On lattices whose minimal vectors form a 6-design.

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dedicated to Eiichi Bannai in occasion of his 60th birthday

Abstract: Let L be a lattice of dimension $n \leq 24$ such that the minimal vectors of L form a 6-design and generate L. Then L is similar to either the root lattice E_8, the Barnes-Wall lattice BW_{16}, the Leech lattice Λ_{24}, or $n = 23$. For $n = 23$ we conjecture that the only possibilities for L are the shorter Leech lattice O_{23} or its even sublattice Λ_{23}.

1 Introduction.

Spherical designs have been introduced in 1977 by Delsarte, Goethals and Seidel [11] and soon afterwards studied by Eiichi Bannai in a series of papers (see [3], [4], [5] to mention only a few of them). A spherical t-design is a finite subset X of the sphere such that every polynomial on $\mathbb{R}^n$ of total degree at most t has the same average over X as over the entire sphere. The theory of lattices has been used quite successfully to classify good designs of minimal possible cardinality (see [6]). In this paper we use the theory of designs to construct good lattices.

Definition 1.1 *A* t-design-lattice *is a lattice* Λ *in Euclidean space such that its minimal vectors*

$$\mathrm{Min}(\Lambda) := \{\lambda \in \Lambda \mid (\lambda, \lambda) = \min(\Lambda)\}$$

form a spherical t*-design and generate the lattice* Λ.

Clearly any t-design-lattice is also a t'-design-lattice for all $t' \leq t$. Note that the 4-design-lattices are exactly the strongly perfect lattices defined in [14] that are generated by their minimal vectors. They are now classified up to dimension 12 (see [14], [12], [13]). From this classification we see:

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Theorem 1.2 *Let $t \geq 4$ be even and let Λ be a t-design-lattice of dimension $n \leq 12$. Then one of the following holds:*

(a) $n = 1$ *and* Λ *is similar to* $\mathbb{Z}$. *Here* t *is arbitrary since the* 0*-dimensional sphere* S^0 *consists only of the two minimal vectors* $\{1, -1\}$ *of* $\mathbb{Z}$.

(b) $n = 2$, Λ *is similar to the hexagonal lattice* A_2, *and* $t = 4$.

(c) $n = 4$, Λ *is similar to the root lattice* D_4, *and* $t = 4$.

(d) $n = 6$, Λ *is similar to the root lattice* E_6 *or its dual lattice* E_6^*, *and* $t = 4$.

(e) $n = 7$, Λ *is similar to the root lattice* E_7 *or its dual lattice* E_7^*, *and* $t = 4$.

(f) $n = 8$, Λ *is similar to the root lattice* E_8, *and* $t \leq 6$.

(g) $n = 10$, Λ *is similar to the lattice* K'_{10} *or its dual lattice* $(K'_{10})^*$, *and* $t = 4$.

(h) $n = 12$, Λ *is similar to the Coxeter-Todd lattice* K_{12}, *and* $t = 4$.

This paper classifies the 6-design-lattices of dimension $23 \neq n \leq 24$. We will show the following theorem

Theorem 1.3 *Let $t \geq 6$ be even and let Λ be a t-design-lattice of dimension $n \leq 24$. Then one of the following holds:*

(a) $n = 1$ *and* Λ *is similar to* $\mathbb{Z}$.

(b) $n = 8$, Λ *is similar to the root lattice* E_8, *and* $t = 6$.

(c) $n = 16$, Λ *is similar to the Barnes-Wall lattice* BW_{16}, *and* $t = 6$.

(d) $n = 23$ *and* $t = 6$. *In this dimension there are at least two* 6*-design lattices, namely the shorter Leech lattice* O_{23} *and its even sublattice* Λ_{23}.

(e) $n = 24$, Λ *is similar to the Leech lattice* Λ_{24}, *and* $t \leq 10$.

In fact all layers of the lattices in Theorem 1.3‘ are spherical t-designs. This is trivial in case (a) and follows from [2, Corollary 3.1] for the remaining cases except for case (d). For case (d) note that the automorphism group of O_{23} and Λ_{23} is $C_2 \times Co_2$ and its first harmonic invariant has degree 8.

We also remark that it is still unknown, whether there are t-design-lattices for $t \geq 12$. The only known 10-design lattices are the known extremal even unimodular lattices of dimension a multiple of 24, namely the Leech lattice Λ_{24} and the three unimodular lattices P_{48p}, P_{48q} and P_{48n} of dimension 48 with minimum 6 (see [10]).

2 Some general remarks on antipodal t-designs.

In the following we assume that $n \geq 2$ to avoid trivialities. Let $X \subset S^{n-1}$ be a finite subset of the $(n-1)$-dimensional unit-sphere such that $X \cap -X = \emptyset$. For any even number $t = 2h$, the condition that $X \cup -X$ be a spherical t-design is equivalent to the existence of some number c_t such that for all $\alpha \in \mathbb{R}^n$

$$(Dt)(\alpha): \sum_{x \in X} (x, \alpha)^t = c_t |X| (\alpha, \alpha)^h.$$

The constant c_t is then uniquely determined and easily calculated by applying t times the Laplace operator Δ with respect to α (see [14]) as

$$c_t = \prod_{j=1}^{h} \frac{2j-1}{n+2j-2} \text{ (where } t = 2h).$$

Note that

$$\Delta(Dt)(\alpha) = (D(t-2))(\alpha) .$$

If we apply these equalities to the minimal vectors $X \dot{\cup} -X = \mathrm{Min}(\Lambda)$ of a t-design-lattice Λ and some minimal vector $\alpha \in \mathrm{Min}(\Lambda^*)$ of the dual lattice we get lower bounds on the Bergé-Martinet invariant

$$\gamma'(\Lambda)^2 := \gamma(\Lambda)\gamma(\Lambda^*) = \min(\Lambda) \min(\Lambda^*)$$

of a t-design lattice as follows.

Since $(x, \alpha) \in \mathbb{Z}$ for all $x \in X$ and $\alpha \in \Lambda^*$ and the product of $t-1$ consecutive integers is divisible by $(t-1)!$ we get that

$$\frac{1}{(t-1)!} \prod_{j=0}^{h-1} ((x, \alpha)^2 - j^2) \in \mathbb{Z}_{\geq 0} \text{ for all } x \in X, \alpha \in \Lambda^*.$$

Summing over X and applying the equalities (Dt') for all even $0 \leq t' \leq t$ we obtain $Q_{n,t}(z) = |X|\frac{z}{n}P_{n,t}(z)$ for a polynomial $P_{n,t}(z)$ of degree $h-1 = t/2-1$ in $z := (\alpha,\alpha)\min(\Lambda)$. Note that

$$\frac{1}{(t-1)!}Q_{n,t}((\gamma,\gamma)\min(\Lambda)) \in \mathbb{Z}_{\geq 0}$$

for all $\gamma \in \Lambda^*$ in particular $P_{n,t}(z) \geq 0$ for $z = \min(\Lambda^*)\min(\Lambda) = \gamma'(\Lambda)^2$.

For small t, the polynomials $P_{n,t}$ are as follows:

$$\begin{array}{ll} P_{n,2}(z) & = 1 \\ P_{n,4}(z) & = \frac{3}{n+2}z - 1 \\ P_{n,6}(z) & = \frac{3\cdot 5}{(n+2)(n+4)}z^2 - 5\frac{3}{n+2}z + 4 \\ P_{n,8}(z) & = \frac{3\cdot 5\cdot 7}{(n+2)(n+4)(n+6)}z^3 - 14\frac{3\cdot 5}{(n+2)(n+4)}z^2 + 49\frac{3}{n+2}z - 36 \end{array}$$

Remark 2.1 *Let Λ be a 6-design lattice of dimension $n > 1$. Then $\min(\Lambda)\min(\Lambda^*) > \frac{n+2}{3}$, hence Λ is not a strongly perfect lattice of minimal type in the sense of [14, Définition 10.5],*

Proof. Since Λ is strongly perfect, we have $P_{n,4}(\min(\Lambda)\min(\Lambda^*)) \geq 0$ and hence $\min(\Lambda)\min(\Lambda^*) \geq \frac{n+2}{3}$ (see [14, Théorème 10.4]). Assume that $\min(\Lambda)\min(\Lambda^*) = \frac{n+2}{3}$. Then $(\alpha, x) \in \{0, \pm 1\}$ for all $\alpha \in \mathrm{Min}(\Lambda^*)$, $x \in X$ since $P_{n,4}((x,x)(\alpha,\alpha)) = 0$. Hence $\frac{n+2}{3}$ is also a zero of $P_{n,6}(t)$ which implies that $5(n+2) = 3(n+4)$ whence $n = 1$. □

Continuing with an arbitrary antipodal t-design $X \,\dot\cup\, -X \subset S^{n-1}$ where $t = 2h$ we may evaluate $(Dt)(\xi\alpha + \chi\beta)$ for vectors $\alpha, \beta \in \mathbb{R}^n$ and arbitrary $\xi, \chi \in \mathbb{R}$ to find

$$\sum_{x\in X}(x, \xi\alpha+\chi\beta)^t = c_t|X|(\xi\alpha+\chi\beta, \xi\alpha+\chi\beta)^h = c_t|X|(\xi^2(\alpha,\alpha)+2\chi\xi(\alpha,\beta)+\chi^2(\beta,\beta))^h.$$

With the trinomial coefficient

$$\binom{h}{i,j} := \frac{h!}{i!j!(h-i-j)!}$$

and comparing the coefficient at $\xi^\ell\chi^{t-\ell}$ we find the equalities

$$D_{\ell,t-\ell}(\alpha,\beta) : \sum_{x\in X}(x,\alpha)^\ell(x,\beta)^{t-\ell} = \frac{c_t|X|}{\binom{t}{\ell}} \sum_{2i+j=\ell} \binom{h}{i,j} 2^j(\alpha,\alpha)^i(\alpha,\beta)^j(\beta,\beta)^{h-i-j}.$$

In particular if β is orthogonal to α then

$$D_{\ell,t-\ell}(\alpha,\beta) = 0 \text{ if } \ell \text{ is odd and } D_{2\ell,t-2\ell}(\alpha,\beta) = \frac{c_t|X|}{\binom{2h}{2\ell}}\binom{h}{\ell}(\alpha,\alpha)^\ell(\beta,\beta)^{h-\ell}. \tag{1}$$

Important for the classification of t-design lattices Λ are the sets

$$N_i(\alpha) := \{x \in \mathrm{Min}(\Lambda) \mid (x,\alpha) = i\}$$

for $\alpha \in \Lambda^*$.

Theorem 2.2 *Let $t = 2h$ and Λ be a t-design lattice. Let $\alpha \in \Lambda^*$ and $d \in \mathbb{Z}_{\geq 0}$ such that $(\alpha, x) \in \{0, \pm 1, \ldots, \pm(h-d)\}$ for all $x \in \mathrm{Min}(\Lambda)$. If $N_{h-d}(\alpha) \neq \emptyset$ then the projection of $N_{h-d}(\alpha)$ to $\alpha^\perp$ is a spherical $(2d+1)$-design in $\mathbb{R}^{n-1}$.*

Proof. Write $\mathrm{Min}(\Lambda) = X \mathbin{\dot\cup} -X$ and let

$$\overline{N_{h-d}(\alpha)} := \{\overline{x} := x - \frac{h-d}{(\alpha,\alpha)}\alpha \mid x \in N_{h-d}(\alpha)\}$$

denote the projection of $N_{h-d}(\alpha)$ to $\alpha^\perp$. For $\beta \in \alpha^\perp$ and $\ell \in \{0,\ldots,d\}$ the polynomial

$$f_{\ell,h-d}(\beta) := \sum_{x \in X} \prod_{j=0}^{h-d-1}((x,\alpha)^2 - j^2)(x,\beta)^{2\ell} = \sum_{x \in N_{h-d}(\alpha)} (h-d)(2(h-d)-1)!(x,\beta)^{2\ell}$$

Since $\mathrm{Min}(\Lambda)$ is a $2h$-design and the degree of $f_{\ell,h-d}$ is $2(h-d)+2\ell \leq 2h$ this sum is a constant multiple of $(\beta,\beta)^\ell$. Using the fact that $(x,\beta) = (\overline{x},\beta)$ for $\beta \in \alpha^\perp$ we get for all $0 \leq \ell \leq d$

$$\sum_{\overline{x} \in \overline{N_{h-d}(\alpha)}} (\overline{x},\beta)^{2\ell} = c_{n-1,\ell}(\beta,\beta)^\ell$$

and

$$\sum_{\overline{x} \in \overline{N_{h-d}(\alpha)}} (\overline{x},\beta)^{2\ell+1} = 0$$

by (1). This shows that $\overline{N_{h-d}(\alpha)}$ is a spherical $(2d+1)$-design. □

Corollary 2.3 *Let α satisfy the conditions of Theorem 2.2 with $d = 0$ and put $m := \min(\Lambda)$.*

(a) $|N_h(\alpha)| = \frac{1}{h\cdot((t-1)!)} \sum_{x\in X} \prod_{j=0}^{h-1}((x,\alpha)^2 - j^2) = \frac{1}{h\cdot((t-1)!)} \frac{|X|(\alpha,\alpha)m}{n} P_{n,t}((\alpha,\alpha)m)$ *and the cardinalities of the other $N_i(\alpha)$ are determined similarly.*

(b) $\sum_{x\in N_h(\alpha)} x = \frac{|N_h(\alpha)|h}{(\alpha,\alpha)}\alpha$.

Proof. (a) is clear and (b) follows since the projection of $N_h(\alpha)$ is a 1-design and hence the sum $\sum_{\overline{x}\in\overline{N_h(\alpha)}} \overline{x} = 0$ which is equivalent to $\sum_{x\in N_h(\alpha)} x = c\alpha$ for some constant c which is calculated by taking the scalar product with α. □

3 6-design-lattices.

General assumption: Throughout the rest of the paper we assume that Λ is a 6-design-lattice of dimension n, $m := \min(\Lambda)$ and choose $X \subset \mathrm{Min}(\Lambda)$ such that $X \cup -X = \mathrm{Min}(\Lambda)$ and $X \cap -X = \emptyset$. Put $s := |X|$ and $r := \min(\Lambda^*)$.

Then Λ is a 6-design-lattice if and only if for all $\alpha \in \mathbb{R}^n$

$$(D6)(\alpha): \quad \sum_{x\in X}(x,\alpha)^6 = \frac{3\cdot 5sm^3}{n(n+2)(n+4)}(\alpha,\alpha)^3.$$

Applying the Laplace operator to $(D6)(\alpha)$ one obtains

$$(D4)(\alpha): \quad \sum_{x\in X}(x,\alpha)^4 = \frac{3sm^2}{n(n+2)}(\alpha,\alpha)^2 \text{ and}$$

$$(D2)(\alpha): \quad \sum_{x\in X}(x,\alpha)^2 = \frac{sm}{n}(\alpha,\alpha).$$

Substituting $\alpha = \sum_{i=1}^6 \xi_i\alpha_i$ in $(D6)$ (resp. $(D4)$ and $(D2)$) we find that for all $\alpha, \beta \in \mathbb{R}^n$:

$$\begin{array}{lll}
(D11) & \sum_{x\in X}(x,\alpha)(x,\beta) & = \frac{sm}{n}(\alpha,\beta) \\
(D13) & \sum_{x\in X}(x,\alpha)(x,\beta)^3 & = \frac{3sm^2}{n(n+2)}(\alpha,\beta)(\beta,\beta) \\
(D22) & \sum_{x\in X}(x,\alpha)^2(x,\beta)^2 & = \frac{sm^2}{n(n+2)}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \\
(D15) & \sum_{x\in X}(x,\alpha)(x,\beta)^5 & = \frac{3\cdot 5sm^3}{n(n+2)(n+4)}(\beta,\beta)^2(\alpha,\beta) \\
(D24) & \sum_{x\in X}(x,\alpha)^2(x,\beta)^4 & = \frac{3sm^3}{n(n+2)(n+4)}((\beta,\beta)^2(\alpha,\alpha) + 4(\alpha,\beta)^2(\beta,\beta)) \\
(D33) & \sum_{x\in X}(x,\alpha)^3(x,\beta)^3 & = \frac{3sm^3}{n(n+2)(n+4)}(2(\alpha,\beta)^3 + 3(\alpha,\alpha)(\beta,\beta)(\alpha,\beta))
\end{array}$$

From Theorem 2.2 we find the following corollary.

Corollary 3.1 *Let $\alpha \in \Lambda^*$ such that $|(x,\alpha)| \leq 2$ for all $x \in X$. Then*

$$\overline{N_2(\alpha)} := \{\overline{x} := x - \frac{2}{(\alpha,\alpha)}\alpha \mid x \in N_2(\alpha)\} \subset \alpha^\perp \cong \mathbb{R}^{n-1}$$

is a spherical 3-design. In particular $|N_2(\alpha)| \geq 2(n-1)$.

There is one special and important case, where equality follows.

Lemma 3.2 *Assume that $mr = 8$. Then any $\alpha \in \mathrm{Min}(\Lambda^*)$ satisfies the conditions of Corollary 3.1,*

$$|N_2(\alpha)| = 2(n-1) \text{ and } n = \dim(\Lambda) = 16.$$

Moreover $N_2(\alpha) = \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$ such that $\frac{m}{2}\alpha = x_i + y_i$ for all $i = 1, \ldots, n-1$.

Proof. For $x \in N_2(\alpha)$ let $\overline{x} := x - \frac{2}{(\alpha,\alpha)}\alpha \in \alpha^\perp$. Then for all $x, y \in N_2(\alpha)$ we get

$$(\overline{x}, \overline{y}) = (x,y) - \frac{m}{2} = \begin{cases} \frac{m}{2} & x = y \\ \leq 0 & x \neq y \end{cases}$$

since x and y are minimal vectors of a lattice. Hence $\overline{N_2(\alpha)}$ is a set of vectors of length $\frac{m}{2}$ in an $(n-1)$-dimensional space such that distinct vectors have non-positive inner products. Therefore $|\overline{N_2(\alpha)}| \leq 2(n-1)$ (see [13, Lemma 2.10]). Using Corollary 3.1 we get $|N_2(\alpha)| = 2(n-1)$ and again by [13, Lemma 2.10] there is a partition of $N_2(\alpha)$ as claimed.

The only possibilities for n are 8 or 16, since n must be a zero of $P_{n,6}(8) = \frac{5\cdot 3}{(n+2)(n+4)}8^2 - \frac{5\cdot 3}{(n+2)}8 + 4$. Since $\gamma_8 = 2$ the only possibility is that $n = 16$. □

For $n \leq 23$ the Hermite constant $\gamma_n \leq 3.9 < 4$. Therefore

$$\min(\Lambda)\min(\Lambda^*) \leq \gamma_n^2 < 16$$

and hence $(x,\alpha) \in \{0, \pm 1, \pm 2, \pm 3\}$ for all $\alpha \in \mathrm{Min}(\Lambda^*)$, $x \in \mathrm{Min}(\Lambda)$. In particular we get

$$|N_3(\alpha)| = \frac{smr}{360n}\left(\frac{5\cdot 3m^2r^2}{(n+2)(n+4)} - \frac{5\cdot 3mr}{(n+2)} + 4\right) \in \mathbb{Z}.$$

For $n = 24$ one knows by [9] that either Λ is the Leech lattice or $\min(\Lambda)\min(\Lambda^*) < 16$ and hence Corollary 2.3 may also be applied in this situation (keeping in mind that the Leech lattice Λ_{24} is a 11-design-lattice).

As in [13, Lemma 2.4] one gets

Lemma 3.3 *Let $m := \min(\Lambda)$, choose $\alpha \in \Lambda^*$ and put $a := (\alpha, \alpha)$. If $am < 18$ and $|(x, \alpha)| \leq 3$ for all $x \in \mathrm{Min}(\Lambda)$, then*

$$|N_3(\alpha)| \leq \frac{am}{18 - am}.$$

<u>Proof.</u> Let $N_3(\alpha) = \{x_1, \ldots, x_k\}$ and $c := \frac{3k}{a}$. Then $\sum_{x \in N_3(\alpha)} x = c\alpha$. Also $(x_i, x_i) = m$ and $(x_i, x_j) \leq \frac{m}{2}$ because the x_i are minimal vectors in Λ. Hence

$$3c = \frac{9k}{a} = (x_1, c\alpha) = (x_1, x_1) + \sum_{i=2}^{k} (x_1, x_i) \leq m + \frac{m(k-1)}{2} = \frac{m(k+1)}{2}$$

which yields that $k = |N_3(\alpha)| \leq \frac{am}{18-am}$. □

Corollary 3.4 *If $\alpha \in \Lambda^*$ with $|(x, \alpha)| \leq 3$ for all $x \in \mathrm{Min}(\Lambda)$, and $|N_3(\alpha)| = 1$ then $m(\alpha, \alpha) = 9$.*

<u>Proof.</u> Let $N_3(\alpha) = \{x\}$. Then $x = c\alpha$ with $c = \frac{3}{(\alpha,\alpha)}$. In particular

$$m = (x, x) = \frac{3}{(\alpha, \alpha)}(x, \alpha) = \frac{9}{(\alpha, \alpha)}.$$

□

4 Exclusion of most cases.

To perform the first computations we rescale the hypothetical 6-design-lattice Λ such that $m = \min(\Lambda) = 1$. Since Λ is a perfect lattice, we then get $r = \min(\Lambda^*) \in \mathbb{Q}$ is a rational number bounded from above by γ_n^2. For each $n \in \{13, \ldots, 24\}$ the known bounds on the maximal kissing number of n-dimensional lattices as given in [1] yield a finite number of possibilities for s. The number

$$a := \frac{sr}{12n}(\frac{3r}{n+2} - 1) = \frac{1}{12} \sum_{x \in X} (x, \alpha)^2((x, \alpha)^2 - 1)$$

is a positive integer bounded from above by $\frac{s\gamma_n^2}{12n}(\frac{3\gamma_n^2}{n+2}-1)$.

Going through all possibilities for s and a using the fact that r is a positive rational solution of $\frac{sr}{12n}(\frac{3r}{n+2}-1)-a=0$ and that

$$\frac{sr}{n},\ \frac{3sr^2}{n(n+2)},\ \text{and}\ \frac{3\cdot 5sr^3}{n(n+2)(n+4)}$$

as well as

$$\frac{sr}{360n}(\frac{5\cdot 3r^2}{(n+2)(n+4)}-\frac{5\cdot 3r}{(n+2)}+4)$$

are all non-negative integers together with the bounds in [13, Lemma 2.4], Lemma 3.3, Corollary 3.1, Corollary 3.4 and Lemma 3.2 and also the bounds on γ_n given by [8] and the fact that the Leech lattice is the unique 24-dimensional lattice L with $\min(L)\min(L^*)=16$ we find the following theorem.

Theorem 4.1 *Let Λ be a 6-design-lattice of dimension n with $13\leq n\leq 24$. Then*

$$(n,s,mr)=(\dim(\Lambda),\frac{1}{2}|\operatorname{Min}(\Lambda)|,\min(\Lambda)\min(\Lambda^*))$$

are one of the following triples:

(a) $(n=16, s=2160=2^4 3^3 5, mr=8)$.

(b) $(n=23, s=2300, mr=9)$.

(c) $(n=23, s=23\cdot 25\cdot s_1, mr=12)$, *with* $4\leq s_1\leq 96$.

(d) $(n=23, s=23\cdot s_1, mr=15)$, *with* $44\leq s_1\leq 2415$.

(e) $(n=24, s=32760=2^3 3^2 5\cdot 7\cdot 13, mr=12)$

(f) $(n=24, s=98280=2^3 3^3 5\cdot 7\cdot 13, mr=16)$

In case (f), the lattice Λ is the Leech lattice by [9].

Lemma 4.2 *Case (e) of Theorem 4.1 is impossible.*

Proof. Let Λ be a 6-design-lattice of dimension 24 rescaled such that $\min(\Lambda) = 2$. Assume that Λ satisfies the condition (e) of Theorem 4.1 and let $\alpha \in \Lambda^*$. Then $(D6)$ implies that

$$\sum_{x\in X}(x,\alpha)^6 = 3^2 5^2 (\alpha,\alpha)^3$$

in particular $(\alpha,\alpha) \in \mathbb{Z}$. Moreover for $\alpha, \beta \in \Lambda^*$ we get

$$\sum_{x\in X}(x,\alpha)^3(x,\beta)^3 = 3^2 5(3(\alpha,\alpha)(\beta,\beta)(\alpha,\beta) + 2(\alpha,\beta)^3)$$

which shows that $\Gamma := \Lambda^*$ is an integral lattice with minimum $\min(\Gamma) = 6$ and $\min(\Gamma^*) = 2$. Fix some $\alpha \in \mathrm{Min}(\Gamma)$ and choose X such that $(x,\alpha) \geq 0$ for all $x \in X$. Then $X = X_0 \cup X_1 \cup X_2 \cup X_3$ with $X_i := \{x \in X \mid (x,\alpha) = i\}$. By Corollary 2.3 $X_3 = \{x_3, y_3\}$ with $(x_3, y_3) = 1$ and $x_3 + y_3 = \alpha$. Equalities $(D2)$, $(D4)$ and $(D6)$ yield that $|X_2| = 513$ and $|X_1| = 14310$. For all $x_2 \in X_2$ we have $2 = (x_2,\alpha) = (x_2,x_3) + (x_2,y_3)$ and therefore $(x_2,x_3) = (x_2,y_3) = 1$ since both scalar products are ≤ 1. The equalities $(D22)$ and $(D24)$ for x_3 and α read as

$$\begin{array}{lll} \sum_{x\in X}(x,x_3)^2(x,\alpha)^2 & = S_1 + 4S_2 + 9(4+1) & = 2\cdot 3\cdot 5\cdot 7(2\cdot 9 + 6\cdot 2) = 6300 \\ \sum_{x\in X}(x,x_3)^2(x,\alpha)^4 & = S_1 + 16S_2 + 81(4+1) & = 3^2 5(6^2 2 + 4\cdot 9\cdot 6) = 12960 \end{array}$$

where

$$S_1 := \sum_{x\in X_1}(x,x_3)^2 \text{ and } S_2 := \sum_{x\in X_2}(x,x_3)^2.$$

This system has the unique solution

$$S_1 = 4155,\ S_2 = 525$$

contradicting the fact that $S_2 = |X_2| = 513$. □

5 Dimension 16

In this section we deal with the first case in Theorem 4.1. We show

Theorem 5.1 *Let Λ be a 6-design lattice of dimension 16. Then Λ is similar to the Barnes-Wall lattice.*

Proof. Rescale Λ such that $\min(\Lambda) = 2$ and let $\Gamma := \Lambda^*$. Then by Theorem 4.1

$$s(\Lambda) = 2160, \min(\Gamma) = 4 \ .$$

From Equation (D6) we find that

$$\sum_{x \in X} (x, \alpha)^6 = 3^2 5 (\alpha, \alpha)^3$$

hence $(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Gamma$. Moreover

$$\frac{1}{120} \sum_{x \in X} (x, \alpha)^2 ((x, \alpha)^2 - 1)((x, \alpha)^2 - 4) = \frac{1}{120} (D6(\alpha) - 5D4(\alpha) + 4D2(\alpha)) \in \mathbb{Z}$$

for all $\alpha \in \Gamma$ yields

$$\frac{3}{8} (\alpha, \alpha)((\alpha, \alpha) - 4)((\alpha, \alpha) - 6) \in \mathbb{Z}$$

for all $\alpha \in \Gamma$, hence Γ is an even lattice.

Now we fix $\alpha \in \Gamma$ with $(\alpha, \alpha) = 4$. By Lemma 3.2 we find that

$$L := \langle N_2(\alpha), \alpha \rangle \cong D_{16}$$

is the root lattice D_{16}. Moreover we have $L \leq \Lambda$ and $\Gamma = \Lambda^* \leq L^*$. Since Γ is an even lattice, we even get that $\Gamma \leq M$, where M is the unique maximal even sublattice of L^*, M is isometric to the even unimodular lattice D_{16}^+,

We now want to show that $2L \subseteq \Gamma$. Since Λ is generated by X, it suffices to show that

$$(x, \beta) \in \{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$$

for all $x \in X$ and $\beta \in N_2(\alpha)$. Fix some $\beta \in N_2(\alpha)$. Then $\alpha = \beta + \beta'$ for some $\beta' \in N_2(\alpha)$ and $(\beta, x) = 1$ for all $x \in N_2(\alpha) - \{\beta, \beta'\}$. Choose X such that $(x, \alpha) \geq 0$ for all $x \in X$ and $(x, \beta) \geq 0$ for all $x \in N_0(\alpha)$. Since we know the scalar products of β with all elements of $N_2(\alpha)$ the equalities $(D11), (D22), (D13), (D24), (D15)$ applied to α and β yield

$$\begin{array}{lll} S1 & := \sum_{x \in N_1(\alpha)} (x, \beta) & = 480 \\ S2 & := \sum_{x \in N_1(\alpha)} (x, \beta)^2 & = 352 \\ S3 & := \sum_{x \in N_1(\alpha)} (x, \beta)^3 & = 288 \\ S4 & := \sum_{x \in N_1(\alpha)} (x, \beta)^4 & = 256 \\ S5 & := \sum_{x \in N_1(\alpha)} (x, \beta)^5 & = 240 \end{array}$$

Since β and β' are shortest vectors of Λ, and $(x, \beta + \beta') = (x, \alpha) = 1$ for all $x \in N_1(\alpha)$, we get

$$0 \le (x, \beta) \le 1 \text{ for all } x \in N_1(\alpha) \ .$$

In particular

$$(x, \beta)((x, \beta) - \frac{1}{2})^2((x, \beta) - 1)^2 \ge 0 \text{ for all } x \in N_1(\alpha).$$

Summing over all $x \in N_1(\alpha)$ we find

$$S5 - 3S4 + \frac{13}{4}S3 - \frac{3}{2}S2 + \frac{1}{4}S1 = 0.$$

Hence $(x, \beta) \in \{0, 1/2, 1\}$ for all $x \in N_1(\alpha)$. We also obtain the exact cardinalities $m_i := |\{x \in N_1(\alpha) \mid (x, \beta) = i\}|$ as

$$m_0 = 224, m_{1/2} = 512, m_1 = 224 \ .$$

We now consider the elements in $X_0 := \{x \in X \mid (x, \alpha) = 0\}$. Explicit calculations show that $Y_0 := X_0 \cap L$ contains 210 elements, 28 of which have scalar product 1 with β, the remaining 182 are perpendicular to β. Let $Z_0 := X_0 - Y_0$. From equalities $(D2), (D4), (D6)$ applied to β (using the fact that we know the inner products (β, x) for all $x \in X - Z_0$) we obtain

$$\begin{array}{lll} T2 & := \sum_{x \in Z_0} (x, \beta)^2 & = 128 \\ T4 & := \sum_{x \in Z_0} (x, \beta)^4 & = 32 \\ T6 & := \sum_{x \in Z_0} (x, \beta)^6 & = 8 \end{array}$$

The square $(x, \beta)^2((x, \beta)^2 - 1/4)^2$ is non-negative for all $x \in Z_0$. Summing up we obtain

$$\sum_{x \in Z_0} (x, \beta)^2((x, \beta)^2 - 1/4)^2 = T6 - \frac{1}{2}T4 + \frac{1}{16}T2 = 0$$

which shows that $(x, \beta) \in \{0, \pm 1/2\}$ for all $x \in Z_0$.

Therefore

$$2M \subset 2L \subset \Gamma \subset M \cong D_{16}^+ \ .$$

Starting with $N_0 := M$, we now successively calculate the $\mathrm{Aut}(N_i)$-orbits on the sublattices N_{i+1} of index 2 in N_i. In each step there is a unique orbit of sublattices N_{i+1} such that the minimum of the dual lattice is $\min(N_{i+1}^*) \ge 2$ (for $0 \le i \le 3$). The unique lattice with minimum 4 is $N_4 \cong \mathrm{BW}_{16}$. □

6 Dimension 23

From the classification of tight 7-designs in [7] we see

Theorem 6.1 *Let* Λ *be a 6-design lattice of dimension* 23. *Then* $s(\Lambda) \geq 2300$ *and if* $s(\Lambda) = 2300$ *then* Λ *is similar to* O_{23}.

To finish the proof of Theorem 1.3 it remains to show that any 6-design lattice Λ of dimension 23 is not an 8-design lattice. If Λ satisfies case (b) of Theorem 4.1, then the minimal vectors of Λ form a tight 7-design and hence cannot be an 8-design. In the other two cases ((c) and (d) of Theorem 4.1) $\gamma'(\Lambda)^2 \in \{12, 15\}$ and hence $P_{23,6}(\gamma'(\Lambda)^2) = 0$ so $(x, \alpha) \in \{0, \pm 1, \pm 2\}$ for all $x \in \mathrm{Min}(\Lambda)$ and $\alpha \in \mathrm{Min}(\Lambda^*)$. If $\mathrm{Min}(\Lambda)$ is an 8-design, then also $P_{23,8}(\gamma'(\Lambda)^2) = 0$. But this polynomial has no rational roots. This finishes the proof of Theorem 1.3.

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