $B(v_i, v_j') = \delta_{ij}$. Then for all $w_1, \ldots, w_{n-2} \in V$ we put

$$\varphi_{ij}(w_1 \otimes \ldots \otimes w_{n-2}) := \sum_{k=1}^{N} (w_1 \otimes \ldots \otimes v_k \otimes \ldots \otimes v_k' \otimes w_{n-2})$$

where v_k is inserted in position i and v_k' in position j. The compositions $\varphi_{ij} \circ \pi_{ij}$ are $O(B)$-invariant endomorphism of $\otimes^n V$ giving rise to generators of the Brauer algebra, the endomorphism algebra of the $O(B)$-module $\otimes^n V$. See also [2] for an application of these ideas to a Schur-Weyl-duality for orthogonal groups in odd characteristic.

Proposition 7.1. *For all* $1 \leq i < j \leq n$ *the maps*

$$\varphi_{ij} : (\otimes^{n-2} V, N \otimes^{n-2} B) \to (\otimes^n V, \otimes^n B)$$

are $O(B)$*-invariant isometric embeddings.*

Proof. For a pair $(v_1, \ldots, v_N)$ and $(v'_1, \ldots, v'_N)$ of dual basis of V the Gram matrices are inverse to each other, so

$$((B(v_i, v_j)_{1\leq i,j\leq N})((B(v'_i, v'_j)_{1\leq i,j\leq N}) = I_N.$$

Now observe that

$$\otimes^2 B(\sum_{k=1}^N v_k \otimes v'_k, \sum_{k=1}^N v_k \otimes v'_k) = \sum_{k,\ell} B(v_k, v_\ell) B(v'_k, v'_\ell) = \sum_{k,\ell} B(v_k, v_\ell) B(v'_\ell, v'_k) = N$$

as the trace of the product of these two Gram matrices. □

Note that the same proof also works for a non-degenerate skew-symmetric bilinear form B, where interchanging v'_k and v'_ℓ introduces a minus sign, and hence we need to replace N by $-N$ in the formula of the proposition.

Remark 7.2. For the full symmetrizations there was an orthogonal decomposition according to the different n-element multi-subsets

$$C = \{i_1^{x_1}, \ldots, i_k^{x_k}\} \subset \{1, \ldots, N\}$$

where the dimension of the orthogonal summand with Gram matrix X_C did only depend on the composition $(x_1, \ldots, x_k)$ of n. For the refined symmetrization, the dimensions of the naturally occurring orthogonal summands grow with $N = \dim(V)$. Following [3], who uses [8] to determine the absolutely irreducible $O(B)$-submodules of $\mathrm{Sym}_\lambda(V)$ for partitions λ of n and $n \leq 6$, we write

$$\mathrm{Sym}_\lambda(V) \cong \mathrm{Sym}'_\lambda(V) \oplus \bigoplus_\gamma m(\lambda, \gamma) \, \mathrm{Sym}'_\gamma(V)$$

where γ runs over certain partitions of $n-2$, $n-4$, ... Here $m(\lambda, \gamma) \in \mathbb{N}$ is the multiplicity of $\mathrm{Sym}'_\gamma(V)$ as a composition factor of $\mathrm{Sym}_\lambda(V)$. As $\mathrm{Sym}'_\gamma(V)$ are absolutely irreducible, there is a one-dimensional space of $O(B)$-invariant quadratic forms on these modules. So there are symmetric invertible matrices

$$c(\lambda, \gamma) \in K^{m(\lambda,\gamma)\times m(\lambda,\gamma)}$$

such that

$$\mathrm{Sym}_\lambda(B) \cong \mathrm{Sym}'_\lambda(B) \oplus \bigoplus_\gamma c(\lambda,\gamma) \otimes \mathrm{Sym}'_\gamma(B).$$

To determine the values of $c(\lambda,\gamma)$ for a partition γ of $n-2$ (with $m(\lambda,\gamma)=1$) it suffices to choose a suitable embedding $\varphi_{ij} : \otimes^{n-2}V \to \otimes^n V$ and compute

$$c(\lambda,\gamma) = (\otimes^n B(e_\lambda \varphi_{ij}(v), e_\lambda \varphi_{ij}(v)))/(\otimes^{n-2}B(v,v))$$

for a suitable $v \in \mathrm{Sym}'_\gamma(V)$, e.g. $v = e_\gamma(v_1 \otimes \ldots \otimes v_{n-2})$. For partitions γ of smaller n we need to iterate this procedure, i.e. consider

$$\varphi_{k,l} \circ \varphi_{i,j} : \otimes^{n-4}V \to \otimes^n V$$

and so on.

The following sections contain some examples illustrating an elementary way to compute determinants of refined symmetrizations for non-degenerate symmetric bilinear forms. As in the previous sections we fix an orthogonal basis $(v_1,\ldots,v_N)$ of V and put $a_i := B(v_i,v_i)$.

7.1 The refined symmetrizations for (1^n) and (n)

Remark 7.3. For the exterior power all $\mathrm{Sym}_{(1^n)}(V)_{ij}$ are zero. As $\dim(\mathrm{Sym}_{(1^n)}(V)) = \binom{N}{n}$, this dimension is even, if and only if $c((1^n),N) = (n!)^{\binom{N}{n}}$ is a square. Then

$$\det(\mathrm{Sym}_{(1^n)}(B)) = \det(B)^{\binom{N}{n}\frac{n}{N}} = \det(B)^{\binom{N-1}{n-1}}.$$

Proposition 7.4.

$$\mathrm{Sym}_{(n)}(V) \cong \bigoplus_{j=0}^{\lfloor n/2 \rfloor} \mathrm{Sym}'_{(n-2j)}(V)$$

with

$$c((n),(n-2)) = 2n!(n-2)!(N+2(n-2))$$

and

$$c((n),(n-4)) = 8n!(n-4)!(N+2(n-4))(N+2(n-3)).$$

Proof. Let $I = \{1,\ldots,n-2\}$ and $w := e_{n-2}(v_1 \otimes \ldots \otimes v_{n-2}) \in \mathrm{Sym}'_{(n-2)}(V)$. Then $\otimes^{n-2}B(w,w) = (n-2)!q(I)$ and

$$\begin{aligned}
&e_{(n)}\varphi_{12}(w) = \textstyle\sum_{k=1}^N \frac{1}{a_k} e_{(n)} v_k \otimes v_k \otimes w = \\
&(n-2)! \textstyle\sum_{k=1}^N \frac{1}{a_k} e_{(n)} v_k \otimes v_k \otimes v_1 \otimes \ldots \otimes v_{n-2} = \\
&(n-2)! \textstyle\sum_{k=1}^{n-2} \frac{1}{a_k} e_{(n)} v_k \otimes v_k \otimes v_1 \otimes \ldots \otimes v_{n-2} \\
&+(n-2)! \textstyle\sum_{k=n-1}^N \frac{1}{a_k} e_{(n)} v_k \otimes v_k \otimes v_1 \otimes \ldots \otimes v_{n-2}.
\end{aligned}$$

The norms of the pure tensors in the first sum are $3!n!q(I)$ whereas the pure tensors in the second sum have norm $2!n!q(I)$. So in total we compute $\otimes^n B(e_{(n)}\varphi_{12}(w)) =$

$$(n-2)!^2((n-2)3!n! + (N-(n-2))2!n!)q(I) = (n-2)!^2 2n!(N+2(n-2))q(I)$$

and hence

$$c((n),(n-2)) = 2(n-2)!n!(N+2(n-2)) = (n-2)!^2(2(n(n-1)))(N+2n-4).$$

To compute $c((n),(n-4))$ we choose $I = \{1,\ldots,n-4\}$ and $w := e_{(n-4)}v_1 \otimes \ldots \otimes v_{n-4} \in \mathrm{Sym}'_{(n-4)}(V)$ of norm $(n-4)!$. Now

$$e_{(n)}\varphi_{12} \circ \varphi_{34}(w) = e_{(n)} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{1}{a_k}\frac{1}{a_\ell} v_k \otimes v_k \otimes v_\ell \otimes v_\ell \otimes w$$

has summands of 5 different types:

case	norm / norm(w)	anz
$k = \ell \in I$	$n!(n-4)!5!$	$n-4$
$k \neq \ell \in I$	$n!(n-4)!3!3!2$	$(n-4)(n-5)$
$k \in I, \ell \notin I$	$n!(n-4)!3!2!2$	$(n-4)(N-(n-4))$
$k \notin I, \ell \in I$	$n!(n-4)!2!3!2$	$(n-4)(N-(n-4))$
$k = \ell \notin I$	$n!(n-4)!4!$	$(N-(n-4))$
$k \neq \ell, k, \ell \notin I$	$n!(n-4)!2!2!2$	$(N-(n-4))(N-(n-3))$

So

$$\begin{aligned}
&c((n),(n-4)) = \\
&n!(n-4)!(5!(n-4) + 2 \cdot 3!^2(n-4)(n-5) + \\
&2^2(2!3!(n-4)(N-(n-4))) + 4!(N-(n-4)) + 8(N-(n-4))(N-(n-3)) = \\
&n!(n-4)!8(N+2n-8)(N+2n-6)
\end{aligned}$$

□

Remark 7.5. Of course it is possible to continue like this and compute $c((n),(n-2j))$ for $j \geq 3$. However this becomes more and more tedious, I think that there should be a better way, as $\mathrm{Sym}'_{(n)}(V)$ is just the space of harmonic polynomials for the Laplace operator $\sum_{i=1}^{N} a_i \frac{d^2}{dx_i^2}$ associated with the quadratic form defined by B.

7.2 The refined symmetrizations for $(2,1)$ and $(3,1)$

Proposition 7.6.

$$(\mathrm{Sym}_{(2,1)}(V), \mathrm{Sym}_{(2,1)}(B)) \cong (V, 8(N-1)B) \perp (\mathrm{Sym}'_{(2,1)}(V), \mathrm{Sym}'_{(2,1)}(B)).$$

Hence $\dim(\mathrm{Sym}'_{(2,1)}(V)) = \frac{1}{3}N(N-2)(N+2)$ *and*

$$\det(\mathrm{Sym}'_{(2,1)}(B)) = 3^{\binom{N}{3}} 2^N (N-1)^N \det(B)^{N^2-2}.$$

In particular the dimension of the refined symmetrization $\mathrm{Sym}'_{(2,1)}(V)$ *is even, if and only if,* $\dim(V) = N$ *is even. In this case* $\binom{N}{3}$ *is also even and hence* $\det(\mathrm{Sym}'_{(2,1)}(B))$ *is a square.*

Proof. A basis of $e_{(2,1)}\varphi_{12}(V)$ is given by

$$(b_i := \sum_{k=1}^{N} \frac{1}{a_k} e_{(2,1)}(v_k \otimes v_k \otimes v_i) \mid 1 \le i \le N).$$

Now $b_i = \sum_{k\neq i} \frac{2}{a_k}(v_k \otimes v_k \otimes v_i - v_i \otimes v_k \otimes v_k)$ satisfies

$$\otimes^3 B(b_i, b_i) = 8 \sum_{k\neq i} \frac{1}{a_k^2} B(v_k, v_k) B(v_k.v_k) B(v_i, v_i) = 8(N-1)B(v_i, v_i).$$

□

Proposition 7.7.

$$\mathrm{Sym}_{(3,1)}(V) \cong \mathrm{Sym}_{(1,1)}(V) \oplus \mathrm{Sym}'_{(2)}(V) \oplus \mathrm{Sym}'_{(3,1)}(V),$$

so

$$\dim(\mathrm{Sym}'_{(3,1)}(V)) = \frac{1}{8}(N-2)(N-1)(N+1)(N+4).$$

Up to squares we get $c((3,1),(1,1)) = 2(N+2)$ *and* $c((3,1),(2)) = N$.

Proof. Put

$$b_1 := \sum_{k=1}^{N} \frac{1}{a_k} e_{(3,1)}(v_k \otimes v_k \otimes v_1 \otimes v_2) \text{ and } b_2 := \sum_{k=1}^{N} \frac{1}{a_k} e_{(3,1)}(v_k \otimes v_k \otimes v_2 \otimes v_1).$$

Then

$$\otimes^4 B(b_1, b_1) = \otimes^4 B(b_2, b_2) = (24(N-2) + 6^2 \cdot 2 + 2^2 \cdot 2)a_1 a_2 = 8(3N+4)a_1 a_2$$

and

$$\otimes^4 B(b_1, b_2) = (-8(N-2) - 2 \cdot 8 \cdot 3)a_1 a_2 = -8(N+4)a_1 a_2.$$

So $b_1 - b_2 = e_{(3,1)}\varphi_{1,2}(e_{(1,1)}(v_1 \otimes v_2))$ has norm $64(N+2)a_1 a_2$ giving $c((3,1),(1,1)) = 2(N+2)$ and $b_1 + b_2 = e_{(3,1)}\varphi_{1,2}(e_{(2)}(v_1 \otimes v_2))$ has norm $32N a_1 a_2$ which yields $c((3,1),(2)) = N$. □

7.3 The refined symmetrizations for $(2,1^{n-2})$

Proposition 7.8. $\mathrm{Sym}_{(2,1^{n-2})}(V) \cong \mathrm{Sym}_{(1^{n-2})}(V) \oplus \mathrm{Sym}'_{(2,1^{n-2})}(V)$, *so*

$$\dim(\mathrm{Sym}'_{(2,1^{n-2})}(V) = (n-2)\binom{N}{n} + (n-1)\binom{N}{n-1}.$$

Up to squares

$$c((2,1^{n-2}),(1^{n-2})) = (n-1)(N-(n-2)).$$

Proof. Let $I = \{1,\ldots,n-2\}$ and put

$$w := e_{(1^{n-2})} v_1 \otimes \ldots \otimes v_{n-2} \in \mathrm{Sym}_{(1^{n-2})}(V).$$

Then the norm of w is $(n-2)!$. For $\lambda := (2,1^{n-2})$ we compute

$$\begin{array}{l} e_\lambda \varphi_{1,2}(w) = \sum_{k=1}^N \frac{1}{a_k} e_\lambda v_k \otimes v_k \otimes w = \\ (n-2)! \sum_{k=1}^N \frac{1}{a_k} e_\lambda v_k \otimes v_k \otimes v_1 \otimes \ldots \otimes v_{n-2} = \\ (n-2)! \sum_{k=1}^{n-2} \frac{1}{a_k} e_\lambda v_k \otimes v_k \otimes v_1 \otimes \ldots \otimes v_{n-2} + \\ (n-2)! \sum_{k=n-1}^N \frac{1}{a_k} e_\lambda v_k \otimes v_k \otimes v_1 \otimes \ldots \otimes v_{n-2}. \end{array}$$

Now the first summand is 0 whereas the last $(N-(n-2))$ summands have norm $(n-2)!^2 2^2 (n-1)! q(I)$. Hence $c(\lambda,(1^{n-2})) = 4(n-1)!(n-2)!(N-(n-2)) = (n-1)(2(n-2))^2(N-(n-2))$. □

7.4 Determinants of refined symmetrizations

This section gives tables of the $c(\lambda,\gamma)$ for partitions λ of n and all $n \leq 6$. Here (V,B) is a non-degenerate symmetric bilinear space over a field of characteristic 0, $N = \dim(V)$ is assumed to be $\geq n$. For fixed partition λ of n we display the partitions γ of $m \leq n$ and give $c(\lambda,\gamma) \in \mathbb{N}$ (up to squares) such that

$$\mathrm{Sym}_\lambda(B) \cong \mathrm{Sym}'_\lambda(B) \oplus_\gamma c(\lambda,\gamma)\,\mathrm{Sym}'_\gamma(B).$$

We omit the rows for 1^n since $\mathrm{Sym}_{(1^n)}(V) = \mathrm{Sym}'_{(1^n)}(V)$. The composition factors of $\mathrm{Sym}_\lambda(V)$ are taken from the table in [3, p. 157]. The only composition factor that occurs with multiplicity > 1 is $\mathrm{Sym}'_{(2)}(V)$ in $\mathrm{Sym}_{(4,2)}(V)$, where the multiplicity is 2. Here

$$A := c((4,2),(2)) = 2^8 \begin{pmatrix} 12N^2 - 32 & 2N^2 - 32 \\ 2N^2 - 32 & 7N^2 + 20N - 32 \end{pmatrix} \in \mathbb{Z}[N]^{2\times 2}$$

is of determinant

$$2^{20} 5 (N-2) N (N+1)(N+4).$$

To compute A we chose the embeddings

$$f_1 := e_{(4,2)} \circ \varphi_{1,2} \circ \varphi_{3,4} \text{ and } f_2 := e_{(4,2)} \circ \varphi_{1,2} \circ \varphi_{5,6} : \mathrm{Sym}'_{(2)}(V) \to \mathrm{Sym}_{(4,2)}(V).$$

For $v = e_{(2)}(v_1 \otimes v_2) \in \mathrm{Sym}'_{(2)}(V)$ we computed $A_{ij} = \otimes^6 B(f_i(v), f_j(v))$.

λ	γ	$c(\lambda, \gamma)$
(2)	$()$	N
(3)	(1)	$3(N+2)$
$(2,1)$	(1)	$2(N-1)$
(4)	$(2),\ ()$	$6(N+4),\ 3N(N+2)$
$(3,1)$	$(2),\ (1^2)$	$N,\ 2(N+2)$
(2^2)	$(2),\ ()$	$N-2,\ 2N(N-1)$
$(2,1^2)$	(1^2)	$3(N-2)$
(5)	$(3),\ (1)$	$10(N+6),\ 15(N+2)(N+4)$
$(4,1)$	$(3),\ (2,1),\ (1)$	$6(N+1),\ 5(N+4),\ 6(N-1)(N+2)$
$(3,2)$	$(3),\ (2,1),\ (1)$	$(N-2),\ 3(N+1),\ (N-1)(N+2)$
$(3,1^2)$	$(2,1),\ (1^3)$	$6(N-1),\ 15(N+2)$
$(2^2,1)$	$(2,1),\ (1)$	$6(N-3),\ 6(N-1)(N-2)$
$(2,1^3)$	(1^3)	$(N-3)$
(6)	$(4),\ (2)$	$15(N+8),\ 5(N+4)(N+6)$
	$()$	$15N(N+2)(N+4)$
$(5,1)$	$(4),\ (3,1)$	$2(N+2),\ N+6$
	$(2),\ (1,1)$	$6N(N+4),\ (N+2)(N+4)$
$(4,2)$	$(4),\ (3,1),\ (2,2)$	$N-2,\ 10(N+2),\ 30(N+4)$
	$2\cdot(2),\ ()$	$A,\ N(N-1)(N+2)$
$(4,1,1)$	$(3,1),\ (2,1,1),\ (1,1)$	$N,\ 2(N+4),\ (N-2)(N+2)$
$(3,3)$	$(3,1),\ (1,1)$	$3N,\ 3(N+1)(N+2)$
$(3,2,1)$	$(3,1),\ (2,2),\ (2,1,1)$	$6(N-3),\ 2(N-1),\ 15(N+1)$
	$(2),\ (1,1)$	$N(N-2), 15(N+2)(N-2)$
(2^3)	$(2,2),\ (2),\ ()$	$N-4,\ 2(N-2)(N-3),\ 6N(N-1)(N-2)$
$(3,1^3)$	$(2,1^2),\ (1^4)$	$2(N-2),\ 6(N+2)$
$(2^2,1^2)$	$(2,1^2),\ (1^2)$	$2(N-4),\ 3(N-2)(N-3)$
$(2,1^4)$	(1^4)	$5(N-4)$

Note that the same programs prove that the refined symmetrization $\mathrm{Sym}'_\gamma(V)$ does occur as an orthogonal summand of the $O(B)$-module $\mathrm{Sym}_\lambda(V)$, whenever we compute $c(\lambda,\gamma) \neq 0$. We can also obtain the multiplicity (at least a lower bound m) by finding $c(\lambda,\gamma) \in K^{m\times m}$ of full rank m.

7.5 An example: the refined symmetrizations of the 24-dimension representation of $2.Co_1$

The covering group of the sporadic simple Conway group $2.Co_1$ is a subgroup of index 2 of the automorphism group of the 24-dimensional extremal unimodular lattice, the Leech lattice. So this group has a 24-dimensional rational representation of determinant 1. The refined symmetrizations for partitions of even numbers hence yield representations of the simple group Co_1. The table below lists the ones that are orthogonally stable together with their decomposition into irreducibles and the corresponding determinants as predicted by the table in Section 7.4. For partitions of odd numbers, the refined symmetrizations are faithful representations of $2.Co_1$ and hence have orthogonal determinant 1 (see for instance [1, Theorem 4]) which is confirmed by the computation of the determinants of the refined symmetrizations.

Determinants of orthogonally stable characters for Co_1:

partition	χ	$\chi(1)$	det
(2)	2	276	1
$(2,1,1)$	8	37674	1
$(2,2)$	7	27300	253
(4)	6	17250	91
$(3,1^3)$	$15+23$	$483000+1771000$	1
$(3,2,1)$	31	4100096	161
$(5,1)$	24	1821600	65
$(4,2)$	29	2816856	13
(2^3)	$12+19$	$313950+822250$	77
(6)	$10+14$	$80730+376740$	35

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