A generalisation of Turyn's construction of self-dual codes.

Gabriele Nebe

Lehrstuhl D für Mathematik, RWTH Aachen University
52056 Aachen, Germany
nebe@math.rwth-aachen.de

Abstract. In [17] Turyn constructed the famous binary Golay code of length 24 from the extended Hamming code of length 8 (see also [10, Theorem 18.7.12]). The present note interprets this construction as a sum of tensor products of codes and uses it to construct certain new extremal (or at least very good) self-dual codes (for example an extremal doubly-even binary code of length 80). The lattice counterpart of this construction has been described by Quebbemann [13]. It was used recently to construct an extremal even unimodular lattice in dimension 72 ([12]).

1 Introduction.

A linear code is a subspace C of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ denotes the field with q elements. The vector space $\mathbb{F}_q^n$ is equipped with the standard inner product $(x,y) := \sum_{i=1}^n x_i y_i$. We call this the standard Euclidean inner product to distinguish it from the Hermitian inner product $h(x,y) := \sum_{i=1}^n x_i \overline{y}_i$ where $x \mapsto \overline{x} = x^r$ is the field automorphism of $\mathbb{F}_q$ of order 2 and $q = r^2$. For $C \leq \mathbb{F}_q^n$ the dual code is

$$C^{\perp} := \{x \in \mathbb{F}_q^n \mid (x,c) = 0 \text{ for all } c \in C\}.$$

Analogously the hermitian dual code $C^{\perp,h}$ is the orthogonal space with respect to h. The code C is called (hermitian) self-orthogonal if $C \subseteq C^{\perp(,h)}$ and (hermitian) self-dual if $C = C^{\perp(,h)}$.

For $x \in \mathbb{F}_q^n$ the weight of x is $wt(x) := |\{i \mid x_i \neq 0\}|$ the number of non-zero entries in x. The error correcting properties of a code C are measured by the minimum weight $d(C) := \min\{wt(c) \mid 0 \neq c \in C\}$. A code C is called m-divisible, if the weight of any codeword is a multiple of m. For $q = 2, 3$ the square of any non-zero element in $\mathbb{F}_q$ is 1 and hence any self-orthogonal code in $\mathbb{F}_q^n$ is q-divisible. Similarly $x\overline{x} = 1$ for any $0 \neq x \in \mathbb{F}_4$ so any hermitian self-orthogonal code in $\mathbb{F}_4^n$ is 2-divisible. The Gleason-Pierce theorem shows that there are essentially four interesting families of self-dual m-divisible linear codes over finite fields: The self-dual binary codes (Type I codes) with $m = 2$, the self-dual ternary codes (Type III codes) with $m = 3$, the hermitian self-dual quaternary codes (Type IV codes) with $m = 2$ and the doubly-even self-dual binary codes (Type II codes) with $m = 4$.

Invariant theory of finite complex matrix groups gives the following bounds on the minimum weight of Type T codes of length n:

$$d(C) \leq \begin{cases} 2+2\lfloor\frac{n}{8}\rfloor & \text{if T=I} \\ 4+4\lfloor\frac{n}{24}\rfloor & \text{if T=II} \\ 3+3\lfloor\frac{n}{12}\rfloor & \text{if T=III} \\ 2+2\lfloor\frac{n}{6}\rfloor & \text{if T=IV} \end{cases}$$

Using the notion of the shadow of a code, Rains [14] improved the bound for Type I codes

$$d(C) \leq 4+4\lfloor\frac{n}{24}\rfloor + a$$

where $a = 2$ if $n \pmod{24} = 22$ and 0 otherwise. Self-dual codes that achieve these bounds are called extremal. The monograph [11] gives a framework to define the notion of a Type of a self-dual code in much more generality and shows how to apply invariant theory to find upper bounds on the minimum weight of codes of a given Type.

Motivated by the article [13] and the construction of extremal 80-dimensional even unimodular lattices in [2] a generalisation of a construction used by Turyn to construct the Golay code of length 24 from the Hamming code of length 8 is given in this paper. The new codes discovered in this paper are an extremal Type II code of length 80 (at least 15 such codes have been known before) and 5 Euclidean self-dual codes in $\mathbb{F}_4^{36}$ with minimum weight 11. All computations are done with MAGMA [4].

2 A construction for self-dual codes.

Theorem 2.1. *Let $C = C^\perp, D = D^\perp \leq \mathbb{F}_q^n$ and $X \leq \mathbb{F}_q^m$ such that $X \cap X^\perp = \{0\}$. Then*

$$\mathcal{T} := \mathcal{T}(C, D, X) := C \otimes X + D \otimes X^\perp \leq \mathbb{F}_q^{nm} = \mathbb{F}_q^n \otimes \mathbb{F}_q^m$$

is a self-dual code.
If $q = 2$ and C and D are doubly-even, then $\mathcal{T}$ is also doubly-even.

Proof. Let $c, c' \in C$, $d, d' \in D$, $x, x' \in X$ and $y, y' \in X^\perp$. Then

$$\begin{array}{ll} (c \otimes x, c' \otimes x') = 0 & \text{since } C \subseteq C^\perp \\ (d \otimes y, d' \otimes y') = 0 & \text{since } D \subseteq D^\perp \\ (c \otimes x, d \otimes y) = 0 & \text{since } x \in X, y \in X^\perp \end{array}$$

so $\mathcal{T} \subset \mathcal{T}^\perp$. Moreover

$$\dim(\mathcal{T}) = \dim(C \otimes X) + \dim(D \otimes X^\perp) - \dim(C \otimes X \cap D \otimes X^\perp) = nm/2 - 0$$

since $X \cap X^\perp = \{0\}$. This implies that $\mathcal{T}$ is self-dual.
If C and D are doubly-even, then the weights of all generators of $\mathcal{T}$ are multiples of 4 and so also $\mathcal{T}$ is doubly-even. □

Remark 2.2. A similar result holds for hermitian self-dual codes: Let $C = C^{\perp,h}, D = D^{\perp,h} \leq \mathbb{F}_q^n$ and $X \leq \mathbb{F}_q^m$ such that $X \cap X^{\perp,h} = \{0\}$. Then

$$\mathcal{T}_h := \mathcal{T}_h(C, D, X) := C \otimes X + D \otimes X^{\perp,h} \leq \mathbb{F}_q^{nm} = \mathbb{F}_q^n \otimes \mathbb{F}_q^m$$

is a hermitian self-dual code.

Remark 2.3. Clearly $X + X^{\perp} = \mathbb{F}_q^m$ has minimum weight 1 and therefore $d(\mathcal{T}(C, D, X)) \leq d(C \cap D)$. For $q = 2$, any self-dual code contains the all-one vector $\mathbf{1}$, so the maximum possible minimum weight for binary codes is $d(\mathcal{T}(C, D, X)) \leq d(C \cap D) \leq d(\langle \mathbf{1} \rangle) = n$.

Example 2.4. *(binary codes)*

1) Turyn's construction of the Golay-code ([17], see [10, Theorem 18.7.12]).
Let $C \cong D \cong h_8 = h_8^{\perp} \leq \mathbb{F}_2^8$ both to be equivalent to the extended Hamming code h_8 of length 8, the unique doubly-even binary self-dual code of length 8. Up to the action of S_8 there is a unique such pair satisfying $C \cap D = \langle \mathbf{1} \rangle$. Let $X := \langle (1, 1, 1) \rangle$. Then $\mathcal{T}(C, D, X)$ is a doubly-even self-dual code of length 24. From the explicit description

$$\mathcal{T}(C, D, X) = \{(c + d_1, c + d_2, c + d_3) \mid c \in C, d_i \in D, d_1 + d_2 + d_3 \in C \cap D = \langle \mathbf{1} \rangle\}$$

one easily sees that the minimum weight of $\mathcal{T}(C, D, X)$ is ≥ 8, so $\mathcal{T}(C, D, X)$ is equivalent to the Golay code: Any non-zero word $w \in \mathcal{T}(C, D, X)$ has either

   1) 1 non-zero component: Then up to permutation w is of the form $(d, 0, 0)$ with $d = \mathbf{1} \in \mathbb{F}_2^8$ and has weight 8.
   2) 2 non-zero components: Then w is equivalent to $(d_1, d_2, 0)$ with non-zero $d_1, d_2 \in D \cong h_8$ and has weight $\geq d(h_8) + d(h_8) = 4 + 4 = 8$.
   3) 3 non-zero components: Since all components of w lie in $C + D = \langle \mathbf{1} \rangle^{\perp}$ they all have even weight, so $wt(w) \geq 2 + 2 + 2 = 6$. The code $\mathcal{T}$ is doubly-even, so the weight of w is a multiple of 4, therefore $wt(w) \geq 8$.

2) Let $X \leq \mathbb{F}_2^{10}$ be the code with generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(see [1]). Then X is equivalent to its dual code, $X \cap X^{\perp} = \langle \mathbf{1} \rangle$ and the minimum weight of X (and of $X^{\perp}$) is 4. Let C and D be as in 1) and put

$$\mathcal{T} := X \otimes C + X^{\perp} \otimes D \leq \mathbb{F}_2^{80}.$$

Then $\mathcal{T}$ is self-orthogonal of dimension

$$\dim(X \otimes C) + \dim(X^\perp \otimes D) - \dim((X \otimes C) \cap (X^\perp \otimes D)) = 20 + 20 - 1 = 39.$$

The three codes T_1, T_2, T_3 with $\mathcal{T} \subsetneq T_i \subsetneq \mathcal{T}^\perp$ are all self-dual, two of them are doubly-even and one of these doubly-even self-dual codes has minimum weight 16, hence is an extremal doubly-even code of length 80. Its automorphism group is isomorphic to $PSL_2(7) \times S_6 : 2$, which can be seen as follows:
Let S be stabiliser of D in $\mathrm{Aut}(C)$. Then $S \cong PSL_2(7)$. The two codes C and D are the only self-dual S-invariant submodules of $\mathbb{F}_2^8$, they are interchanged by the normalizer of S in S_8 which is isomorphic to $PGL_2(7)$. Hence there is $\tau \in S_8$ interchanging C and D.
The automorphism group A of X is isomorphic to S_6, it also fixes the dual code $X^\perp$. The two codes X and $X^\perp$ are the only A-invariant subspaces of $\mathbb{F}_2^{10}$ which have dimension 5, therefore they are interchanged by the normalizer of A in S_{10}, which contains A of index 2. So there is $\sigma \in S_{10}$ with $\sigma(X) = X^\perp$ and $\sigma(X^\perp) = X$. One therefore gets an obvious action of

$$H := \langle A \otimes S, \sigma \otimes \tau \rangle \cong PSL_2(7) \times S_6 : 2$$

on $\mathcal{T}$. Since the three self-dual codes T_1, T_2, T_3 are not equivalent, the automorphism group of $\mathcal{T}$ also stabilizes all codes T_i. With MAGMA one checks that $\mathrm{Aut}(T_1) = H$. To the author's knowledge this code is not described before in the literature.

Example 2.5. Ternary codes:
Let $C \leq \mathbb{F}_3^{12}$ be the linear ternary self-dual code with generator matrix

$$\begin{pmatrix} 1&0&0&0&0&0&0&1&1&1&1&1 \\ 0&1&0&0&0&0&1&0&1&2&2&1 \\ 0&0&1&0&0&0&1&1&0&1&2&2 \\ 0&0&0&1&0&0&1&2&1&0&1&2 \\ 0&0&0&0&1&0&1&2&2&1&0&1 \\ 0&0&0&0&0&1&1&1&2&2&1&0 \end{pmatrix}$$

Then C is equivalent to the ternary Golay code of length 12. Let $h \in S_{12}$ be the permutation $(1,4,6,12,3,9,8)(2,11,7,10)$ and let $D = h(C)$. Then $C \cap D$ is of dimension 1 and minimum weight 12.
Choose $X = \langle (1,1) \rangle \leq \mathbb{F}_3^2$. Then $\mathcal{T}(C,D,X)$ is a self-dual code of minimum weight 9. The extremal ternary codes of length 24 are classified in [8]. There are two such codes, one of them is the extended quadratic residue code, the other one is equivalent to $\mathcal{T}(C,D,X)$.

Example 2.6. *Euclidean self-dual quaternary codes:*
Let $C \leq \mathbb{F}_4^{12}$ be the code with generator matrix

$$\begin{pmatrix} 1&0&0&0&0&0&1&\omega^2&1&1&\omega&1 \\ 0&1&0&0&0&0&\omega&0&1&\omega^2&\omega&\omega^2 \\ 0&0&1&0&0&0&\omega&\omega^2&\omega&\omega^2&0&1 \\ 0&0&0&1&0&0&0&\omega&\omega^2&\omega&\omega^2&1 \\ 0&0&0&0&1&0&\omega^2&\omega&1&0&\omega^2&\omega \\ 0&0&0&0&0&1&\omega^2&1&1&\omega&1&1 \end{pmatrix}.$$

Then C is a euclidean self-dual code equivalent to the extended quadratic residue code of length 12 over $\mathbb{F}_4$. Putting $D = \pi(C)$ for permutations $\pi \in S_{12}$ running through a right transversal of $\mathrm{Aut}(C)$ in S_{12}, $X = \langle(1, \omega)\rangle \leq \mathbb{F}_4^2$ and $X^\perp = \langle(1, \omega + 1)\rangle$ one constructs 20 monomially inequivalent euclidean self-dual codes in $\mathbb{F}_4^{24}$ with minimum weight 8.
Taking $X = \langle(1, 1, 1)\rangle$ one obtains five monomially inequivalent euclidean self-dual codes in $\mathbb{F}_4^{36}$ with minimum weight 11: T_1, T_2 (108 minimum words) and T_3, T_4 and T_5 (1188 minimum words each). These codes are not equivalent to the ones given in [3]. Permutations π_i yielding these codes T_i are

$$\begin{array}{rc}
\pi_1 = & (1,10,7,2,11,8,5)(3,4,12,9) \\
\pi_2 = & (1,10,6,4,12,9,5)(2,11,8,7) \\
\pi_3 = & (1,3,4,5,7,8,9,11)(2,10,12) \\
\pi_4 = & (1,6,11)(2,5,8,12,4,7,10)(3,9) \\
\pi_5 = & (1,10,2,8)(3,11,12,6)(4,7,5,9)
\end{array}$$

The permutation groups are $S_3 \times A_5$ for T_i (i=1,2,3,4) and $S_3 \times PSL_2(11)$ for T_5.

3 An application to lattices.

In [13] Quebbemann describes a construction of integral lattices that is the lattice counterpart of the construction described in the last section. Here a lattice (L, Q) is an even positive definite lattice, i.e. a free $\mathbb{Z}$-module L equipped with a quadratic form $Q : L \to \mathbb{Z}$ such that the bilinear form

$$(\cdot, \cdot) : L \times L \to \mathbb{Z}, (x, y) := Q(x + y) - Q(x) - Q(y)$$

is positive definite on the real space $\mathbb{R} \otimes L$. The dual lattice

$$L^{\#} := \{x \in \mathbb{R} \otimes L \mid (x, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}$$

contains L and the finite abelian group $L^{\#}/L =: D(L, Q)$ is called the discriminant group.

L is called unimodular, if $L = L^{\#}$. Note that unimodular quadratic lattices are usually called even unimodular lattices. They correspond to regular positive definite integral quadratic forms.

The minimum of a lattice (L, Q) is

$$\min(L, Q) := \min\{Q(\ell) \mid 0 \neq \ell \in L\}$$

which is half of the usual minimum of the lattice.

The theory of modular forms allows to show that the minimum of a unimodular quadratic lattice of dimension n is always

$$\min(L, Q) \leq \lfloor \frac{n}{24} \rfloor + 1.$$

Lattices achieving this bound are called extremal.

For any prime p not dividing the order of $D(L,Q)$ the quadratic form Q induces a non-degenerate quadratic form

$$\overline{Q} : L/pL \to \mathbb{Z}/p\mathbb{Z}, \overline{Q}(\ell + pL) := Q(\ell) + p\mathbb{Z}.$$

From the theory of integral quadratic forms (see for instance [15]) it is well known that this quadratic space $(L/pL, \overline{Q})$ is hyperbolic, so there are maximal isotropic subspaces $A = A^{\perp}$ and $A' = (A')^{\perp}$ such that

$$L/pL = A \oplus A', \overline{Q}(A) = \overline{Q}(A') = \{0\}.$$

If M and N are the full preimages of A and A', then $L = M + N, pL = N \cap M$ and $(M, \frac{1}{p}Q)$ and $(N, \frac{1}{p}Q)$ are again integral lattices with the same discriminant group as L. The pair (M,N) is called a polarisation of L (for the prime p).

Theorem 3.1. *([13, Proposition]) Let (L,Q), p, A, A' be as above and let $B \leq A^n$ be a subgroup of A^n. Put*

$$B' := (A')^n \cap B^{\perp} = \{z = (z_1, \dots, z_n) \in (A')^n \mid \sum_{i=1}^{n} \overline{(b_i, z_i)} = 0 \text{ for all } (b_1, \dots, b_n) \in B\}.$$

Then $C := B \oplus B' \leq (L/pL)^n$ satisfies $\overline{Q}^n(C) = \{0\}$ and $C = C^{\perp}$. The lattice

$$\Lambda := \Lambda(L, A, A', B) := \{\ell \in L^n \mid \overline{\ell} \in C\}$$

is integral with respect to $\tilde{Q} := \frac{1}{p}Q^n$ and satisfies $D(\Lambda, \tilde{Q}) \cong D(L,Q)^n$.

Of particular interest is the case where

$$B = \{(x, \dots, x) \mid x \in A\}$$

is the diagonal subgroup of A^n. Then

$$B' = \{(z_1, \dots, z_n) \mid z_i \in A' \text{ and } \sum z_i = 0\}$$

and $\Lambda(L, A, A', B)$ will be denoted by $\Lambda(L, A, A', n)$ or equivalently $\Lambda(L, M, N, n)$, where M, N are the full preimages of A, A' respectively.

Lemma 3.2. *Let (N,M) be a polarisation of L modulo 2 and assume that $d = \min(L,Q) = \min(N, \frac{1}{2}Q) = \min(M, \frac{1}{2}Q)$. Then*

$$\lceil \frac{3d}{2} \rceil \leq \min(\Lambda(L, M, N, 3), \tilde{Q}) \leq 2d.$$

<u>Proof.</u> The lattice $\Lambda := \Lambda(L, M, N, 3)$ has the following description

$$\Lambda = \{(m + n_1, m + n_2, m + n_3) \mid m \in M, n_1, n_2, n_3 \in N, n_1 + n_2 + n_3 \in 2L\}.$$

We write any element of λ of Λ as $\lambda = (a, b, c)$ and distinguish according to the number of non-zero components:

1) One non-zero component: Then $\lambda = (a, 0, 0)$ with $a = 2\ell \in 2L$ so $\tilde{Q}(\lambda) = \frac{1}{2}Q(2\ell) = 2\mathbb{Q}(\ell) \geq 2d$.

2) Two non-zero components: Then $\lambda = (a, b, 0)$ with $a, b \in N$ so $\tilde{Q}(\lambda) = \frac{1}{2}Q(a) + \frac{1}{2}Q(b) \geq 2d$.

3) Three non-zero components: Then $\tilde{Q}(\lambda) = \frac{1}{2}(Q(a) + Q(b) + Q(c)) \geq \frac{3}{2}d$.

□

Examples for $p = 2$ and $n = 3$

1) Take $(L, Q) = E_8$ the unique (even) unimodular lattice of dimension 8. Then for $p = 2$, the quadratic space $L/2L$ has a unique polarisation $L/2L = A \oplus A'$ up to the action of the orthogonal group of L. By Lemma 3.2 the lattice $\Lambda(E_8, A, A', 3)$ is an even unimodular lattice of minimum 2, therefore isomorphic to the Leech lattice, the unique unimodular lattice of dimension 24 with minimum 2. This has been remarked independently in [16], [9], [13].

2) Take $L = \Lambda_{24}$ to be the Leech lattice and take a polarization $L = M + N$, $M \cap N = 2L$ such that $(M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$. Bob Griess [7] remarked that $\Lambda(L, M, N, 3)$ is a 72-dimensional unimodular lattice of minimum 3 or 4 (this also follows from Lemma 3.2). In [6] the number of sublattices $M \leq \Lambda_{24}$ such that $(M, \frac{1}{2}Q) \cong \Lambda_{24}$ is computed. There are 5,163,643,468,800,000 such sublattices, about 1/68107 of all maximal isotropic subspaces. Each maximal isotropic subspace A has 2^{66} complements (the number of alternating 12×12 matrices over $\mathbb{F}_2$). Assuming that approximately 1/68107 of these complements correspond to lattices that are similar to the Leech lattice, the number of pairs (M, N) such that $M + N = \Lambda_{24}$, $M \cap N = 2\Lambda_{24}$ and $(M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$ is about $5.6 \cdot 10^{30}$. Dividing by the order of the Conway group, $\mathrm{Aut}(\Lambda_{24})/\{\pm 1\}$, one gets a rough estimate of 10^{12} orbits of such polarisations of the Leech lattice. Presumably most of these orbits will give rise to lattices of minimum 3. In [12] I found one lattice $\Gamma := \Lambda(\Lambda_{24}, M, N, 3)$ to be an extremal unimodular lattice of dimension 72. Here the sublattices $M = \alpha\Lambda_{24}$ and $N = (\alpha + 1)\Lambda_{24}$ are obtained using a hermitian structure of the Leech lattice over the ring of integers $\mathbb{Z}[\alpha]$ in the imaginary quadratic number field of discriminant -7, where $\alpha^2 + \alpha + 2 = 0$. The Leech lattice has nine such Hermitian structures and one of them defines a polarisation giving rise to an extremal unimodular lattice. Γ can also be constructed as the tensor product of the Leech lattice with the unique unimodular $\mathbb{Z}[\alpha]$-lattice P_b or dimension 3, $\Gamma = \Lambda_{24} \otimes_{\mathbb{Z}[\alpha]} P_b$. This construction allows to find the subgroup $\mathrm{SL}_2(25) \times PSL_2(7) : 2$ of the automorphism group of Γ. For more details on this lattice see my preprint [12].

The extremal 72-dimensional lattice Γ described above is constructed using a polarization (M, N) of Λ_{24} that is invariant under $\mathrm{SL}_2(25)$. This group contains an element g of order 13, acting as a primitive 13th root of unity on $L/2L$ and it is interesting to investigate all g-invariant polarisations:

Remark 3.3. Take $L := \Lambda_{24}$ to be the Leech lattice and let $g \in \mathrm{Aut}(L)$ be an element of order 13 (there is a unique conjugacy class of such elements). Then g acts fixed point free on $L/2L$ and hence there are $2^{12} + 1$ subspaces of dimension 12 that are invariant under $\langle g \rangle$. The preimage M in L of 41 of these invariant subspaces is similar to the Leech lattice. The normalizer G in $\mathrm{Aut}(L)$ of $\langle g \rangle$ acts on these lattices with orbits of length 36, 4, and 1. In total we obtain 31 representatives (M, N) of G-orbits on the ordered polarizations (M, N) of L modulo 2 such that

$$gN = N, gM = M, (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong (L, Q) \cong \Lambda_{24}.$$

Only one such pair yields a lattice $L(M, N, 3)$ that has minimum 4. This lattice is necessarily isometric to Γ.
I did a similar computation for an element $g \in \mathrm{Aut}(\Lambda_{24})$ acting as a primitive 21st root of 1. All 71 orbits of the normalizer on the ordered "good" polarisations (M, N) yield lattices $L(M, N, 3)$ that contain vectors of norm 3.

Example.
In [2] we used the code $X \leq \mathbb{F}_2^{10}$ from example 2.4 2) to construct two 80-dimensional extremal unimodular lattices from the E_8-lattice.

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