A fourth extremal even unimodular lattice of dimension 48.

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Abstract. We show that there is a unique extremal even unimodular lattice of dimension 48 which has an automorphism of order 5 of type $5-(8,16)-8$. Since the three known extremal lattices do not admit such an automorphism, this provides a new example of an extremal even unimodular lattice in dimension 48.
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1 Introduction

A lattice L in Euclidean space $(\mathbb{R}^n, (,))$ is a free $\mathbb{Z}$-module of rank n containing a basis of $\mathbb{R}^n$. The lattice L is called *even*, if the associated quadratic form $Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, x \mapsto Q(x) := \frac{1}{2}(x,x)$ is integral on L, so $Q(L) \subseteq \mathbb{Z}$. Then L is contained in its dual lattice

$$L^{\#} := \{y \in \mathbb{R}^n \mid (y, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}.$$

L is *unimodular*, if $L = L^{\#}$. Any lattice L defines a sphere packing, whose density measures its error correcting properties. One of the main goals in lattice theory is to find dense lattices. This is a very difficult problem, the densest lattices are known only in dimension $n \leq 8$ and in dimension 24 [3], for $n = 8$ and $n = 24$ the densest lattices are even unimodular lattices. The density of a unimodular lattice is measured by its *minimum*,

$$\min(L) := \min\{2Q(\ell) \mid 0 \neq \ell \in L\}.$$

For even unimodular lattices the theory of modular forms allows one to bound this minimum $\min(L) \leq 2+2\lfloor\frac{n}{24}\rfloor$ and *extremal lattices* are those even unimodular lattices L that achieve equality. Of particular interest are extremal even unimodular lattices L in the *jump dimensions* $24m$. For $m = 1$ there is a unique extremal even unimodular lattice, the *Leech lattice*, which is the densest 24-dimensional lattice [3]. By [4], its automorphism group is a covering group of the sporadic simple Conway group Co_1. The 196560 minimal vectors of the Leech lattice form the unique tight spherical

11-design and realise the maximal kissing number in dimension 24. In dimension 72 one knows one extremal unimodular lattice [11]. The existence of such a lattice was a longstanding open problem. In dimension 48 there are at least four extremal even unimodular lattices. They are the densest known lattices in their dimension and realise the maximal known kissing number $52,416,000$. It is a very interesting problem to classify all 48-dimensional extremal even unimodular lattices. To get an idea of how many such lattices might exist, I started a program to find all extremal lattices L whose automorphism group

$$\mathrm{Aut}(L) = \{g \in \mathrm{GL}(L) \mid Q(xg) = Q(x) \text{ for all } x\}$$

is not too small. In [12] I classified all 48-dimensional extremal lattices that have an automorphism of order a whose Euler phi value is $\varphi(a) > 24$. All these lattices are isometric to one of the lattices P_{48p}, P_{48q}, or P_{48n}, which were known before. The present paper classifies all extremal lattices invariant under a certain automorphism of order 5. It turns out that there is a unique such lattice, P_{48m}, and this lattice is not isometric to one of the lattices above.

Table 1: The known extremal even unimodular lattices in the jump dimensions

name	autom. group	order	ref.
Λ_{24}	$2.Co_1$	8315553613086720000 $= 2^{22}3^9 5^4 7^2 11\ 13\ 23$	[8], [4]
P_{48p}	$(\mathrm{SL}_2(23) \times S_3) : 2$	$72864 = 2^5 3^2 11\ 23$	[5], [12]
P_{48q}	$\mathrm{SL}_2(47)$	$103776 = 2^5 3\ 23\ 47$	[5], [12]
P_{48n}	$(\mathrm{SL}_2(13) \mathsf{Y} \mathrm{SL}_2(5)).2^2$	$524160 = 2^7 3^2 5\ 7\ 13$	[10], [12]
P_{48m}	$(C_5 \times C_5 \times C_3) : (D_8 Y C_4)$	$1200 = 2^4 3\ 5^2$	this paper
Γ_{72}	$(\mathrm{SL}_2(25) \times \mathrm{PSL}_2(7)) : 2$	$5241600 = 2^8 3^2 5^2 7\ 13$	[11], [12]

2 The type of an automorphism

The notion of the type of an automorphism of a lattice L was introduced in [12]. It was motivated by the analogous notion of a type of an automorphism of a code.

Let $\sigma \in \mathrm{GL}_n(\mathbb{Q})$ be an element of prime order p. Let $K := \ker(\sigma - 1)$ and $I := \mathrm{im}(\sigma - 1)$. Then K is the fixed space of σ and the action of σ on I gives rise to a vector space structure on I over the p-th cyclotomic number field $\mathbb{Q}[\zeta_p]$. In particular $n = d + z(p-1)$, where $d := \dim_{\mathbb{Q}}(K)$ and $z = \dim_{\mathbb{Q}[\zeta_p]}(I)$.

If L is a σ-invariant $\mathbb{Z}$-lattice, then L contains a sublattice M with

$$L \geq M = (L \cap K) \oplus (L \cap I) =: L_K(\sigma) \oplus L_I(\sigma) \geq pL$$

of finite index $[L : M] = p^s$ where $s \leq \min(d, z)$.

Definition 2.1. *The triple* $p-(z,d)-s$ *is called the* type *of the element* $\sigma \in \mathrm{GL}(L)$.

Remark 2.2. *Let* (L,Q) *be an even unimodular lattice and* $\sigma \in \mathrm{Aut}(L)$ *be of type* $p-(z,d)-s$. *Then* $L_I^{\#}(\sigma)/L_I(\sigma) \cong (\mathbb{Z}[\zeta_p]/(1-\zeta_p))^s$ *and* $L_K^{\#}(\sigma)/L_K(\sigma) \cong (\mathbb{Z}/p\mathbb{Z})^s$ *as* $\mathbb{Z}[\sigma]$*-modules. In particular* $0 \le s \le \min(z,d)$. *If* $z = s$ *then* $(1-\sigma)L_I^{\#}(\sigma) = L_I(\sigma)$ *and hence* $(L_I^{\#}(\sigma), pQ) = (X, trace_{\mathbb{Q}[\zeta_p]/\mathbb{Q}}(h))$ *is the trace lattice of an Hermitian unimodular* $\mathbb{Z}[\zeta_p]$ *lattice* (X,h) *of rank* $\dim_{\mathbb{Z}[\zeta_p]}(X) = z$.

Remark 2.3. *If* L *is an even lattice and* p *is odd, then* $L_K(\sigma)$ *and* $L_I(\sigma)$ *are also even lattices, because* $L_K(\sigma) \oplus L_I(\sigma)$ *is a sublattice of odd index in* L.

In [12] we narrowed down the possible types of prime order automorphisms of an extremal even unimodular lattice in dimension 48. By Remark 2.3 the fixed lattice of an element of order 3 cannot be $\sqrt{3}D_{12}^{+}$, as this is an odd lattice. So Type $3-(18,12)-12$ is not possible and the possible types are among the ones in Table 2.

Table 2: The possible types of automorphisms $\sigma \neq -1$ of prime order

Type	$L_K(\sigma)$	$L_I(\sigma)$	example	complete
47-(1,2)-1	unique	unique	P_{48q}	[12, Thm 5.6]
23-(2,4)-2	unique	at least 2	P_{48q}, P_{48p}	
13-(4,0)-0	$\{0\}$	at least 1	P_{48n}	
11-(4,8)-4	unique	at least 1	P_{48p}	
7-(8,0)-0	$\{0\}$	at least 1	P_{48n}	
7-(7,6)-5	$\sqrt{7}A_6^{\#}$	not known	not known	
5-(12,0)-0	$\{0\}$	at least 2	P_{48n}, P_{48m}	
5-(10,8)-8	$\sqrt{5}E_8$	at least 1	P_{48m}	
5-(8,16)-8	$[2.\mathrm{Alt}_{10}]_{16}$	Λ_{32}	P_{48m}	Thm. 3.2
3-(24,0)-0	$\{0\}$	at least 3	P_{48p}, P_{48n}, P_{48m}	
3-(20,8)-8	$\sqrt{3}E_8$	not known	not known	
3-(16,16)-16	$\sqrt{3}(E_8 \perp E_8)$	at least 4	P_{48p}, P_{48q}, P_{48n}	
3-(16,16)-16	$\sqrt{3}D_{16}^{+}$	at least 4	not known	
3-(15,18)-15	unique	two	not known	
3-(14,20)-14	?	unique	not known	
3-(13,22)-13	?	unique	not known	
2-(24,24)-24	$\sqrt{2}\Lambda_{24}$	$\sqrt{2}\Lambda_{24}$	P_{48n}	
2-(24,24)-24	$\sqrt{2}O_{24}$	$\sqrt{2}O_{24}$	P_{48n}, P_{48p}, P_{48m}	

Remark 2.4. *Table 2 lists the possible types of prime order automorphisms* $\sigma \neq -1$. *The type usually determines the genus of the* $\mathbb{Z}$*-lattice* $L_K(\sigma)$ *and the* $\mathbb{Z}[\zeta_p]$*-lattice* $L_I(\sigma)$. *If these genera are either classified in the literature (in particular the*

unimodular genera) or easily computed in Magma, we give the names or the number of lattices of minimum ≥ 6 in these genera. Column "example" lists the known examples and the last column gives the two instances where the classification of the lattices is known to be complete.

3 Automorphisms of type $5-(8,16)-8$.

Proposition 3.1. *There is a unique Hermitian unimodular $\mathbb{Z}[\zeta_5]$-lattice $(\mathcal{L}, h)$ of dimension $\dim_{\mathbb{Z}[\zeta_5]}(\mathcal{L}) = 8$ such that the dual lattice*

$$\Lambda_{32} := (\mathcal{L}, \frac{1}{5} trace(h))^{\#}$$

of the rescaled trace lattice of $\mathcal{L}$ has minimum ≥ 6.

Proof. A complete enumeration of the genus using the Kneser neighboring method [6] which is described for Hermitian lattices in [13] shows that the genus of Hermitian unimodular $\mathbb{Z}[\zeta_5]$ lattices of rank 8 contains 207 isometry classes of Hermitian unimodular lattices. The completeness of the enumeration may also be verified using Shimura's mass formula, the mass is

$$\frac{46956347226527}{108864000000000} = \frac{46956347226527}{2^{15}3^5 5^9 7} \sim 0.43.$$

Only for one of the 207 lattices the dual, say Λ_{32}, of the trace lattice has minimum ≥ 6. $\square$

The automorphism group of the $\mathbb{Z}$-lattice Λ_{32} is a soluble group of order $2^8 3\ 5^2 = 19200$.

Theorem 3.2. *There is a unique extremal even unimodular lattice of dimension 48 that admits an automorphism σ of type $5-(8,16)-8$. The lattice is available under the name P_{48m} in [7].*

Proof. Let L be such a lattice and let σ be an automorphism of Type $5-(8,16)-8$ of L. Then $L_K(\sigma)$ is a 16-dimensional lattice of minimum 6 in the genus of the 5-modular lattices. It has been shown in [1, Theorem 8.1] that this genus contains a unique lattice, say Λ_{16}, of minimum 6, which was denoted by $[2.\mathrm{Alt}_{10}]_{16}$ in [9].
Since $s = z$ and by Remark 2.2, the dual $D = L_I(\sigma)^{\#}$ of the lattice $L_I(\sigma)$ is the trace lattice of a Hermitian unimodular lattice over $\mathbb{Z}[\zeta_5]$ of dimension 8. So $L_I(\sigma) \cong \Lambda_{32}$ by Proposition 3.1. Therefore (up to isometry) L contains a sublattice $M := \Lambda_{32} \oplus \Lambda_{16}$ of index 5^8.

With extensive computations in MAGMA [2] we construct a set of lattices that contains representatives of all $\mathrm{Aut}(M)$-orbits on

$$\{X \mid M \leq X \leq M^{\#}, \min(X) = 6, X = X^{\#}\}.$$

To this aim we first compute orbit representatives of the 1-dimensional subspaces of $\Lambda_{32}^{\#}/\Lambda_{32}$ under the action of $\mathrm{Aut}(\Lambda_{32})$ to see that no proper integral overlattice of Λ_{32} has minimum ≥ 6. We then choose a suitable basis $\overline{b} := (\overline{b}_1, \ldots, \overline{b}_8)$ of $\Lambda_{32}^{\#}/\Lambda_{32}$ such that $(b_i, b_i) = \frac{12}{5} = \min(\Lambda_{32}^{\#})$ and

$$(b_1, b_i) \in \frac{1}{5} + \mathbb{Z}, (b_2, b_j) \in \mathbb{Z}, 2 \leq i \leq 8, 3 \leq j \leq 8.$$

More precisely the Gram matrix (b_i, b_j) is

$$F := \begin{pmatrix} 12/5 & 6/5 & 6/5 & 6/5 & 6/5 & 6/5 & 6/5 & 1/5 \\ 6/5 & 12/5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6/5 & 1 & 12/5 & 1 & 1/5 & 1 & 1/5 & 3/5 \\ 6/5 & 1 & 1 & 12/5 & 4/5 & 1/5 & 2/5 & 1/5 \\ 6/5 & 1 & 1/5 & 4/5 & 12/5 & 4/5 & 4/5 & 3/5 \\ 6/5 & 1 & 1 & 1/5 & 4/5 & 12/5 & 2/5 & 2/5 \\ 6/5 & 1 & 1/5 & 2/5 & 4/5 & 2/5 & 12/5 & 3/5 \\ 1/5 & 1 & 3/5 & 1/5 & 3/5 & 2/5 & 3/5 & 12/5 \end{pmatrix}.$$

Fixing the basis above we hence know that $L = \langle M, (b_i, c_i) \mid 1 \leq i \leq 8 \rangle$ where $(\overline{c}_1, \ldots, \overline{c}_8)$ is a basis of $\Lambda_{16}^{\#}/\Lambda_{16}$ so that that the Gram matrix of c is congruent to $-F$ mod $\mathbb{Z}$. Computing orbit representatives of the action of $2.\mathrm{Alt}_{10}$ we find that all non-zero classes in $\Lambda_{16}^{\#}/\Lambda_{16}$ are represented by vectors of norm ≤ 4. In particular $\overline{c}_i = c_i + \Lambda_{16}$ has minimum $18/5$ for $i = 1, \ldots, 8$. To narrow down the possibilities for c_1 and c_2 we use the fact that $\langle \overline{b}_1, \overline{b}_2 \rangle$ contains six classes of minimum $12/5$. So we compute the orbit representatives of the 2-dimensional subspaces of $\Lambda_{16}^{\#}/\Lambda_{16}$ under $2.\mathrm{Alt}_{10}$ to find that there are exactly 47 orbits of such 2-dimensional spaces that contain at least 6 classes of minimum $18/5$. These are our candidates for $\langle \overline{c}_1, \overline{c}_2 \rangle$. All of them contain exactly 6 such classes $(\pm\overline{x}, \pm\overline{y}, \pm\overline{z})$, where the signs are chosen such that (x, y) and (x, z) are in $\frac{-1}{5} + \mathbb{Z}$.

We now fix one of these 47 orbit representatives. As $\mathrm{Aut}(\Lambda_{32})$ acts transitively on the minimal vectors of $\Lambda_{32}^{\#}$ we may assume that $c_1 = x$ is fixed. Then our candidates for c_2 are y and z. For ten of the spaces, the stabiliser of x in the stabiliser of the 2-dimensional subspace interchanges y and z, so we may assume that $c_2 = y$ in these cases and the program splits naturally into 84 subroutines each starting with a different (c_1, c_2).

For a fixed (c_1, c_2) we compute the set $S(c_1, c_2)$ of all vectors v of norm 18/5 in $\Lambda_{16}^{\#}$ so that $(v, c_1) \in -\frac{1}{5} + \mathbb{Z}$ and $(v, c_2) \in \mathbb{Z}$. Then for $i = 3, \ldots, 8$ the list of candidates for c_i is

$$C_i := \{v \in S(c_1, c_2) \mid \min(\langle M, (b_1, c_1), (b_2, c_2), (b_i, v)\rangle) = 6\}.$$

We get sets of different cardinalities varying between $1119 \leq |C_i| \leq 2469$. Usually the smallest set is C_3 with a cardinality between 1119 and 1261. For all we successively construct

$$L_k := \langle M, (b_1, c_1), (b_2, c_2), (b_3, c_3), (b_4, c_4), \ldots, (b_k, c_k)\rangle$$

where the c_i run through C_i and we stop if $\min(L_k) \leq 6$ or if L_k is not integral. If we reach $k = 8$ then we have found an extremal unimodular lattice. It takes between 1 and 2 hours to check all possibilities for a fixed c_3 (depending on the computer and on the deepness of the recursions). In total this algorithm constructs six extremal even unimodular lattices X.

We check that all these six lattices are indeed in the same orbit under $\mathrm{Aut}(M)$. Let P_{48m} be one representative of this orbit. Then $\mathrm{Stab}_{\mathrm{Aut}(M)}(P_{48m})$ has order 1200 and is identified with MAGMA with group number $(1200, 573)$. $\square$

Remark 3.3. *With Bill Unger's improvement of Bernd Souvignier's automorphism group program,* MAGMA *verifies in about two months of computation that the above construction reveals the full automorphism group of the lattice,* $\mathrm{Aut}(P_{48m}) \cong (C_5 \times C_5 \times C_3) : (D_8 Y C_4)$ *of order* 1200.

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