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Block Squares

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Abstract. A block square is a certain 1-block design. We show that there are exactly two block squares in which each block consists of 5 elements and each element is contained in 6 blocks.

1. Introduction

The notion of block squares arose in connection of a better understanding of a mathematical structure which was called *combinatorial magic square* in [1]. This structure had its origin in a study of extremal codes of length 48, see [1] for details.

A block square is a finite geometrical structure whose points are the blocks of a 1-block design. (See e.g. [1, p. 88] for the notion of 1-design.) At least for small parameters block squares are rare phenomena. Let λ be the number of blocks containing a fixed element of the base set S. From our axioms in Section 2 follows $\lambda \equiv 0 \pmod 2$ and $\lambda \geq 4$. For $\lambda = 4$ block squares exist if and only if the cardinality k of the blocks is of the form $k = 2^i - 1$, $i = 2, 3, \ldots$ and for given i such block squares are unique up to equivalence (Section 3). If $\lambda = 6$ then k must be greater or equal to 5. The magic squares of [1] appear for $\lambda = 6$ and $k = 5$. The main result of the paper at hand consists in showing that up to equivalence there are exactly two block squares with $\lambda = 6$ and $k = 5$ (see Table 2 and Table 3). A block square with $\lambda = 6$ and $k = 5$ consists of 36 blocks which are arranged in 6 rows of 6 blocks. They are the subject of conditions described in Definition 2.3. We study block squares by considering first pairs of lines. One finds that there are up to equivalence 41 pairs of lines. Then we use a computer program to show that only 5 of these pairs lead to block squares.

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2. Block Squares

2.1. Block Lines

Definition 2.1. A 1-block design $\mathcal{L}$ with base set R is called a block line if the following two conditions are satisfied:

a) $|A \cap B| = 1$ for any pair $A, B \in \mathcal{L}$ with $A \neq B$.

b) Any $a \in R$ is contained in two and only two blocks of $\mathcal{L}$.

The following proposition is an immediate consequence of the definition above.

Proposition 2.2. *Let k be the cardinality of the blocks of a block line $\mathcal{L}$. Then $\mathcal{L}$ consists of $k+1$ blocks and $|R| = (1/2)(k+1)k$. The block line $\mathcal{L}$ is up to equivalence uniquely determined by k. There exists a block line for any $k > 1$.*

We call $k+1$ the *size* of the line $\mathcal{L}$. Now, let M be any finite set. A *line of size* $k+1$ in M is a block line $(\mathcal{L}, S)$ with $S \subset M$ and $|\mathcal{L}| = k+1$.

2.2. Definition of Block Squares

Let $\mathcal{D}$ be a 1-block design with base set S. A block line in $\mathcal{D}$ is a subsystem $\mathcal{L}$ of $\mathcal{D}$ with base set $R \subseteq S$ such that $\mathcal{L}$ is a block line in the sense of Definition 2.1.

Definition 2.3. A 1-block design $\mathcal{D}$ is called a block square if the following conditions are satisfied:

a) $|A \cap B| \in \{0, 1, 2\}$ for any pair $A, B \in \mathcal{D}$ with $A \neq B$.

b) For any pair $A, B \in \mathcal{D}$ with $|A \cap B| = 1$ there is one and only one block line $\mathcal{L}$ in $\mathcal{D}$ with $A, B \in \mathcal{L}$. The block line containing A and B will be denoted by AB.

c) For any $A \in \mathcal{D}$ there are three and only three block lines containing A.

d) Let $\mathcal{L}$ be a block line of $\mathcal{D}$ and $A \in \mathcal{D} - L$. Then there are two and only two blocks B, C in $\mathcal{L}$ such that $|A \cap B| = |A \cap C| = 1$.

Proposition 2.4. *Let $\mathcal{L}$ be a block line of $\mathcal{D}$ and $A \in \mathcal{D} - L$. Then there is one and only one block line $\mathcal{L}'$ such that $A \in \mathcal{L}'$ and $\mathcal{L} \cap \mathcal{L}' = \emptyset$.*

Proof. By d) and b) we have block lines AB and AC with $B, C \in \mathcal{L}$ which are different since otherwise by b) $AB = AC = BC = \mathcal{L}$. Hence by c) there is one and only one further block line $\mathcal{L}'$ through A, and $\mathcal{L}'$ has no block in common with $\mathcal{L}$. □

We call $\mathcal{L}'$ the parallel line to $\mathcal{L}$ through A.

Proposition 2.5. *Let $\mathcal{L}_1, \mathcal{L}_2$ be block lines with $\mathcal{L}_1 \cap \mathcal{L}_2 = \{A\}$.
Let $B, C \in \mathcal{L}_1 - \{A\}$, $B \neq C$ and let $\mathcal{L}_B$ resp. $\mathcal{L}_C$ be the parallel to $\mathcal{L}_2$ through B resp. C. Then $\mathcal{L}_B \cap \mathcal{L}_C = \emptyset$.*

P r o o f . Assume that $\mathcal{L}_B \cap \mathcal{L}_C$ is not empty and $D \in \mathcal{L}_B \cap \mathcal{L}_C$. Then $\mathcal{L}_B = \mathcal{L}_C$ by Proposition 2.4. This implies $\mathcal{L}_B = \mathcal{L}_C = BC = \mathcal{L}_1$ in contradiction to the assumptions of Proposition 2.5. □

Proposition 2.6. *Let $\mathcal{L}$ be a block line and let $\{\mathcal{L}_i | i \in I\}$ be the system of all block lines parallel to $\mathcal{L}$ (including $\mathcal{L}$). Then $\bigcup_{i \in I} \mathcal{L}_i = \mathcal{D}$ and $|I| = |\mathcal{L}|$.*

P r o o f . $\bigcup_{i \in I} \mathcal{L}_i = \mathcal{D}$ follows immediately from Proposition 2.4. Let $\mathcal{L}'$ be an arbitrary block line which is not parallel to $\mathcal{L}$. Then $\mathcal{L}'$ is not parallel to any $\mathcal{L}_i$ by Proposition 2.5. Hence $\mathcal{L}' \cap \mathcal{L}_i = \{A_i\}$, $\mathcal{L}' = \{A_i | i \in I\}$. □

The set of all block lines of $\mathcal{D}$ splits in three classes $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$ of parallel lines. We fix $\mathfrak{K}_i, \mathfrak{K}_j$ with $i \neq j$. Then any $A \in \mathcal{D}$ lies in one and only one line of $\mathfrak{K}_i$ and of $\mathfrak{K}_j$. This explains the name block square.

Let a be a fixed element in the base set S of $\mathcal{D}$. If a appears in a block line $\mathcal{L}$ of $\mathcal{D}$ then a appears exactly in two blocks of $\mathcal{L}$. This implies that the number λ of blocks of $\mathcal{D}$ containing a is even and greater or equal to 4.

2.3. The Connection with Binary Codes

Let $(S, \mathcal{D})$ be a block square, let T be the set of lines of $(S, \mathcal{D})$ and let $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ be the three classes of parallel lines.

We consider the disjoint union of S and T as the set of places of a binary code $C(\mathcal{D}) \subset \mathbb{F}_2^{S \cup T}$ with generating code words which are defined as follows:

We consider the words of $C(\mathcal{D})$ as subsets of $S \cup T$. To $w \in \mathbb{F}_2^{S \cup T}$ corresponds the subset of places a of $S \cup T$ with $w(a) = 1$.

Define

$$1 := S \cup T, \ \ w_{ij} := \mathfrak{K}_i \cup \mathfrak{K}_j \ \text{ for } \ \{i,j\} \subseteq \{1,2,3\}, i \neq j.$$

For any $B \in \mathcal{D}$ we have a $w_B \in C(\mathcal{D})$:

$$w_B := B \cup \mathfrak{K}_1 - \{\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3\},$$

where $\mathcal{L}_i$ is the line in $\mathfrak{K}_i$ through B.

Then $C(\mathcal{D})$ is generated by $1, w_{ij}, (i \neq j \in \{1,2,3\})$ and $w_B (B \in \mathcal{D})$.

In the case $\lambda = 6$, $k = 5$ one has $|S \cup T| = 48$. The two nonequivalent block squares $(S, \mathcal{D})$ of Table 2 and 3 lead to self-dual doubly even codes $C(\mathcal{D})$ of minimal weight 8. Note that an extremal self-dual doubly even code of length 48 has minimal weight 12.

2.4. The Case $\lambda = 4$

In this section we consider first the case $\lambda = 4$.

Let A_1, A_2, A_3, A_4 be the four blocks containing a fixed element a of S. Then axiom c) implies $|A_i \cap A_j| = 1$ for $i,j \in \{1,2,3,4\}$, $i \neq j$. It follows $|A \cap B| \in \{0,1\}$ for all $A, B \in \mathcal{D}$, $A \neq B$.

Proposition 2.7. *A block square with $\lambda = 4$ exists if and only if the cardinality k of the blocks has the form $k = 2^i - 1$, $i = 2, 3, 4, \ldots$, and the block square for given k of this form is unique up to equivalence.*

Proof. We construct block squares $\mathcal{D}_i$ with $k = 2^i - 1$ with a fixed order of the classes $\mathfrak{K}_{1i}, \mathfrak{K}_{2i}, \mathfrak{K}_{3i}$ of parallel lines of $\mathcal{D}_i$ by induction over i.

We begin with $i = 1$; $S_1 = \{e\}$, $A_{00} = A_{01} = A_{10} = A_{11} = \{e\}$,

$$\begin{aligned} \mathcal{D}_1 &= \{A_{00}, A_{01}, A_{10}, A_{11}\}, \\ \mathfrak{K}_{11} &= \{A_{00}A_{01}, A_{10}A_{11}\}, \\ \mathfrak{K}_{21} &= \{A_{00}A_{10}, A_{01}A_{11}\}, \\ \mathfrak{K}_{31} &= \{A_{00}A_{11}, A_{01}A_{10}\}. \end{aligned}$$

We consider $A_{00}, A_{01}, A_{10}, A_{11}$ as four distinct points which are identical as sets. $\mathcal{D}_1$ is of course not a block square, but it serves as first step of our induction procedure.

Now assume that $\mathcal{D}_i$ is already constructed for a certain i and has base set S_i.

The blocks of $\mathcal{D}_{i+1}$ will be denoted by $(A)_{00}$, $(A)_{01}$, $(A)_{10}$, $(A)_{11}$ for $A \in \mathcal{D}_i$. The base set S_{i+1} of $\mathcal{D}_{i+1}$ is the union of two disjoint sets

$$\{(a)_{\nu\mu} | a \in S_i, \nu, \mu \in \{0,1\}\}$$

and

$$\{a_{AB} | A, B \in \mathcal{D}_i, |A \cap B| = 1, AB \in \mathfrak{K}_{3i}\} \cup \{a_{AA} | A \in \mathcal{D}_i\}.$$

The sets $(A)_{\nu\mu}$ are defined as follows:

$$\begin{aligned} (A)_{00} &= \{(a)_{00} | a \in A\} \cup \{a_{AB} | B \in \mathcal{D}_i, AB \in \mathfrak{K}_{3i}\} \cup \{a_{AA}\}, \\ (A)_{01} &= \{(a)_{01} | a \in A\} \cup \{a_{CD} | CA \in \mathfrak{K}_{1i}, DA \in \mathfrak{K}_{2i}\} \cup \{a_{AA}\}, \\ (A)_{10} &= \{(a)_{10} | a \in A\} \cup \{a_{CD} | CA \in \mathfrak{K}_{2i}, DA \in \mathfrak{K}_{1i}\} \cup \{a_{AA}\}, \\ (A)_{11} &= \{(a)_{11} | a \in A\} \cup \{a_{BA} | B \in \mathcal{D}_i, AB \in \mathfrak{K}_{3i}\} \cup \{a_{AA}\}. \end{aligned}$$

With this definition it is easy to see that $\mathcal{D}_{i+1}$ is a block square. The order of $\mathfrak{K}_{1,i+1}$, $\mathfrak{K}_{2,i+1}$, $\mathfrak{K}_{3,i+1}$ is defined as the order corresponding to $\mathfrak{K}_{1,i}$, $\mathfrak{K}_{2,i}$, $\mathfrak{K}_{3,i}$.

The block square $\mathcal{D}_2$ looks as follows:

$(e)_{00}$ $a_{A_{00}A_{00}}, a_{A_{00}A_{11}}$	$(e)_{00}$ $a_{A_{01}A_{01}}, a_{A_{01}A_{10}}$	$(e)_{01}$ $a_{A_{00}A_{00}}, a_{A_{01}A_{10}}$	$(e)_{01}$ $a_{A_{01}A_{01}}, a_{A_{00}A_{11}}$
$(e)_{00}$ $a_{A_{10}A_{10}}, a_{A_{10}A_{01}}$	$(e)_{00}$ $a_{A_{11}A_{11}}, a_{A_{11}A_{00}}$	$(e)_{01}$ $a_{A_{10}A_{10}}, a_{A_{11}A_{00}}$	$(e)_{01}$ $a_{A_{11}A_{11}}, a_{A_{10}A_{01}}$
$(e)_{10}$ $a_{A_{00}A_{00}}, a_{A_{10}A_{01}}$	$(e)_{10}$ $a_{A_{01}A_{01}}, a_{A_{11}A_{00}}$	$(e)_{11}$ $a_{A_{00}A_{00}}, a_{A_{11}A_{00}}$	$(e)_{11}$ $a_{A_{01}A_{01}}, a_{A_{10}A_{01}}$
$(e)_{10}$ $a_{A_{10}A_{10}}, a_{A_{00}A_{11}}$	$(e)_{10}$ $a_{A_{11}A_{11}}, a_{A_{01}A_{10}}$	$(e)_{11}$ $a_{A_{10}A_{10}}, a_{A_{01}A_{10}}$	$(e)_{11}$ $a_{A_{11}A_{11}}, a_{A_{00}A_{11}}$

Table 1

The uniqueness and non-existence statement of Proposition 2.7 can be proved by induction, too. If $2^i - 1 < k \leq 2^{i+1} - 1$ and $\mathcal{D}$ is a block square with base set S, $r = 4$ and block cardinality k, one finds in $\mathcal{D}$ a subsquare $\mathcal{D}'_i \subset \mathcal{D}$ such that the elements of S which appear in $\mathcal{D}'_i$ in four blocks are the base set S''_i for a block square $\mathcal{D}''_i := \{A \cap S'' | A \in \mathcal{D}'_i\}$. Then $\mathcal{D}''_i$ is equivalent to $\mathcal{D}_i$ by induction. Furthermore, $\mathcal{D}''_i$ can be continued to $\mathcal{D}$ only if $k = 2^{i+1} - 1$ and in a unique way up to equivalence. □

2.5. The Case $\lambda = 6$

Let $\mathcal{D}$ be a block square with $\lambda = 6$. Since λ divides $k(k+1)^2$ the smallest possible values for k are $k = 2, 3, 5$. It is easily to be seen that the case $k = 2$ is impossible. In Section 3.6 we show that $k = 3$ is impossible as well.

Our main result concerns the case $k = 5$. We prove the following

Theorem 2.8. *Up to equivalence there are exactly two block squares with $\lambda = 6$ and $k = 5$. With the base set $S = \{1, 2, \ldots, 30\}$ these block squares have the following form*

1,3, 6,7,8	2,5 7,9,10	2,3, 11,12,13	4,8, 9,11,14	1,4, 5,12,15	6,10, 13,14,15
1,4, 16,17,18	2,3, 16,19,20	3,6, 17,21,22	2,4, 5,21,23	5,18, 19,22,24	1,6, 20,23,24
4,6, 9,10,21	9,16, 25,26,27	6,13, 20,25,28	14,21, 26,28,29	4,13, 14,16,19	10,19, 20,27,29
7,10, 18,22,29	10,15, 20,25,30	2,17, 18,20,23	2,7, 11,25,26	14,15 17,22,26	11,14, 23,29,30
3,16, 21,27,29	3,7, 12,15,26	12,21, 23,28,30	7,8, 23,24,29	1,16, 24,26,28	1,8, 15,27,30
8,9, 17,22,27	5,12, 19,27,30	11,18, 22,25,30	5,9, 24,25,28	12,13, 17,18,28	8,11 13,19,24

Table 2

1,3, 4,5,6	1,2, 7,8,9	2,3, 10,11,12	4,7, 10,13,14	5,8, 11,13,15	6,9, 12,14,15
1,2, 16,17,18	2,3, 19,20,21	1,3, 22,23,24	4,6, 16,19,22	4,5, 17,20,23	5,6, 18,21,24
2,3, 25,26,27	1,3, 13,14,15	1,2, 28,29,30	13,22, 24,25,28	15,22 23,26,29	14,23, 24,27,30
4,7, 9,17,26	9,15, 21,28,29	10,17 18,24,28	7,19, 21,24,27	4,10, 11,19,29	11,15, 18,26,27
5,7, 8,18,27	7,14, 19,29,30	11,16 18,22,29	12,14, 16,25,27	8,19, 20,22,25	5,11 12,20,30
6,8, 9,16,25	8,13, 20,28,30	12,16, 17,23,30	6,10, 12,21,28	10,13, 17,25,26	9,20 21,23,26

Table 3

The classification has been accomplished by the following method which led us to the block squares of Table 2 and 3: One fixes two classes $\mathfrak{K}_1, \mathfrak{K}_2$ of parallels. The lines of $\mathfrak{K}_1$ are called rows and the lines of $\mathfrak{K}_2$ are called columns. We consider two rows $\mathcal{L}_1, \mathcal{L}_2$. The blocks of the rows in the same column have one element in common. For the six columns we get six elements $a_1, \ldots, a_6$. Each $a_1, \ldots, a_6$ appears a second time in $\mathcal{L}_1$ and $\mathcal{L}_2$. The blocks of $\mathcal{L}_1$ and of $\mathcal{L}_2$ have no other element of S in common. Hence the lines $\mathcal{L}_1, \mathcal{L}_2$ are given up to equivalence by a two-coloured directed graph Γ with 6 vertices $a_1, \ldots, a_6$ and 6 edges for both colours. By definition there is an edge from a_i to a_j if a_i stands in the block given by a_j of $\mathcal{L}_1$ resp. $\mathcal{L}_2$. The axioms for block squares imply several conditions restricting the possibilities for Γ.

Starting from $\mathcal{L}_1, \mathcal{L}_2$ given by some Γ one tries to complete the two lines to a block square.

In the next chapter we describe the procedure leading from two parallel lines to two-coloured directed graphs in detail. In Chapter 4 we give the classification of these graphs and in Chapter 5 we describe the computer program which has the block squares of Table 2 and 3 as its result.

3. Two Parallel Lines

In this section we investigate pairs of parallel lines in a block square with $\lambda = 6$. We formulate conditions which such pairs have to satisfy and call two parallels which satisfy those conditions admissible. We then give the full classification of admissible parallel lines in the case $k = 5$.

3.1. Admissible Pairs of Lines in Block Squares

Let $\mathcal{L}_r$ and $\mathcal{L}_g$ be two parallel lines of the block square $(M, \mathcal{B})$. Furthermore, let $\mathfrak{K}$ be the system of parallels belonging to $\mathcal{L}_r$ and let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be the other systems (Section 2.2).

If $\mathcal{L} \in \mathfrak{K}_1$, then $\mathcal{L} \cap \mathcal{L}_r = \{B_r\}$, and one has a one-to-one mapping of $\mathfrak{K}_1$ onto $\mathcal{L}_r$ given by $\mathcal{L} \to B_r$. Correspondingly $\mathcal{L} \cap \mathcal{L}_g = \{B_g\}$ determines a one-to-one mapping of $\mathfrak{K}_1$ onto $\mathcal{L}_g$. Furthermore,

$$B_r \cap B_g = \{a\}, \;\; a \in M.$$

We call $a = \alpha(\mathcal{L})$ the characteristic element of $(\mathcal{L}_r, \mathcal{L}_g)$ with respect to $\mathcal{L}$.

Proposition 3.1. *The map α from $\mathfrak{K}_1$ into M given by $\mathcal{L} \to \alpha(\mathcal{L})$ is injective.*

Proof. Suppose $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{K}_1$, $\mathcal{L}_1 \neq \mathcal{L}_2$, and $a = \alpha(\mathcal{L}_1) = \alpha(\mathcal{L}_2)$. This contradicts the distribution of the blocks B of $\mathcal{B}$ with $a \in B$. In fact, beside the four such blocks in the rows $\mathcal{L}_r$ and $\mathcal{L}_g$ we have two further blocks B_1, B_2 containing a. Since B_1 and B_2 belong to a parallel line of $\mathcal{L}_r$, it follows that in the line of $\mathfrak{K}_1$ through B_1 there is no other block containing a in contradiction to the axioms of block squares. □

Let C be the image of α. Proposition 3.1 shows that we have one-to-one maps α_r and α_g from $\mathcal{L}_r$ and $\mathcal{L}_g$ onto C given by

$$\mathcal{L} \cap \mathcal{L}_r \to \alpha_r(\mathcal{L}), \;\; \mathcal{L} \cap \mathcal{L}_g \to \alpha_g(\mathcal{L}).$$

In the following we put

$$B_a^r := \alpha_r^{-1}(a), \;\; B_a^g := \alpha_g^{-1}(a) \;\text{ for }\; a \in C.$$

Proposition 3.2.

$$B_a^r \cap B_b^g \subset C \;\; \textit{for} \;\; a, b \in C.$$

Proof. Suppose $s \in B_a^r \cap B_b^g$ with $s \in M - C$. Then s appears twice in blocks of $\mathcal{L}_r$ and of $\mathcal{L}_g$. This contradicts the distribution of blocks B with $s \in B$. The situation is illustrated in the following example with $k = 5$ and $C = \{1, 2, 3, 4, 5, 6\}$. We write the blocks of $\mathcal{L}_r$ as boxes in a first row and the blocks of $\mathcal{L}_g$ as boxes in a second row such that the boxes belonging to the same $a \in C$ form the columns of the rectangle:

$1, s$	$2, s$	3	4	5	6
1	2	$3, s$	$4, s$	5	6

□

Proposition 3.3. *There are permutations π and ε of C such that $\pi(c) \neq c$ for $c \in C$ and*

$$B_c^r \cap B_{\pi(c)}^g = \{\varepsilon(c)\} \;\text{ for }\; c \in C. \tag{3.1}$$

If c, d are such elements of C that $d \notin \{c, \pi(c)\}$, then $B_c^r \cap B_d^g$ has cardinality 0 or 2.

Proof. $\pi(c)$ is given by the line $\mathcal{L}$ in $\mathfrak{K}_2$ containing B_c^r:

$$\{B_{\pi(c)}^g\} = \mathcal{L} \cap \mathcal{L}_g. \tag{3.2}$$

By Definition 2.1 we have $c \neq \pi(c)$ and $|(B_c^r \cap B_{\pi(c)}^g)| = 1$. Proposition 3.2 shows that the unique element in $B_c^r \cap B_{\pi(c)}^g$ is in C. Now considering Proposition 3.1 for $\mathfrak{K}_2$ instead of $\mathfrak{K}_1$ we see that ε defined by equation 3.1 as well as π is a permutation of C. □

If we make the construction above for $\mathfrak{K}_2$ instead of $\mathfrak{K}_1$ we get mappings of $\mathcal{L}_r$ resp. $\mathcal{L}_g$ onto C, which we denote by β_r resp. β_g.

We put

$$D_a^r := \beta_r^{-1}(a), \;\; D_a^g := \beta_g^{-1}(a).$$

For $\mathcal{L} \in \mathfrak{K}_2$ with $\mathcal{L} \cap \mathcal{L}_r = \{B_c^r\}$ we have $\mathcal{L} \cap \mathcal{L}_g = \{B_{\pi(c)}^g\}$. Hence equation 3.1 shows that

$$\beta_r(B_c^r) = \varepsilon(c), \;\; \beta_g(B_{\pi(c)}^g) = \varepsilon(c),$$

therefore

$$D^r_a = B^r_{\varepsilon^{-1}(a)}, \ \ D^g_a = B^g_{\pi\varepsilon^{-1}(a)}. \tag{3.3}$$

Example. We use the notation of the example in the proof of Proposition 3.2:

(3.4)

1,3,4,5	2,1,6	3,2	4	5	6
1	2	3	4,1,6	5,4	6,2,3,5

We put in the boxes only the elements which are common in $\mathcal{L}_r$ and $\mathcal{L}_g$, i.e. the elements of C. To get the full blocks of $\mathcal{L}_r$ and $\mathcal{L}_g$ one has to add 9 elements in $\mathcal{L}_r$ and in $\mathcal{L}_g$. We denote them by $7, 8, \dots, 15$; $16, 17, \dots, 24$. Then we get

1,3,4,5,7	2,1,6,8,9	3,2,10,11,12	4,8,10,13,14	5,9,11,13,15	6,7,12,14,15
1 16,17,18,19	2 16,20,21,22	3 17,20,23,24	4,1,6 21,23	5,4 18,22,24	6,2,3,5 19

One finds easily

$$\pi = \begin{pmatrix} 1,2,3,4,5,6 \\ 3,1,2,5,6,4 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1,2,3,4,5,6 \\ 3,1,2,4,5,6 \end{pmatrix}$$

3.2. Definition of Admissible Pairs of Lines

Now we consider pairs of parallel lines $(\mathcal{L}_r, \mathcal{L}_g)$ in a more abstract manner.

Let M be a finite set and C a subset of M of cardinality l. Let $(\mathcal{L}_r, S_r)$ and $(\mathcal{L}_g, S_g)$ be lines in M of size l with $C \subseteq S_r$, $C \subseteq S_g$.

Then a *pair of lines* $\mathcal{L}_r, \mathcal{L}_g$ *associated to* C is a pair of one-to-one mappings

$$\alpha_r : \mathcal{L}_r \to C, \ \ \alpha_g : \mathcal{L}_g \to C.$$

The block corresponding to $c \in C$ in $\mathcal{L}_r$ resp. $\mathcal{L}_g$ will be denoted by B^r_c resp. B^g_c. Such a pair $(C(\mathcal{L}_r, \mathcal{L}_g))$ is called *admissible* if the following conditions are fulfilled:

a) $B^r_c \cap B^g_c = \{c\}$ for $c \in C$.

b) There are permutations π and ε of C such that $\pi(c) \neq c$ for $c \in C$ and $B^r_c \cap B^g_{\pi(c)} = \{\varepsilon(c)\}$ for $c \in C$.

c) Let $c, d \in C$ such that $d \notin \{c, \pi(c)\}$, then $B^r_c \cap B^g_d$ is contained in C and its cardinality is 0 or 2.

Two pairs (α_r, α_g) and (α'_r, α'_g) for the same basis set M are called *equivalent* if there is a permutation φ of M such that $C' = \varphi(C)$, $\mathcal{L}'_r = \varphi(\mathcal{L}_r)$, $\mathcal{L}'_g = \varphi(\mathcal{L}_g)$, $\varphi\alpha_r = \alpha'_r\varphi$.

Since the blocks of $\mathcal{L}_r$ and $\mathcal{L}_g$ have no elements in common beside elements of C the pairs (α_r, α_g) and (α'_r, α'_g) are equivalent if and only if the intersections with C are mapped by φ onto the intersections with C'. Therefore we have to consider only these intersections and we will do this in the following.

Proposition 3.4. *Let $\mathcal{L}_r$ and $\mathcal{L}_g$ be two parallel lines of a block square, let C be the set of characteristic elements of $(\mathcal{L}_r, \mathcal{L}_g)$ with respect to $\mathfrak{K}_1$, and let α_r, α_g be the maps introduced after Proposition 3.2. Then (α_r, α_g) is an admissible pair of lines associated to C.*

Proof. This follows from Proposition 3.3. □

Let $\phi = (\alpha_r, \alpha_g)$ be an admissible pair of lines associated to C with lines $\mathcal{L}_r$ and $\mathcal{L}_g$. We associate to ϕ the reciprocal pair ϕ_* and the dual pair ϕ^*.

ϕ_* is the pair (α_r, α_g) associated to C with maps α_g and α_r. Hence, in the reciprocal pair, only the roles of $\mathcal{L}_g, \mathcal{L}_r$ are interchanged.

ϕ^* is the pair (β_r, β_g) associated to C with the maps β_r, β_g given by

$$\beta_r(B_c^r) = \varepsilon(c), \quad \beta_g(B_{\pi(c)}^g) = \varepsilon(c).$$

It follows immediately from the axioms of pairs that ϕ^* is admissible.

Proposition 3.5. *Let ϕ be an admissible pair of lines $\mathcal{L}_r, \mathcal{L}_g$ associated to C. Then*

$$(\phi_*)^* = (\phi^*)_*.$$

Proof. This follows from the fact that in the definition of ϕ^* the lines $\mathcal{L}_r$ and $\mathcal{L}_g$ play a reciprocal role. □

If we consider a set of two parallel lines in a block square, then we associate to it three admissible pairs ϕ, ϕ_* and ϕ^* on equal rights.

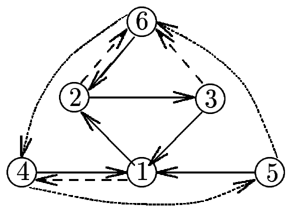
3.3. The Graph Associated to Admissible Pairs

Let (α_r, α_g) be an admissible pair of lines $\mathcal{L}_r, \mathcal{L}_g$ associated to C as defined in Section 3.2. We are going to express such pairs by a doubly coloured directed graph Γ. The advantage is that one can see of two such graphs easily whether they are equivalent or not.

The vertices of Γ are the elements of C. From every point $x \in C$ there goes one red resp. one green edge to a point x^r resp. x^g of C, both x^r and x^g are distinct from x and they are defined as follows:
By the definition of lines there is beside B_x^r exactly one second block B in $\mathcal{L}_r$ resp. $\mathcal{L}_g$ such that $x \in B$. Then x^r is the uniquely determined element of C such that $B = B_{x^r}^r$. The red edge from x then goes to x^r and will be denoted by xx^r. The green edge from x is defined correspondingly starting with B_x^g and it determines the second block of $\mathcal{L}_g$ containing x.

From this definition and from what was said with respect to the equivalence of admissible pairs it is immediately clear that the graph Γ determines the admissible pair and that equivalence of graphs corresponds to the equivalence of admissible pairs.

As an example we give the graph of the pair (3.4) in the example in Section 3.2. We write the red edges as solid arrows and the green edges as dashed arrows. For technical reasons, the appearance of the dashed arrows that are not straight lines differs from the one of the other dashed arrows. The reader should note, that all the curved arrows that occur in this paper are dashed (i.e. green) ones.



3.4. The Local Structure of the Graph

In this section we study the structure of the graph of an admissible pair at a vertex x and at neighbouring vertices with respect to outgoing and incoming edges. We denote the number of red resp. green edges going to x with $i_r(x)$ resp. $i_g(x)$.

Proposition 3.6. *If $y^r = x$, then $y^g \neq x$ and $x^r \neq y$.*

Proof. Let $y^r = x$. Hence $B^r_x \supset \{x, y\}$. Then $y^g = x$ means $B^g_x \supset \{x, y\}$ and $x^r = y$ means $B^r_y \supset \{x, y\}$. Both are impossible by the definition of admissible pairs of lines. □

Proposition 3.7. *If $i_r(x) = 0$, then $i_g(x) > 0$.*

Proof. $i_r(x) = 0$ and $i_g(x) = 0$ means

$$B^r_x \cap C = \{x\}, \quad B^g_x \cap C = \{x\}.$$

It follows

$$B^r_x \cap B^g_{\pi(x)} = \{x\}, \quad B^r_{\pi(x)} \cap B^g_x = \{x\},$$

hence

$$x = \varepsilon(x) = \varepsilon(\pi(x)), \text{ contradiction.}$$

□

Proposition 3.8. *Let $i_r(x) = 0$. Then $i_r(x^r) \geq 2$.*

Proof. We have $B^r_x \cap B^g_{\pi(x)} = \{x\}$ hence $\pi(x) = x^g$. Furthermore $x \in B^r_{x^r} \cap B^g_x$, $x \in B^r_{x^r} \cap B^g_{\pi(x)}$, and $x \neq x^r$ implies

$$|B^r_{x^r} \cap B^g_x| \geq 2, \quad |B^r_{x^r} \cap B^g_{\pi(x)}| \geq 2.$$

But x^r can not be in $B^g_x \cap B^g_{\pi(x)}$. Hence beside x and x^r there must be a third element of C in $B^r_{x^r}$. □

Proposition 3.9. *Let $i^r(x) = 1$ and $y^r = x$. Then $x^g = y$, $\pi(x) = y^g$, $\varepsilon(x) = y$ or $x^g = y^g$, $\pi(x) = \varepsilon(x) = y$*

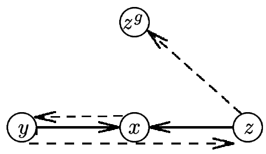
(3.5)

Proof . $B_x^r \cap C = \{x, y\}$ by assumption. If $x \in B_y^g$, then $\pi(x) \neq y$ and $x \notin B^g_{\pi(x)}$ hence $B_x^r \cap B^g_{\pi(x)} = \{y\}$, $y = \varepsilon(x)$. If $x \notin B_y^g$, then $B_x^r \cap B_y^g = \{y\}$. Hence $y = \pi(x) = \varepsilon(x)$ and $|B_x^r \cap B^g_{x^g}| \geq 2$. This implies $y \in B^g_{x^g}$. □

Proposition 3.10. *Let $i^r(x) = 2$ and let y, z be the vertices with $y^r = z^r = x$. Then one has the following possibilities:*

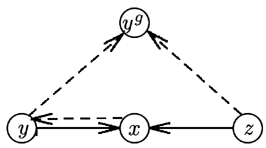
a) $x^g = y,\ \ y^g = z,\ \ z^g = \pi(x),\ \ \varepsilon(x) = z,$

(3.6)



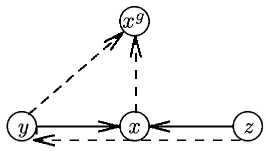
b) $x^g = y,\ \ y^g = z^g,\ \ \pi(x) = \varepsilon(x) = z$

(3.7)



c) $x^g = y^g \neq z,\ \ z^g = y,\ \ \pi(x) = \varepsilon(x) = z$

(3.8)



Proof . $B_x^r \cap C = \{x, y, z\}$ by assumption. If $x^g = y$, then $B_y^g \supset \{x, y\}$ hence $\pi(x) \neq y$. If $\pi(x) = z$, then $B_x^r \cap B_z^g = \{z\}$, $y \notin B_z^g$. Hence $y^g \notin \{x, y, z\}$. Furthermore $y \in B_x^r \cap B^g_{y^g}$ implies $|B_x^r \cap B^g_{y^g}| = 2$, hence $z \in B^g_{y^g}$. We are in the situation of b).

If $\pi(x) \neq z$, then $|B_x^r \cap B_z^g| = 2$, therefore $y \in B_z^g$ and $B_x^r \cap B^g_{\pi(x)} = \{z\}$. We are in the situation of a).

Since the case $x^g = z$ is the same as $x^g = y$ by symmetry it remains the case that $x^g \notin \{y, z\}$. Then $x \notin B^g_y$, $x \notin B^g_z$, but $|B^r_x \cap B^g_y| \geq 1$, $|B^r_x \cap B^g_z| \geq 1$. Without loss of generality, let $|B^r_x \cap B^g_y| = 2$. then $z \in B^g_y$ and $B^r_x \cap B^g_z = \{z\}$, hence $z = \pi(x) = \varepsilon(x)$. Furthermore, $|B^r_x \cap B^g_{x^g}| = 2$ and therefore $y \in B^g_{x^g}$. We are in the situation of c). □

3.5. One-Coloured Graphs

In this section we consider admissible one-coloured graphs, i.e. the graphs Γ_r with vertices x and edges $xx^r, x \in C$, associated to a pair (α_r, α_g). Beside Proposition 3.8 we have the following restriction for the structure of such graphs:

Proposition 3.11. *Let* $x, y, z \in C$ *with* $z = x^r = y^r$, $i_r(x) = i_r(y) = 0$. *Then* $i_r(z) \geq 3$.

Proof . Suppose $B^r_z \cap C = \{x, y, z\}$. Then

$$B^r_x \cap B^g_{\pi(x)} = \{x\}, B^r_y \cap B^g_{\pi(y)} = \{y\}, B^r_z \cap B^g_{\pi(z)} = \{z\}.$$

Hence $\pi(z) \notin \{x, y\}$ and therefore

$$|B^r_z \cap B^g_x| = 2, \quad |B^r_z \cap B^g_y| = 2.$$

Since $z \in B^g_z$, $z \in B^g_{\pi(z)}$, one has $z \notin B^g_x$, $z \notin B^g_y$ and therefore

$$y \in B^g_x, \;\; x \in B^g_y, \;\; B^g_x \cap B^g_y \supseteq \{x, y\}$$

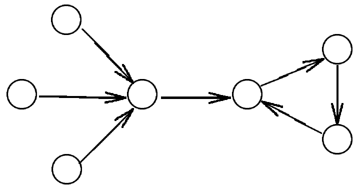
in contradiction to the definition of $\mathcal{L}_g$. □

The restrictions to one-coloured graphs given by Propositions 3.8 and 3.11 lead for small k to a small number of possibilities which we list in the following:

We call a directed graph to be of type A^m, if it consists of an m-gon and simple ends: If $x_1, \ldots, x_m$ are the vertices of the m-gon, then a simple end is an $x \in C$ with $x \neq x_i$, $i = 1, \ldots, m$, and $\iota_r(x) = 0$.

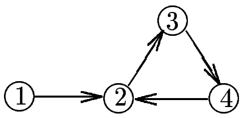
For the two smallest possible values of k, i.e. $k = 3$, $k = 5$ all connected admissible one-coloured graphs are of type A^m, $m = 3, 4, \ldots, k + 1$.

For $k = 6$ one has one exception:



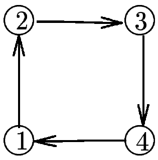
3.6 The Case $k = 3$

In this section we prove the non-existence of block squares with $k = 3$ and $\lambda = 6$. In this case we have two possibilities for Γ^r:



(A^3)

and



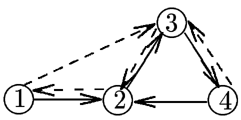
(A^4)

Corresponding to the rules in Section 3.4, A^3 can be completed to a two-coloured graph as follows:

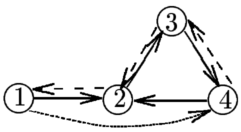
The edge from 2 has to go to 1 because by Proposition 3.7 there is an edge going to 1, and if there is an edge from 3 or 4 to 1 then there is also an edge from 2 to 1 by Proposition 3.6.

The remaining possibilities are as follows:

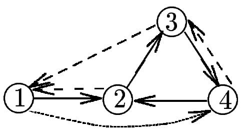
(3.9)



(3.10)

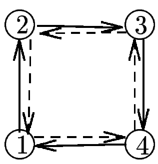


(3.11)



By symmetry of Γ^r and Γ^g we have for the completion of Γ^r of type A^4 only to look for Γ^g of type A^4. Hence there is only one two-coloured graph

(3.12)



Now we show that the four two-coloured graphs do not lead to block squares. We denote the elements of the basis set by $\{1, 2, \ldots, 8\}$ and the block in the i-th row and j-th column by B_{ij}. To (3.9) correspond the lines

156	214	325	436
127	238	314	478

Without loss of generality we assume that 1 is in the third line, i.e. $1 \in B_{32}, 1 \in B_{33}$. Then 4 has to be in the fourth line, i.e. $4 \in B_{42}, 4 \in B_{43}$.We cannot have $8 \in B_{33} \cap B_{43}$ because we have also an 8 in the second column. This implies $8 \in B_{31} \cap B_{41}$. Correspondingly $5 \in B_{34} \cap B_{44}$.

Now we consider the case $8 \in B_{32}$. Our block square looks as follows:

156	214	325	436
127	238	314	478
8	18	1	5
8	4	4	5

It follows $8 \in B_{44}$, $3 \in B_{42}$, $3 \in B_{34}$, $6 \in B_{44}$, $6 \in B_{31}$, $7 \in B_{34}$, $7 \in B_{41}$.

Now we see that 5 or 6 has to be in B_{41}. this is impossible because of $B_{44} = \{5, 6, 8\}$. The case $8 \in B_{42}$ can be treated in the same manner.

To (3.10) correspond the lines

156	214	325	436
127	238	347	418

We may assume $1 \in B_{32} \cap B_{34}$, then $3 \in B_{42} \cap B_{44}$, $4 \in B_{42}$, $4 \in B_{33}$, $7 \in B_{43}$, $7 \in B_{31}$, $2 \in B_{41}$, $2 \in B_{33}$, $5 \in B_{43}$, $5 \in B_{31}$, $6 \in B_{41}$, $6 \in B_{34}$, $8 \in B_{44}$, $8 \in B_{32}$.

156	214	325	436
127	238	347	418
75	18	42	16
26	34	75	38

5 cannot be placed in the fourth columns because in that case we would have $B_{11} = B_{34}$, hence $5 \in B_{32} \cap B_{42}$. It follows $7 \in B_{34} \cap B_{44}$, $6 \in B_{33} \cap B_{34}$, $8 \in B_{31} \cap B_{41}$.

156	214	325	436
127	238	347	418
758	185	426	167
268	345	756	387

Comparing the first and third line, we see that our square does not satisfy the rules of a block square

To (3.11) correspond the lines

156	214	325	436
123	278	347	418
	17	4	1
	84	7	8

Now, we consider the case $2 \in B_{31}$. Then $2 \in B_{43}$, $3 \in B_{41}$, $3 \in B_{34}$, $6 \in B_{44}$, $6 \in B_{31}$, $5 \in B_{41}$, $5 \in B_{33}$. We get the square

156	214	325	436
123	278	347	418
26	17	45	13
35	84	72	86

5 cannot be in the second column because then we would have 7 in the fourth column and $|B_{12} \cap B_{24}| = |B_{12} \cap B_{44}| = 1$. Hence $5 \in B_{34} \cap B_{44}$, $6 \in B_{32} \cap B_{42}$, $7 \in B_{31} \cap B_{41}$, $8 \in B_{33} \cap B_{43}$.

156	214	325	436
123	278	347	418
267	176	458	135
357	846	728	865

This is not a block square because of $|B_{12} \cap B_{31}| = |B_{12} \cap B_{34}| = 1$. In the case $2 \in B_{41}$ we get

156	214	325	436
123	278	347	418
358	175	426	167
268	845	756	837

This is not a block square because of $|B_{12} \cap B_{41}| = |B_{13} \cap B_{42}| = 1$.

It remains the case (3.12). The corresponding two lines are

145	216	325	436
127	238	347	418

We may assume $1 \in B_{32} \cap B_{34}$, then $3 \in B_{42} \cap B_{44}$, $8 \in B_{32} \cap B_{44}$, $6 \in B_{42} \cap B_{34}$.

The permutation (13)(24) permutes the corresponding columns. Therefore, we may assume without loss of generality $8 \in B_{31} \cap B_{41}$, $6 \in B_{33} \cap B_{34}$. Our square looks now as follows

145	216	325	436
127	238	347	418
8	18	6	16
8	63	6	83

The permutation (13)(68)(57) permute the first and third column but leaves the second and fourth line unchanged. It permutes the first row with the second and the third row with the fourth. Hence we may put $5 \in B_{34} \cap B_{44}$, $7 \in B_{32} \cap B_{42}$.

Now there is only one possibility to complete the third and fourth row.

145	216	325	436
127	238	347	418
825	187	672	165
874	637	645	835

But this is not a block square since we have through B_{12} the fourth line B_{11} B_{13} B_{21} B_{23}. Hence we have proved the following

Proposition 3.12. *There is no block square with parameters $k = 3$ and $\lambda = 6$.*

Still the square which we get by permuting the second and third columns could be considered as a combinatorial magic square since all rows and columns and both diagonals are lines:

(3.13)

145	325	216	436
127	347	238	418
825	627	178	165
847	645	637	835

For the understanding of the rather easy combinatorial structure of the design (15) we notice that the blocks contain all pairs of elements in $\{1, 2, \ldots, 8\}$ beside the pairs

$$\{1, 3\}, \ \{2, 4\}, \ \{5, 7\}, \ \{6, 8\} \tag{3.14}$$

and that the Boolean algebra generated by the blocks of (15) contain four one element sets: $\{2\}, \{3\}, \{5\}, \{8\}$. Now we see that (15) consists of all subsets of cardinality 3 of $\{2, 3, 5, 8\}$ and of the triples which contain one element a of $\{2, 3, 5, 8\}$ and a pair of elements in $\{1, 4, 6, 7\}$ which does not contain the element which form a pair of (16) with a.

Hence the permutation $\pi = (1684532)$ transforms our design $(M, \mathcal{B})$ in $(M, \pi\mathcal{B})$ with blocks being either the subsets of $\{1, 2, 3, 4\}$ of cardinality 3 or the sets $\{a, b, c\}$ with $a \in \{1, 2, 3, 4\}$ and $b, c \in \{5, 6, 7, 8\}$ but $a + 4 \notin \{b, c\}$.

3.6. Characterization of Block Squares by Means of Pairs of Parallel, Admissible Lines

Let $(M, \mathcal{B})$ be a 1-block design with parameters k, $\lambda = 6$ with the following properties:

a) There are three systems $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$ of parallel lines such that for all pairs $\{i, j\} \subset \{1, 2, 3\}$ the intersection of two lines $\mathcal{L}_i \in \mathfrak{K}_i$ and $\mathcal{L}_j \in \mathfrak{K}_j$ consists of one block B and deliver a parametrization of all blocks of $\mathcal{B}$ by pairs of lines $(\mathcal{L}_i, \mathcal{L}_j)$.

b) For any $i \in \{1, 2, 3\}$ two lines in $\mathfrak{K}_i$ are admissible in the sense of Section 3.2.

Proposition 3.13. *Let $(M, \mathcal{B})$ be a 1-block design with parameters $k > 3, \lambda = 6$, which satisfies the conditions* a) *and* b)*. Then $(M, \mathcal{B})$ is a block square.*

Proof. We have to show that there are no other lines beside the lines in $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$. Assume that A is a block of $\mathcal{B}$ and $A \in \mathcal{L}$, where $\mathcal{L}$ is a line of $\mathcal{B}$ not in $\mathfrak{K}_1, \mathfrak{K}_2$ or $\mathfrak{K}_3$. Let $\mathcal{L}_1, \mathcal{L}_2$ resp. $\mathcal{L}_3$ be the uniquely determined lines through A in $\mathfrak{K}_1, \mathfrak{K}_2$ resp. $\mathfrak{K}_3$. Then $\mathcal{L}$ consists of blocks B in $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ because only these blocks have the property $|A \cap B| = 1$. Without loss of generality assume that $\mathcal{L}$ contains a second block B_1 resp. B_2 in $\mathcal{L}_1$ resp. $\mathcal{L}_2$ beside A. Then there is no further block of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{L}$ because $|B_1 \cap C_2| \neq 1$ for $C_2 \in \mathcal{L}_2 - \{A, B_2\}$ and $|B_2 \cap C_1| \neq 1$ for $C_1 \in \mathcal{L}_1 - \{A, B_1\}$. The same is true for $(\mathcal{L}_1, \mathcal{L}_3)$. It follows that the line $\mathcal{L}$ has at most 4 blocks but the case $k = 3$ was excluded. This proves the proposition. □

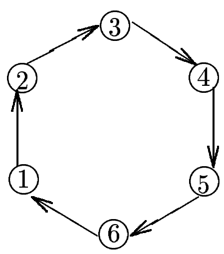
The example in Section 3.1 shows that Proposition 3.13 is not true for $k = 3$.

4. Two-Coloured Graphs in the Case $k = 5$, $\lambda = 6$

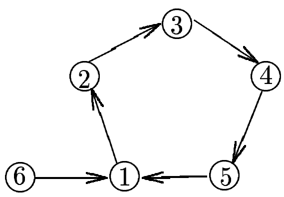
In this chapter we consider the case $k = 5$, $\lambda = 6$.

4.1. One-Coloured Graphs

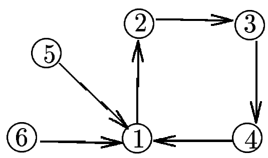
First we list all possible one-coloured graphs according to Section 3.5. We will see that they all can be completed to two-coloured graphs of admissible pairs. There are ten such one-coloured graphs which we list below starting with type A_6 and ending with type A_3: The vertices will be denoted by $1, \ldots, 6$.



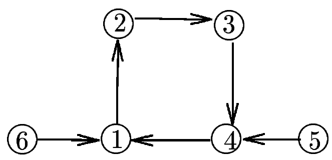
(Γ1)



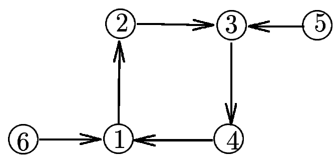
(Γ2)



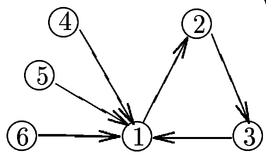
(Γ3)



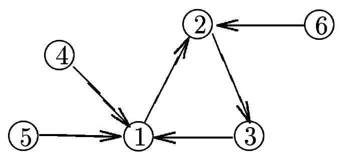
(Γ4)



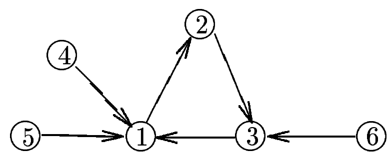
(Γ5)



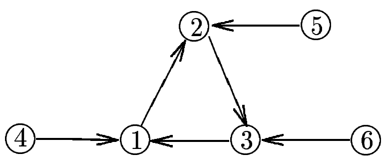
(Γ6)



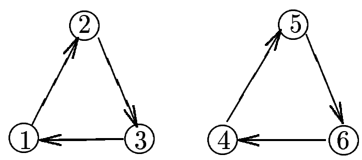
(Γ7)



(Γ8)



(Γ9)

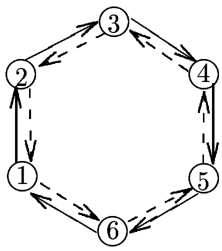


(Γ10)

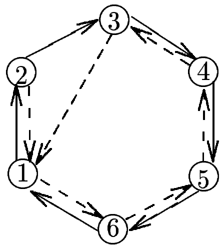
Let ϕ be an admissible pair of parallel lines. In Section 3.2 we have defined the reciprocal pair ϕ^* and the dual pair ϕ_*. In ϕ^* the roles of the colours are interchanged. Since it is sufficient to consider one of the two pairs ϕ, ϕ^* we list in the following only the graphs of pairs ϕ such that the number a of the red-coloured graph is smaller or equal to the number b of the green-coloured graph. We denote such a graph by $\Gamma a.b$ and if there are several such graphs we distinguish them by adding the first letters of the Latin alphabet.

4.2. Γ1

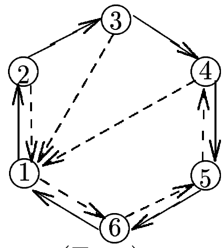
We begin with (Γ1). To find the possible two-coloured graphs one has only to take into account Proposition 3.9 which shows that there are two possibilities for the green edge starting from a vertex x: Either it goes to the neighbouring vertex or it goes to a different vertex, then all vertices in between have edges that go to that vertex as well. Starting with the vertex 1, we get the following two-coloured graphs:



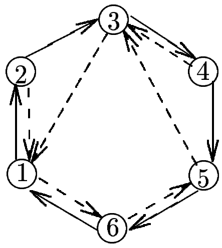
(Γ1.1)



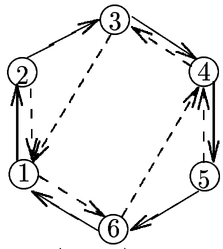
(Γ1.2)



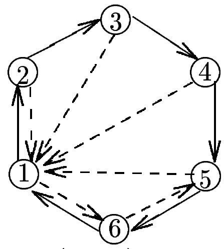
(Γ1.3)



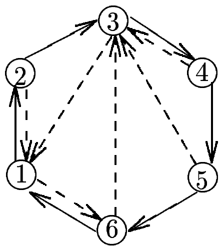
(Γ1.4)



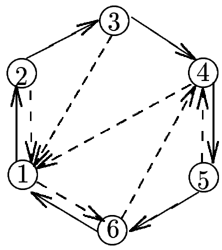
(Γ1.5)



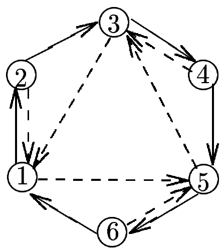
(Γ1.6)



(Γ1.7)



(Γ1.8)



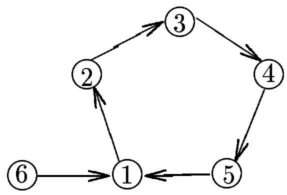
(Γ1.9)

One checks that the pairs of lines corresponding to (Γ1.1) – (Γ1.9) are admissible. All these graphs are self-dual.

4.3. Γ2

In this section we consider the type (Γ2). We have the following

Proposition 4.1. *If the red-coloured graph has the form*

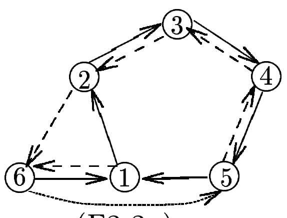


(Γ2)

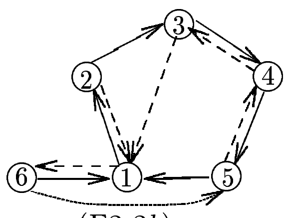
then the corresponding two-coloured graph has the edge 16.

Proof . By Proposition 3.7 there is a green edge going to 6. If it starts at a with $2 \leq a \leq 5$, then, by Proposition 3.9, there is an edge from any b with $1 \leq b \leq a$ to 6. □

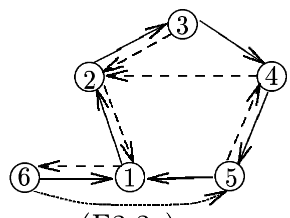
With respect to $x = 1$ we are in Proposition 3.10 in the situation (3.6) or (3.7). ($\Gamma 2$) can be completed to a two-coloured graph of type ($\Gamma 2a$), $a = 2, 3$, as follows.



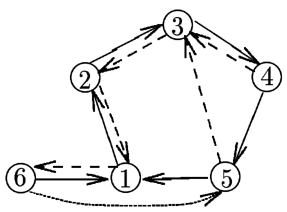
($\Gamma 2.2a$)



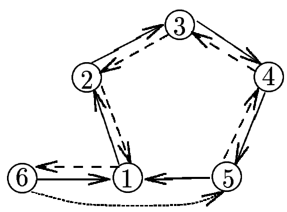
($\Gamma 2.2b$)



($\Gamma 2.2c$)



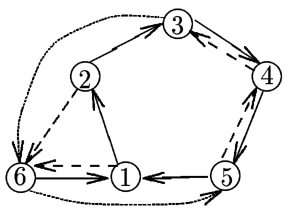
($\Gamma 2.2d$)



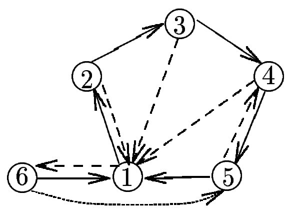
($\Gamma 2.2e$)

One has $(\Gamma 2.2x)_* = (\Gamma 2.2x)$ for $x \in \{a, b, c, d, e\}$. With respect to dual graphs one finds

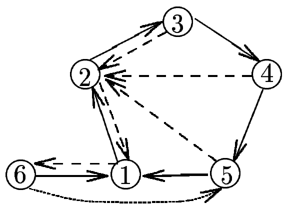
$$\begin{aligned} (\Gamma 2.2a)^* &= (\Gamma 2.2e), \\ (\Gamma 2.2b)^* &= (\Gamma 2.2d), \\ (\Gamma 2.2c)^* &= (\Gamma 2.2c). \end{aligned}$$



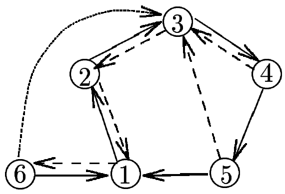
($\Gamma 2.3a$)



($\Gamma 2.3b$)

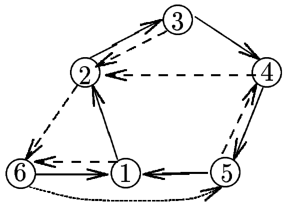


$(\Gamma 2.3c)$

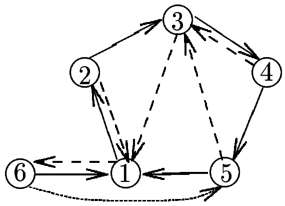


$(\Gamma 2.3d)$

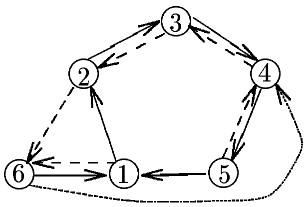
$$\begin{aligned}(\Gamma 2.3a)^* &= (\Gamma 2.3d)\\ (\Gamma 2.3b)^* &= (\Gamma 2.3c).\end{aligned}$$



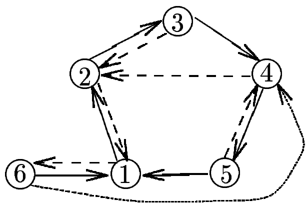
$(\Gamma 2.4a)$



$(\Gamma 2.4b)$

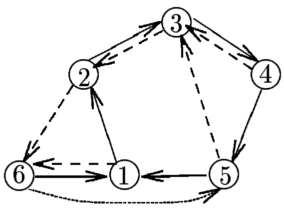


$(\Gamma 2.4c)$

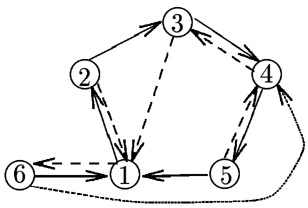


$(\Gamma 2.4d)$

$$\begin{aligned}(\Gamma 2.4a)^* &= (\Gamma 2.4d),\\ (\Gamma 2.4b)^* &= (\Gamma 2.4b),\\ (\Gamma 2.4c)^* &= (\Gamma 2.4c).\end{aligned}$$

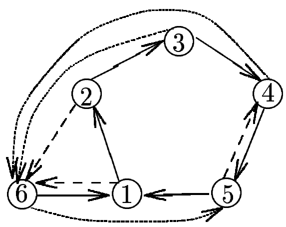


$(\Gamma 2.5a)$

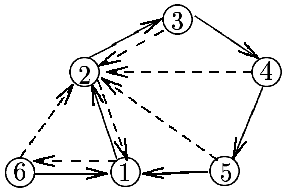


$(\Gamma 2.5b)$

$$(\Gamma 2.5a)^* = (\Gamma 2.5b)$$

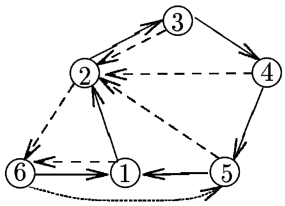


$(\Gamma 2.6a)$

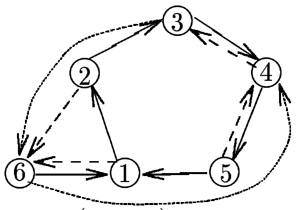


$(\Gamma 2.6b)$

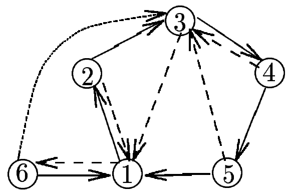
$$(\Gamma 2.6a)^* = (\Gamma 2.6b)$$



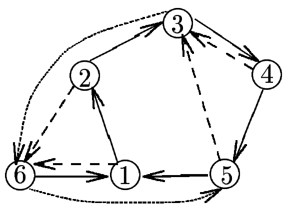
$(\Gamma 2.7a)$



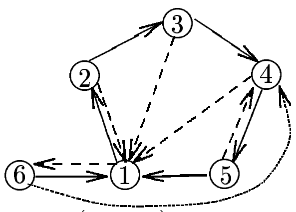
$(\Gamma 2.7b)$



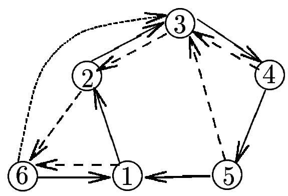
$(\Gamma 2.7c)$



$(\Gamma 2.8a)$

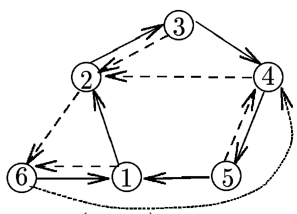


$(\Gamma 2.8b)$



$(\Gamma 2.8c)$

$$\begin{aligned} (\Gamma 2.7a)^* &= (\Gamma 2.8b), \\ (\Gamma 2.7b)^* &= (\Gamma 2.8c), \\ (\Gamma 2.7c)^* &= (\Gamma 2.8a). \end{aligned}$$



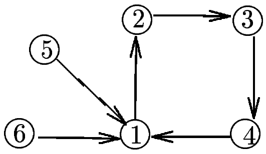
$(\Gamma 2.9)$

$$(\Gamma 2.9)^* = (\Gamma 2.9).$$

This finishes the consideration of the types with red-coloured graphs of type 2.

4.4. $\Gamma 3$

We come to the case of a red-coloured graph of type 3:

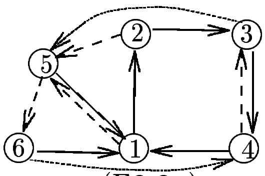


(Γ3)

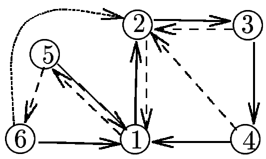
Here we have the following rule:

Proposition 4.2. *In* (Γ3) *there is an edge from* 1 *to* 5 *or* 6 *in the green-coloured graph. If* 15*, then* 56*. Furthermore,* 4 *cannot go to* 5 *or* 6*.*

Proof. Since there is a green edge to 5 and 6 by Proposition 3.7 we can assume that 1,2,3 or 4 goes to 5. But then 15 and 56. If 45, then 35, 25, 15. Hence $B_1^r \supseteq \{1,4,5,6\}$, $B_5^g \supseteq \{1,2,3,4,5\}$, contradiction. □

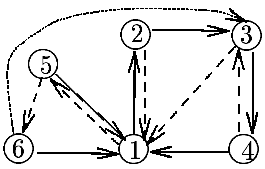


(Γ3.3a)

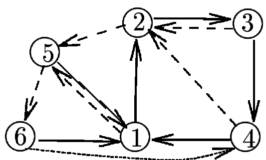


(Γ3.3b)

$$(\Gamma 3.3a)_* = (\Gamma 3.3a),$$
$$(\Gamma 3.3b)_* = (\Gamma 3.3b),$$
$$(\Gamma 3.3a)^* = (\Gamma 3.3b).$$

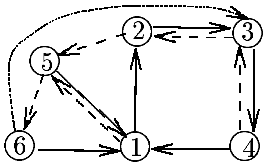


(Γ3.4a)



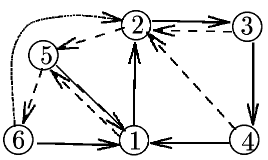
(Γ3.4b)

$$(\Gamma 3.4a)^* = (\Gamma 3.4b).$$

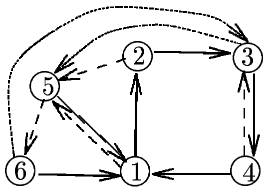


(Γ3.5)

$(\Gamma 3.5)^* = (\Gamma 3.5)$



(Γ3.7)

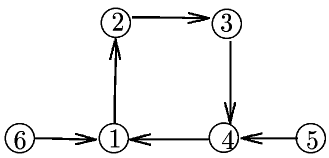


(Γ3.8)

$(\Gamma 3.7)^* = (\Gamma 3.8)$

4.5. Γ4

Now we consider red-coloured graphs of type Γ4:



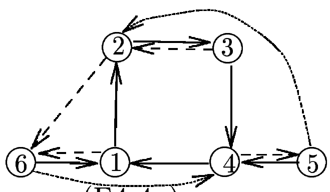
(Γ4)

Proposition 4.3. *In the green-coloured graph belonging to a red-coloured graph of type* Γ4 *one has edges* 16 *and* 45.

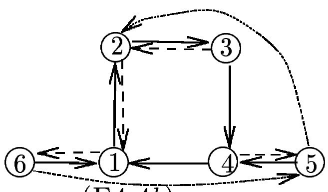
Proof. First we prove 16: There is an edge going to 6. a) If 26 or 36, then 16 by Proposition 3.10. b) If 46, we are in the situation (3.8) with respect to $x = 4$. Hence 36 or 56. Since 36 leads to 16 we may assume 56. Then 35 which implies 25 and 15. Now follows 45 from (3.8) with respect to $x = 1$. Hence $B_5^g \supset \{1, 2, 3, 4, 5\}$, $B_4^r \supseteq \{3, 4, 5\}$. This is a contradiction. c) If 56 and (3.6) with respect to $x = 4$, then 35 and 43. This implies 25 and 15. But this contradicts (3.8) with respect to $x = 1$. d) If 56 and (3.7) with respect to 4, then 36 hence 16. e) If 56 and (3.8) with respect to 4, then 46 and we are in the case b).

Now we prove 45. Since we know already that we have 16, the edge going to 5 can only be 45 or 65. Assume 65. Then we are in the situation (3.7) with respect to $x = 1$ and this implies 45. □

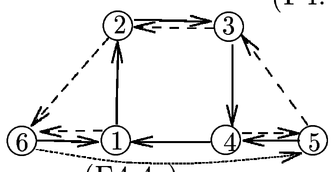
(3.7) with respect to $x = 1$ implies 64 or 65. We first consider 64 and then 65.



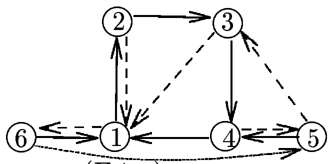
(Γ4.4*a*)



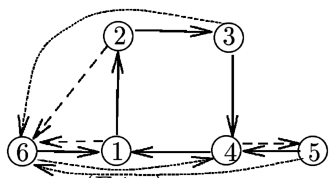
(Γ4.4*b*)



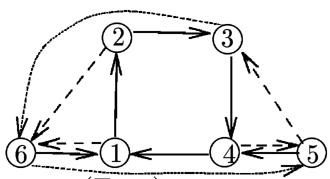
(Γ4.4*c*)



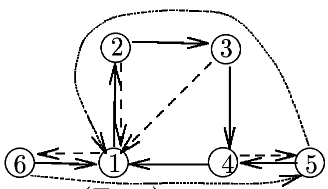
(Γ4.5)



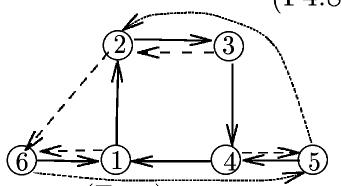
(Γ4.6)



(Γ4.7)



(Γ4.8)



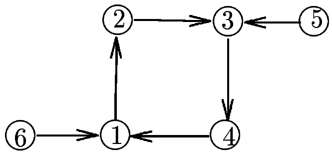
(Γ4.9)

The graphs (Γ4.4*a*) – (Γ4.4*c*) are self reciprocal.

$$
\begin{aligned}
(\Gamma 4.4a)^* &= (\Gamma 4.4a),\\
(\Gamma 4.4b)^* &= (\Gamma 4.4c),\\
(\Gamma 4.5)^* &= (\Gamma 4.5),\\
(\Gamma 4.6)^* &= (\Gamma 4.6),\\
(\Gamma 4.7)^* &= (\Gamma 4.8),\\
(\Gamma 4.9)^* &= (\Gamma 4.9).
\end{aligned}
$$

4.6. Γ5

In the case of a red-coloured graph of type



(Γ5)

we have the following

Proposition 4.4. *In the green-coloured graphs belonging to a red-coloured graph of type* Γ5 *one has edges* 35 *and* 16.

Proof. By symmetry it is sufficient to prove 35. Since there is an edge going to 5 we consider the possible cases:

a) 15 implies that we have (3.8) with respect to $x = 1$, hence 45 or 65. Since 45 implies 35 we assume 65 and therefore 46, which implies 36. Now (3.8) with respect to $x = 3$ implies 26 and 52. But 26 implies 16, contradiction.

b) 25 implies 15.

c) 45 implies 35.

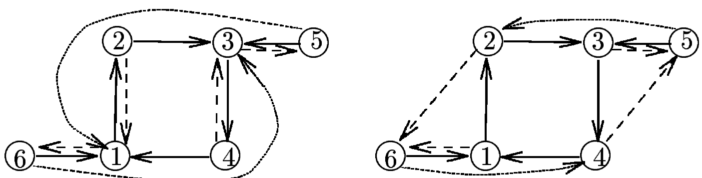
d) 65 and (3.6) with respect to $x = 1$ implies 46 and 16. But then $B_1^r \supset \{1,4,6\}$, $B_6^g \supset \{1,4,6\}$.

e) 65 and (3.7) with respect to $x = 1$ implies 45 hence 35.

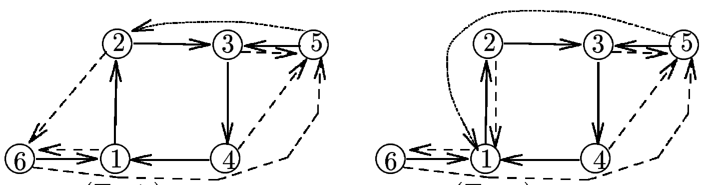
f) 65 and (3.8) with respect to $x = 1$ implies 15.

□

We have the following cases:



(Γ5.5*a*) (Γ5.5*b*)



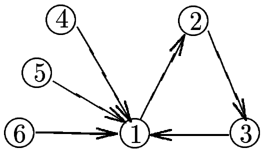
(Γ5.7) (Γ5.8)

All these graphs are self reciprocal and we have

$$(\Gamma 5.5a)^* = (\Gamma 5.5b),$$
$$(\Gamma 5.7)^* = (\Gamma 5.8).$$

4.7. $\Gamma 6$

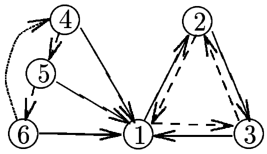
In the case



($\Gamma 6$)

one has $B_a^r \supset \{13456\}$. It follows that $i^g(x) \geq 2$ only for two vertices x and that $i^g(x) \geq 3$ for no vertex x. Hence we have to consider only the case ($\Gamma 6.10$).

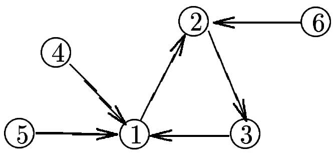
In case ($\Gamma 6.10$) Proposition 3.9 implies 21g and 32g, hence 13g. It follows up to equivalence 45g, 56g, 64g:



($\Gamma 6.10$)

4.8. $\Gamma 7$

In the case



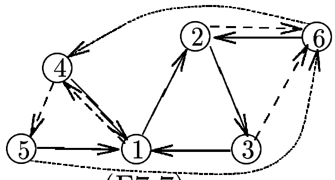
($\Gamma 7$)

we use in addition to our previous methods the following

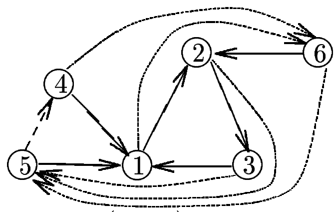
Proposition 4.5. *Let ($\Gamma a.b$) be a two-coloured graph with $7 \leq a \leq b$. Then the green edges xy, yz, zw imply wy.*

Proof. Otherwise the green-coloured graph of ($\Gamma a, b$) would contain no triangle in contradiction to the assumption $7 \leq a \leq b$. □

By means of this proposition and using Proposition 3.7 with respect to $x = 4$ we derive that there are only two possibilities for two-coloured graphs with red-coloured graph ($\Gamma 7$):



(Γ7.7)



(Γ7.8)

4.9. Γ8

The case (Γ8) is dual to (Γ7) in the following sense:

Proposition 4.6. *Let* Γ *be a graph whose red-coloured graph is of type* (Γ7). *Then the red-coloured graph of* Γ^* *is of type* (Γ8).

Proof. To (Γ7) corresponds the line

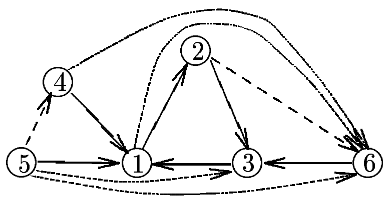
1,345	2,16	3,2	4	5	6

It follows from Propositions 3.9 and 3.10 that the characteristic numbers in [216] and [32] have to change if we go over to the dual graph. Since 4,5,6 remain characteristic numbers, we see that the corresponding red line of Γ^* is

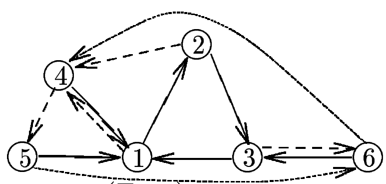
1,26	2,3	3,145	4	5	6

which is of type (Γ8). □

Now we get following graphs:



(Γ8.7)

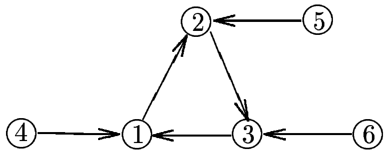


(Γ8.8)

$$(\Gamma 7.7)^* = (\Gamma 8.8),$$
$$(\Gamma 7.8)^* = (\Gamma 8.7).$$

4.10. $\Gamma 9$

In the case



($\Gamma 9$)

we have the following

Proposition 4.7. *Let Γ be a graph with red-coloured graph of type* ($\Gamma 9$)*. Then the green-coloured graph of Γ has no edge in the triangle.*

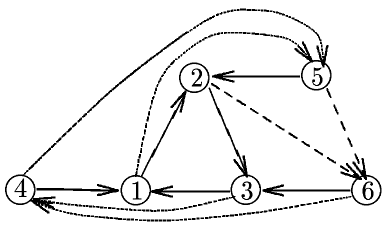
Proof . Assume $2^g = 1$ and consider local conditions in $x = 3$ (Proposition 3.10). Only (3.6) or (3.7) are possible.

a) (3.6): Then $3^g = 6$ and $6^g = 2$. Now applying Proposition 3.10 to $x = 1$ we see that 1 and 4 go to 3 and 6. Hence there is no green edge to 5 in contradiction to Proposition 3.7.

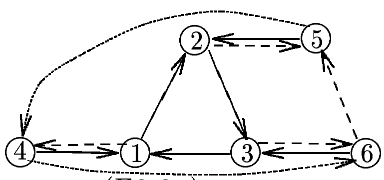
b) (3.7): Then $6^g = 1$ and $3^g \in \{2.6\}$ and Proposition 3.10 applied to $x = 1$ again shows that there is no green edge to 5.

□

Now one can see by using Proposition 3.10 that the assumption of $1^g = 4$ leads to the following type ($\Gamma 9.9a$); the assumption $1^g = 5$ leads to the type ($\Gamma 9.9b$) and $1^g = 6$ leads to a contradiction



($\Gamma 9.9b$)

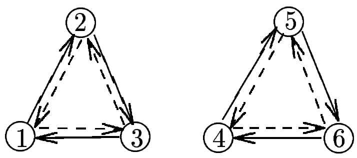


($\Gamma 9.9a$)

Both graphs are self reciprocal and self dual.

4.11. $\Gamma 10$

It remains the case ($\Gamma 10.10$). It is immediately clear from Proposition 3.9 that there is only one such graph:



(Γ10.10)

5. Completing the block squares

To prove Theorem 2.8 it remains to go through all the 41 representatives of $\langle\ ^*,\ _*\rangle$ orbits on the two-coloured graphs, construct the corresponding first two rows, and try to complete them in all possible ways to block squares with the computer.

A block square contains 3 different systems of parallel lines: the rows, the columns and a third system where the lines are indicated by the letters a,b,...,f, which we also call types.

Note that an admissible two-coloured graph determines the first two rows of a possible block square up to equivalence as follows: We may assume that the first two rows start as follows:

a, 1	b, 2	c, 3	d, 4	e, 5	f, 6
1	2	3	4	5	6

The positions of the other entries in $C := \{1, \ldots, 6\}$ are then given by the two-coloured graphs, which also determine the types of the blocks in the second row. Then these two rows can be completed to an admissible pair of lines by filling in the numbers $\{7, \ldots, 15\}$ resp. $\{16, \ldots, 24\}$. Up to equivalence this completion is unique.

Let $s \in S = \{1, \ldots, 30\}$. Since we are in the case $\lambda = 6$, there are in each system of parallel lines, exactly 3 lines $\mathcal{L}$ that contain a block that contains s. In this case there are exactly 2 blocks in $\mathcal{L}$ that contain s. So the distribution of the blocks that contain s is equivalent to:

s	s				
s		s			
	s	s			

Therefore each number s in C occurs in three columns c_1, c_2, c_3 and in three different types t_1, t_2, t_3 which are already determined by the first two rows. There is one distinguished column (resp. type), say c_1 resp. t_1, where it occurs twice in the first two rows. The two other blocks that contain s are in one row in column c_2 and c_3 and of type t_2 resp. t_3. This gives restrictions on the possibilities where to insert those numbers s during the completion of the block square.

We illustrate this by discussing the case (Γ2.4b) and showing that (Γ2.7a) is impossible:

For the first two rows of (Γ2.4b) one finds:

a, 1, 5, 6, 7, 8	b, 2, 1, 9, 10, 11	c, 3, 2, 7, 12, 13	d, 4, 3, 8, 9, 14	e, 5, 4, 10, 12, 15	f, 6, 11, 13, 14, 15
d, 1, 2, 3, 16, 17	c, 2, 18, 19, 20, 21	a, 3, 4, 5, 18, 22	e, 4, 16, 19, 23, 24	f, 5, 6, 17, 20, 23	b, 6, 1, 21, 22, 24

Therefore the numbers 2, 3, 5, 6 are missing in the first column, where the type of the block that contains 2 is either b or d, for 3 one finds a or c, etc. Since the first column already contains blocks of type a and d, one concludes that the new block in the first column that contains 2 is of type b and for 3 one finds c. This can be visualized as follows:

2, b, d 3, a, c 5, e, f 6, a, b	1, a, d	2, b, d 4, a, d 5, e, f	3, a, c	4, a, d 6, a, b	1, a, d

It is now easy to see that the third, fourth and fifth row can be assumed to contain

b, 2, 6		d, 2, 4		a, 4, 6	
c, 3			a, 3		
	1				1

Using this starting information, the computer tries to complete the blocks row by row using a backtrack algorithm. After about 11 sec. the computer stops without having found a new block square.

To see that ($\Gamma 2.7a$) is impossible, note that the first two rows are

a, 1, 5, 6, 7, 8	b, 2, 1, 9, 10, 11	c, 3, 2, 7, 12, 13	d, 4, 3, 8, 9, 14	e, 5, 4, 10, 12, 15	f, 6, 11, 13, 14, 15
b, 1, 16, 17, 18, 19	a, 2, 3, 4, 5, 16	d, 3, 17, 20, 21, 22	e, 4, 18, 20, 23, 24	f, 5, 6, 19, 21, 23	c, 6, 1, 2, 22, 24

Therefore the second column contains a block of type c with entries 1 and 3. In particular, 1 and 3 are in the same row. But 3 has to be also in column 4 with type a and 1 has to be in the last column with type a, which is impossible, since each row contains a unique block of type a.

Analogously one treats the other graphs. The most difficult graph is ($\Gamma 1.1$), where the computer needs 1 hour to exclude this case. Only the cases ($\Gamma 2.4a$), ($\Gamma 2.4c$), ($\Gamma 6.10$), ($\Gamma 7.7$), and ($\Gamma 9.9a$) (and their $\langle\ ^*,\ _*\rangle$ orbits) can be completed to block squares. The computer constructs nearly 300 different block squares. With MAGMA [3] one checks that these fall into two isomorphism classes yielding the two non isomorphic block squares of Theorem 2.8.

References

[1] Conway,J.H., Sloane, N.J.A., Sphere Packings, Lattices and Groups, Springer-Verlag New York 1988

[2] Koch, H., Unimodular lattices and self-dual codes, Proceedings of the ICM'86, Berkeley

[3] The MAGMA homepage: http://www.maths.usyd.edu.au:8000/u/magma/

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