

# BOLYTROPE ORDERS

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ABSTRACT. Bolytropes are bounded subsets of an affine building that consist of all points that have distance at most  $r$  from some polytrope. We prove that the points of a bolytrope describe the set of all invariant lattices of a bolytrope order, generalizing the correspondence between polytropes and graduated orders.

## 1 Introduction

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The work of this paper finds its purpose within the framework of a larger joint project involving the three authors, Marvin Anas Hahn, and Bernd Sturmfels, and regarding the investigation of the interplay between orders  $\Lambda$  over discrete valuation rings (in split simple algebras) and certain bounded convex subsets  $Q$  of affine buildings. The set of 0-simplices,  $\mathcal{B}^0$ , of the building is in bijection with the maximal orders [2, 8]. Closed orders are intersections of finitely many maximal orders. These are exactly the Plesken-Zassenhaus orders  $\Lambda = \text{PZ}(\mathcal{L})$  of the finite subsets  $\mathcal{L}$  of the building; cf. Definition 3.10 and Proposition 3.13. Any order  $\Lambda$  defines a bounded convex set  $Q(\Lambda) \subset \mathcal{B}^0$  corresponding to the maximal orders that contain  $\Lambda$ . A set of the form  $Q(\Lambda)$  is called Plesken-Zassenhaus-closed (PZ-closed, for short). The PZ-closed sets that are contained in one apartment are exactly the polytropes and the corresponding PZ-orders are the graduated orders; cf. [7, 14] and references therein. In this paper, we define the new class of bolytrope orders  $\Lambda$  for which  $Q(\Lambda)$  is a bolytrope. In turn, a bolytrope is defined as the set of all elements in  $\mathcal{B}^0$  that have distance at most  $r$  from a polytrope, the associated central polytrope. The special case of ball orders, i.e. bolytrope orders where the central polytrope consists of one point, is treated in Section 5.

The word “bolytrope” is of our own invention and is the fusion of the words “ball” and “polytrope”. Polytropes are the tropical analoga of polytopes (see for instance [9]) and describe the sets of lattices that are invariant under graduated orders. A bolytrope order is the intersection of a ball order and a graduated order (see Lemma 6.2) and a bolytrope is the set of invariant lattices of a bolytrope order.

One of the main results of this paper is Theorem 6.6, stating that bolytropes and bolytrope orders are PZ-closed and closed, respectively.

Closed orders in quaternion algebras have been extensively studied in the context of class groups, Brandt matrices, and Hecke operators, cf. [3, 4, 5, 6, 15]. In Section 7, we apply our results to reprove and extend the work from [15], showing in particular that all closed split quaternion orders are bolytrope orders.

In Section 4, we present our main tool: the radical idealizer chain of an order. This allows for an inductive procedure to handle bolytrope orders as explained in Lemma 6.5 and also shows that the central polytrope  $Q$  is uniquely determined by the bolytrope  $B$ ; its Plesken-Zassenhaus order  $\text{PZ}(Q)$  is the first term in the radical idealizer chain applied to the bolytrope order  $\text{PZ}(B)$  that happens to be a graduated order.

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1.1. **Notation.** Throughout the paper let  $K$  be a discretely valued field with valuation ring  $\mathcal{O}_K$ , uniformizer  $\pi$ , and maximal ideal  $\mathfrak{m}_K = \mathcal{O}_K\pi \neq \{0\}$ . If  $K$  is commutative, which we assume henceforth, there is no need for  $K$  to be complete: in particular,  $K = \mathbb{Q}$  with some  $p$ -adic valuation is allowed. Let, moreover,  $d$  be a positive integer. We write  $\mathbf{1}$  for the vector  $(1, \dots, 1) \in \mathbb{Z}^d$  and  $J_d$  for the matrix, in  $\mathbb{Z}^{d \times d}$ , with zeros on the diagonal and ones elsewhere.

## 2 Graduated orders and apartments

An  $\mathcal{O}_K$ -lattice (or simply lattice) in  $K^d$  is a free  $\mathcal{O}_K$ -submodule of maximal rank  $d$ . The (homothety) class of a lattice  $L$  in  $K^d$  is

$$[L] = \{cL \mid c \in K \setminus \{0\}\} = \{\pi^n L \mid n \in \mathbb{Z}\},$$

while  $\text{End}_{\mathcal{O}_K}(L)$  denotes the endomorphism ring of  $L$  as an  $\mathcal{O}_K$ -module. Note that any two homothetic lattices have the same endomorphism ring.

An  $\mathcal{O}_K$ -order  $\Lambda$  in the matrix ring  $K^{d \times d}$  is an  $\mathcal{O}_K$ -lattice that is also a ring, i.e.  $\Lambda$  is multiplicatively closed and contains the identity element  $\text{Id}_d$  of  $K^{d \times d}$ . If  $\Lambda$  is an order in  $K^{d \times d}$ , then a  $\Lambda$ -lattice is a lattice in  $K^d$  that is also a  $\Lambda$ -module.

2.1. **Graduated orders and polytropes.** In the present section, we define graduated orders following [11] and collect some related results from [7, 11].

**Definition 2.1.** An  $\mathcal{O}_K$ -order  $\Lambda$  in  $K^{d \times d}$  is called *graduated* if  $\Lambda$  contains a complete set of orthogonal primitive idempotents  $\epsilon_1, \dots, \epsilon_d$  of  $K^{d \times d}$ .

The primitive idempotents of  $K^{d \times d}$  are exactly the projections onto 1-dimensional subspaces of  $K^d$ , so each set  $\{\epsilon_1, \dots, \epsilon_d\}$  as in Definition 2.1 defines a *frame*

$$K^d = \epsilon_1 K^d \oplus \dots \oplus \epsilon_d K^d = K e_1 \oplus \dots \oplus K e_d$$

i.e. a decomposition of  $K^d$  as a direct sum of 1-dimensional subspaces. In any frame basis  $(e_1, \dots, e_d)$  the idempotents are diagonal matrices with exactly one entry 1 on the diagonal. The projection onto the  $ij$ -matrix entry  $\epsilon_i \Lambda \epsilon_j$  is an  $\mathcal{O}_K$ -submodule of  $\epsilon_i K^{d \times d} \epsilon_j \cong K$ . Hence, writing matrices with respect to the frame basis  $(e_1, \dots, e_d)$ , there exists a matrix  $M = (M_{ij}) \in \mathbb{Z}^{d \times d}$  such that the graduated order  $\Lambda$  is of the form

$$\Lambda(M) = \{X = (X_{ij}) \in K^{d \times d} \mid X_{ij} \in \mathfrak{m}_K^{M_{ij}} \text{ for all } i, j = 1, \dots, d\}.$$

The matrix  $M$  is called the *exponent matrix* of  $\Lambda$ .

**Remark 2.2.** When we state that a given order  $\Lambda$  is contained in some graduated order  $\Lambda(M)$  we always mean that there exists a suitable basis such that this graduated overorder of  $\Lambda$  has the form  $\Lambda(M)$ . How to find such a basis is usually indicated in the proofs.

The fact that  $\Lambda = \Lambda(M)$  is a ring is equivalent to having, for all  $i, j, k \in \{1, \dots, d\}$ , that

$$(2.1) \quad M_{ii} = 0 \quad \text{and} \quad M_{ij} + M_{jk} \leq M_{ik}.$$

With the *polytrope region*  $\mathcal{P}_d$  as defined in [7, Section 4], we have that (2.1) is equivalent to the condition  $M \in \mathcal{P}_d \cap \mathbb{Z}^{d \times d}$ . Putting

$$\mathcal{P}_d(\mathbb{Z}) = \{M \in \mathbb{Z}^{d \times d} \mid M_{ii} = 0, \quad M_{ij} + M_{jk} \leq M_{ik} \text{ for all } i, j, k = 1, \dots, d\},$$

we can see that (2.1) is equivalent to  $M \in \mathcal{P}_d(\mathbb{Z})$ .

**Remark 2.3.** For  $M \in \mathcal{P}_d(\mathbb{Z})$ , the  $\Lambda(M)$ -lattices  $L$  are of the form  $L = \bigoplus_{i=1}^d \epsilon_i L$  and hence there exists  $u = (u_1, \dots, u_d) \in \mathbb{Z}^d$  such that

$$L = L_u := \mathcal{O}_K \pi^{u_1} e_1 \oplus \dots \oplus \mathcal{O}_K \pi^{u_d} e_d.$$

The tuple  $u$  is called the *exponent vector* of the lattice  $L$ . Moreover,  $L = L_u$  is a  $\Lambda(M)$ -lattice if and only if, for any choice of  $1 \leq i, j \leq d$ , one has  $M_{ij} + u_j \geq u_i$  and two  $\Lambda(M)$ -lattices  $L_u$  and  $L_v$  are isomorphic if and only if  $u - v \in \mathbb{Z}\mathbf{1}$ . Put

$$Q_M := \{[u] \in \mathbb{R}^d/\mathbb{R}\mathbf{1} \mid M_{ij} + u_j \geq u_i \text{ for all } i, j = 1, \dots, d\}.$$

Then  $Q_M$  is a *polytrope* and the integral points of  $Q_M$  parametrize the  $\Lambda(M)$ -stable lattices in  $K^d$ .

**2.2. Buildings and apartments.** In line with the content of this paper, we define the affine building of  $\mathrm{SL}_d(K)$  via its lattice class model and refer the interested reader to [1] for the more general description.

**Definition 2.4.** The *affine building*  $\mathcal{B}_d(K)$  is an infinite simplicial complex such that

- (1) the vertex set is  $\mathcal{B}_d^0 = \{[L] \mid L \text{ is an } \mathcal{O}_K\text{-lattice in } K^d\}$ .
- (2)  $\{[L_1], \dots, [L_s]\}$  is a simplex in  $\mathcal{B}_d(K)$  if and only if, up to permutation of the indices and choice of representatives, one has  $L_1 \supset L_2 \supset \dots \supset L_s \supset \pi L_1$ .

An *apartment* of  $\mathcal{B}_d(K)$  is any subset  $\mathcal{A}(E)$  of  $\mathcal{B}_d^0(K)$  of the form

$$\mathcal{A}(E) := \{[L_u] \mid u = (u_1, \dots, u_d) \in \mathbb{Z}^d\}$$

where  $E$  is a frame  $K^d = \epsilon_1 K^d \oplus \dots \oplus \epsilon_d K^d = K e_1 \oplus \dots \oplus K e_d$  of  $K^d$ .

Strictly speaking,  $\mathcal{A}(E)$  consists only of the set of 0-simplices in the apartment. Note that, whereas the exponent vector  $u$  depends on the choice of the basis  $(e_1, \dots, e_d)$ , the whole apartment  $\mathcal{A}(E)$  only depends on the frame or the corresponding set  $\{\epsilon_1, \dots, \epsilon_d\}$  of orthogonal primitive idempotents. An explicit choice of the basis gives an identification of the lattice classes  $[L_u] \in \mathcal{A}(E)$  with the integral points  $[u] \in \mathbb{Z}^d/\mathbb{Z}\mathbf{1}$  in the  $(d-1)$ -dimensional space  $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ .

**2.3. Closed orders.**

**Definition 2.5.** Let  $\Lambda$  be an order in  $K^{d \times d}$ . Then

$$Q(\Lambda) := \{[L] \in \mathcal{B}_d^0(K) \mid \Lambda L = L\}$$

denotes the set of homothety classes of  $\Lambda$ -lattices in  $K^d$ . The order  $\Lambda$  is called *closed* if

$$\Lambda = \bigcap_{[L] \in Q(\Lambda)} \mathrm{End}_{\mathcal{O}_K}(L).$$

Notice that the closed orders are exactly the ones that are determined by their sets of invariant lattices. This is not the case in general as the following example shows.

**Example 2.6.** Let  $M := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P}_2(\mathbb{Z})$ . Then  $\Lambda(M)$  is a graduated order with  $Q(\Lambda(M)) = \{[L_{(0,1)}], [L_{(0,0)}]\}$ . Let  $\Lambda := \{X \in \Lambda(M) \mid X_{11} \equiv X_{22} \pmod{\pi}\}$ . Then  $\Lambda$  is an order in  $K^{2 \times 2}$  satisfying  $Q(\Lambda) = Q(\Lambda(M))$ . It follows that  $\Lambda$  is not closed.

### 3 The distance, balls and bolytropes

In this section, we define a distance on  $\mathcal{B}_d^0(K)$  and use it to define balls and bolytropes in the building  $\mathcal{B}_d(K)$ . Balls are a special type of bolytropes and bolytropes can be thought of as balls “around polytropes”.

**3.1. The distance.** The work in this paper heavily depends on the following notion of distance on  $\mathcal{B}_d^0(K)$ .

**Definition 3.1.** Let  $[L_1], [L_2] \in \mathcal{B}_d^0(K)$  be two homothety classes of lattices. Then

$$\text{dist}([L_1], [L_2]) := \min\{s \mid \text{there are } L'_1 \in [L_1], L'_2 \in [L_2] \text{ with } \pi^s L'_1 \subseteq L'_2 \subseteq L'_1\}.$$

For a subset  $\mathcal{L} \subseteq \mathcal{B}_d^0(K)$ , we put  $\text{dist}([L], \mathcal{L}) := \min\{\text{dist}([L], [L']) \mid [L'] \in \mathcal{L}\}$ . The set  $\mathcal{L}$  is called *bounded*, if  $\sup\{\text{dist}([L], [L']) \mid [L], [L'] \in \mathcal{L}\}$  is finite.

The following result ensures that  $\text{dist}$  is in fact a distance.

**Lemma 3.2.** *The map  $\text{dist} : \mathcal{B}_d^0(K) \times \mathcal{B}_d^0(K) \rightarrow \mathbb{Z}$  is a distance on  $\mathcal{B}_d^0(K)$ .*

*Proof.* We check that the defining properties of a distance hold. For this, let  $[L_1], [L_2] \in \mathcal{B}_d^0(K)$  with  $\text{dist}([L_1], [L_2]) = s$  and let  $L'_1, L'_2$  be as in Definition 3.1. Then

- (1)  $\text{dist}([L_1], [L_2]) = 0$  if and only if  $L'_1 \subseteq L'_2 \subseteq L'_1$ , equivalently  $[L_1] = [L_2]$ .
- (2) If  $\pi^s L'_1 \subseteq L'_2 \subseteq L'_1$  then  $\pi^s L'_2 \subseteq \pi^s L'_1 \subseteq L'_2$ , so  $\text{dist}([L_1], [L_2]) = \text{dist}([L_2], [L_1])$ .
- (3) Let  $[L_3] \in \mathcal{B}_d^0(K)$  and set  $s' = \text{dist}([L_2], [L_3])$ . Let, moreover  $L'_3 \in [L_3]$  and  $L''_2 \in [L_2]$  be such that  $\pi^{s'} L''_2 \subseteq L'_3 \subseteq L''_2$ . Write  $L''_2 = \pi^t L'_2$ . Then

$$\pi^t L'_1 \supseteq \pi^t L'_2 \supseteq L'_3 \supseteq \pi^{s'+t} L'_2 \supseteq \pi^{s+s'+t} L'_1 = \pi^{s+s'}(\pi^t L'_1),$$

yielding that  $\text{dist}([L_1], [L_2]) + \text{dist}([L_2], [L_3]) \geq \text{dist}([L_1], [L_3])$ .

The choices of  $[L_1], [L_2], [L_3]$  being arbitrary, the proof is complete.  $\square$

Thanks to the elementary divisor theorem for modules over PIDs, we know that any two lattices in  $K^d$  have compatible bases, i.e. for any two lattice classes  $[L_1]$  and  $[L_2]$ , there is always an apartment containing both. So, to compute their distance, we may choose a frame basis  $(e_1, \dots, e_d)$  of  $K^d$ , so that  $L_1 = L_{(0, \dots, 0)}$  and  $L_2 = L_{(u_1, \dots, u_d)}$  with  $u_1 \geq \dots \geq u_d$ . With this choice, we obtain that  $\text{dist}([L_1], [L_2]) = u_1 - u_d$ .

**Remark 3.3.** The distance between lattice classes  $[L_u]$  and  $[L_v]$  in the same apartment  $\mathcal{A}(E)$  is given by

$$\text{dist}([L_u], [L_v]) = \max_{1 \leq i \leq d} (v_i - u_i) - \min_{1 \leq j \leq d} (v_j - u_j).$$

In particular, any bounded subset of an apartment is finite. For a connection to tropical geometry, see for instance [9, Section 5.3].

Note that the distance from Definition 3.1 coincides with the 1-skeleton distance on  $\mathcal{B}_d^0(K)$ , as the following result shows. For  $L$  and  $L'$  lattices with  $\pi L \subset L' \subset L$ , write  $([L], [L'])$  for the 1-simplex with ends  $[L]$  and  $[L']$ .

**Lemma 3.4.** *Let  $[L_1], [L_2] \in \mathcal{B}_d^0(K)$  be distinct and set  $s = \text{dist}([L_1], [L_2])$ . Then  $s > 0$ ,*

- (1) *there exist  $[L_1] = [X_0], [X_1], \dots, [X_{s-1}], [X_s] = [L_2] \in \mathcal{B}_d^0(K)$  such that  $([X_{i-1}], [X_i])$  are 1-simplices for all  $1 \leq i \leq s$ , and*
- (2) *there is no shorter sequence connecting  $[L_1]$  and  $[L_2]$  in the 1-skeleton of  $\mathcal{B}_d(K)$ .*

*Proof.* The number  $s$  is positive as a consequence of Lemma 3.2. Without loss of generality, assume that  $\pi^s L_1 \subseteq L_2 \subseteq L_1$  and put  $X_1 := \pi L_1 + L_2$ . Then  $\pi L_1 \subseteq X_1 \subseteq L_1$  and so  $([L_1], [X_1])$  is a 1-simplex in  $\mathcal{B}_d(K)$ . For  $i = 2, \dots, s$ , put  $X_i := \pi X_{i-1} + L_2 = \pi^i L_1 + L_2$ . Then  $X_s = L_2$  and all  $([X_{i-1}], [X_i])$  are 1-simplices in the building. We have proven (1), while (2) follows from the triangle inequality and the fact that two lattice classes in a 1-simplex have distance at most 1.  $\square$

### 3.2. Balls and bolytropes.

**Definition 3.5.** Let  $\mathcal{L}$  be a bounded subset of  $\mathcal{B}_d^0(K)$ . Then the *closed ball of radius  $r$  and center  $\mathcal{L}$*  is

$$B_r(\mathcal{L}) := \{[L] \in \mathcal{B}_d^0(K) \mid \text{dist}([L], \mathcal{L}) \leq r\}.$$

If  $\mathcal{L} = \{[L]\}$  consists of one element only, then

$$B_r([L]) := B_r(\mathcal{L})$$

is the *ball with center  $[L]$  and radius  $r$* . If  $\mathcal{L} = Q(\Lambda(M))$ , then

$$B_r(M) := B_r(Q(\Lambda(M)))$$

is called the *bolytrope with center  $Q(\Lambda(M))$  and radius  $r$* .

In particular, the ball  $B_r([L])$  consists of all lattice classes  $[L']$  that are represented by some lattice  $L'$  such that  $\pi^r L \subseteq L' \subseteq L$ . We close the section by computing the intersection of a bolytrope with an apartment. Recall that  $J_d \in \mathcal{P}_d(\mathbb{Z})$  is the matrix with all 1s outside of the main diagonal.

**Lemma 3.6.** *Let  $\mathcal{A}$  be an apartment containing  $Q(\Lambda(M))$ . Then*

$$B_r(M) \cap \mathcal{A} = Q(\Lambda(M + rJ_d)).$$

*Proof.* Let  $(e_1, \dots, e_d)$  be a frame basis defining  $\mathcal{A}$  and put  $Q = Q(\Lambda(M + rJ_d))$ . We will use Remark 2.3 with respect to this basis. Since  $\pi^r \Lambda(M) \subseteq \Lambda(M + rJ_d) \subseteq \Lambda(M)$ , we have the inclusion  $Q \subseteq B_r(M)$ . Now we show the other inclusion. Let  $[L_u]$  in  $\mathcal{A}$  be of distance at most  $r$  from some lattice  $[L_v] \in Q(\Lambda(M))$ . Suppose that  $[L_u] \notin Q(\Lambda(M + rJ_d))$ . This means that there exist  $1 \leq i \neq j \leq d$  such that  $u_i - u_j > M_{ij} + r$ . However, since  $[L_v] \in Q(\Lambda(M))$ , we have  $v_i - v_j \leq M_{ij}$  and hence  $u_i - u_j > v_i - v_j + r$ . In other words

$$(u_i - v_i) - (u_j - v_j) > r, \text{ so } \text{dist}([L_u], [L_v]) > r.$$

This is a contradiction and so the proof is complete.  $\square$

**3.3. Plesken-Zassenhaus closed sets.** We have seen that closed orders are determined by the collection of their stable lattices; such sets are thus of fundamental importance for the study of closed orders.

**Definition 3.7.** A subset  $\mathcal{L}$  of  $\mathcal{B}_d^0(K)$  is called *PZ-closed* if  $\mathcal{L} = Q(\Lambda)$  for some order  $\Lambda$ .

For the study of PZ-closed subsets it clearly suffices to consider closed orders  $\Lambda$ . Note that the bijection  $\Lambda \leftrightarrow Q(\Lambda)$  is a Galois correspondence between

$$\{ \text{closed orders in } K^{d \times d} \} \longleftrightarrow \{ \text{PZ-closed subsets of } \mathcal{B}_d^0(K) \}.$$

As shown in [7] (see also Remark 3.12), the PZ-closed subsets of one apartment  $\mathcal{A}$  are exactly the finite and convex subsets of  $\mathcal{B}_d^0(K)$ , i.e. the polytropes. In general, being bounded and convex is a necessary but not sufficient condition for a subset of  $\mathcal{B}_d^0(K)$  to be closed.

**Proposition 3.8.** *Let  $\Lambda$  be an order in  $K^{d \times d}$ . Then  $Q(\Lambda)$  is a non-empty bounded convex subset of  $\mathcal{B}_d^0(K)$ .*

*Proof.* As any order is contained in a maximal order, there is some maximal order  $\Gamma$ , with  $\Lambda \subseteq \Gamma$ . Both lattices  $\Lambda$  and  $\Gamma$  have full rank in  $K^{d \times d}$ , so there is  $r \in \mathbb{Z}_{\geq 0}$  such that  $\pi^r \Gamma \subseteq \Lambda \subseteq \Gamma$ . If  $[L]$  is the unique class of  $\Gamma$ -lattices, then  $[L] \in Q(\Lambda)$  and hence  $Q(\Lambda)$  is not empty. Moreover, all lattice classes in  $Q(\Lambda)$  have a representative between  $L$  and  $(\pi^r \Gamma)L = \pi^r L$ , so  $Q(\Lambda)$  is contained in the ball of radius  $r$  around  $[L]$ . In particular,  $Q(\Lambda)$  is bounded. To see convexity, let  $[L'], [L''] \in Q(\Lambda)$ . Then there is an apartment containing

both lattice classes, so  $\Gamma' := \text{End}_{\mathcal{O}_K}(L') \cap \text{End}_{\mathcal{O}_K}(L'')$  is a graduated order containing  $\Lambda$ . But then the convex set  $Q(\Gamma') \subseteq Q(\Lambda)$  contains both lattice classes  $[L']$  and  $[L'']$ , and,  $[L']$  and  $[L'']$  being arbitrary,  $Q(\Lambda)$  is convex.  $\square$

**Remark 3.9.** Let  $\Lambda$  be an order in  $K^{d \times d}$  and let  $\mathcal{A}$  be an apartment in  $\mathcal{B}_d(K)$  such that  $Q(\Lambda) \cap \mathcal{A} \neq \emptyset$ . Then

$$Q(\Lambda) \cap \mathcal{A} = Q(\Gamma)$$

for a unique graduated overorder  $\Gamma$  of  $\Lambda$ . Indeed, if  $\mathcal{A} = \mathcal{A}(E)$  and  $\mathcal{E} = \{\epsilon_1, \dots, \epsilon_d\}$  is the set of projections on the frame  $E$ , then there are only finitely many maximal overorders of  $\Lambda$  that contain  $\mathcal{E}$ . Their intersection is the desired graduated order  $\Gamma$ .

### 3.4. The degree of a closed order.

**Definition 3.10.** Let  $\mathcal{L}$  be a bounded subset of  $\mathcal{B}_d^0(K)$ . The *Plesken-Zassenhaus order* associated to  $\mathcal{L}$  is

$$\text{PZ}(\mathcal{L}) := \bigcap_{[L] \in \mathcal{L}} \text{End}_{\mathcal{O}_K}(L).$$

**Proposition 3.11.** *The Plesken-Zassenhaus order  $\text{PZ}(\mathcal{L})$  of a bounded subset  $\mathcal{L}$  of  $\mathcal{B}_d^0(K)$  is an  $\mathcal{O}_K$ -order in  $K^{d \times d}$*

*Proof.* Put  $\Lambda = \text{PZ}(\mathcal{L})$ . Then  $\Lambda$  is an  $\mathcal{O}_K$ -module that is closed under multiplication and contains  $\text{Id}_d$ . It remains to show that  $\Lambda$  is of full rank in  $K^{d \times d}$ . As  $\mathcal{L}$  is bounded, there are  $[L] \in \mathcal{L}$  and  $r \in \mathbb{Z}_{\geq 0}$  such that  $\mathcal{L} \subseteq B_r([L])$ . For the maximal order  $\Gamma = \text{End}_{\mathcal{O}_K}(L)$  we hence have that  $\pi^r \Gamma L' \subseteq L'$  for all  $L' \in \mathcal{L}$ . So  $\pi^r \Gamma \subseteq \Lambda \subseteq \Gamma$  and, as  $\pi^r \Gamma$  contains a  $K$ -basis of  $K^{d \times d}$ , the same is true for  $\Lambda$ .  $\square$

The following remark illustrates the notions introduced in Section 2.3 and Definition 3.10 for the special case of graduated orders.

**Remark 3.12.** ([7, Proposition 6 and 7, Corollary 9]) If  $M \in \mathcal{P}_d(\mathbb{Z})$ , then the graduated order  $\Lambda(M)$  is closed and

$$Q(\Lambda(M)) = \{[L_u] \mid u \in \mathbb{Z}^d, u + \mathbb{R}\mathbf{1} \in Q_M\}$$

is a finite set which we can identify with the integral points of the polytrope  $Q_M$ . Moreover, the projective  $\Lambda(M)$ -lattices are given by the columns of  $M$  in the following way: if  $M^{(1)}, \dots, M^{(d)}$  denote the columns of  $M$ , then, for each projective  $\Lambda(M)$ -lattice  $L$ , there exists  $i \in \{1, \dots, d\}$  such that  $L$  is homothetic to

$$P_i := \Lambda(M)\epsilon_i = L_{M^{(i)}}.$$

The polytrope  $Q_M$  is the min-convex hull of the set  $\{M^{(1)} + \mathbb{R}\mathbf{1}, \dots, M^{(d)} + \mathbb{R}\mathbf{1}\}$  and has dimension  $\dim(Q_M) = |\{[P_1], \dots, [P_d]\}| - 1$ . The order  $\Lambda(M) = \text{PZ}([P_1], \dots, [P_d])$  is the Plesken-Zassenhaus order of its projective lattices in  $K^d$ .

The next proposition shows that closed orders are always an intersection of finitely many maximal orders.

**Proposition 3.13.** *Let  $\mathcal{L} \subseteq \mathcal{B}_d^0(K)$  be bounded and let  $\Lambda = \text{PZ}(\mathcal{L})$  denote its Plesken-Zassenhaus order. Then there exists a finite subset  $\{[L_1], \dots, [L_n]\}$  of  $\mathcal{L}$  such that  $\Lambda = \text{PZ}([L_1], \dots, [L_n])$ .*

*Proof.* Choose  $[L_1] \in \mathcal{L}$  arbitrarily and put  $\Gamma = \text{End}_{\mathcal{O}_K}(L_1)$ . As  $\mathcal{L}$  is bounded, there is  $r \in \mathbb{Z}_{\geq 0}$  such that  $\mathcal{L} \subseteq B_r([L_1])$  and so

$$\pi^r \Gamma \subseteq \Lambda \subseteq \Gamma.$$

In particular, the  $\mathcal{O}_K$ -module  $\Gamma/\Lambda$  has finite composition length (at most the composition length  $d^2r$  of  $\Gamma/\pi^r\Gamma$ ). We proceed by induction on this composition length. If  $\Gamma = \Lambda$  then we are done, otherwise there is some  $[L_2] \in \mathcal{L}$  such that  $[L_2] \notin Q(\Gamma)$ . Replace  $\Gamma$  by  $\Gamma \cap \text{End}_{\mathcal{O}_K}(L_2) = \text{PZ}([L_1], [L_2])$  to decrease the composition length of  $\Gamma/\Lambda$ . After finitely many steps this process constructs the finite set  $\{[L_1], \dots, [L_n]\}$  with  $\Lambda = \text{PZ}([L_1], \dots, [L_n])$ .  $\square$

For a closed order  $\Lambda$ , the minimal cardinality of a set  $\mathcal{L}$  such that  $\Lambda = \text{PZ}(\mathcal{L})$  is hence an interesting invariant.

**Definition 3.14.** Let  $\Lambda$  be a closed order. Then the *degree* of  $\Lambda$  is

$$\text{deg}(\Lambda) := \min\{|\mathcal{L}| - 1 \mid \mathcal{L} \subseteq \mathcal{B}_d^0(K) \text{ with } \Lambda = \text{PZ}(\mathcal{L})\}.$$

Thanks to Proposition 3.13, any closed order is a finite intersection of maximal orders, so the degree of a closed order is always finite. The closed orders of degree 0 are exactly the maximal orders and the ones of degree 1 are certain graduated orders. In general, the degree of a graduated order  $\Lambda(M)$  is equal to  $\dim(Q_M)$ , cf. Remark 3.12. In the coming sections, we will see that, for ball orders and bolytrope orders, the degree is always bounded from above by  $d$ , cf. Theorems 5.6 and 6.9, though such bound need not always be sharp, cf. Remark 5.7.

## 4 The radical idealizer process

Let  $\Lambda$  be an order in  $K^{d \times d}$ . In this section, we describe the radical idealizer chain of  $\Lambda$ , a construction that will be at the foundation of the proofs of our main results.

**Definition 4.1.** Let  $\Lambda$  and  $L$  be an order and a lattice in  $K^{d \times d}$ , respectively.

- The *Jacobson radical*  $\text{Jac}(\Lambda)$  of  $\Lambda$  is the intersection of all maximal left ideals of  $\Lambda$ .
- The *idealizer* of  $L$  is  $\text{Id}(L) := \{X \in K^{d \times d} \mid XL \subseteq L \text{ and } LX \subseteq L\}$ .

**Remark 4.2.** If  $\Lambda$  is an order in  $K^{d \times d}$ , then  $\text{Jac}(\Lambda)$  is a two-sided ideal of  $\Lambda$  that contains  $\pi\Lambda$ . The quotient  $\Lambda/\text{Jac}(\Lambda)$  is a semisimple  $\mathcal{O}_K/\mathfrak{m}_K$ -algebra and, for some  $n$ , one has  $\text{Jac}(\Lambda)^n \subseteq \pi\Lambda$ . Moreover,  $\text{Jac}(\Lambda)$  is the unique pro-nilpotent ideal with semisimple quotient ring. For this and more, see for instance [12, Chapter 1, Section 6].

**Definition 4.3.** Let  $\Lambda$  be an order in  $K^{d \times d}$ . The *radical idealizer chain*  $(\Omega_i)_{i \geq 0}$  of  $\Lambda$  is recursively defined by

$$\Omega_0 := \Lambda \text{ and } \Omega_{i+1} = \text{Id}(\text{Jac}(\Omega_i)).$$

**Remark 4.4.** The radical idealizer chain of an order  $\Lambda$  is an ascending finite chain  $\Omega_0 \subset \Omega_1 \subset \dots \subset \Omega_s (= \Omega_{s+1} = \dots)$ ; cf. [10, Remark 3.8]. Moreover, as  $\pi\Lambda \subseteq \text{Jac}(\Lambda)$ , we have

$$\Lambda \subseteq \Omega_1 = \text{Id}(\text{Jac}(\Lambda)) \subset \frac{1}{\pi}\Lambda.$$

This yields an efficient algorithm to compute the radical idealizer chain for orders based on solving linear equations in the residue field; cf. [10]. The sets of invariant lattices  $\mathcal{L}_i := Q(\Omega_i)$  form a descending chain

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots \supset \mathcal{L}_s,$$

where the last element  $\mathcal{L}_s = Q(\Omega_s)$  is known to be a simplex in the building  $\mathcal{B}_d(K)$ ; cf. [12, Theorem (39.14)]. The length  $s \geq 0$  of the radical idealizer chain is called the *radical idealizer length* of the order  $\Lambda$ .

**Lemma 4.5.** *Let  $\Lambda$  be an order in  $K^{d \times d}$  and put  $\Omega_1 := \text{Id}(\text{Jac}(\Lambda))$ . Then*

$$Q(\Omega_1) \subseteq Q(\Lambda) \subseteq B_1(Q(\Omega_1))$$

*and all lattices in  $Q(\Lambda)$  have distance at most one from  $Q(\Omega_1)$ .*

*Proof.* As  $\Omega_1 \supseteq \Lambda$ , we know that  $Q(\Omega_1) \subseteq Q(\Lambda)$  and thus we get  $Q(\Omega_1) = \{[\Omega_1 L] \mid [L] \in Q(\Lambda)\}$ . Moreover, by Remark 4.4, we have  $\Lambda \subseteq \Omega_1 \subseteq \frac{1}{\pi}\Lambda$ , so  $L \subseteq \Omega_1 L \subseteq \frac{1}{\pi}L$  and hence  $\text{dist}([L], [\Omega_1 L]) \leq 1$ , for all  $[L] \in Q(\Lambda)$ .  $\square$

**Lemma 4.6.** *Let  $M \in \mathcal{P}_d(\mathbb{Z})$ . Then  $\text{Id}(\text{Jac}(\Lambda(M + J_d))) = \Lambda(M)$ .*

*Proof.* As  $\dim(Q_{M+J_d}) = d - 1$ , we know by [7, Example 23] that the Jacobson radical of  $\Lambda(M + J_d)$  is equal to  $\pi\Lambda(M)$ . This is a 2-sided principal ideal in the order  $\Lambda(M)$ , so  $\text{Id}(\pi\Lambda(M)) = \Lambda(M)$ .  $\square$

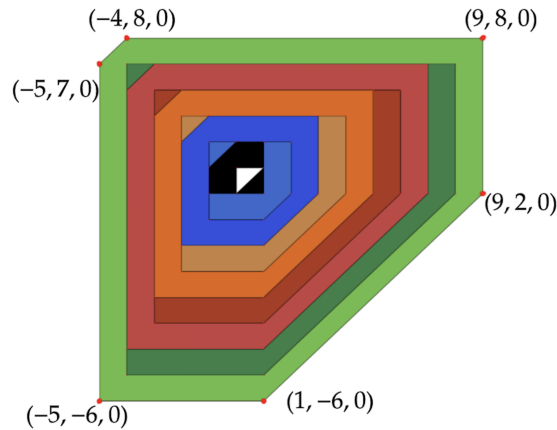


FIGURE 1. The radical idealizer process for the order  $\Lambda(M)$  in Example 4.7

**Example 4.7.** Consider the configuration of lattice classes  $[L_{u_1}]$ ,  $[L_{u_2}]$  and  $[L_{u_3}]$  where

$$u_1 = (0, 12, 5) \sim (-5, 7, 0), \quad u_2 = (7, 0, 6) \sim (1, -6, 0), \quad \text{and} \quad u_3 = (9, 8, 0).$$

In the notation of [7], this configuration corresponds to the matrix

$$M = \begin{pmatrix} 0 & 7 & 9 \\ 12 & 0 & 8 \\ 5 & 6 & 0 \end{pmatrix}$$

and the decreasing sequence of polytropes  $(Q(\Omega_i))_{i \geq 0}$  corresponding to the radical idealizer process for the order  $\Lambda(M)$  is depicted in Figure 1. As expected, the last polytope (in white) is indeed a simplex.

## 5 Ball Orders

In this section, we define and study a first subfamily of the bolytrope orders, namely closed orders whose set of invariant lattices is a ball in  $\mathcal{B}_d^0(K)$ .

**Definition 5.1.** A *ball order* in  $K^{d \times d}$  is an order of the form  $\mathbb{B}_r([L]) := \text{PZ}(\text{B}_r([L]))$ , where  $L$  is a lattice in  $K^d$  and  $r$  is a non-negative integer.

**Theorem 5.2.** *Let  $L$  be a lattice in  $K^d$  and let  $(e_1, \dots, e_d)$  be a basis of  $L$ . Let, moreover,  $r$  be a non-negative integer. Then, with respect to  $(e_1, \dots, e_d)$ , we have*

$$\mathbb{B}_r([L]) = \{X \in \Lambda(rJ_d) \mid X_{11} \equiv \dots \equiv X_{dd} \pmod{\pi^r}\}.$$

*Moreover,  $Q(\mathbb{B}_r([L])) = \text{B}_r([L])$  and the ball  $\text{B}_r([L])$  is PZ-closed.*



*Proof.* Put  $\Lambda = \{X \in \Lambda(rJ_d) \mid X_{11} \equiv \dots \equiv X_{dd} \pmod{\pi^r}\}$  and  $\Gamma = \text{End}_{\mathcal{O}_K}(L) = \Lambda(0)$ . It follows from the definition of  $\Lambda$  that  $\pi^r\Gamma \subseteq \Lambda$ . If  $L'$  is another lattice such that  $\pi^r L \subseteq L' \subseteq L$ , then  $\pi^r\Gamma L' \subseteq \pi^r\Gamma L = \pi^r L \subseteq L'$ , which yields  $\pi^r\Gamma \subseteq \mathbb{B}_r([L])$ . Now the lattice classes at distance at most  $r$  from  $[L]$  can be described as submodules of  $V_r = L/\pi^r L$ . In particular, the image  $\overline{\mathbb{B}_r([L])}$  of  $\mathbb{B}_r([L])$  in the endomorphism ring  $\text{End}_{\mathcal{O}_K}(V_r) \cong (\mathcal{O}_K/\mathfrak{m}_K^r)^{d \times d}$  is equal to the collection of all endomorphisms stabilizing every submodule of  $V_r$ . This ensures that

$$\overline{\mathbb{B}_r([L])} = (\mathcal{O}_K/\mathfrak{m}_K^r) \text{Id}_d = \overline{\Lambda}.$$

As both orders  $\mathbb{B}_r([L])$  and  $\Lambda$  contain the kernel  $\pi^r\Gamma$  of the projection  $\Gamma \rightarrow \text{End}_{\mathcal{O}_K}(V_r)$ , we conclude that  $\Lambda = \mathbb{B}_r([L]) = \text{PZ}(\mathbb{B}_r([L]))$ . We now show that  $Q(\mathbb{B}_r([L])) = \mathbb{B}_r([L])$ . To this end, let  $[L'] \in Q(\Lambda)$ . Then  $[\Gamma L'] \in Q(\Gamma) = \{[L]\}$ . Replacing  $L'$  by some homothetic lattice we hence may assume that  $\Gamma L' = L$ . But  $\pi^r\Gamma \subseteq \Lambda \subseteq \text{End}_{\mathcal{O}_K}(L')$  so  $\pi^r\Gamma L' = \pi^r L \subseteq L'$  so  $[L'] \in \mathbb{B}_r([L])$ .  $\square$

**Remark 5.3.** (Radical idealizer chain of ball orders) Let  $r$  be a positive integer. Then the Jacobson radical of the ball order  $\mathbb{B}_r([L]) = \text{PZ}(\mathbb{B}_r([L]))$  is  $\text{Jac}(\mathbb{B}_r([L])) = \pi\mathbb{B}_{r-1}([L])$ , because  $\pi\mathbb{B}_{r-1}([L])$  is a pro-nilpotent ideal of  $\mathbb{B}_r([L])$  with simple quotient  $\mathbb{B}_r([L])/\pi\mathbb{B}_{r-1}([L])$  isomorphic to  $\mathcal{O}_K/\mathfrak{m}_K$ . Now  $\pi\mathbb{B}_{r-1}([L])$  is a principal 2-sided ideal of  $\mathbb{B}_{r-1}([L])$  so

$$\text{Id}(\text{Jac}(\mathbb{B}_r([L]))) = \text{Id}(\pi\mathbb{B}_{r-1}([L])) = \mathbb{B}_{r-1}([L])$$

and the radical idealizer chain for ball orders is thus

$$\mathbb{B}_r([L]) \subset \mathbb{B}_{r-1}([L]) \subset \dots \subset \mathbb{B}_1([L]) \subset \mathbb{B}_0([L]) = \text{End}_{\mathcal{O}_K}(L).$$

The corresponding chain of PZ-closed subsets of  $\mathcal{B}_d^0(K)$  is

$$\mathbb{B}_r([L]) \supset \mathbb{B}_{r-1}([L]) \supset \dots \supset \mathbb{B}_1([L]) \supset \mathbb{B}_0([L]) = \{[L]\}.$$

The knowledge of the radical idealizer chain of ball orders allows to prove strong properties of ball orders, like the following.

**Proposition 5.4.** *Let  $r$  be a positive integer and  $\Lambda$  a closed order in  $K^{d \times d}$  such that  $\text{Id}(\text{Jac}(\Lambda)) = \mathbb{B}_{r-1}([L])$ . Then one has  $\mathbb{B}_r([L]) \subseteq \Lambda \subseteq \mathbb{B}_{r-1}([L])$ .*

*Proof.* It follows from the hypotheses and the combination of Lemma 4.5 with Theorem 5.2 that  $\mathbb{B}_{r-1}([L]) \subseteq Q(\Lambda) \subseteq \mathbb{B}_r([L])$ . The orders being closed, Remark 5.3 yields that  $\mathbb{B}_r([L]) \subseteq \Lambda \subseteq \mathbb{B}_{r-1}([L])$ .  $\square$

**Definition 5.5.** Let  $r$  be a non-negative integer and  $L$  a lattice in  $K^d$ . A *star configuration*  $\star_r([L])$  with center  $[L]$  and radius  $r$  is a set

$$\star_r([L]) = \{[L_1], \dots, [L_d], [L_{d+1}]\}$$

such that the following hold:

- (1)  $\pi^r L \subseteq L_1, \dots, L_{d+1} \subseteq L$ ,
- (2) for each  $i \in \{1, \dots, d+1\}$ , one has  $L_i/\pi^r L \cong \mathcal{O}_K/\mathfrak{m}_K^r$ ,
- (3) for each  $i \in \{1, \dots, d+1\}$ , one has  $L = \sum_{j \neq i} L_j$ .

When  $r = 1$ , i.e. when  $\mathcal{O}_K/\mathfrak{m}_K^r$  is a field, the 1-dimensional free  $\mathcal{O}_K/\mathfrak{m}_K^r$ -modules  $L_i/\pi^r L$  of  $L/\pi^r L$  form a projective basis. In this sense, Definition 5.5 generalizes the definition of a projective basis to modules over rings.

**Theorem 5.6.** *Let  $r$  be a non-negative integer and let  $L$  be a lattice in  $K^d$ . Let, moreover,  $\star_r([L])$  denote a star configuration with center  $[L]$  and radius  $r$ . Then one has*

$$\mathbb{B}_r([L]) = \text{PZ}(\star_r([L])) \quad \text{and} \quad \text{deg}(\mathbb{B}_r([L])) \leq d.$$

*Proof.* Write  $\Lambda := \text{PZ}(\star_r([L]))$  and  $\star_r([L]) =: \{[L_1], \dots, [L_{d+1}]\}$ . Since  $\star_r([L])$  has radius  $r$ , we have that  $\star_r([L]) \subseteq \mathbb{B}_r([L])$ , so  $\Lambda \supseteq \mathbb{B}_r([L])$ . We now claim that  $\Lambda$  stabilizes all lattices  $L'$  with  $\pi^r L \subseteq L' \subseteq L$ . To this end, write  $\bar{L} = L/\pi^r L$  and use the bar notation for the submodules of  $\bar{L}$ . For  $1 \leq i \leq d$  let  $e_i \in L_i$  be such that  $\overline{\mathcal{O}_K e_i} = \bar{L}_i$ . Since  $L_1 + \dots + L_d = L$ , the set  $\{\bar{e}_1, \dots, \bar{e}_d\}$  is a basis of the free module  $\bar{L}$ . So there are  $a_i \in \mathcal{O}_K$  such that  $\mathcal{O}_K \sum_{i=1}^d a_i e_i = \overline{L_{d+1}}$ . Since  $\star_r([L])$  is a star configuration, all  $a_i$ 's are units, so, replacing  $e_i$  by  $a_i e_i$ , we assume, without loss of generality, that  $L_{d+1} = \mathcal{O}_K(e_1 + \dots + e_d) + \pi^r L$ . Since each  $L_i$  is  $\Lambda$ -stable, the image of  $\Lambda$  in  $\text{End}(\bar{L}) \cong (\mathcal{O}_K/\mathfrak{m}_K^r)^{d \times d}$  consists of scalar matrices and so all submodules of  $\bar{L}$  are stable. This yields the claim and so  $\mathbb{B}_r([L]) = \text{PZ}(\star_r([L]))$ . The order  $\mathbb{B}_r([L])$  has degree at most  $d$ , because a star configuration has cardinality  $d+1$ .  $\square$

The following remark shows that ball orders in  $K^{d \times d}$  can have degree smaller than  $d$ .

**Remark 5.7.** The degree of  $\mathbb{B}_r([\mathcal{O}_K^4])$  is at most 3, because  $\mathbb{B}_r([\mathcal{O}_K^4])$  is equal to the Plesken-Zassenhaus order of the following lattices (where the columns of the matrices are the basis elements):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & \pi^r & 0 \\ 0 & 0 & 0 & \pi^r \end{pmatrix}, \begin{pmatrix} \pi^r & 0 & 0 & 0 \\ 0 & \pi^r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & \pi^r & 0 \\ 1 & 0 & 0 & \pi^r \end{pmatrix}, \text{ and } \begin{pmatrix} \pi^r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi^r & 0 \\ 0 & 1 & 0 & \pi^r \end{pmatrix}.$$

Via change of coordinates, one obtains that any ball order in  $K^{4 \times 4}$  has degree at most 3.

## 6 Bolytrope Orders

Let  $M \in \mathcal{P}_d(\mathbb{Z})$ . Recall, from Definition 3.5, that the bolytrope  $\mathbb{B}_r(M)$  is defined to be  $\mathbb{B}_r(Q(\Lambda(M)))$ .

**Definition 6.1.** A *bolytrope order* is an order of the form  $\mathbb{B}_r(M) := \text{PZ}(\mathbb{B}_r(M))$ , where  $M$  is an element of  $\mathcal{P}_d(\mathbb{Z})$  and  $r$  is a non-negative integer.

Until the end of the present section, fix  $M \in \mathcal{P}_d(\mathbb{Z})$  and an apartment  $\mathcal{A}$  containing  $Q(\Lambda(M))$ . Let, moreover,  $r$  be a non-negative integer. Then, by Lemma 3.6, we have that  $\mathbb{B}_r(M) \cap \mathcal{A} = Q(\Lambda(M + rJ_d))$ , in particular  $\mathbb{B}_r(M) \subseteq \Lambda(M + rJ_d)$ . Put

$$\Lambda_r(M) = \{X \in \Lambda(M + rJ_d) \mid X_{11} \equiv \dots \equiv X_{dd} \pmod{\pi^r}\}.$$

We will show that  $\Lambda_r(M) = \mathbb{B}_r(M)$  and  $Q(\Lambda_r(M)) = \mathbb{B}_r(M)$  is PZ-closed; cf. Theorem 6.6.

**Lemma 6.2.** *Let  $[L]$  be a lattice class in  $Q(\Lambda(M))$ . Then  $\Lambda_r(M) = \Lambda(M + rJ_d) \cap \mathbb{B}_r([L])$  and  $\Lambda_r(M)$  is a closed order.*

*Proof.* Let  $(e_1, \dots, e_d)$  be a basis of  $L$  that is also a frame basis defining the apartment  $\mathcal{A}$ . Then, with respect to this basis,  $[L] = [\mathcal{O}_K^d]$  and thus  $\Lambda(M) \subseteq \text{End}_{\mathcal{O}_K}(L) = \mathcal{O}_K^{d \times d} = \Lambda(0)$ . It follows in particular that  $M$  has non-negative entries. The explicit description of the ball order in Theorem 5.2 allows to deduce that  $\Lambda_r(M) = \Lambda(M + rJ_d) \cap \mathbb{B}_r([L])$ . Since  $\Lambda(M + rJ_d)$  and  $\mathbb{B}_r([L])$  are closed orders, then so is  $\Lambda_r(M)$ .  $\square$

**Lemma 6.3.** *One has  $\mathbb{B}_r(M) \subseteq Q(\Lambda_r(M))$  and  $\Lambda_r(M) \subseteq \mathbb{B}_r(M)$ .*

*Proof.* We first show that  $\mathbb{B}_r(M) \subseteq Q(\Lambda_r(M))$ . For this, let  $[L'] \in \mathbb{B}_r(M)$  and let  $[L] \in Q(\Lambda(M))$  be such that  $\text{dist}([L'], [L]) \leq r$ . Then the combination of Remark 5.3 and Lemma 6.2 yields that

$$[L'] \in \mathbb{B}_r([L]) = Q(\mathbb{B}_r([L])) \subseteq Q(\Lambda_r(M)).$$

To conclude, note that, thanks to Lemma 6.2, the order  $\Lambda_r(M)$  is closed and thus the inclusion  $\mathbb{B}_r(M) \subseteq Q(\Lambda_r(M))$  implies that  $\Lambda_r(M) \subseteq \mathbb{B}_r(M)$ .  $\square$

To prove that  $\mathbb{B}_r(M) = \Lambda_r(M)$  we use the radical idealizer chain of  $\Lambda_r(M)$ , which we describe in the following remark.

**Remark 6.4.** Assume that  $r \geq 1$ . Then, similarly to what is done in Remark 5.3, one sees that  $\text{Jac}(\Lambda_r(M)) = \pi\Lambda_{r-1}(M)$  is a 2-sided principal ideal of  $\Lambda_{r-1}(M)$  and hence  $\text{Id}(\text{Jac}(\Lambda_r(M))) = \Lambda_{r-1}(M)$ .

**Lemma 6.5.** *One has  $Q(\Lambda_r(M)) = \mathbb{B}_r(M)$ .*

*Proof.* Lemma 6.3 shows that  $\mathbb{B}_r(M) \subseteq Q(\Lambda_r(M))$ . For the opposite inclusion, we rely on Remark 6.4 to proceed by induction on  $r$ . Assume first that  $r = 0$ . Then  $Q(\Lambda_0(M)) = Q(\Lambda(M)) = \mathbb{B}_0(M)$  and so we are done. Now assume that  $r > 0$  and that  $Q(\Lambda_{r-1}(M)) = \mathbb{B}_{r-1}(M)$ . The fact that  $\Lambda_{r-1}(M) = \text{Id}(\text{Jac}(\Lambda_r(M)))$  together with Lemma 4.5 then yields that

$$Q(\Lambda_r(M)) \subseteq \mathbb{B}_1(Q(\Lambda_{r-1}(M))) = \mathbb{B}_1(\mathbb{B}_{r-1}(M)) \subseteq \mathbb{B}_r(M).$$

This concludes the proof.  $\square$

The following is the main result of this section and of the paper.

**Theorem 6.6.** *The following hold:*

$$\Lambda_r(M) = \mathbb{B}_r(M) \quad \text{and} \quad Q(\mathbb{B}_r(M)) = \mathbb{B}_r(M).$$

*In particular bolytrope orders are closed and bolytropes are PZ-closed.*

*Proof.* As a consequence of Lemma 6.2, both  $\Lambda_r(M)$  and  $\mathbb{B}_r(M)$  are closed orders. We are now done thanks to Lemma 6.5.  $\square$

**Corollary 6.7.** *The beginning of the radical idealizer chain for bolytrope orders is*

$$\mathbb{B}_r(M) \subset \mathbb{B}_{r-1}(M) \subset \dots \subset \mathbb{B}_1(M) \subset \mathbb{B}_0(M) = \Lambda(M).$$

*The first  $r + 1$  elements in the corresponding chain of PZ-closed subsets of  $\mathcal{B}_d(K)$  are*

$$\mathbb{B}_r(M) \supset \mathbb{B}_{r-1}(M) \supset \dots \supset \mathbb{B}_1(M) \supset Q(\Lambda(M)).$$

Note that  $\Lambda(M)$  is the first term in the radical idealizer process that is a graduated order. The polytrope  $Q(\Lambda(M))$  is hence canonically determined by the bolytrope  $\mathbb{B}_r(M)$  and called the *central polytrope* of  $\mathbb{B}_r(M)$ .

In analogy with ball orders, we obtain the following stronger property of bolytrope orders.

**Corollary 6.8.** *Assume that  $r \geq 1$  and let  $\Lambda$  be a closed order in  $K^{d \times d}$  such that  $\text{Id}(\text{Jac}(\Lambda)) = \mathbb{B}_{r-1}(M)$ . Then one has  $\mathbb{B}_r(M) \subseteq \Lambda \subseteq \mathbb{B}_{r-1}(M)$ .*

*Proof.* Analogous to the proof of Proposition 5.4.  $\square$

**Theorem 6.9.** *Let  $[P_1], \dots, [P_d]$  be the distinct classes of projective  $\Lambda(M + rJ_d)$ -lattices. Then there is a lattice class  $[L_{d+1}] \in \mathbb{B}_r(M)$ , such that*

$$\mathbb{B}_r(M) = \text{PZ}([P_1], \dots, [P_d], [L_{d+1}]).$$

*Moreover, the degree of  $\mathbb{B}_r(M)$  is at most  $d$ .*

*Proof.* As a consequence of Remark 3.12, we have that  $\Lambda(M + rJ_d) = \text{PZ}([P_1], \dots, [P_d])$ . In particular, for any lattice class  $[L_{d+1}] \in \mathbb{B}_r(M)$ , Lemma 6.2 and Theorem 6.6 imply that

$$\mathbb{B}_r(M) \subseteq \text{PZ}([P_1], \dots, [P_d], [L_{d+1}]) \subseteq \Lambda(M + rJ_d).$$

To construct  $L_{d+1}$  such that the inclusion  $\mathbb{B}_r(M) \supseteq \text{PZ}([P_1], \dots, [P_d], [L_{d+1}])$  holds, choose  $[L] \in Q(\Lambda(M))$  and a lattice basis  $(e_1, \dots, e_d)$  of  $L$  that is also a frame basis for some apartment containing  $Q(\Lambda(M + rJ_d))$ . Define  $L_{d+1} := \mathcal{O}_K(e_1 + \dots + e_d) + \pi^r L$  and, for each  $i = 1, \dots, d$ , put  $L_i := \mathcal{O}_K e_i + \pi^r L \in Q(\Lambda(M + rJ_d))$ . Then  $\{[L_1], \dots, [L_d], [L_{d+1}]\}$  is a star configuration with center  $[L]$  and radius  $r$ . By Theorem 5.6, we thus have

$$\text{PZ}([L_1], \dots, [L_d], [L_{d+1}]) = \mathbb{B}_r([L]),$$

which, together with Lemma 6.2 and Theorem 6.6, implies that  $\text{PZ}([P_1], \dots, [P_d], [L_{d+1}])$  is contained in  $\Lambda(M + rJ_d) \cap \mathbb{B}_r([L]) = \Lambda_r(M) = \mathbb{B}_r(M)$ .  $\square$

## 7 When the building is a tree

Throughout this section, assume that  $d = 2$ . Then the building  $\mathcal{B}_2(K)$  is an infinite tree. Apartments correspond to infinite paths in the tree and the bounded convex subsets of  $\mathcal{B}_2(K)$  are the bounded subtrees. For more on this and other trees, see for instance [13].

The following is the main result of this section, which extends [15, Theorem 2] beyond the case of finite residue fields.

**Theorem 7.1.** *Let  $\Lambda$  be a closed order in  $K^{2 \times 2}$ . Then there are  $r, m \in \mathbb{Z}_{\geq 0}$  such that*

$$\Lambda = \mathbb{B}_r \left( \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right) = \{X \in \mathcal{O}_K^{2 \times 2} \mid X_{12} \in \mathfrak{m}_K^{m+r}, X_{21} \in \mathfrak{m}_K^r, X_{11} \equiv X_{22} \pmod{\pi^r}\}.$$

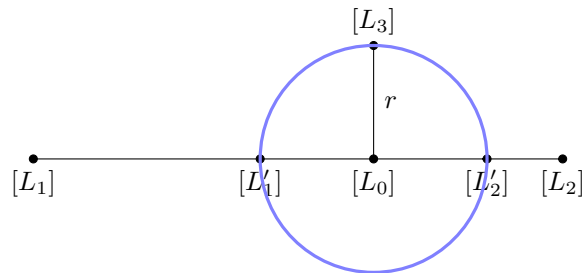
*Proof.* Put  $R := \max\{\text{dist}([L], [L']) \mid [L], [L'] \in Q(\Lambda)\}$  and let  $[L_1], [L_2] \in Q(\Lambda)$  be such that  $R = \text{dist}([L_1], [L_2])$ . Then the convex hull

$$\mathcal{L} = Q(\text{PZ}([L_1], [L_2])) \subseteq Q(\Lambda)$$

is a line segment and is hence contained in an apartment  $\mathcal{A}$ . Define

$$r := \max\{\text{dist}([L], \mathcal{L}) \mid [L] \in Q(\Lambda)\}$$

and let  $[L_3] \in Q(\Lambda)$  be such that  $r = \text{dist}([L_3], \mathcal{L})$ . Let, moreover,  $[L_0] \in \mathcal{L}$  denote the unique lattice class in  $\mathcal{L}$  satisfying  $\text{dist}([L_3], [L_0]) = r$ .



Now choose a frame basis  $(e_1, e_2)$  for  $\mathcal{A}$  such that, with respect to this basis, there exists an integer  $m$  such that  $[L_1] = [L_{(0,r)}]$  and  $[L_2] = [L_{(m+r,0)}]$ . It follows from the definition of  $R$  that

$$R + 1 = |\mathcal{L}| = m + 2r + 1.$$

With respect to the chosen basis, note now that  $\mathcal{L} = Q(\Lambda(M + rJ_2))$  and hence

$$\Lambda \subseteq \Lambda(M + rJ_2).$$

Moreover, if  $[L'_1]$  and  $[L'_2] \in \mathcal{L}$  are the two lattice classes at distance  $r$  from  $[L_0]$  and such that  $\text{dist}([L'_1], [L'_2]) = 2r$ , then the set  $\{[L_3], [L'_1], [L'_2]\}$  is a star configuration with radius

$r$  and center  $[L_0]$ . As a consequence of the definition of  $\mathcal{L}$ , such lattice classes  $[L'_1], [L'_2]$  exist and thus Theorem 5.6 ensures that

$$\Lambda \subseteq \mathbb{B}_r([L_0]).$$

We have proven that  $\Lambda \subseteq \mathbb{B}_r([L_0]) \cap \Lambda(M+rJ_2)$  and so  $\Lambda \subseteq \mathbb{B}_r(M)$ , thanks to Lemma 6.2. As  $Q(\Lambda) \subseteq \mathbb{B}_r(M) = Q(\mathbb{B}_r(M))$ , we obtain  $\Lambda = \mathbb{B}_r(M)$  as stated in the theorem.  $\square$

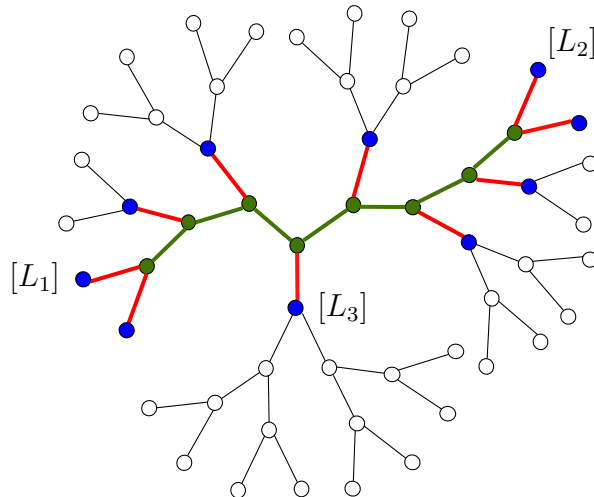


FIGURE 2. The bolytrope  $B_1(Q) = B_1\left(\begin{pmatrix} 07 \\ 00 \end{pmatrix}\right)$  in the Bruhat Tits tree of  $SL_2(\mathbb{Q}_2)$ . The green segment is the central polytrope  $Q := Q\left(\begin{pmatrix} 07 \\ 00 \end{pmatrix}\right) = \{[L_{(i,0)} \mid 0 \leq i \leq 7]\}$ . The set  $\mathcal{L} = Q\left(\begin{pmatrix} 08 \\ 10 \end{pmatrix}\right)$  is the convex hull of  $[L_1] = [L_{(0,1)}]$  and  $[L_2] = [L_{(8,0)}]$ . The blue vertices are the points at distance 1 from  $Q$ . The PZ-order of the lattice classes  $[L_1], [L_2]$  and  $[L_3]$  is the same as the PZ-order of all the colored vertices.

**Corollary 7.2.** *The PZ-closed subset of  $\mathcal{B}_2^0(K)$  are precisely the bolytropes.*

**Corollary 7.3.** *The degree of a closed order  $\Lambda$  in  $K^{2 \times 2}$  is 0, 1, or 2. Orders of degree 0 are the maximal orders, whereas the closed orders of degree 1 are precisely the graduated non-maximal orders. All non-graduated closed orders in  $K^{2 \times 2}$  have degree 2.*

**Remark 7.4.** Theorem 6.9 implies [15, Theorems 1 and 8]. To see this, note that, by taking  $[L_1], [L_2], [L_3]$  as in the proof of Theorem 7.1, we get that  $\Lambda = \text{PZ}([L_1], [L_2], [L_3])$ .

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