Lattice chain models for affine buildings of classical type

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ABSTRACT. A concrete lattice chain model for the buildings of the classical groups over non archimedean complete skew fields is given. The building axioms are proved in a uniform way using hereditary orders with involution instead of lattice chains.

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1 Introduction

By the theory of Bruhat and Tits (see [BrT 72], [BrT 84a]), there is an affine building associated to each reductive group over a local field. This group theoretic construction of affine buildings is, in general, highly complicated. However, in special cases "concrete" descriptions for these buildings were developed (and applied), which do not refer to the general Bruhat-Tits theory. The most well-known example of this type is the model of the affine building associated to $SL_n(\mathcal{D})$, where we denote by $\mathcal{D}$ a discretely valuated skew field and by $\mathfrak{M}$ the corresponding valuation ring. This model, which for $n = 2$ was introduced in [Ser 77], uses $\mathfrak{M}$-lattices in the vector space $\mathcal{D}^n$. The simplices of the building are either described (as in [Ser 77]) as certain sets of (homothety) classes of $\mathfrak{M}$-lattices or alternatively as chains of $\mathfrak{M}$-lattices which are stable under the action of $\mathcal{D}^*$ (cf. for instance [Gra 80] or [Gar 97, Chapter 19]).

The other classical groups are described as unitary groups of an algebra with involution, see [Wei 61], and hence naturally act on a hermitian vector space. From this point of view it is natural to look for lattice class (or chain) models in the case of affine buildings associated to other classical groups over local fields as well. This is partly done in [Gar 97, Chapter 20]. In [BrT 84b], [BrT 87], Bruhat and Tits discuss in detail a reinterpretation of their affine buildings for classical groups over complete, discretely valuated (and even more general) skew fields. This discussion is based on the fact that the points of the Bruhat-Tits building (viewed as a metric space) associated to $SL_n(\mathcal{D})$ are in one-to-one correspondence with the homothety classes of the "normes scindables" (cf. [BrT 84b, 2.13 and 5.1]), which in their turn are closely related to lattice chains (cf. [BrT 84b, 1.7 and 1.8]). In [BrT 87, Section 2] it is then shown that the other classical affine buildings can always be obtained as subspaces (with "normes maximinorantes" as points) of this model for the affine building of $SL_n(\mathcal{D})$.

In the present paper, we develop an approach towards "concrete" models for classical affine buildings, where we concentrate on constructing the latter as simplicial

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complexes, or more precisely as chamber complexes with a distinguished (maximal) system of apartments. Our approach is less general than that of Bruhat and Tits but (as we hope) more easily accessible and, this being our main point, completely independent of the Bruhat-Tits theory. In fact, we shall show that certain simplicial complexes, on which the corresponding unitary groups act in a natural way, are affine buildings, and we shall deduce from this the existence of affine BN-pairs in these groups. (In the Bruhat-Tits theory, these affine BN-pairs are constructed previous to the affine buildings, cf. [BrT 72, 6.5].) The simplicial complexes in question can again be described by lattice chains. In the case of unitary groups (and we include symplectic as well as orthogonal groups here), the appropriate lattice chains (called #-admissible in this paper) are self dual. So the simplicial complexes associated to the unitary groups arise as fixed point structures (but not necessarily as subcomplexes!) of the standard model for the affine building of $SL_n(\mathcal{D})$, this latter building being naturally acted upon by an involution # which is induced by the hermitian form (cf. Section 4). Using a criterion of Mühlherr (cf. [Mue 94, Theorem 1.8.22]), we show that these fixed point structures are (weak) buildings. At the same time, we are exhibiting concrete (and very natural) systems of apartments for these buildings. All this depends on a new description - given in Section 7 - of our simplicial complexes, where the simplices are interpreted as certain orders with involution instead of lattice chains. In particular this allows to treat all the unitary groups in a uniform way, without distinguishing the different types of hermitian forms.

Self dual lattice chains have already been used in the study of parahoric subgroups of p-adic algebraic groups (see [Mor 91a], [Mor 91b], [Kar 98], [KLP 97]). They come up naturally since the parahoric subgroups are the stabilizers of lattice chains. In the present paper we will not focus on the group point of view, because this poses additional difficulties, but use orders with involution which are much closer to lattices than the groups.

The paper is organized as follows. In Section 3, we briefly recall some facts concerning the affine building $\hat{\Delta}$ associated to $SL_n(\mathcal{D})$. The simplicial complex Δ consisting of all "#-admissible" chains of lattices in a given hermitian space V is introduced in Section 4. First properties of the thin subcomplexes associated to "hyperbolic frames" as well as of the natural action of the unitary group $U(V)$ on Δ are derived in Section 6. As a preparation, we deduce some facts about maximal lattices in anisotropic spaces in Section 5. The heart of this paper is Section 7, where important properties of Δ are derived by using orders with involution and in particular by studying their idempotents. The main result is Theorem 26, which yields that Δ is a weak building (of type $\tilde{C}_r$, where r is the Witt index of V) admitting the thin subcomplexes introduced in Section 6 as a system of apartments. In order to obtain thick buildings also in those cases when there are panels in Δ which are contained in two chambers only, we apply a generalization of the well-known "oriflamme construction" (cf. [Tit 74, Chapter 7]) in Section 8. It follows that $U(V)$ possesses a normal subgroup $U(V)_0$ with $[U(V) : U(V)_0] \leq 4$, which acts strongly transitively on a thick building Δ_0 of type $\tilde{C}_r$, $\tilde{B}_r$, or $\tilde{D}_r$ associated to Δ ($\Delta_0 = \Delta$ and $U(V)_0 = U(V)$ if Δ is already thick). This is used in order to construct an affine BN-pair in $U(V)_0$, and it also shows that Δ_0 can be described algebraically as the (Bruhat-Tits) building associated to this BN-pair.

2 Preliminaries concerning buildings

In this section we collect some notions and (well-known) facts from building theory which are used throughout this paper. Our standard references are [Tit 74] and [Bro 89].

We start by remarking that from our point of view, buildings are special finite dimensional simplicial complexes. Recall that a (combinatorial) simplicial complex can be considered in the usual way, where the simplices are finite sets of vertices, or equivalently as a partially ordered set satisfying certain conditions (cf. [Bro 89, Chapter I, Appendix]). The rank of a simplex, when considered as a set of vertices, is by definition the cardinality of this set. The rank of a simplicial complex Δ is defined to be the supremum of the ranks of its simplices, which we always require to be finite. To any simplex a of Δ, we associate a simplicial complex $\mathrm{lk}_\Delta(a)$, called the link of a in Δ (deviating from [Tit 74], where this complex is called the star of a) consisting of all simplices of Δ containing a. If Δ is considered as a poset, then we have $\mathrm{lk}_\Delta(a) = \{b \in \Delta \mid a \leq b\}$. If simplices are considered as sets of vertices, then the link of a is given as a simplicial complex by $\mathrm{lk}_\Delta(a) = \{b - a \mid b \in \Delta \text{ and } a \subseteq b\}$.

A simplex of a simplicial complex Δ of rank r is called a chamber if its rank is r, and it is called a panel if its rank is $r - 1$. Two chambers are called adjacent if their intersection is a panel. A finite sequence $(\mathcal{C}_0, \ldots, \mathcal{C}_l)$ of chambers is called a gallery, and $\mathcal{C}_0$ and $\mathcal{C}_l$ are said to be joined by this gallery, if $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are adjacent for all $0 \leq i \leq l - 1$. The length of this gallery is l by definition, and it is called minimal if there does not exist any gallery of length smaller than l starting in $\mathcal{C}_0$ and ending in $\mathcal{C}_l$.

A chamber complex is a simplicial complex Δ with the property that any simplex is contained in a chamber and any two chambers can be joined by a gallery. The (gallery) distance $d(\mathcal{C}, \mathcal{D})$ between two chambers $\mathcal{C}$ and $\mathcal{D}$ is by definition the length of a minimal gallery joining them. The distance $d(a, b)$ between two simplices a, b of Δ is defined to be the minimum of all $d(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ runs over the chambers containing a and $\mathcal{D}$ runs over the chambers containing b. We define the diameter $diam(\Delta)$ of Δ as the supremum of all $d(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}, \mathcal{D}$ run over the chambers of Δ. The chamber complex Δ is called thick, respectively weak, respectively thin if any of its panels is contained in at least 3 chambers, respectively at least 2 chambers, respectively exactly 2 chambers.

A folding ϕ of a thin chamber complex Δ is an endomorphism of Δ (i. e. a simplicial map from Δ to Δ mapping chambers onto chambers) such that $\phi^2 = \phi$ and the preimage of any chamber $\mathcal{C}$ of Δ consists of 0 or 2 chambers. The thin chamber complex Δ is called a Coxeter complex if there exists, for any two adjacent chambers $\mathcal{C}, \mathcal{D}$ of Δ, a folding ϕ of Δ satisfying $\phi(\mathcal{C}) = \mathcal{D}$. A root (or half-apartment) of a Coxeter complex Δ is a subcomplex of Δ which is the image of a folding of Δ. The wall of a root α, denoted by $\partial\alpha$, is by definition the subcomplex of α consisting of all simplices a of α such that there exists a chamber $\mathcal{C} \notin \alpha$ containing a.

Coxeter complexes and Coxeter groups are equivalent concepts (cf. [Tit 74, Chapter 2]). The Coxeter matrix $M = (m_{i,j})_{1\leq i,j\leq r}$ associated to a Coxeter complex Δ of rank r, which is also the the Coxeter matrix of the corresponding Coxeter group, can be obtained as follows (cf. [Tit 74, Subsection 2.11]). Choose a chamber $\mathcal{C}_0$ of Δ, number its vertices by $1, \ldots, r$, let $a_{i,j}$ be the face of $\mathcal{C}_0$ consisting of all vertices with numbers in $\{1, \ldots, r\} - \{i, j\}$ and set $m_{i,j} = diam(\mathrm{lk}_\Delta(a_{i,j}))$.

Throughout this paper, the notion of a building is used as it is defined in [Bro 89, Section IV.1]. So a building is, from our point of view, a chamber complex possessing a system of subcomplexes, called apartments, which are Coxeter complexes and such that any two chambers of Δ are contained in an apartment, and for any two apartments Σ, Σ', there is an isomorphism from Σ onto Σ' fixing all elements in $\Sigma \cap \Sigma'$. This definition slightly deviates from the definition of a building given in [Tit 74] in so far as we do not assume that a building is always thick. If we want to remind the reader of this fact, we use the term "weak building".

The apartment system occurring in the definition of a building Δ is in general not uniquely determined by Δ. However, there always exists a unique maximal apartment system of Δ, cf. [Bro 89, Section IV.4]. In particular, all possible apartments of Δ are isomorphic to each other. The Coxeter matrix (or type) of Δ is by definition the Coxeter matrix of any of its apartments. We say that Δ is a spherical building if its apartments are finite Coxeter complexes (and hence its Coxeter matrix is that of a finite Coxeter group). Δ is called an (irreducible) affine building if its Coxeter matrix is that of an affine Weyl group associated to an irreducible root system or equivalently if any of its apartments is a simplicial complex arising from the tessellation of a Euclidean space by the reflection hyperplanes of an irreducible affine reflection group (cf. [Bro 89, Chapter VI], where affine buildings are called Euclidean buildings).

Let Δ be a building of rank r and I an index set of cardinality r. We consider the power set $\mathcal{P}(I)$ as a simplicial complex. Then there exists a simplicial map type : $\Delta \to \mathcal{P}(I)$ mapping any chamber of Δ onto I (cf. [Tit 74, Subsection 3.8]), and this map is unique up to a bijection of I. The function type will be called a numbering or labelling of Δ. If a numbering over the index set I is fixed, the type of a simplex a of Δ is by definition the set $\mathrm{type}(a)$, and the cotype of a is the set $\mathrm{cotype}(a) := I - \mathrm{type}(a)$. By abuse of notation, we shall also write $\mathrm{type}(a) = i$, respectively $\mathrm{cotype}(a) = i$, if a is a vertex, respectively a panel, and strictly speaking $\mathrm{type}(a) = \{i\}$, respectively $\mathrm{cotype}(a) = \{i\}$.

In Corollary 27 and in Section 8 below, we shall also need projections in buildings. Let a chamber complex Δ, a chamber $\mathcal{C}$ and a simplex a of Δ be given. If there exists a unique chamber $\mathcal{D}$ containing a such that $d(\mathcal{C}, \mathcal{D}') = d(\mathcal{C}, \mathcal{D}) + d(\mathcal{D}, \mathcal{D}')$ for all chambers $\mathcal{D}'$ containing a, then $\mathcal{D}$ is called the projection of $\mathcal{C}$ onto a and denoted by $\mathrm{proj}_a(\mathcal{C})$. It is well known that for any building Δ, any $a \in \Delta$ and any chamber $\mathcal{C}$ of Δ, the projection $\mathrm{proj}_a(\mathcal{C})$ exists (cf. [Tit 74, Subsection 3.19]). If b is another simplex of Δ, we define the projection $\mathrm{proj}_a(b)$ of b onto a as the intersection of all chambers $\mathrm{proj}_a(\mathcal{D})$, where $\mathcal{D}$ runs over the chambers containing b. The following characterization of projections in buildings is implied by [DrS 87, Section 3].

Lemma 1 *Given two simplices a, b of a building Δ and a chamber $\mathcal{C}$ of Δ containing a, then $\mathcal{C}$ contains* $\text{proj}_a(b)$ *if and only if* $d(\mathcal{C}, b) = d(a, b)$.

3 The affine building associated to V

This paper gives concrete models for certain affine buildings. In this section, we recall the standard model of the affine building associated to the general linear group over a discretely valuated skew field. This model will also be fundamental for the construction of the other classical affine buildings.

Let $\mathcal{D}$ be a discretely valuated skew field with valuation ν, valuation ring $\mathfrak{M}$ and π a prime element of $\mathfrak{M}$. Let V be an n-dimensional left vector space over $\mathcal{D}$.

Definition 2 *An $\mathfrak{M}$-*lattice *L in V is a free $\mathfrak{M}$-module $L \leq V$ with $\mathcal{D}L = V$. If $L = \mathfrak{M}b_1 \oplus \ldots \oplus \mathfrak{M}b_n$ then the $\mathcal{D}$-basis $(b_1, \ldots, b_n)$ of V is called a* lattice basis *of L.*

A chain $\ldots \subseteq L_s \subseteq L_{s+1} \subseteq \ldots$ of $\mathfrak{M}$-lattices L_i is called admissible, if the set $\{L_i\}$ is closed under multiplication by integral powers of π.

A $\mathcal{D}$-basis $(b_1, \ldots, b_n)$ of V is called a chain basis for the chain $(L_i)_{i\in\mathbb{Z}}$ if for each L_i there are $\alpha_{ij} \in \mathbb{Z}$ such that $(\pi^{\alpha_{ij}} b_j, 1 \leq j \leq n)$ is a lattice basis for L_i.

Definition 3 *Denote by $\hat{\Delta}$ the partially ordered (by inclusion) set of all admissible chains of lattices in V.*

A frame of V is a collection $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ of 1-dimensional $\mathcal{D}$-subspaces spanning V. The subcomplex $\hat{\Sigma}(v_1, \ldots, v_n)$ of $\hat{\Delta}$ associated with the frame $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ of V is the set of all admissible chains admitting $(v_1, \ldots, v_n)$ as a chain basis.

It is well known that $\hat{\Delta}$ is a thick affine building of type $\tilde{A}_{n-1}$ and that $\{\hat{\Sigma}(v_1, \ldots, v_n)\}$, where $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ runs through all frames of V, forms a system of apartments of $\hat{\Delta}$, see for instance [Gar 97], [BrT 84b]. Also one has a natural numbering on $\hat{\Delta}$ which can be described as follows: Fix a lattice L_0 in V. Then for an arbitrary lattice $L =: L_0 g$ for $g \in GL(V)$, the vertex $\{\pi^m L \mid m \in \mathbb{Z}\}$ receives the number $\nu(det(g))$ (mod n) $\in \{0, \ldots, n-1\}$, where $det(g)$ is the Dieudonné determinant of g.

From now on we assume that $\mathcal{D}$ is finite dimensional over its center $K := Z(\mathcal{D})$. Let $A := \text{End}_{\mathcal{D}}(V)$ be the endomorphism ring of V and $R := \mathfrak{M} \cap K$. There is a close connection between lattices in V and certain R-orders in A. An R-order Λ in A is an R-lattice that is closed under multiplication such that $K\Lambda = A$. By [Rei 75, Corollary (1.7.4)] the maximal R-orders in A are endomorphism rings $\text{End}_{\mathfrak{M}}(L) := \{\lambda \in A \mid L\lambda = L\}$ of $\mathfrak{M}$-lattices L in V. An R-order Λ of A is called a chain order or hereditary order, if Λ is the intersection $\cap_{L\in\mathcal{L}} \text{End}_{\mathfrak{M}}(L)$ of all endomorphism rings of lattices in an admissible chain $\mathcal{L}$. Its invariant lattices are precisely the ones in this chain (cf. [Rei 75, Theorem 39.23]). Therefore one gets the following remark (which seems to hold without the assumption that $\dim_K(\mathcal{D}) < \infty$).

Remark 4 *The map $\mathcal{L} \mapsto \cap_{L\in\mathcal{L}}\mathrm{End}(L)$ defines an inclusion reversing isomorphism between $\hat{\Delta}$ and the poset of all chain orders in A.*

The projections $e_1, \ldots, e_n$ with respect to the decomposition

$$V = \langle v_1 \rangle \oplus \ldots \oplus \langle v_n \rangle$$

are orthogonal idempotents of A. The condition that an admissible chain $\mathcal{L}$ lies in the apartment $\hat{\Sigma}(v_1, \ldots, v_n)$ is equivalent to $e_1, \ldots, e_n \in \cap_{L\in\mathcal{L}}\mathrm{End}_{\mathfrak{M}}(L)$.

4 The affine building of the hermitian space V

Now let $K := Z(\mathcal{D})$ and $R = K \cap \mathfrak{M}$. We shall assume from now on that $\mathcal{D}$ is complete and finite dimensional over K and that $k := \mathfrak{M}/\pi\mathfrak{M}$ has characteristic $\neq 2$.

Let $^-$ be an involution on $\mathcal{D}$ with $\overline{\mathfrak{M}} = \mathfrak{M}$, $\epsilon = \pm 1$ and

$$(,) : V \times V \to \mathcal{D}$$

be a non degenerate $(^-, \epsilon)$-hermitian form on V, i.e. $(,)$ is $\mathcal{D}$-linear in the first argument and $(v, w) = \epsilon\overline{(w, v)}$ for all $v, w \in V$. Denote the Witt index of $(,)$ by r. In this paper, we will always assume that $r > 0$.

Then $(,)$ induces an involution $\,^\circ$ on $A = \mathrm{End}_{\mathcal{D}}(V)$: Let $a \in A$. Since $(,)$ is non degenerate, there is a unique $a^\circ \in A$ with $(va, w) = (v, wa^\circ)$ for all $v, w \in V$. Then the unitary group

$$U(V) = \{a \in A \mid aa^\circ = 1\}$$

is the full isometry group of $(,)$, that is the full orthogonal, symplectic, or unitary group of V.

Definition 5 *If L is an $\mathfrak{M}$-lattice in V then its dual lattice is*

$$L^\# := \{v \in V \mid (v, l) \in \mathfrak{M} \text{ for all } l \in L\}.$$

Note that $L^\#$ is again an $\mathfrak{M}$-lattice since $\mathfrak{M}$ is stable under $^-$. Moreover $\mathrm{End}_{\mathfrak{M}}(L^\#) = \mathrm{End}_{\mathfrak{M}}(L)^\circ$. Since $(L^\#)^\# = L$ and the duals of the lattices in an admissible chain $\mathcal{L}$ form again an admissible chain $\mathcal{L}^\#$, $\#$ induces an automorphism of order 2 on the affine building $\hat{\Delta}$. We call an admissible chain $\mathcal{L}$ #-admissible, if $\mathcal{L} = \mathcal{L}^\#$.

Definition 6 *Denote by Δ the partially ordered (by inclusion) set of all #-admissible chains of lattices in V.*

The isometry group $U(V)$ acts on Δ. For a #-admissible chain $\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$ and $g \in U(V)$ let $\mathcal{L}g$ be the #-admissible chain $\mathcal{L}g := \{L_i g \mid i \in \mathbb{Z}\}$.

Remark 7 *Let $c \in \mathcal{D}^*$ with $\overline{c} = \pm c$. Then the form f_c defined by $f_c(v, w) := (v, w)c$ for all $v, w \in V$ is an $(^{-'}, \epsilon')$-hermitian form on V, where $\epsilon' = \epsilon\overline{c}^{-1}c$ and the involution $^{-'}$ on $\mathcal{D}$ is defined by $\overline{a}' := c^{-1}\overline{a}c$. The group $U(V)$ as well as the set Δ remain the same.*

Remark 8 *As a set Δ clearly consists of the fixed points of $\#$ in $\hat{\Delta}$. As one easily shows, the partially ordered set Δ is in fact a simplicial complex. However much more holds true according to [Mue 94, Theorem 1.8.22] which implies that Δ is a totally gated chamber complex.*

In Section 7 below, we shall show that Δ is in fact a weak building.

Definition 9 *(i) A* hyperbolic frame *of V is a collection $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ of 1-dimensional subspaces of V such that $(v_i, v_j) \neq 0$ if and only if $j = 2r + 1 - i$.*

(ii) The subcomplex $\Sigma(v_1, \ldots, v_{2r})$ of Δ associated with the frame $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ of V is the simplicial complex consisting of all $\#$-admissible chains that admit a chain basis which is contained in $\{v_1, \ldots, v_{2r}\} \cup \langle v_1, \ldots, v_{2r}\rangle^{\perp}$.

Since r is the Witt index of $(\,,)$, the hermitian space $\langle v_1, \ldots, v_{2r}\rangle^{\perp}$ is anisotropic. To show that the $\Sigma(v_1, \ldots, v_{2r})$ are apartments, we need some general facts about anisotropic spaces.

5 Maximal lattices in anisotropic spaces

Let $\mathcal{D}$, $^-$, $\mathfrak{M}$ be as in Section 4, in particular $\bar{\mathfrak{M}} = \mathfrak{M}$ and $2 \in \mathfrak{M}^*$.

Lemma 10 *Let W be a vector space over $\mathcal{D}$, $\epsilon \in Z(\mathcal{D}) = K$ with $\epsilon\bar{\epsilon} = 1$ and $\phi : W \times W \to \mathcal{D}$ a $(^-, \epsilon)$-hermitian form. Assume that there exist $v_0, w \in W$ satisfying $\phi(v_0, v_0) \in \pi\mathfrak{M}$, $\phi(w, w) \in \mathfrak{M}$ and $\phi(w, v_0) \in \mathfrak{M}^*$. Then W contains an isotropic vector.*

Proof: We construct inductively $v_m \in \mathcal{D}w + \mathcal{D}v_0$ satisfying

$$\phi(v_m, v_m) \in \pi^{2^m}\mathfrak{M},\ \phi(w, v_m) \in \mathfrak{M}^*,\ v_{m+1} - v_m \in \pi^{2^m}\mathfrak{M}w$$

for all $m \in \mathbb{N}_0$. Given v_m, we set

$$\alpha := \phi(v_m, v_m) = \epsilon\overline{\alpha} \in \pi^{2^m}\mathfrak{M},\ \eta := \phi(w, v_m) \in \mathfrak{M}^*,\ \beta := \phi(w, w) \in \mathfrak{M}.$$

With $\lambda := -\frac{1}{2}\alpha\eta^{-1} \in \pi^{2^m}\mathfrak{M}$ let $v_{m+1} := v_m + \lambda w$. Then

$$\phi(w, v_{m+1}) = \phi(w, v_m) + \phi(w, w)\overline{\lambda} \in \mathfrak{M}^* + \pi^{2^m}\mathfrak{M} \subseteq \mathfrak{M}^*$$

and

$$\phi(v_{m+1}, v_{m+1}) = \alpha + \lambda\eta + \epsilon\overline{\lambda\eta} + \lambda\beta\overline{\lambda} = \alpha - \frac{1}{2}(\alpha + \epsilon\overline{\alpha}) + \lambda\beta\overline{\lambda} = \lambda\beta\overline{\lambda} \in \pi^{2^{m+1}}\mathfrak{M}.$$

Since $\mathcal{D}$ is complete, the Cauchy sequence $(v_m)_{m\in\mathbb{N}_0}$ converges in $\mathcal{D}v_0 + \mathcal{D}w$. Then $v := \lim_{m\to\infty} v_m$ obviously satisfies $\phi(v, v) = 0$ and $\phi(w, v) \in \mathfrak{M}^*$. In particular $v \neq 0$ and W is isotropic. □

Corollary 11 *Assume that W is anisotropic and finite dimensional and set $X := \{x \in W \mid \phi(x,x) \in \mathfrak{M}\}$. Then the following holds*

(i) $\phi(X,X) \subseteq \mathfrak{M}$.

(ii) X *is an $\mathfrak{M}$-lattice in W.*

(iii) $\pi X^{\#} \subseteq X \subseteq X^{\#}$.

(iv) If L is an $\mathfrak{M}$-lattice in W satisfying $\pi L^{\#} \subseteq L \subseteq L^{\#}$ then $L = X$.

(v) X has an orthogonal basis.

Proof: (i) Suppose there are $x, y \in X$ such that $\phi(x,y) \in \pi^{-m}\mathfrak{M}^*$ for some $m \geq 1$. Then $v_0 := \pi^m x$ and $w := y$ satisfy the conditions of Lemma 10 implying that W is isotropic in contradiction to our assumption.

(ii) From (i) it follows that X is closed under addition. So X is obviously an $\mathfrak{M}$-module. Clearly $\mathcal{D}X = W$. Now choose an $\mathfrak{M}$-lattice L in W which is contained in X. Applying (i) once more, we obtain $X \subseteq L^{\#}$. Since $L^{\#}$ is an $\mathfrak{M}$-lattice, X is an $\mathfrak{M}$-lattice as well.

(iii) $X \subseteq X^{\#}$ follows from (i). In order to show $\pi X^{\#} \subseteq X$, we shall verify $\phi(\pi x, \pi x) \in \mathfrak{M}$ for all $x \in X^{\#}$. In fact, we even get $\phi(x,x) \in \pi^{-1}\mathfrak{M}$: Assume that $x \in X^{\#}$ with $\phi(x,x) \in \pi^{-m}\mathfrak{M}^*$ for some $m \geq 2$. Let $l := \left\lceil \frac{m+1}{2} \right\rceil$ and $x' := \pi^l x$. Then $\phi(x', x') \in \mathfrak{M}$, hence $x' \in X$. Now $x \in X^{\#}$ implies $\phi(x', x) \in \mathfrak{M}$, in contradiction to $\phi(x', x) = \pi^l \phi(x,x) \in \pi^{l-m}\mathfrak{M}^*$.

(iv) In view of (iii) and the definition of X, $\pi L^{\#} \subseteq L \subseteq L^{\#}$ implies

$$\pi X^{\#} \subseteq \pi L^{\#} \subseteq L \subseteq X \subseteq X^{\#} \subseteq L^{\#}.$$

Suppose $L \neq X$. Then also $\pi L^{\#} \neq \pi X^{\#}$, and we can choose $v_0 \in \pi L^{\#} - \pi X^{\#}$. It follows that $\phi(v_0, v_0) \in \phi(\pi L^{\#}, L) \subseteq \pi\mathfrak{M}$. Furthermore, since $v_0 \notin \pi X^{\#}$, there is $w \in X$ such that $\phi(w, v_0) \in \mathfrak{M}^*$. Now Lemma 10 implies that W is isotropic which contradicts our assumption.

(v) First we observe that there exists $v \in X$ satisfying $\phi(X,X) = \phi(v,v)\mathfrak{M}$. Note that the inclusion $\phi(v,v)\mathfrak{M} = \phi(v, \mathfrak{M}v) \subseteq \phi(X,X)$ is true for any $v \in X$. The opposite inclusion is also clear for any $v' \in X$ such that $\phi(v', v') \in \mathfrak{M}^*$. If such a v' does not exist, then there must be a $v \in X$ such that $\phi(v,v) \in \pi\mathfrak{M}^*$. ($\phi(x,x) \in \pi^2\mathfrak{M}$ for all $x \in X$ would lead to the contradiction $\pi^{-1}X \subseteq X$). Furthermore, Lemma 10 immediately implies $\phi(X,X) \subseteq \pi\mathfrak{M} = \phi(v,v)\mathfrak{M}$ in this case. Now if $x \in X$ is given arbitrarily, $x - \lambda v$ is in the orthogonal complement of v for $\lambda := \phi(x,v)\phi(v,v)^{-1} \in \mathfrak{M}$. This shows that $X = \mathfrak{M}v + (X \cap v^{\perp})$ is an orthogonal decomposition of the lattice X. The claim now follows by induction on $\dim_{\mathcal{D}}(W)$. □

6 Apartments and the action of $U(V)$

Lemma 12 *Let $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ be a hyperbolic frame of V. Then there is a basis $(v_1, \ldots, v_n)$ of V such that the apartment $\hat{\Sigma}(v_1, \ldots, v_n)$ (cf. Section 3) is invariant under # and*

$$\hat{\Sigma}(v_1, \ldots, v_n) \cap \Delta = \Sigma(v_1, \ldots, v_{2r}).$$

Proof: Let $(v_{2r+1}, \ldots, v_n)$ be an orthogonal basis of the maximal integral lattice X in the anisotropic space $W := \langle v_1, \ldots, v_{2r}\rangle^\perp$ (see Corollary 11). Then it is clear that $\hat{\Sigma}(v_1, \ldots, v_n)$ is invariant under #. By definition $\hat{\Sigma}(v_1, \ldots, v_n) \cap \Delta \subseteq \Sigma(v_1, \ldots, v_{2r})$. To see the other inclusion, let $L \in \mathcal{L} \in \Sigma(v_1, \ldots, v_{2r})$. Then

$$L = \oplus_{i=1}^{2r} \pi^{m_i} \mathfrak{M} v_i \perp Y$$

for some $m_i \in \mathbb{Z}$ and a lattice Y in W. Since $\mathcal{L}^\# = \mathcal{L}$, there is some $m \in \mathbb{Z}$ such that

$$\pi^m Y \subseteq Y^\# \subseteq \pi^{m-1} Y.$$

One easily shows using Corollary 11 that either Y or $Y^\#$ is a multiple of the maximal integral lattice X. Therefore $\mathcal{L} \in \hat{\Sigma}(v_1, \ldots, v_n)$. □

6.1 A combinatorial model for $\Sigma(v_1, \ldots, v_{2r})$

Let $\mathcal{L} \in \Delta$ be a vertex, which means that $\mathcal{L}$ is a minimal non-empty #-admissible chain. Then it is straightforward to see that there is a unique lattice $L \in \mathcal{L}$ which satisfies $L \subseteq L^\# \subseteq \pi^{-1} L$. This lattice is called the standard representative of $\mathcal{L}$. One obtains an identification of the set of vertices in Δ with the set of standard lattices L in V, which are the lattices $L \subset V$ with $L \subseteq L^\# \subseteq \pi^{-1} L$.

Let $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ be a hyperbolic frame of V such that $(v_i, v_{2r+1-i}) \in \mathfrak{M}^*$ for all $1 \leq i \leq r$ and let $W := \langle v_1, \ldots, v_{2r}\rangle^\perp$. Let L be a lattice of the form $L = \oplus_{i=1}^{2r} \pi^{n_i} \mathfrak{M} v_i \perp Y$ with $n_i \in \mathbb{Z}$ and a lattice $Y \subset W$.

Then $L^\# = \oplus_{i=1}^{2r} \pi^{-n_{2r+1-i}} \mathfrak{M} v_i \perp Y^\#$, and hence L is a standard lattice if and only if firstly $n_i + n_{2r+1-i} \in \{0, 1\}$ for all $1 \leq i \leq r$ and secondly $Y \subseteq Y^\# \subseteq (1/\pi) Y$. By Corollary 10 the second condition is equivalent to $Y = X$, where X is the maximal integral lattice in W. The vertices $\mathcal{L}, \mathcal{L}' \in \Sigma(v_1, \ldots, v_{2r})$ defined by the standard lattices $L = \oplus_{i=1}^{2r} \pi^{n_i} \mathfrak{M} v_i \perp X$ and $L' = \oplus_{i=1}^{2r} \pi^{n'_i} \mathfrak{M} v_i \perp X$ are adjacent, if and only if the multiples of L, $L^\#$, L', and $L'^\#$ form a chain. In view of $L \subseteq L^\# \subseteq (1/\pi) L$ and $L' \subseteq L'^\# \subseteq (1/\pi) L'$ this is equivalent to $L \subseteq L'$ (i.e. $n_i \geq n'_i$ for all $1 \leq i \leq 2r$) or $L' \subseteq L$ (i.e. $n_i \leq n'_i$ for all $1 \leq i \leq 2r$). Summarizing this, we get the following proposition.

Proposition 13 *Denote by N_{2r} the set*

$$N_{2r} := \{(n_1, \ldots, n_{2r}) \in \mathbb{Z}^{2r} \mid n_i + n_{2r+1-i} \in \{0, 1\} \text{ for all } 1 \leq i \leq r\}$$

and endow it with a partial ordering by

$$(n_1, \dots, n_{2r}) \le (n'_1, \dots, n'_{2r}) :\Leftrightarrow n_i \le n'_i \text{ for all } 1 \le i \le 2r.$$

Let Σ_{2r} be the simplicial complex with set of vertices N_{2r} and set of simplices equal to the set of all subsets of N_{2r} which are totally ordered. Then $\Sigma(v_1, \dots, v_{2r})$ is isomorphic to Σ_{2r}.

The isomorphism mentioned in Proposition 13 depends on the choice of $(v_1, \dots, v_{2r})$ and is obtained by identifying a vertex $\mathcal{L} \in \Sigma(v_1, \dots, v_{2r})$ with standard representative $L = \oplus_{i=1}^{2r} \pi^{n_i} \mathfrak{M} v_i \perp X$ with the $2r$-tuple $(n_1, \dots, n_{2r})$. By a slight abuse of notation, we will also speak of "the standard lattice $L = (n_1, \dots, n_{2r})$" or of "the vertex $\mathcal{L} = (n_1, \dots, n_{2r})$".

Proposition 13 can be used in order to derive properties of $\Sigma(v_1, \dots, v_{2r})$ in a combinatorial way. For instance, it is clear that all maximal simplices in Σ_{2r} contain precisely $r + 1$ vertices. One could also show directly that Σ_{2r} is a thin chamber complex with "sufficiently many" foldings and hence a Coxeter complex. However, we shall not carry out the corresponding computations here, since this also follows from a general building theoretic result (see Corollary 14).

6.2 The standard chamber

Now we fix $b_1, \dots, b_{2r} \in V$ with $(b_i, b_j) = \delta_{j,2r+1-i}$ for all $1 \le i \le j \le 2r$ and consider the standard apartment $\Sigma_0 = \Sigma(b_1, \dots, b_{2r})$. Let X_0 be the maximal integral lattice in the anisotropic space $W_0 := \langle b_1, \dots, b_{2r} \rangle^{\perp}$. For $j = 0, \dots r$ we consider the standard lattices

$$L_j := \oplus_{i=1}^{j} \pi \mathfrak{M} b_i \oplus \oplus_{i=j+1}^{2r} \mathfrak{M} b_i \oplus X_0 = (\underbrace{1, \dots, 1}_{j \text{ times}}, 0 \dots, 0)$$

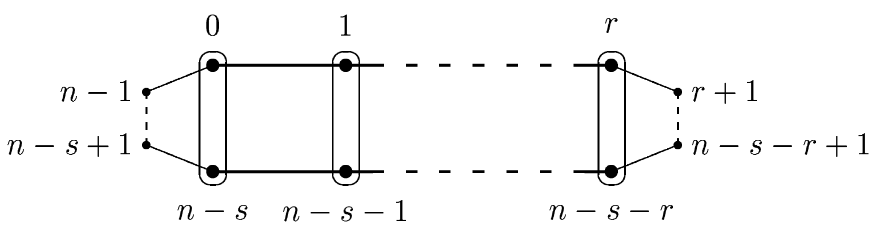
and the corresponding vertices $\mathcal{L}_j \in \Sigma_0$. Then $\mathcal{C}_0 := \cup_{j=0}^{r} \mathcal{L}_j$ is a maximal #-admissible chain in Σ_0, called the standard chamber.

The type of $\mathcal{C}_0 \in \hat{\Delta}$ is $J := \{0, \dots, r\} \cup \{n-s, \dots, n-s-r\} \subseteq \mathbb{Z}/n\mathbb{Z} = \{0, \dots, n-1\}$ where $s = \nu(\det(g)) \in \{0, \dots, n-2r\}$ for some $g \in GL(W_0)$ with $X_0^{\#} g = X_0$.

The types of the Δ-vertices in $\mathcal{C}_0$, which are vertices or edges in $\hat{\Delta}$, are the elements of

$$\{\{j, n-s-j\} \mid j = 0, \dots, r\} \text{ which we identify with } \tilde{J} := \{0, \dots, r\}.$$

This can be visualized by the following diagram:



By Lemma 12 $\Sigma_0 = \Sigma(b_1, \dots, b_{2r})$ can be considered as the fixed complex of # acting on the Coxeter complex $\hat{\Sigma}(b_1, \dots, b_n)$. In [Mue 94, Proposition 2.3.1 (7.)] a

general criterion involving diagrams and types is given which guarantees that the fixed complex of an involution (or more generally: a group of automorphisms) acting on a Coxeter complex is again a Coxeter complex. In our situation, this criterion reads as follows (and is immediately checked):
Set $I := \{0, \dots, n-1\}, \overline{J} := I - J$ and $\tilde{j} := \{j, n-s-j\}$ for any $j \in J$. Then for any $j \in J$, the subdiagram of the Coxeter diagram (of type $\tilde{A}_{n-1}$) over I of $\hat{\Sigma}(b_1, \dots, b_n)$ generated by $\overline{J} \cup \tilde{j}$ is a spherical Coxeter diagram (i. e. corresponds to a finite Coxeter group), and the involution of this spherical Coxeter diagram induced by the opposition involution of the corresponding spherical Coxeter complex maps $\tilde{j}$ onto itself.
Therefore [Mue 94, Proposition 2.3.1 (7.)] immediately implies the following Corollary.

Corollary 14 *Σ_0 is a Coxeter complex with numbering $\tau : \Sigma_0 \to \mathcal{P}(\tilde{J})$, induced by the numbering of $\hat{\Sigma}(b_1, \dots, b_n)$.*

Applying Proposition 13 it is easy to determine the Coxeter matrix $M = (m_{i,j})$ of $\Sigma_0 \cong \Sigma_{2r}$. We first assume $r \geq 2$ and calculate the diameter m_{ij} of the link of the face of $\mathcal{C}_0$ of type $\tilde{J} - \{i, j\}$ $(0 \leq i < j \leq r)$ by examining the vertices $\mathcal{L} = (n_1, \dots, n_{2r})$ in this link. If $\{i, j\} = \{0, 1\}$ then $(n_1, \dots, n_{2r}) < (1, 1, 0, \dots, 0)$. Therefore $n_3 = \dots = n_{2r-2} = 0$ and $(n_1, n_2; n_{2r-1}, n_{2r}) \in \{(0,0;0,0), (1,0;0,0), (1,0;0,-1), (0,1;0,0), (0,1;-1,0), (1,1;0,-1), (1,1;-1,0), (1,1;-1,-1)\}$. Hence $m_{01} = 4$. Similarly, $m_{r-1,r} = 4$. If $1 \leq i \leq r-2$ and $j = i+1$, then the standard representative L of $\mathcal{L}$ satisfies $L_{i-1} \supset L \supset L_{i+2}$. Therefore $n_1 = \dots = n_{i-1} = 1$, $n_{i+3} = \dots = n_{2r} = 0$, and $(n_i, n_{i+1}, n_{i+2}) \in \{0,1\}^3 - \{(0,0,0), (1,1,1)\}$. Hence $m_{i,i+1} = 3$ for all $1 \leq i \leq r-2$. Finally one easily verifies $m_{ij} = 2$ for $j - i \geq 2$, since the corresponding rank 2 links contain precisely 4 vertices.

If $r = 1$, one easily checks that every vertex is joined by an edge with precisely two other vertices. Hence, since Σ_2 is infinite, it is a Coxeter complex of type $\tilde{A}_1$. Thus we have shown the following remark.

Remark 15 *The Coxeter complex Σ_0 is of affine type $\tilde{C}_r$ if $r \geq 2$ and $\tilde{A}_1$ if $r = 1$.*

6.3 The action of $U(V)$

Remark 16 *The same arguments used in [Mue 94, Proposition 2.3.1] in order to derive a numbering on Σ_0 show that the natural numbering on $\hat{\Delta}$ induces a numbering $\tau : \Delta \to \mathcal{P}(\tilde{J})$. (Alternative argument: Since Δ is in fact a building, as we shall show in Corollary 27, the numbering τ of Σ_0 obtained in Corollary 14 uniquely extends to a numbering $\tau : \Delta \to P(\tilde{J})$ of Δ.)*

Since the determinants of the elements in $U(V)$ are units in $\mathfrak{M}$, $U(V)$ acts type preservingly on $\hat{\Delta}$ and hence on Δ.

Lemma 17 *$U(V)$ acts transitively on the set of pairs $(\mathcal{C}, \Sigma(v_1, \dots, v_{2r}))$ where $\langle v_1 \rangle$, …, $\langle v_{2r} \rangle$ is a hyperbolic frame of V and $\mathcal{C} \in \Sigma(v_1, \dots, v_{2r})$ a maximal #-admissible chain.*

Proof: First we note that $U(V)$ clearly acts transitively on the set of hyperbolic frames of V and hence on the set of apartments of Δ. So it remains to show that the stabilizer N in $U(V)$ of Σ_0 acts transitively on the chambers in Σ_0. For $j = 1, \ldots, r-1$ let $g_j \in N$ be the element of $U(V)$ that interchanges b_j with b_{j+1} and b_{2r-j} with b_{2r-j+1} and is the identity on the orthogonal complement $\langle b_1, b_{j+1}, b_{2r-j}, b_{2r-j+1}\rangle^\perp$. Let $g_0 \in N$ be the element mapping b_1 to $\epsilon\pi^{-1}b_{2r}$ and b_{2r} to $\overline{\pi}b_1$ and fixing the orthogonal complement of $\langle b_1, b_{2r}\rangle$ and $g_r \in N$ be the element mapping b_r to ϵb_{r+1} and b_{r+1} to b_r and fixing the orthogonal complement of $\langle b_r, b_{r+1}\rangle$. Then

$$\mathcal{C}_0 g_0, \ldots, \mathcal{C}_0 g_r$$

are precisely the neighbours of $\mathcal{C}_0$ in Σ_0 and $\mathcal{C}_0 g_j^2 = \mathcal{C}_0$ for all j. This shows that the g_j act as generating reflections of the Weyl group of the Coxeter complex Σ_0. □

7 Orders with involution

The automorphism # of $\hat{\Delta}$ is induced by the involution on A, as one easily sees identifying $\hat{\Delta}$ with the poset of all chain orders in A (cf. Remark 4).

A chain order Λ is called admissible if $\Lambda = \Lambda^\circ$, or equivalently if the chain of Λ-invariant lattices is #-admissible.

Remark 18 *The map $\mathcal{L} \mapsto \cap_{L\in\mathcal{L}}\mathrm{End}_{\mathfrak{M}}(L)$ defines an inclusion reversing isomorphism between Δ and the poset of all admissible chain orders in A.*

Next we want to study some connections between hyperbolic frames of V and certain idempotents in A (or in admissible chain orders).

Definition 19 *Given a hyperbolic frame $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ of V the set of orthogonal idempotents $\{e_1, \ldots, e_{2r}\}$ of A defined by $v_i e_j = \delta_{ij} v_i$, $\langle v_1, \ldots, v_{2r}\rangle^\perp e_j = 0$ for all $1 \leq i, j \leq 2r$ is said to be* associated *to this frame.*

Lemma 20 *(i) If the set $\{e_1, \ldots, e_{2r}\}$ of orthogonal idempotents of A is associated to the hyperbolic frame $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ of V then $e_i^\circ = e_{2r+1-i}$ for all $1 \leq i \leq 2r$.*

(ii) If the set $\{e_1, \ldots, e_{2r}\}$ of orthogonal idempotents of A satisfies $e_i^\circ = e_{2r+1-i}$ for all $1 \leq i \leq 2r$, then $Ve_1, \ldots, Ve_{2r}$ is a hyperbolic frame of V to which $\{e_1, \ldots, e_{2r}\}$ is associated.

(iii) If $\mathcal{L}$ is a #-admissible chain, $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ a hyperbolic frame of V and $\{e_i \mid 1 \leq i \leq 2r\}$ the associated set of idempotents of A, then the following equivalence holds:

$$\mathcal{L} \in \Sigma(v_1, \ldots, v_{2r}) \Leftrightarrow \{e_1, \ldots, e_{2r}\} \subseteq \mathrm{End}_{\mathfrak{M}}(L) \text{ for all } L \in \mathcal{L}$$

Proof: (i) This follows from the Definitions 9 (i) and 19, from the fact that $(ve_i^\circ, w) = (v, we_i)$ for all $v, w \in V$ and from the non-degeneracy of $(\cdot,\cdot)$.

(ii) Setting $e_0 := 1 - \Sigma_{i=1}^{2r} e_i$, we obtain the partition $1 = \Sigma_{i=0}^{2r} e_i$ of 1 into orthogonal idempotents. Hence $V = \oplus_{i=0}^{2r} Ve_i$. From $(Ve_i, Ve_j) = (V, Ve_j e_i^\circ) = \delta_{i,2r+1-i}\mathcal{D}$ for all $1 \le i, j \le 2r$ we deduce that the subspaces $\oplus_{i=1}^{r} Ve_i$ and $\oplus_{i=r+1}^{2r} Ve_i$ are totally isotropic. Since r is the Witt index of V, it follows that $\dim_{\mathcal{D}} Ve_i = 1$ for all $1 \le i \le 2r$. Hence $\{Ve_i \mid 1 \le i \le 2r\}$ is a hyperbolic frame of V. In order to see that $\{e_i \mid 1 \le i \le 2r\}$ is associated to this frame, we still have to verify $(\Sigma_{i=1}^{2r} Ve_i)^\perp e_j = 0$ for all $1 \le j \le 2r$. But this follows from $e_0^\circ = e_0$ which implies that $(\Sigma_{i=1}^{2r} Ve_i)^\perp = Ve_0$.

(iii) If $\mathcal{L} \in \Sigma(v_1, \ldots, v_{2r})$ and $L \in \mathcal{L}$ then by Definition 9 (ii)

$$L = \oplus_{i=1}^{2r}(L \cap \mathcal{D}v_i) \oplus (L \cap \langle v_1, \ldots, v_{2r}\rangle^\perp).$$

Hence the projection $Le_j = L \cap \mathcal{D}v_j \subseteq L$ lies again in L, or equivalently $e_j \in \mathrm{End}_{\mathfrak{M}}(L)$ for all $1 \le j \le 2r$. Now assume that $\{e_1, \ldots, e_{2r}\} \subseteq \mathrm{End}_{\mathfrak{M}}(L)$ and set again $e_0 := 1 - \Sigma_{i=1}^{2r} e_i$. Then

$$L = \oplus_{i=0}^{2r} Le_i = \oplus_{i=1}^{2r}(L \cap \mathcal{D}v_i) \oplus (L \cap \langle v_1, \ldots, v_{2r}\rangle^\perp)$$

for all L in $\mathcal{L}$, and this shows that we can choose a chain basis for $\mathcal{L}$ in $\{v_1, \ldots, v_{2r}\} \cup \langle v_1, \ldots, v_{2r}\rangle^\perp$. □

Chain orders are examples of **graduated orders**. An R-order Λ in A is called **graduated**, if Λ contains a full system $x_1, \ldots, x_n$ of orthogonal primitive idempotents of A such that $x_i \Lambda x_i \cong \mathfrak{M}$ is the maximal order in $x_i A x_i \cong \mathcal{D}$. The lattices of graduated orders are well understood and described in [Ple 83, Remark (II.4)]: Namely let L be a lattice in V with $L\Lambda = L$ for some graduated order Λ in A and let $x_1, \ldots, x_n \in \Lambda$ be a system of orthogonal primitive idempotents of A. Then $L = L \cdot 1 = L \cdot (\Sigma_{i=1}^{n} x_i) = \oplus_{i=1}^{n} Lx_i$. Let $b_i \in V$ be a generator of Vx_i $(1 \le i \le n)$. Then for each Λ-lattice L there are integers $\alpha_1, \ldots, \alpha_n$ such that $L = \mathfrak{M}\pi^{\alpha_1} b_1 \oplus \ldots \oplus \mathfrak{M}\pi^{\alpha_n} b_n$.

Lemma 21 *Let $\Gamma_1 \supseteq \Gamma_2$ be two graduated orders in A. Let $S_1, \ldots, S_a$ represent the isomorphism classes of simple Γ_1-modules. Then the Γ_2-composition factors of the S_i ($1 \le i \le a$) form a system of representatives of the isomorphism classes of the simple Γ_2-modules.*

Proof: Let $x_1, \ldots, x_n \in \Gamma_2$ be a full system of orthogonal primitive idempotents in A. Let L be a Γ_1-lattice in the simple A-module V and $\mathcal{L}: \ L = L_0 > L_1 > \ldots > L_b = \pi L$ a chain of Γ_1-lattices in V such that $M_i := L_{i-1}/L_i$ $(1 \le i \le b)$ are simple Γ_1-modules. It follows from the fact that the central primitive idempotents of $\Gamma_1/J(\Gamma_1)$ can be lifted to orthogonal idempotents in Γ_1 (see [Rei 75, Theorem 6.19]) and that L is a faithful Γ_1-module, that $M_1, \ldots, M_b$ represent all isomorphism classes of simple Γ_1-modules. Choosing a composition series of M_i as Γ_2-module, we can refine the chain $\mathcal{L}$ to a chain $\mathcal{L}'$: $L = L'_0 > L'_1 > \ldots > L'_c = \pi L$ of Γ_2-lattices such that $M'_i := L'_{i-1}/L'_i$ $(1 \leq i \leq c)$ represent all isomorphism classes of simple Γ_2-modules. Since $x_1, \ldots, x_n \in \Gamma_2$, $L'_i = \oplus_{j=1}^n L'_i x_j$ for all $0 \leq i \leq c$ and $(b_1, \ldots, b_n)$ is a chain basis of $\mathcal{L}'$, where $\mathfrak{M} b_i = L x_i$. So for each $1 \leq i \leq n$ there is exactly one $j \in \{1, \ldots, c\}$ with $M'_j x_i \neq 0$. In particular the composition factors of $L/\pi L$ are pairwise non isomorphic Γ_2-modules. □

Lifting idempotents

Let Λ be an R-order in A that is stable under the involution, $\Lambda^\circ = \Lambda$.

It is well known that one may lift idempotents of $\Lambda/J(\Lambda)$ to idempotents of Λ (see [Rei 75, Theorem 6.18]). Here $J(\Lambda)$ is the Jacobson radical of Λ, i.e. the intersection of all maximal (left or right) ideals of Λ. One important property of $J(\Lambda)$ is that $\Lambda/J(\Lambda)$ is the biggest semisimple quotient of Λ. It follows that the isomorphism classes of the simple Λ-modules (regarded as $\Lambda/J(\Lambda)$ modules) are in bijection with the central primitive idempotents of $\Lambda/J(\Lambda)$.

Since $\Lambda^\circ = \Lambda$, the involution $^\circ$ preserves $J(\Lambda)$ and therefore permutes the central primitive idempotents of $\Lambda/J(\Lambda)$. We want to lift these idempotents to a $^\circ$-invariant set of orthogonal idempotents of Λ.

Remark 22 *(i) If $\epsilon \in \Lambda$ maps onto an idempotent of $\Lambda/J(\Lambda)$ then the Newton-Hensel-iteration $e_0 := \epsilon$, $e_{i+1} := e_i + (e_i^2 - e_i)(1 - 2e_i) = 3e_i^2 - 2e_i^3$, $i = 0, 1, 2, \ldots$ yields an idempotent $e = \lim(e_i)$ in Λ, with $e \equiv \epsilon\ (J(\Lambda))$. (cf. [Rei 75, Theorem 6.18])*

(ii) If $e_0 = e_0^\circ$, then it holds for all e_i that $e_i^\circ = e_i$. In particular $e^\circ = e$.

(iii) If $\epsilon \equiv \epsilon^\circ\ (J(\Lambda))$, then $e_0 := \epsilon\epsilon^\circ$ is equivalent to ϵ modulo $J(\Lambda)$ and invariant under the involution $^\circ$.

The remark above says that one may lift involution invariant idempotents of $\Lambda/J(\Lambda)$ to involution invariant idempotents of Λ. Since $2 \in R^*$, one gets a similar result if the involution $^\circ$ interchanges two orthogonal idempotents of $\Lambda/J(\Lambda)$.

Lemma 23 *Let $\epsilon \in \Lambda$ be congruent to an idempotent of $\Lambda/J(\Lambda)$ such that $\epsilon\epsilon^\circ$ and $\epsilon^\circ\epsilon \in J(\Lambda)$. Then there is an idempotent e' in Λ with $e' \equiv \epsilon\ (J(\Lambda))$ and $e'e'^\circ = 0$.*

<u>Proof:</u> Let $e_0 := \epsilon + \epsilon^\circ$. Then $e_0^2 = \epsilon^2 + (\epsilon^\circ)^2 + \epsilon\epsilon^\circ + \epsilon^\circ\epsilon \equiv e_0\ (J(\Lambda))$. The Newton-Hensel-iteration in Remark 22 (i) yields an involution invariant idempotent $e = e^2 \equiv e_0\ (J(\Lambda))$.

Now we are looking for an element $f \in \Lambda$ satisfying $f \equiv \epsilon - \epsilon^\circ\ (J(\Lambda))$, $f^\circ = -f$, $f^2 = e$ and $ef = fe = f$. If we find such an f, then $e' := \frac{1}{2}(e + f)$ obviously proves the lemma. So let us start by setting $f_0 := e(\epsilon - \epsilon^\circ)e$. Observe that $ef_0 = f_0 e$ and that $e \equiv \epsilon + \epsilon^\circ$, $\epsilon\epsilon^\circ \equiv \epsilon^\circ\epsilon \equiv 0$, and $\epsilon^2 \equiv \epsilon\ (J(\Lambda))$ imply that $f_0^\circ = -f_0$ and $f_0^2 \equiv e\ (J(\Lambda))$. For $i \geq 0$ let $f_{i+1} := f_i - \frac{1}{2}(f_i^2 - e)f_i$. Then $ef_{i+1} = f_{i+1}e$ and $f_{i+1}^2 - e = \frac{1}{4}(f_i^2 - e)^2(f_i^2 - 4e)$ and hence, by induction, $f_i^2 - e \in (J(\Lambda))^{2^i}$ for all i. In particular $(f_i)_{i\in\mathbb{N}_0}$ is a Cauchy sequence, therefore converging to an $f := \lim(f_i) \in \Lambda$. Since we have $f_i \equiv f_0 \equiv \epsilon - \epsilon^\circ\ (J(\Lambda))$, $f_i^\circ = -f_i$, $f_i^2 \equiv e\ (J(\Lambda)^{2^i})$, $ef_i = f_ie = f_i$ for all $i \in \mathbb{N}_0$, f possesses the desired properties. □

Corollary 24 *Assume that $\epsilon_1, \ldots, \epsilon_s \in \Lambda$ satisfy the following conditions:*

(i) $\epsilon_1 + J(\Lambda), \ldots, \epsilon_s + J(\Lambda)$ is a system of orthogonal idempotents in $\Lambda/J(\Lambda)$.

(ii) There is a permutation $^* : \{1, \ldots, s\} \to \{1, \ldots, s\}$ such that $\epsilon_{i^*} \equiv \epsilon_i^\circ\ (J(\Lambda))$ $1 \leq i \leq s$.

Then there exist orthogonal idempotents $e_1, \ldots, e_s \in \Lambda$ satisfying $e_i \equiv \epsilon_i\ (J(\Lambda))$ and $e_{i^*} = e_i^\circ$ for all $1 \leq i \leq s$.

Proof: We remark that the assumptions imply $i^{**} = i$ for all i and deduce the assertion by induction on s. If $1^* = 1$, Remark 22 yields an idempotent $e_1 \in \Lambda$ satisfying $e_1 \equiv \epsilon_1\ (J(\Lambda))$ and $e_1^\circ = e_1$. If $1^* \neq 1$, Lemma 23 provides idempotents $e_1, e_{1^*} := e_1^\circ \in \Lambda$ satisfying $e_1 \equiv \epsilon_1, e_{1^*} \equiv \epsilon_1^\circ \equiv \epsilon_{1^*}\ (J(\Lambda))$, with $e_1e_{1^*} = e_{1^*}e_1 = 0$. Set $f := 1 - e_1$ in the first case, $f = 1 - e_1 - e_{1^*}$ in the second and consider the involution invariant R-order $\Lambda' := f\Lambda f$. Setting $\epsilon_i' := f\epsilon_i f$ for all $i \in \{1, \ldots, s\} - \{1, 1^*\}$, we deduce $\epsilon_i' \equiv \epsilon_i(J(\Lambda))$ from condition (i). Furthermore, our assumptions together with the identity $J(\Lambda) \cap f\Lambda f = fJ(\Lambda)f = J(\Lambda')$ (which is proved in [Jac 64, Ch. III, §7, Proposition 1]) imply that Λ' and the ϵ_i' again satisfy the conditions (i) and (ii). Now the induction hypothesis yields a system of orthogonal idempotents $e_i \in \Lambda'$ such that $e_i \equiv \epsilon_i'(J(\Lambda'))$ and $e_i^\circ = e_{i^*}$ $(i \in \{1, \ldots, s\} - \{1, 1^*\})$. Since f is orthogonal to e_1 and e_{1^*}, $\{e_i \mid 1 \leq i \leq s\}$ is a system of orthogonal idempotents as well, and it has the required properties. □

Lemma 25 *Let $\Gamma \subseteq \Lambda$ be graduated $^\circ$-invariant R-orders in A. Assume that there exists a system of orthogonal idempotents $\{f_i \mid 1 \leq i \leq 2r\}$ in Λ such that $f_i^\circ = f_{2r+1-i}$ and $f_i + J(\Lambda)$ is central primitive in $\Lambda/J(\Lambda)$ for all i. Then Γ also contains a system of orthogonal idempotents $\{e_i \mid 1 \leq i \leq 2r\}$ satisfying $e_i^\circ = e_{2r+1-i}$ and $e_i + J(\Lambda) = f_i + J(\Lambda)$.*

Proof: Let S_i be the simple Λ-module associated with f_i. Since $\dim(Vf_i) = 1$ the module S_i is of dimension 1 over $\mathfrak{M}/\pi\mathfrak{M}$. Now the simple components of $\Gamma/J(\Gamma)$ are isomorphic to matrix rings over $\mathfrak{M}/\pi\mathfrak{M}$, so the S_i stay simple Γ-modules. By Lemma 21 the S_i are pairwise non isomorphic as Γ-modules. Let $\epsilon_i + J(\Gamma)$ be the central primitive idempotent of $\Gamma/J(\Gamma)$ that induces the identity on S_i and 0 on the other S_j $(j \neq i \in \{1, \ldots, 2r\})$. The involution $^\circ$ preserves Γ and $J(\Gamma)$ and therefore permutes the central primitive idempotents of $\Gamma/J(\Gamma)$. Since $\epsilon_i^\circ + J(\Lambda)$ acts like $\epsilon_{2r+1-i} + J(\Lambda)$ on the simple Λ-modules one gets $\epsilon_i^\circ + J(\Gamma) = \epsilon_{2r+1-i} + J(\Gamma)$.

By Lemma 23 there are orthogonal idempotents $e_i \in \Gamma$ with $e_i^\circ = e_{2r+1-i}$ and $e_i + J(\Gamma) = \epsilon_i + J(\Gamma)$. Since e_i induces the identity on S_i and annihilates the other simple Γ-modules one gets $e_i + J(\Lambda) = f_i + J(\Lambda)$. □

Theorem 26 *For any two simplices in Δ there is an apartment $\Sigma(v_1,\dots,v_{2r})$ of Δ containing both.*

Proof: We first show that for any simplex $\mathcal{L}$ in Δ, there is an apartment $\Sigma(v_1,\dots,v_{2r})$ of Δ containing $\mathcal{L}$ and the standard chamber $\mathcal{C}_0$. It is convenient to use the language of orders. So let $\Lambda := \bigcap_{L\in\mathcal{L}} \mathrm{End}_{\mathfrak{M}}(L)$ be an admissible chain order in A and $\Lambda_0 := \bigcap_{L\in\mathcal{C}_0} \mathrm{End}_{\mathfrak{M}}(L)$ be the standard minimal admissible chain order in A. It is well known (cf. for instance [Gar 97, p. 326-327]) that there is a basis of V that is a chain basis for both chains of Λ and Λ_0. This is the same as saying that $\Gamma := \Lambda \cap \Lambda_0$ is a graduated order. Since Λ_0 contains a system $\{f_1,\dots,f_{2r}\}$ of orthogonal idempotents associated to a hyperbolic frame of V, Lemma 25 implies that there are orthogonal idempotents $e_1,\dots,e_{2r} \in \Gamma$ with $e_i^\circ = e_{2r+1-i}$ $(1 \le i \le 2r)$. Hence Λ and Λ_0 both belong to the apartment defined by the frame Ve_i, $i = 1,\dots,2r$.

In particular every chamber in Δ corresponds to a minimal admissible chain order Λ in A that contains orthogonal idempotents $f_1,\dots,f_{2r} \in \Lambda$ belonging to a hyperbolic frame of V such that $f_i + J(\Lambda)$ are central in $\Lambda/J(\Lambda)$. Now the same proof as above shows the Theorem for an arbitrary pair of simplices in Δ. □

This readily implies the following

Corollary 27 *Δ is a weak building of type $\tilde{C}_r$ (if $r > 1$), respectively $\tilde{A}_1$ (if $r = 1$), and*

$$\mathcal{A} := \{\Sigma(v_1,\dots,v_{2r}) \mid \langle v_1\rangle,\dots,\langle v_{2r}\rangle \text{ is a hyperbolic frame of } V\}$$

is a system of apartments in Δ. The group $U(V)$ acts strongly transitively on $(\Delta,\mathcal{A})$.

Proof: The last claim follows from the first and Lemma 17. In view of Theorem 26 and the fact that the $\Sigma(v_1,...,v_{2r})$ are Coxeter complexes of type $\tilde{C}_r$, respectively $\tilde{A}_1$ (cf. Corollary 14 and Remark 15), we just have to verify the following building axiom.

(*) If Σ and Σ' are elements of $\mathcal{A}$ with a common chamber, then there exists an isomorphism $\alpha : \Sigma \to \Sigma'$ which fixes $\Sigma \cap \Sigma'$ pointwise.

Choose a chamber $\mathcal{C} \in \Sigma\cap\Sigma'$ and a type preserving isomorphism α between Σ and Σ' fixing $\mathcal{C}$ (note that Σ and Σ' are isomorphic by Proposition 13 and that their Weyl groups are transitive on their chamber sets). By Lemma 12, there exist apartments $\hat{\Sigma}$, $\hat{\Sigma}'$ of $\hat{\Delta}$ satisfying $\hat{\Sigma} \cap \Delta = \Sigma$, respectively $\hat{\Sigma}' \cap \Delta = \Sigma'$. Since $\hat{\Sigma} \cap \hat{\Sigma}'$ is a convex subcomplex of $\hat{\Delta}$, one has $\mathrm{proj}_a(b) \in \hat{\Sigma} \cap \hat{\Sigma}'$ for any $a, b \in \hat{\Sigma} \cap \hat{\Sigma}'$. Now it is shown (among other things) in [Mue 94, Theorem 1.8.22] that the complex of all simplices of a building fixed by an involution (and more generally: by a group of automorphisms of this building) is a chamber complex possessing projections (hence is a "totally gated chamber complex" in the sense of Mühlherr), and that for elements a, b of the fixed complex the projection $\mathrm{proj}_a(b)$ is the same, whether taken in the original building or in the fixed complex. Applied to the present situation, where Δ is the fixed complex in $\hat{\Delta}$ under the involution $\#$, we obtain $\mathrm{proj}_a(b) \in \Sigma \cap \Sigma'$ for any $a, b \in \Sigma \cap \Sigma'$. In particular, $\mathrm{proj}_a(\mathcal{C})$ is in $\Sigma \cap \Sigma'$ for any $a \in \Sigma \cap \Sigma'$, which shows that all the maximal simplices of $\Sigma \cap \Sigma'$ are in fact chambers of Δ. So in order to prove (*), it suffices to show that any chamber $\mathcal{C}'$ of Δ, which is contained in $\Sigma \cap \Sigma'$, is fixed by α. So let $\gamma = (\mathcal{C} = \mathcal{C}_0, \ldots, \mathcal{C}_l = \mathcal{C}')$ be a minimal gallery in Δ connecting $\mathcal{C}$ and $\mathcal{C}'$. Set $p_i := \mathcal{C}_i \cap \mathcal{C}_{i+1}$, and note that $\mathcal{C}_{i+1} = \mathrm{proj}_{p_i}(\mathcal{C}')$ since γ is minimal. So the above argument about $\mathrm{proj}_a(b)$ yields by induction that $\mathcal{C}_i \in \Sigma \cap \Sigma'$ for all i. Finally, since α is an isomorphism between the thin chamber complexes Σ and Σ' fixing $\mathcal{C}$, it again follows by induction on i that α has to fix all chambers $\mathcal{C}_i$ of γ, hence in particular $\mathcal{C}'$. □

Remark 28 *One can also use orders with involution again in order to derive the above building axiom (*). One essentially shows that two sets of orthogonal idempotents as in Lemma 20 in a graduated order $\Gamma = \Gamma^\circ$ that lift the same central primitive idempotents of $\Gamma/J(\Gamma)$ are already conjugate in $U(\Gamma) = \Gamma^* \cap U(V)$.*

If one is only interested in the fact that the simplicial complex Δ is a building (and not in an apartment system), then this can already be derived from the fact that any element of Δ is contained in a subcomplex of type $\Sigma(v_1, ..., v_{2r})$ by applying [Mue 94, Theorem 1.8.22], [SchR 85] and our Lemma 17, which implies the "homogeneity" assumption of [SchR 85].

8 Thick buildings of type $\tilde{B}_r$, $\tilde{C}_r$, $\tilde{D}_r$ and associated BN-pairs

In the previous sections we have shown that Δ is a (weak) building of type $\tilde{C}_r$. We now analyze, when this building is already thick. If it is not, one has to apply a building theoretical construction, a generalization of the well known "oriflamme" construction in [Tit 74, 7.12] for the C_r-case, in order to obtain thick buildings.

8.1 The thin panels

First we determine the thin panels of Δ, i.e. those panels which are contained in exactly two chambers. Since $U(V)$ acts transitively on the chambers in Δ, it is enough to consider the panels of the standard chamber $\mathcal{C}_0$ described on page 10.

The chambers that have a panel of cotype $i \in \{0, \ldots, r\}$ in common with the standard chamber $\mathcal{C}_0$ are the maximal admissible lattice chains, in which all multiples of L_i and $L_i^\#$ are replaced by multiples of suitable lattices L and $L^\#$. For $1 \le i \le r-1$ these i-neighbours of $\mathcal{C}_0$ are in bijection with the 1-dimensional subspaces in the 2-dimensional space L_{i-1}/L_{i+1}. In particular the panels of cotype i with $1 \le i \le r-1$ are thick. So the crucial cotypes are $i = 0$ and $i = r$. The chambers in Δ, that contain the panel of cotype 0 of $\mathcal{C}_0$, are in bijection with the isotropic subspaces of $L_1^\#/L_1$: Since $\mathrm{char}(k) \neq 2$ there is a prime element π of $\mathfrak{M}$ satisfying $\overline{\pi} = \epsilon'\pi$ for some $\epsilon' = \pm 1$. Then we define $f : L_1^\#/L_1 \times L_1^\#/L_1 \to k = \mathfrak{M}/\pi\mathfrak{M}$ via $f(x+L_1, y+L_1) := (x,y)\pi + \pi\mathfrak{M}$. One easily checks, that f is a $(\epsilon\epsilon', \iota)$-hermitian form for the involution $\iota : k \to k, x \mapsto \pi^{-1}\overline{x}\pi$. Therefore the panels of cotype 0 are thin, iff this hermitian space $L_1^\#/L_1$ is a hyperbolic plane with a symmetric bilinear form, i.e. $X_0 = X_0^\#$, $\epsilon\epsilon' = 1$ and $\iota = id$.

For cotype r one has to consider $L_{r-1}/\pi L_{r-1}^{\#}$ and finds that the panels of cotype r are thin, iff $X_0 = \pi X_0^{\#}$, $\epsilon = 1$ and $^-$ induces the identity on k.

If $^- \neq id$, then there exists $c \in \mathcal{D}^*$ with $\overline{c} = -c$, because $\mathrm{char}(\mathcal{D}) \neq 2$. Rescaling the form as described in Remark 7, we assume that $\epsilon = -1$ if $^- \neq id$. If the involution is the identity on $\mathcal{D}$, then $\mathcal{D}$ is commutative. Choosing $c = \pi^{-1}$ in Remark 7 if necessary, we may also assume that the maximal lattice X_0 in the anisotropic kernel satisfies $X_0 \neq \pi X_0^{\#}$, if $n > 2r$. Hence the panels of cotype r are thin, iff $^- = id$, $\epsilon = 1$ and $n = 2r$. Notice that in this case also the panels of cotype 0 are thin.

8.2 A general "oriflamme" construction

In this subsection let Δ be a weak building of rank $r+1$ endowed with a numbering of its vertices by the integers in $I := \{0, 1, \ldots, r\}$ and a function type : $\Delta \to \mathcal{P}(I)$. If two distinct vertices x, y of Δ are such that {x,y} is an edge of Δ, we say that x is incident with y or that x is a neighbour of y. We assume that the following conditions are satisfied:

(1) For any vertex x of type 1, there exist precisely two vertices of type 0 which are incident with x.

(2) The entry m_{01} of the Coxeter matrix M of Δ is even and ≥ 4.

Then we associate to Δ a simplicial complex $\tilde{\Delta}$. The simplices (considered as sets of vertices) in $\tilde{\Delta}$ are the simplices $\tilde{a}$ where a is a simplex in Δ and

$$\tilde{a} := \begin{cases} a & 1 \notin \mathrm{type}(a) \\ (a - \{x_1\}) \cup \{x_0', x_0''\} & 1 \in \mathrm{type}(a) \end{cases}$$

where $x_1 \in a$ is of type 1 and x_0', x_0'' are the two neighbours of x_1 of type 0. Geometrically, $\tilde{\Delta}$ is obtained from Δ by deleting all panels of cotype 0 so that the two Δ-chambers sharing such a panel become one $\tilde{\Delta}$-chamber. In particular, one may identify the geometric realizations of Δ and $\tilde{\Delta}$.

Let $\mathcal{A}$ be a system of apartments for Δ.

Lemma 29 *(i) $\tilde{\Delta}$ is a weak building and $\tilde{\mathcal{A}} := \{\tilde{\Sigma} \mid \Sigma \in \mathcal{A}\}$ is a system of apartments for $\tilde{\Delta}$.*

(ii) $\tilde{\Delta}$ possesses a numbering $\widetilde{\mathrm{type}} : \tilde{\Delta} \to \mathcal{P}(\tilde{I})$, where $\tilde{I} := \{0', 0'', 2, 3, \ldots, r\}$ such that the $\tilde{\Delta}$-vertices of type i are the Δ-vertices of type i for $i \in \{2, \ldots, r\}$ and the Δ-vertices of type 0 are the $\tilde{\Delta}$-vertices of type $0'$ or $0''$.

(iii) The links in Δ and $\tilde{\Delta}$ are related as follows:
$\mathrm{lk}_{\tilde{\Delta}}(\tilde{a}) = \mathrm{lk}_{\Delta}(a)$ *for* $a \in \Delta$, $0, 1 \in \mathrm{type}(a)$*;*
$\mathrm{lk}_{\tilde{\Delta}}(a) \cong \mathrm{lk}_{\Delta}(a)$ *for* $a \in \Delta$, $0 \in \mathrm{type}(a)$, $1 \notin \mathrm{type}(a)$*;*
$\mathrm{lk}_{\tilde{\Delta}}(a) \cong \widetilde{\mathrm{lk}_{\Delta}(a)}$ *for* $a \in \Delta$, $0, 1 \notin \mathrm{type}(a)$.

(iv) *If $M = (m_{ij})$ is the Coxeter matrix of Δ, then the Coxeter matrix $\tilde{M} = (\tilde{m}_{ij})$ of $\tilde{\Delta}$ is given by $\tilde{m}_{ij} = m_{ij}$ for all $2 \leq i, j \leq r$; $\tilde{m}_{0'j} = \tilde{m}_{0''j} = m_{1j}$ for all $2 \leq j \leq r$; $\tilde{m}_{0'0''} = m_{01}/2$.*

(v) *If a group G acts strongly transitively on $(\Delta, \mathcal{A})$, then its index 2 subgroup $\tilde{G} := \{g \in G \mid \widetilde{\text{type}}(g \cdot a) = \widetilde{\text{type}}(a) \text{ for all } \textit{vertices } a \in \tilde{\Delta}\}$ acts strongly transitively on $(\tilde{\Delta}, \tilde{\mathcal{A}})$.*

Proof: First of all condition (1) implies

$$(1') \quad m_{0i} = 2 \text{ for all } 2 \leq i \leq r.$$

In fact, if $a \in \Delta$ has cotype $\{0, i\}$, then $\text{lk}_\Delta(a)$ contains exactly two vertices of type 0 by (1) and therefore must have diameter 2.

Now we can prove the following:

($\star$) Let $x_1 \in \Delta$ be a vertex of type 1 and x_0', x_0'' its neighbours in Δ of type 0. Then for any vertex $y \in \Delta$ of type $i \neq 0$, $\{x_1, y\}$ is a simplex in Δ if and only if $\{x_0', y\}$ and $\{x_0'', y\}$ are.

If $\{x_1, y\}$ is a simplex, then it is contained in a panel p of cotype 0. Then $p \cup \{x_0'\}$ and $p \cup \{x_0''\}$ are the only chambers in Δ that contain p, which shows that $\{y, x_0'\}$ and $\{y, x_0''\}$ are simplices in Δ. To prove the reverse conclusion, we first show that $x_1 \in \text{proj}_{x_0'}(x_0'')$. Let $a \in \Delta$ be of cotype $\{0, 1\}$ such that $c' := \{x_0', x_1\} \cup a$ (and hence also $c'' := \{x_0'', x_1\} \cup a$) is a Δ-chamber. Choose an apartment $\Sigma \in \mathcal{A}$ which contains c' and c'' and a root $\alpha \in \Sigma$ such that $c' \in \alpha$ and $\{x_0'\} \cup a \in \partial\alpha$ (:= wall bounding α). Since $\text{lk}_\Sigma(a)$ has diameter $m_{01} \geq 4$ by condition (2), x_0'' is not opposite x_0' in $\text{lk}_\Sigma(a)$ and therefore $x_0'' \notin \partial\alpha$. This implies $\text{proj}_{x_0'}(x_0'') \notin \partial\alpha$. On the other hand, (c', c'') is a minimal gallery stretched between x_0' and x_0'' and hence $\text{proj}_{x_0'}(x_0'') \subseteq c'$ by Lemma 1.

Since $\text{proj}_{x_0'}(x_0'') \nsubseteq \{x_0'\} \cup a$, we have shown that $x_1 \in \text{proj}_{x_0'}(x_0'')$. This proves that the vertex x_1 is the unique vertex of Δ of type 1 that is incident with x_0' and x_0''. Now let y be a vertex of type $i > 1$ such that x_0' and x_0'' are both incident with y. Then there exists a minimal gallery $(\mathcal{C}_0, \ldots, \mathcal{C}_l)$ such that $l = d(x_0', x_0'')$, $x_0' \in \mathcal{C}_0$, $x_0'' \in \mathcal{C}_l$ and $y \in \mathcal{C}_k$ for all $0 \leq k \leq l$. By Lemma 1 $\text{proj}_{x_0'}(x_0'')$ and hence in particular x_1 is contained in $\mathcal{C}_0$, as is y, which shows ($\star$).

We are now in a position to prove the claims of the lemma.

(i) Let $\Sigma \in \mathcal{A}$ be an apartment of Δ. We first show that $\tilde{\Sigma}$ is a Coxeter complex. It is clear that the chambers of $\tilde{\Sigma}$ are of the form $\tilde{c}$ with $c \in \text{cham}(\Sigma)$. It is also obvious that $\tilde{c}$ and $\tilde{d}$ are adjacent (i.e. share a panel), if c and d are. Hence $\tilde{\Sigma}$ is a chamber complex. In view of ($\star$) it is thin. Any folding φ of Σ induces a folding of $\tilde{\Sigma}$ provided that the wall $\partial\alpha$ corresponding to φ does not contain any panels of cotype 0. However, the conditions (1') and (2) imply (in view of the fact that $\partial\alpha$ is itself a chamber complex, cf. [Abr 94, Proposition 1]) that all Δ-panels in $\partial\alpha$ have the same cotype if $\partial\alpha$ possesses at least one panel of cotype 0. Since all $\tilde{\Sigma}$-panels contain vertices of type 0, foldings along all $\tilde{\Sigma}$-panels of a $\tilde{\Sigma}$-chamber exist and $\tilde{\Sigma}$ is a Coxeter complex.

Let $c, d \in \text{cham}(\Delta)$. Then there is an apartment $\Sigma \in \mathcal{A}$ containing both chambers. Hence also $\tilde{c}, \tilde{d} \in \tilde{\Sigma}$, because for any vertex x_1 of type 1 in Σ, both neighbors of type 0 must also be contained in Σ. Finally, let $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ be such that $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2$ contains a $\tilde{\Delta}$-chamber $\tilde{c}$. Then $\Sigma_1 \cap \Sigma_2$ contains the Δ-chamber c, and there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ fixing $\Sigma_1 \cap \Sigma_2$ pointwise. In particular f is type preserving and thus induces an isomorphism $\tilde{f} : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, which fixes $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2$ pointwise.

(ii) Choose a $\tilde{\Delta}$-chamber $\tilde{c}$ and define $\widetilde{\text{type}}$ on its vertices such that the vertices of Δ-type $i > 1$ obtain the same $\tilde{\Delta}$-type and the two vertices of $\tilde{c}$ of Δ-type 0 are numbered $0'$ and $0''$ respectively. Since $\tilde{\Delta}$ is a building, it possesses a numbering. This implies that there exists a unique simplicial map $\rho_{\tilde{c}}$ of $\tilde{\Delta}$ onto the complex consisting of $\tilde{c}$ and all its faces such that all vertices of $\tilde{c}$ are fixed by $\rho_{\tilde{c}}$. Define $\widetilde{\text{type}}(x) := \widetilde{\text{type}}(\rho_{\tilde{c}}(x)) \in \mathcal{P}(\tilde{I})$ for all simplices $x \in \tilde{\Delta}$. Checking the construction of $\rho_{\tilde{c}}$ as carried out in connection with numberings in [Tit 74, 2.3, 2.4, 3.3 and 3.8], we see that $\rho_{\tilde{c}}$ preserves the Δ-type of all vertices in $\tilde{\Delta}$, which proves (ii).

(iii) Firstly let $a \in \Delta$ be such that $0, 1 \in \text{type}(a)$, x_1 its vertex of type 1, x_0 its vertex of type 0 and $y_0 \neq x_0$ the other vertex of type 0 in Δ which is incident with x_1. Then $\tilde{a} = a - \{x_1\} \cup \{y_0\}$, and both $b \in \text{lk}_{\tilde{\Delta}}(\tilde{a})$ as well as $b \in \text{lk}_{\Delta}(a)$ imply that $b \in \Delta$ and $\text{type}(b) \subseteq \text{cotype}(a) (\subseteq I - \{0, 1\})$. Since the buildings $\text{lk}_{\tilde{\Delta}}(\tilde{a})$ and $\text{lk}_{\Delta}(a)$ are flag complexes (cf. [Tit 74, 3.16]), it now suffices to show that any vertex $y \in \Delta$ with $\text{type}(y) \subset \text{cotype}(a)$ is incident with all vertices of $\tilde{a}$ if and only if y is incident with all vertices of a. However, this follows from $(\star)$.
Secondly we assume that $a \in \Delta$, $0 \in \text{type}(a)$, $1 \notin \text{type}(a)$. Let x_0 be the vertex of a of type 0. We define a map $f : \{\text{vertices of } \text{lk}_{\Delta}(a)\} \to \{\text{vertices of } \text{lk}_{\tilde{\Delta}}(a)\}$ as follows. f is the identity on the vertices in $\text{lk}_{\Delta}(a)$ of some type $i > 0$. For all vertices x_1 of type 1 that satisfy $\{x_1\} \cup a \in \Delta$ we set $f(x_1) = y_0$, where $y_0 \neq x_0$ is the unique other neighbour of x_1 of type 0. Now again $(\star)$ implies that f induces a simplicial map $\text{lk}_{\Delta}(a) \to \text{lk}_{\tilde{\Delta}}(a)$ which is an isomorphism.
Finally let $a \in \Delta$ be such that $0, 1 \in \text{cotype}(a)$. Then the weak building $\Theta := \text{lk}_{\Delta}(a)$ also satisfies the conditions (1) and (2) above. Hence we already know, that there is a building $\tilde{\Theta}$ associated to Θ as described before. Looking at the definitions, one immediately sees that the buildings $\tilde{\Theta}$ and $\text{lk}_{\tilde{\Delta}}(a)$ are canonically isomorphic.

(iv) This statement follows by specializing (iii) to simplices $\tilde{a}, a \in \tilde{\Delta}$ of codimension 2. Note that the diameter of $\tilde{\Theta}$ is m, if Θ is a generalized $(2m)$-gon.

(v) Let $\tilde{\Sigma}_1, \tilde{\Sigma}_2 \in \tilde{\mathcal{A}}$ and $\tilde{c}_i \in \text{cham}(\tilde{\Sigma}_i)$ be given, where $\Sigma_i \in \mathcal{A}$, $c_i \in \text{cham}(\Sigma_i)$, $i = 1, 2$. Denote by d_i the unique Δ-chamber $d_i \neq c_i$ such that $\tilde{d}_i = \tilde{c}_i$. Then d_i is adjacent to c_i and also contained in Σ_i. Since G acts strongly transitively on $(\Delta, \mathcal{A})$, there exists a $g \in G$ such that $g \cdot \Sigma_1 = \Sigma_2$ and $g \cdot c_1 = c_2$. This implies $g \cdot \tilde{\Sigma}_1 = \tilde{\Sigma}_2$ and $g \cdot \tilde{c}_1 = \tilde{c}_2$. Note that G preserves all Δ-types (this is part of the definition of the strongly transitive action), but g may interchange the $\tilde{\Delta}$-types $0'$ and $0''$. If this is the case, we choose an element $n \in Stab_G(\Sigma_2)$ which induces a reflection on Σ_2 such that $n \cdot c_2 = d_2$ and put $\tilde{g} := ng$, otherwise $\tilde{g} := g$. Then $\tilde{g}$ preserves all $\tilde{\Delta}$-types, $\tilde{g} \cdot \tilde{\Sigma}_1 = \tilde{\Sigma}_2$ and $\tilde{g} \cdot \tilde{c}_1 = \tilde{c}_2$. □

8.3 The thick buildings and their BN-pairs

In our situation the building Δ of admissible lattice chains clearly satisfies the conditions (1) and (2) of the previous subsection, if Δ is not thick.

A lattice class model for the thick building is obtained as follows: Assume that the panels of cotype 0 are thin. Then $L_1^\#/L_1$ is a hyperbolic plane. The two maximal isotropic subspaces of $L_1^\#/L_1$ correspond to the two lattices $L_{0'} := L_0 = (0, \ldots, 0)$ and $L_{0''} := (1, 0, \ldots, 0, -1)$ in the notation of Subsection 6.1. The set of vertices in the new standard apartment $\tilde{\Sigma}(b_1, \ldots, b_{2r})$ corresponds to

$$\tilde{N}_{2r} := \{(n_1, \ldots, n_{2r}) \in N_{2r} \mid \sum_{i=1}^{2r} n_i \neq 1\}.$$

We define the type of $(n_1, \ldots, n_{2r}) \in \tilde{N}_{2r}$ to be $t := \sum_{i=1}^{2r} n_i$ if $t > 0$. If $t = 0$ then by definition the vertex has type $0'$ if $\frac{1}{2}\sum_{i=1}^{2r} |n_i|$ is even and $0''$ if it is odd. Since for any standard lattice L satisfying $L = L^\#$ there is an apartment $\Sigma(v_1, \ldots, v_{2r})$ containing L as well as $L_{0'}$, one can verify that L is of type $0'$ (resp. type $0''$) if $L \cap L_{0'} = L_{0'}g$ for some $g \in GL(V)$ with $\nu(det(g))$ even (resp. odd). The simplicial complex $\tilde{\Sigma}_{2r}$ is obtained from Σ_{2r} by the general procedure described in Subsection 8.2: If $a \subset N_{2r}$ is a simplex in Σ_{2r} then the corresponding simplex in $\tilde{\Sigma}_{2r}$ is $\tilde{a} = a$, if $a \subset \tilde{N}_{2r}$. Otherwise let $x := (n_1, \ldots, n_{2r}) \in a$ be the vertex of type 1 and $i \in \{1, \ldots, r\}$ such that $n_i + n_{2r+1-i} = 1$ and $n_j = -n_{2r+1-j}$ for all $i \neq j \in \{1, \ldots, r\}$. Let x_1 (resp. x_2) be the vertex in $\tilde{N}_{2r}$ replacing n_i by $n_i - 1$ (resp. n_{2r+1-i} by $n_{2r+1-i} - 1$). Then $\tilde{a} := a - \{x\} \cup \{x_1, x_2\}$ is a simplex in $\tilde{\Sigma}_{2r}$. Since $|n_i - 1| + |n_{2r+1-i}| - |n_i| - |n_{2r+1-i} - 1| = \pm 2$ the vertices $x_1, x_2 \in \tilde{N}_{2r}$ have different types (in $\{0', 0''\}$) and the labeling defines a labeling of $\tilde{\Sigma}_{2r}$.

In the lattice chain model, the chambers in $\tilde{\Delta}$ correspond to the sets of lattices $\tilde{\mathcal{C}}$ in V that are closed under multiplication by π and taking duals, which contain lattices $L_{0'} = L_{0'}^\#, L_{0''} = L_{0''}^\# \in \tilde{\mathcal{C}}$ such that $L_1 := L_{0'} \cap L_{0''} \notin \tilde{\mathcal{C}}$ is a maximal $\mathfrak{M}$-sublattice of $L_{0'}$ (and $L_{0''}$) and $\mathcal{C} := \tilde{\mathcal{C}} - \{\pi^i L_{0'} \mid i \in \mathbb{Z}\} \cup \{\pi^i L_1, \pi^i L_1^\# \mid i \in \mathbb{Z}\}$ is a maximal admissible chain of lattices in V.

If also the panels of cotype r are thin, one deals with them analogously (working with the basis $(\pi b_1, \ldots, \pi b_r, b_{r+1}, \ldots, b_{2r})$ and multiplying the bilinear form by π^{-1}) where we assume that $r > 2$. Then the chambers in $\tilde{\tilde{\Delta}}$ correspond to the sets of lattices $\tilde{\tilde{\mathcal{C}}}$ in V that are closed under multiplication by π and taking duals, which contain lattices $L_{0'} = L_{0'}^\#, L_{0''} = L_{0''}^\# L_{r'} = \pi^{-1} L_{r'}^\#, L_{r''} = \pi^{-1} L_{r''}^\# \in \tilde{\tilde{\mathcal{C}}}$ such that $L_1 := L_{0'} \cap L_{0''}, L_{r-1} := L_{r'} + L_{r''} \notin \tilde{\mathcal{C}}$ satisfy $L_{0'}/L_1 \cong L_{r-1}/L_{r'} \cong \mathfrak{M}/\pi\mathfrak{M}$ and $\mathcal{C} := \tilde{\tilde{\mathcal{C}}} - \{\pi^i L_{0'}, \pi^i L_{r'} \mid i \in \mathbb{Z}\} \cup \{\pi^i L_1, \pi^i L_1^\#, \pi^i L_{r-1}, \pi^i L_{r-1}^\# \mid i \in \mathbb{Z}\}$ is a maximal admissible chain of lattices in V.

The corresponding subgroups of $U(V)$ that act type preservingly on the thick building can be constructed as follows: Let B be the stabilizer in $U(V)$ of the standard chamber $\mathcal{C}_0$ and let N be the stabilizer in $U(V)$ of the standard apartment Σ_0. In the notation of Subsection 6.2 let $d := n - 2r$ be the dimension of the anisotropic kernel W_0 and $(x_1, \ldots, x_d)$ be a orthogonal $\mathfrak{M}$-basis of the maximal integral lattice X_0 in W_0 such that $(x_1, x_1), \ldots, (x_t, x_t) \in \mathfrak{M}^*$ and $(x_{t+1}, x_{t+1}), \ldots, (x_d, x_d) \in \pi\mathfrak{M}^*$, where $t := d - s$. With respect to the basis $(b_1, \ldots, b_r, x_1, \ldots, x_t, b_{r+1}, \ldots, b_{2r}, x_{t+1}, \ldots, x_d)$

$$B = \begin{pmatrix} \mathfrak{M} & \pi\mathfrak{M} & \ldots & \pi\mathfrak{M} & \pi\mathfrak{M}^{1\times s} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \mathfrak{M}^{t\times t} & \ddots & \vdots \\ \mathfrak{M} & \ldots & \ldots & \mathfrak{M} & \pi\mathfrak{M}^{1\times s} \\ \mathfrak{M}^{s\times 1} & \ldots & \ldots & \mathfrak{M}^{s\times 1} & \mathfrak{M}^{s\times s} \end{pmatrix} \cap U(V).$$

The group N acts monomially on $(b_1, \ldots, b_{2r})$, i.e.

$$N = \{g \in U(V) \mid b_i g = \pi^{e(g)_i} u_i b_{\sigma_g(i)} \text{ with } e(g)_i \in \mathbb{Z}, u_i \in \mathfrak{M}^*(i = 1, \ldots, r), \sigma_g \in S_{2r}\}.$$

Since $g \in U(V)$, the associated exponent vector satisfies $\sum_{i=1}^{2r} e(g)_i = 0$. Let

$$\tilde{N} := \{g \in N \mid \sum_{i=1}^{2r} |e(g)_i| \equiv 0 \pmod 4\}$$

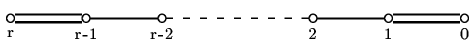
and

$$\tilde{\tilde{N}} := \{g \in \tilde{N} \mid |\{1, \ldots, r\}\sigma_g \cap \{r+1, \ldots, 2r\}| \equiv 0 \pmod 2\}.$$

Then $\tilde{\tilde{N}} \leq \tilde{N} \leq N$ are subgroups with $[N : \tilde{N}] = [\tilde{N} : \tilde{\tilde{N}}] = 2$. We define subgroups $U(V)_0$, N_0 of $U(V)$, N as follows: If Δ is thick, set $U(V)_0 := U(V)$ and $N_0 := N$. If only the panels of cotype 0 are thin, let $U(V)_0 := \tilde{U}(V) := \widetilde{U(V)}$ (cf. the definition in Lemma 29 (v)) and $N_0 := N \cap U(V)_0 = \tilde{N}$. If the panels of cotype 0 and cotype r are thin (and $r > 2$), set $U(V)_0 := \tilde{\tilde{U}}(V) := \widetilde{\widetilde{U(V)}}$ and $N_0 := N \cap U(V)_0 = \tilde{\tilde{N}}$. Observe that in each case $U(V)_0$ acts on a thick building now (cf. Lemma 29 (iii) and the discussion in Subsection 8.1). Therefore [Tit 74, Proposition 3.11], Corollary 27 and Lemma 29 immediately imply the following

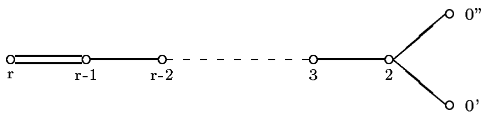
Proposition 30 *The pair (B, N_0) is a BN-pair of the group $U(V)_0$.* □

To deduce properties of the associated BN-pairs for the possible thick buildings, it is helpful to know their Coxeter diagrams. By Remark 15, the weak building Δ has always type $\tilde{C}_r$ (if $r > 1$):

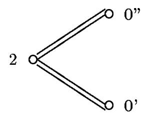


If the panels of cotype 0 are thin (i.e. $X_0 = X_0^{\#}$, $\epsilon\epsilon' = 1$ and $\iota = id$), then one obtains a building $\tilde{\Delta}$ of type $\tilde{B}_r$, if $r \geq 3$ and of type $\tilde{C}_r$, if $r = 2$:

$r > 2$

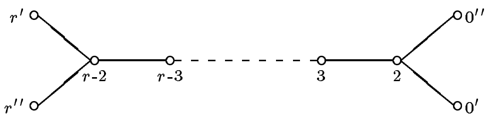


$r = 2$

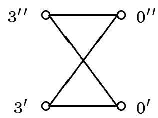


If also the panels of cotype r are thin (i.e. $^- = id$, $\epsilon = 1$ and $n = 2r$) then the associated thick building $\tilde{\tilde{\Delta}}$ is of type $\tilde{D}_r$, if $r \geq 4$ and of type $\tilde{A}_3$, if $r = 3$:

$r > 3$



r=3



If $r = 2$, then the oriflamme construction described in Section 8.2 cannot be applied any longer to the still not thick building $\tilde{\Delta}$ of type $\tilde{C}_2$. In this case, the appropriate "thickening procedure" would be as follows. Take the original building Δ of type $\tilde{C}_2$, and delete in any star of any vertex of type 1 this vertex and the four edges containing it. What one obtains is a thick polysimplicial (in our case: quadratic) complex of type $\tilde{A}_1 \times \tilde{A}_1$. This is precisely the Bruhat-Tits building associated to the non-simple split group SO_4, which is of type $A_1 \times A_1$, over the discretely valuated field K.

Finally, we recall that Δ is of type $\tilde{A}_1$, i. e. a tree without vertices of valency 1, if $r = 1$ (cf. Remark 15). Then either Δ is a thick semihomogeneous tree, or $\tilde{\Delta}$ is a thick homogeneous tree (obtained from Δ by removing the vertices of valency 2) or Δ is a thin tree. The latter happens precisely for two-dimensional hyperbolic quadratic spaces, to which no thick buildings can be associated in a natural way.

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