An Atlas of Orthogonal Representations

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Abstract Let G be a finite group and $\rho : G \to \mathrm{GL}(2n, F)$ be an absolutely irreducible orthogonal representation of even degree over a finite field F. Then $\rho(G)$ embeds into $\mathrm{GO}^+(2n, F)$ or $\mathrm{GO}^-(2n, F)$. We describe methods to decide which case holds for ρ, and use them to determine most of the orthogonal discriminants of the absolutely irreducible orthogonal representations of even degree that are listed in the ATLAS of Finite Groups [CCNPW85].

In memory to our friend and colleague Richard Parker, who sadly passed away after the preparation of this chapter

1 Introduction

The ATLAS of Finite Groups [CCNPW85] and the ATLAS of Brauer Characters [JLPW95] contain the ordinary and modular character tables of finite simple groups, their covering groups and automorphism groups. These characters classify the absolutely irreducible representations ρ of the group G, the building blocks of all group homomorphisms of G into a linear group. Often $\rho(G)$ lies in a smaller classical group, such as the symplectic or unitary group, or an orthogonal group. In even dimension n there are two possible orthogonal groups over a finite field F, $\mathrm{GO}^+(n, F)$ and $\mathrm{GO}^-(n, F)$.

During the past two years, the authors compiled a list of additional data, the *orthogonal discriminants* of the even degree indicator + characters. Over

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finite fields these are $O+$ resp. $O-$ according to whether $\rho(G)$ is a subgroup of GO^+ or GO^-. Note that these questions make sense only if one considers the representations over finite fields (and number fields), contrary to the situation in many representation theoretical results, where one considers only representations over algebraically closed fields.

The computational task is to determine the orthogonal discriminants (as far as possible) of absolutely irreducible representations of Atlas groups.

The results are collected in the text file [23].

The data rely on the notation and the ordering of character tables in the ATLAS of Finite Groups [CCNPW85], in the ATLAS of Brauer Characters [JLPW95], and in the character table library that belongs to the `OSCAR` system, as a part of the `GAP` system. More generally, the names of groups and characters as well as the notation to describe irrational values from character fields in characteristic zero are compatible with the functions in `GAP` and `OSCAR` that deal with characters and character tables.

Section 2 introduces the notion of *orthogonally stable* characters and the necessary facts about characters, quadratic forms, and indicators. The methods for computing orthogonal discriminants are then described in Section 3, and Section 4 shows two examples. Finally, Section 5 lists further applications of our results.

2 Theoretical Background

2.1 Characters

Let G be a finite group. Any group homomorphism $\rho\colon G \to \mathrm{GL}(n, K)$, for some field K, is called a (matrix) *representation* of G.

Put $T_\rho\colon G \to K$, $g \mapsto \mathrm{Tr}(\rho(g))$. If the characteristic of K is zero then $\chi_\rho := T_\rho$ is called an *ordinary character*. In this case, two representations are equivalent if and only if they have the same character. The *character field* of the character χ is $F(\chi) = \mathbb{Q}(\{\chi(g); g \in G\})$. Since each matrix $\rho(g)$ is diagonalizable, where the diagonal entries are roots of unity, $F(\chi)$ is contained in some cyclotomic field $\mathbb{Q}(\zeta_N)$, where $\zeta_N = \exp(2\pi i/N)$ for some divisor N of $|G|$.

If the characteristic of K is a prime p then we consider only the situation that K is a finite extension of its prime field $\mathbb{F}_p$. The map T_ρ is then called a *Frobenius character*, and the character field $F(\chi) = \mathbb{F}_p(\{\chi(g); g \in G\})$ of a Frobenius character χ is a finite field. Frobenius characters do in general not determine their representations up to equivalence.

In order to relate representations in characteristic zero and in finite characteristic p, we define the *Brauer character* of a representation $\rho\colon G \to \mathrm{GL}(n, K)$, where K is a finite extension of $\mathbb{F}_p$, as a map on the set $G_{p'}$ of those elements in G that have order coprime to p, as follows.

For each element $g \in G_{p'}$, the matrix $\rho(g)$ is conjugate to a diagonal matrix $\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$. Let q be a power of p such that $\mathbb{F}_q$ contains all eigenvalues of all $\rho(g)$ for $g \in G_{p'}$. The multiplicative group $\mathbb{F}_q^\times$ is cyclic, we first choose a generator z and define the group isomorphism $\eta_0 \colon \langle \zeta_{q-1} \rangle \to \mathbb{F}_q^\times$ by $\eta_0(\zeta_{q-1}) = z$. Then we define $\eta_q \colon \mathbb{Z}[\zeta_{q-1}] \to \mathbb{F}_q$ as the unique ring homomorphism with the property $\eta_q(\zeta_{q-1}) = z$. The *Brauer character* of ρ at g is defined as $\varphi_\rho(g) = \eta_0^{-1}(\epsilon_1) + \cdots + \eta_0^{-1}(\epsilon_n)$. Note that $\eta_q(\varphi_\rho(g)) = \chi_\rho(g)$, that is, the Brauer character of ρ determines the Frobenius character of ρ.

Note that the Brauer character values depend on our choice of the generator z of $\mathbb{F}_q^\times$. We want to consider many different groups and their Brauer characters at the same time, thus we have to choose the maps η_q compatibly for various powers q of p (see Remark 1).

An ordinary or Brauer character is called *absolutely irreducible* if it is not the sum of two characters. We denote the set of absolutely irreducible ordinary characters of G by $\mathrm{Irr}(G)$, and the set of absolutely irreducible Brauer characters of G in characteristic p by $\mathrm{IBr}_p(G)$. The cardinalities of $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$ are equal to the numbers of conjugacy classes of elements in G and in $G_{p'}$, respectively.

Each character can be written uniquely as a sum of absolutely irreducible characters, with nonnegative integer coefficients. Moreover, the restriction of each ordinary character to $G_{p'}$ yields a Brauer character; this is described by the p-modular *decomposition matrix* $D_p = [d_{\chi,\varphi}]$ of G, whose rows and columns are indexed by $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}_p(G)$, respectively, where $\chi_{G_{p'}} = \sum_{\varphi \in \mathrm{IBr}_p(G)} d_{\chi,\varphi}\varphi$.

If p does not divide $|G|$ then $G_{p'} = G$ holds, in this case regarding ordinary characters as p-Brauer characters defines a bijection from $\mathrm{Irr}(G)$ to $\mathrm{IBr}_p(G)$; thus after reordering $\mathrm{IBr}_p(G)$ we have $D_p = I$ is the unit matrix.

Remark 1 The choice of η_q can be interpreted as the choice of a series of prime ideals in the cyclotomic fields $\mathbb{Q}[\zeta_{q-1}]$, and hence of prime ideals in the character fields of the ordinary characters compatible with the action of the Galois group on $\mathrm{Irr}(G)$ (for more details see [NP23, Section 6]). These prime ideals do play a crucial role when we use the decomposition matrix to deduce restrictions on the orthogonal discriminants as illustrated in [NP23, Section 7.1] and also Section 3.1.2 below.

If the characteristic p divides the group order, then representations are not necessarily (equivalent to) the direct sum of irreducible representations; the Brauer character χ of a representation ρ only determines the composition factors of ρ. Choosing a composition series the matrices in $\rho(G)$ are block triangular matrices where the diagonal blocks give the action of G on the composition factors. In particular we get the following remark.

Remark 2 For any $a \in KG$ the characteristic polynomial of $\rho(a)$ does not depend on the representation ρ of G but only on its character χ. In particular $\det_\chi := \det \circ \rho : KG \to K, a \mapsto \det(\rho(a))$ only depends on the character χ.

2.1.1 Some Notation

We briefly recall the most important abbreviations for character values as they are used in [CCNPW85]. For more details see [CCNPW85, Section 7.10]. Character values are expressed as sums of roots of unity, e.g. $z_N = \zeta_N$ and $y_N = \zeta_N + \zeta_N^{-1}$. The superscript ^{*k} means the same sum where each root of unity is replaced by its k-th power. The names b_N, c_N, ... usually denote irrationalities in the N-th cyclotomic number field that have degree 2, 3, ... over the rationals.

2.2 Quadratic Forms

Let K be a field and V a finite dimensional vector space over K. A *quadratic form* is a map $Q : V \to K$ such that $Q(av) = a^2 Q(v)$ for all $v \in V, a \in K$ and such that its associated *polarisation*

$$B_Q : V \times V \to K, B_Q(v,w) := Q(v+w) - Q(v) - Q(w)$$

is a K-bilinear form. The quadratic form is called *non-degenerate*, if its polarisation is a non-degenerate symmetric bilinear form. As $2Q(v) = B_Q(v,v)$, one recovers the quadratic form from the symmetric bilinear form B_Q if $\mathrm{char}(K) \neq 2$. This can be used to define the *discriminant* of the quadratic form as $(-1)^a \det(B_Q)(K^\times)^2$, where $a = \dim(V)(\dim(V)-1)/2$ and $\det(B_Q)$ is the determinant of a Gram matrix of B_Q. For fields of characteristic 2 the discriminant is replaced by the Arf invariant (see [KMRT98, page xix], [Kne02, Section 10]).

2.2.1 Finite Fields

Over finite fields dimension and discriminant are separating invariants of the isometry classes of quadratic forms. A classification of quadratic forms over finite fields is well known (see [Kne02, Chapter IV]): So let K be a finite field and $Q : V \to K$ a non-degenerate quadratic form. If the characteristic of K is odd, then the space (V, B_Q) has an orthogonal basis and for each even dimension there are exactly two isometry classes of non-degenerate quadratic forms according to their two possible discriminants $\in K^\times/(K^\times)^2$. If the characteristic of K is 2, then B_Q is a non-degenerate symplectic form and hence the dimension of any non-degenerate quadratic space is even.

Over any finite field there are exactly two non-degenerate quadratic spaces of dimension 2, the *hyperbolic plane*

$$\mathbf{H} := (\langle e, f \rangle, Q) \text{ with } Q(ae+bf) = ab$$

and the *norm form* $\mathbf{N} := (F, N_{F/K})$ where F/K is the field extension of degree 2. Every quadratic space of dimension $2n$ is an orthogonal sum of copies of $\mathbf{H}$ and $\mathbf{N}$. As $\mathbf{N} \perp \mathbf{N} \cong \mathbf{H} \perp \mathbf{H}$ there are hence two isometry classes of such quadratic spaces of even dimension

$$Q_{2n}^{+} :=\perp^{n} \mathbf{H} \text{ and } Q_{2n}^{-} :=\perp^{n-1} \mathbf{H} \perp \mathbf{N}.$$

In odd characteristic the discriminant of Q_{2n}^{+} is a square and the discriminant of Q_{2n}^{-} is a non-square.

Definition 1 For all finite fields we denote the discriminant of Q_{2n}^{+} by $O+$ and the discriminant of Q_{2n}^{-} by $O-$.

The *orthogonal groups* of non-degenerate quadratic spaces over a field K with q elements are denoted by

$$\mathrm{GO}_{2n}^{+}(q) = O(Q_{2n}^{+}),\ \mathrm{GO}_{2n}^{-}(q) := O(Q_{2n}^{-}),\ \text{and}\ \mathrm{GO}_{2n+1}(q)$$

where the latter only occurs for odd q, and is the orthogonal group of any odd dimensional quadratic space (V, Q). Note that if $\dim(V) = 2n+1$ is odd, then

$$\mathrm{disc}(V, \epsilon Q) = \epsilon\, \mathrm{disc}(V, Q)$$

and $O(V, Q) = O(V, \epsilon Q)$ for any $\epsilon \in K^{\times}$.

2.2.2 Hermitian Forms

Given a Galois extension L/K of degree 2 and an L-vector space V of finite dimension n. Restriction of scalars turns V into a K-vector space V_K of dimension $2n$. Any Hermitian form $H : V \times V \to L$ defines a quadratic form $Q_H : V \to K, v \mapsto H(v, v)$. The discriminant of this quadratic form is determined directly by the extension L/K (see [Sch85, page 350], [NP23, Proposition 3.12]):

Proposition 1 *Let (V, H) be a non-degenerate Hermitian L-vector space of dimension n.*

(a) If char$(K) \neq 2$ then write $L = K[\sqrt{\delta}]$. Then $\mathrm{disc}(Q_H) = \delta^n (K^{\times})^2$.
(b) If K is a finite field in any characteristic then $\mathrm{disc}(Q_H) = O+$ if n is even and $\mathrm{disc}(Q_H) = O-$ if n is odd.

2.3 The Indicator of an Irreducible Character

Let χ be an irreducible ordinary character or Brauer character and let $\rho : G \to \mathrm{GL}(V)$ be an absolutely irreducible representation with character χ.

Then the character of the contragredient representation $\rho^\vee : G \to \mathrm{GL}(V^*)$ is the complex conjugate character $\overline{\chi}$. If $\chi = \overline{\chi}$ then any isomorphism $\varphi : V \to V^* = \mathrm{Hom}(V, K)$ gives rise to a G-invariant bilinear form on V defined by $B'(v, w) := \varphi(v)(w)$. As the radical of an invariant form is a submodule of V this form $B := B'$ is either skew-symmetric or $B(v, w) := B'(v, w) + B'(w, v)$ is a symmetric non-degenerate G-invariant bilinear form. In characteristic 2 we need to distinguish whether B is the polarisation of a G-invariant quadratic form (indicator $+$) or not (indicator $-$).

Definition 2 The *indicator* of χ is defined as

- $\circ$ if χ takes non real values.
- $+$ if $\chi = \mathbf{1}$ is the trivial character or χ is real and the form B comes from a G-invariant quadratic form on V.
- $-$ if χ is real and B is not the polarisation of a G-invariant quadratic form on V.

2.4 Orthogonally Stable Characters

Given a representation $\rho : G \to \mathrm{GL}(V)$ we use

$$\mathcal{Q}(\rho) := \{Q : V \to K \text{ quad. form } \mid Q(vg) = Q(v) \text{ for all } g \in G, v \in V\}$$

to denote the space of G-invariant quadratic forms in ρ. Then ρ is called *orthogonal*, if $\mathcal{Q}(\rho)$ contains a non-degenerate quadratic form. A character χ of G is called *orthogonal* if there is an orthogonal representation affording χ.

An orthogonal character χ is *orthogonally stable*, if there is a square class Δ of the character field of χ such that for all representations $\rho : G \to \mathrm{GL}_{\chi(1)}(K)$ of G affording the character χ all non-degenerate quadratic forms in $\mathcal{Q}(\rho)$ have discriminant $\Delta(K^\times)^2$. (Note that K may be larger than the character field of χ.) Then $\Delta =: \mathrm{disc}(\chi)$ is called the *orthogonal discriminant* of χ. Clearly orthogonally stable characters and their orthogonal constituents have even degree, but this is the only restriction for being orthogonally stable:

Theorem 1 *(see [NP23, Theorem 5.15]) An orthogonal character χ is orthogonally stable, if and only if all indicator $+$ constituents of χ have even degree.*

The main result of [Neb22b] shows that even though there might be no representation ρ over the character field with character χ, there is always such a square class of the character field that gives the orthogonal discriminant of an orthogonally stable character.

If $\chi = \chi_1 + \chi_2$ is the sum of two orthogonally stable characters then $\mathrm{disc}(\chi) = \mathrm{disc}(\chi_1)\,\mathrm{disc}(\chi_2)$ (see [NP23, Proposition 5.17] for a precise formulation taking into account the different character fields). So it suffices to determine the orthogonal discriminants of the *orthogonally simple* characters ([NP23, Section 5.3]).

Remark 3 The orthogonally simple characters χ are

- $+$ Absolutely irreducible characters χ of even degree and indicator $+$.
- $\circ$ The sum $\chi = \psi + \overline{\psi}$ of a pair of complex conjugate characters of indicator $\circ$: Then $K(\psi) = K(\chi)[\sqrt{\delta}]$ and $\mathrm{disc}(\chi) = \delta^{\psi(1)}(K(\chi)^\times)^2$ by Proposition 1.
- $-$ $\chi = 2\psi$ for an indicator $-$ self-dual character and $\mathrm{disc}(\chi) = 1$.

Starting from the character table of G with all indicators known it hence suffices to compute the orthogonal discriminants of the absolutely irreducible even degree characters of indicator $+$.

3 Methods

3.1 Theoretical Methods

3.1.1 p-Groups

The paper [Neb22a] gives a formula for the orthogonal discriminant of an orthogonally stable ordinary character χ of a p-group P. The idea is described easily for odd primes p. Given a non-trivial absolutely irreducible representation ρ of P, the image $\rho(P)$ is a non-trivial p-group and hence has a non-trivial center. As ρ is absolutely irreducible, the center acts as scalar matrices. Hence the character field of ρ contains the cyclotomic field $\mathbb{Q}[\zeta_p]$ and one may use Proposition 1 to obtain the orthogonal discriminant of $\rho + \overline{\rho}$:

The maximal real subfield of $\mathbb{Q}[\zeta_p]$ is generated by $y_p := \zeta_p + \zeta_p^{-1}$. Choose $\delta_p \in \mathbb{Q}[y_p] =: Z^+$ such that $\mathbb{Q}[\zeta_p] = Z^+[\sqrt{\delta_p}]$. For $p \equiv 3 \pmod 4$ one may choose $\delta_p = -p$, in general the totally negative generator $\delta_p = (\zeta_p - \zeta_p^{-1})^2 = y_p^{*2} - 2$ of the prime ideal over p is a possible choice.

The character χ is orthogonally stable, if and only if χ does not contain the trivial character as a constituent. Let K denote the character field of χ, put $K_1 := K \cap Z^+$, and $a := [Z^+ : K_1]$. Then $2a$ divides $\chi(1)$.

Theorem 2 *(see [Neb22a, Theorem 4.3, Theorem 4.7]) Let χ be an orthogonally stable character of a p-group P and let K_1, a be as above.*

- *If p is odd then* $\mathrm{disc}(\chi) = N_{Z^+/K_1}(\delta_p)^{\chi(1)/(2a)}(K^\times)^2$.
- *For $p \equiv 3 \pmod 4$ this reads as* $\mathrm{disc}(\chi) = (-p)^{\chi(1)/2}$.
- *If $p = 2$ then* $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}$.

3.1.2 Modular Reduction

The discriminant of an ordinary character χ is a square class $\mathrm{disc}(\chi) = \delta(K^\times)^2$ of the character field $K = F(\chi)$. It hence determines a unique field extension $\mathrm{Disc}(\chi) := K[\sqrt{\delta}]$ of degree 1 or 2 of the character field. This field extension is called the *discriminant field* of χ.

Theorem 3 *(see [NP23, Theorem 6.4]) Let χ be an orthogonally stable ordinary character. If the reduction of χ modulo the prime $\wp$ (cf. Remark 1) is orthogonally stable then $\wp$ is unramified in the discriminant field extension* $\mathrm{Disc}(\chi)/K$.

Mild extra conditions allow one to read off $\mathrm{disc}(\chi \pmod{\wp})$ from the decomposition behaviour (split or inert) of $\wp$ in the discriminant field extension $\mathrm{Disc}(\chi)/K$. These extra conditions are always satisfied if $\wp$ does not divide the group order and allow one to determine the modular orthogonal discriminants from the ordinary ones for those primes.

Corollary 1 *The only primes that might ramify in* $\mathrm{Disc}(\chi)/K$ *are the prime divisors of the group order. This yields a finite a priori list of possibilities for* $\mathrm{disc}(\chi)$.

For characters in blocks with cyclic defect group, even more is true. We only give the conclusion for defect 1:

Remark 4 (see [NP23, Theorem 6.10]) If χ is an irreducible character in a block of defect 1, then also the converse of Theorem 3 holds: $\wp$ is ramified in $\mathrm{Disc}(\chi)/K$ if and only if the reduction of χ modulo $\wp$ is not orthogonally stable.

[NP23, Section 7.1] exclusively uses the modular decomposition matrices and the methods described above to determine all orthogonal discriminants for the sporadic simple group J_1. Another example where this strategy works well is given in the next section.

3.1.3 The Orthogonal Discriminants of $R(27)$

The finite simple group $R(27)$ is a twisted group of Lie type, the centraliser of an outer automorphism in $G_2(27)$. The order of $R(27)$ is $2^3 \cdot 3^9 \cdot 7 \cdot 13 \cdot 19 \cdot 37$, and there are no even degree indicator + absolutely irreducible 3-Brauer characters. All modular and ordinary orthogonal discriminants of $R(27)$ are determined by the p-modular decomposition matrices for the primes $p = 2, 7, 13, 19$, and 37 as shown in the following table.

χ	$F(\chi)$	disc(χ)	mod 2	mod 7	mod 13	mod 19	mod 37
$13832abcdef$	f_{37}	1	$O+$	$O+$	$O+$	$O+$	$O+$
$18278a$	$\mathbb{Q}$	-3	$O-$	$O+, O+$	$O+$	$O+$	$O+$
$18278bcd$	y_7	-3	$O-$	$O+$	$O+$	$O+$	$O+$
$19684abcdef$	y_{13}	$3(2-y_{13})$	$O-$	$O-$	$1+19683$	$O-$	$O-$
$19684ghijkl$	y_{13}	$3(2-y_{13})$	$O-$	$O-$	$703+18981$	$O-$	$O-$
$26936abc$	c_{19}	1	$O+$	$O+$	$O+$	$O+, O+, O+$	$O+$

The first column gives the ordinary absolutely irreducible orthogonal character in the form $\chi(1)ab...$, the second one its character field (in ATLAS notation see Section 2.1.1) followed by a representative of the orthogonal discriminant disc(χ). We group the Galois conjugate characters into one row. The next columns, headed by mod p, indicate the p-modular reduction of χ, where we list the orthogonal discriminants of the orthogonally simple constituents.

By Theorem 3 the discriminant field extension is unramified at all primes but possibly at the ones dividing 3 for all absolutely irreducible characters of degree $\neq 19684$. For the 12 characters of degree 19684, Remark 4 implies that the discriminant field extension is ramified at the prime dividing 13 and possibly at the two primes dividing 3. In all cases this yields a unique discriminant field from which one obtains the orthogonal discriminants of the ordinary irreducible characters of indicator $+$. These allow one to read off the modular orthogonal discriminants of their modular reductions and hence all orthogonal discriminants for all irreducible p-Brauer characters χ of indicator $+$ that do lift. Only the following three exceptions do not lift:

(a) $p = 2$, $\chi(1) = 16796$. Here χ occurs with multiplicity 1 in a permutation character of degree 19684 which decomposes as

$$2 \cdot \mathbf{1} + 2 \cdot 702 + 741ab + 16796.$$

The following argument can also be found in [GW97, Section 1]: Let $V \cong \mathbb{F}_2^{19684}$ be the permutation module and $e := v_1 + \ldots + v_{19684}$ the canonical fixed vector in V. The subspace $e^\perp$ consists of even weight vectors and half of the weight mod 2 is an S_{19684}-invariant quadratic form on $e^\perp$ with radical $\langle e \rangle$. Hence it induces a non-degenerate quadratic form Q on $e^\perp/\langle e \rangle$, which is of orthogonal discriminant $O-$, as $19684 \equiv 4 \pmod 8$. Now $e^\perp/\langle e \rangle = 2 \cdot 702 + 741ab + 16796$ is an orthogonally stable module for $R(27)$. The irrationality of $741a$ is z_3, so $741ab$ contributes $O-$ to this sum leaving $O+$ for the orthogonal discriminant of 16796.

(b) $p = 7$, $\chi(1) = 16796$. Here χ occurs in the 7-modular reduction of $\mathcal{X}_{15} = 741ab + 16796$. As $z_3 \in \mathbb{F}_7$, the orthogonal discriminant of $741ab$ is $O+$ and hence the orthogonal discriminant of 16796 is also $O+$.

(c) $p = 19$, $\chi(1) = 19682$. Here χ occurs in the 19-modular reduction of $\mathcal{X}_{33} = 1443ab + 2184ab + 19682$ which is orthogonally stable. The character fields of $1443a$ and $2184a$ are both $\mathbb{F}_{19}[z_3] = \mathbb{F}_{19}$ so the orthogonal discriminant of χ is $O+$.

3.2 Reduction to Simple Groups

3.2.1 Groups with a non-trivial Center

By Schur's Lemma, central elements act as scalars on irreducible representations, in particular, it is enough to consider cyclic central subgroups. If the exponent of the center of G is strictly bigger than 2 then all faithful irreducible characters of G are non-real, i.e. of indicator $\circ$, and Proposition 1 can be used to determine orthogonal discriminants. For central elements of order 2 we use the spinor norm to deduce discriminants:

Given a non-degenerate quadratic form $Q : V \to K$, the *spinor norm* defines a group homomorphism from the orthogonal group of Q into $K^\times/(K^\times)^2$, a group of exponent 2, where the spinor norm of a reflection along vector v equals $Q(v)$ (see [Kne02]). Over a field K of characteristic not 2, the space V has an orthonormal basis $(v_1, \ldots, v_n)$. The orthogonal mapping $-\mathrm{id}_V$ is the product of the reflections along the v_i and hence its spinor norm is $\prod_{i=1}^n Q(v_i) = 2^{-n}\det(Q)$.

Theorem 4 *(see for instance [Neb99, Section 3.1.2]) Let χ be an orthogonally stable character of a finite group G in characteristic not 2 and let ρ be a faithful representation of G affording χ*

- *If there is $g \in G$ with $\rho(g)^2 = -\mathrm{id}$ then $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}$.*
- *If $[G : G']$ is odd and $-\mathrm{id} \in \rho(G)$ then $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}$.*

3.2.2 Split Extensions

Given a finite group G and an outer automorphism α of order 2 the split extension $H := G : 2$ has a pseudo presentation

$$G : \langle \alpha \rangle = \langle G, h \mid hgh^{-1} = \alpha(g), h^2 = 1 \rangle.$$

Given an orthogonal character χ of G such that $\chi \circ \alpha \neq \chi$, Clifford theory shows that there is a unique irreducible character $\mathcal{X}$ of H such that $\mathcal{X}_{|G} = \chi + \chi \circ \alpha$. As $\mathcal{X}(H \setminus G) = \{0\}$, the character field F of $\mathcal{X}$ is contained in the character field K of χ.

Theorem 5 *(see [Neb22b, Theorem 4.3]) Assume that the characteristic is not 2. If $K = F$ then $\mathrm{disc}(\mathcal{X}) = (-1)^{\chi(1)}(F^\times)^2$. Otherwise $K = F[\sqrt{\delta}]$ is a quadratic extension of F and $\mathrm{disc}(\mathcal{X}) = (-\delta)^{\chi(1)}(F^\times)^2$.*

Note that in the case that χ is already orthogonally stable, then $\mathrm{disc}(\chi) = \mathrm{disc}(\chi \circ \alpha)$ and $\mathrm{disc}(\mathcal{X}) = N_{K/F}(\mathrm{disc}(\chi)) \in (K^\times)^2 \cap F$.

3.2.3 Non-split Extensions

The following table lists all those examples of characters of almost simple Atlas groups H of the structure $G.2$, such that our methods (Theorem 5 and restriction to the normal subgroup G) do not suffice to compute the orthogonal discriminant of χ from that of an irreducible constituent ψ of χ_G.

H	G	χ	i	$\mathbb{Q}(\chi)$	$\mathbb{Q}(\psi)$	$\mathrm{disc}(\chi)$
$L_2(16).4$	$L_2(16).2$	$34a$	15	$\mathbb{Q}$	$\mathbb{Q}(b_5)$	-1
$L_2(16).4$	$L_2(16).2$	$34b$	16	$\mathbb{Q}$	$\mathbb{Q}(b_5)$	-5
$U_3(4).4$	$U_3(4).2$	$78a$	10	$\mathbb{Q}$	$\mathbb{Q}(b_5)$	-5
$U_3(4).4$	$U_3(4).2$	$78b$	11	$\mathbb{Q}$	$\mathbb{Q}(b_5)$	-1

The orthogonal discriminants can be computed in these cases as follows.

The group $H = L_2(16).4$ is a subgroup of $S_4(4).2$, the irreducible characters of degree 50 of $S_4(4).2$ have orthogonal discriminant -17, and the restrictions of these characters to G are orthogonally stable, and decompose as $16a + 34a$ and $16c + 34a$, respectively. Both $16a$ and $16c$ have orthogonal discriminant 17, thus $34a$ has orthogonal discriminant -1. Analogously, the irreducible character $34c$ of $S_4(4).2$, which has orthogonal discriminant -5, restricts to $34b$ of H, which thus also has orthogonal discriminant -5.

The group $H = U_3(4).4$ is a subgroup of $G_2(4).2$, the irreducible character $350a$ of $G_2(4).2$ has orthogonal discriminant -13, its restriction to H is orthogonally stable and decomposes as $78a + 52abcd + 64a$, where $52abcd$ and $64a$ have orthogonal discriminants 1 and 65, respectively, thus $78a$ has orthogonal discriminant -5. Analogously, the irreducible character $78a$ of $G_2(4).2$, which has orthogonal discriminant -1, restricts to $78b$ of H, which thus also has orthogonal discriminant -1.

3.3 Direct Methods

Given an orthogonal representation ρ affording the character χ one can determine $\mathcal{Q}(\rho)$ either by solving a system of linear equations or by applying the Reynolds operator (see [PS96] for a more sophisticated approach). Then it is straightforward to compute the orthogonal discriminant $\mathrm{disc}(\chi)$.

If the characteristic of the underlying field K is not 2, there is no need to determine $\mathcal{Q}(\rho)$, as we can compute $\mathrm{disc}(\chi)$ as the discriminant of the adjoint involution:

3.3.1 The Natural Involution on the Group Algebra

Let K be a field of characteristic not 2. Inverting the group elements defines a natural involution $^\circ$ on KG, i.e. $(\sum_{g\in G} a_g g)^\circ = \sum_{g\in G} a_g g^{-1}$. Then $KG = KG^- \oplus KG^+$ where $KG^\epsilon = \{a \in KG \mid a^\circ = \epsilon a\}$. Now let ρ be an orthogonal representation of G and choose a non-degenerate $Q \in \mathcal{Q}(\rho)$. The condition $B_Q(\rho(g)v, \rho(g)w) = B_Q(v,w)$ for all $g \in G, v, w \in V$ shows that $\rho(a^\circ) = \rho(a)^{ad}$ for all $a \in KG$, where ^{ad} is the adjoint involution of B_Q. To see this fix a basis of V and work with matrices. Let B be the Gram matrix of B_Q. Then $\rho(g)B\rho(g)^{tr} = B$ and hence $B\rho(g)^{tr}B^{-1} = \rho(g^{-1})$ for all $g \in G$, thus

$$\rho(a^\circ) = B\rho(a)^{tr}B^{-1} \text{ for all } a \in KG.$$

In particular $XB = -BX^{tr}$ for all $X \in \rho(KG^-)$. As the determinant of a skew symmetric matrix is always a square, we conclude that $\det(X)(K^\times)^2 = \det(B)(K^\times)^2$. By Remark 2, this determinant only depends on the character of ρ, so we conclude the following lemma.

Lemma 1 *The orthogonal character χ is orthogonally stable if and only if there is $X \in KG^-$ with $\det_\chi(X) \neq 0$. Then, $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}\det_\chi(X)$.*

In practice, one finds a suitable X as the sum of at most three matrices $g - g^{-1}$, where g are randomly chosen elements of order at least 3 in $\rho(G)$.

3.3.2 Condensation Methods

Lemma 1 also allows one to compute the orthogonal discriminant of a character using well established condensation techniques (see [Ryb90]). To analyse the composition factors $S_1, \ldots, S_t$ of a KG-module V one computes a suitable idempotent $e \in KG$. The *condensed module* Ve is then a module for $eKGe$ with composition factors $\{S_i e \mid 1 \leq i \leq t\} \setminus \{0\}$. The main problem here is that a K-algebra generating set $\{g_1, \ldots, g_s\}$ of KG does not necessarily condense to a K-algebra generating set $\{eg_ie \mid 1 \leq i \leq s\}$, the map $a \mapsto eae$ is only a vector space homomorphism and even the condensed algebra is in general too big to compute a basis.

In practise we use fixed point condensation in permutation representations V with respect to a suitable subgroup H whose order is not divisible by the characteristic of K. In view of Section 3.1.1, we choose $H = P$ to be either a Sylow p-subgroup of G (for p odd), or $H = P'P^2$, where P is a Sylow 2-subgroup of G, and $e := \frac{1}{|H|}\sum_{h\in H} h$. Then for any orthogonal KG-module

V, the restriction of $V(1-e)$ to the Sylow p-subgroup P is orthogonally stable and its discriminant can be computed with the formula in Section 3.1.1.

We start with a big permutation representation $V := 1_U^G$. Then, a basis for Ve is given by the H-orbit sums $\sum o_1, \ldots, \sum o_m$, and for $g \in G$, the matrix of $ege = (a_{ij})_{i,j=1}^m$ satisfies

$$a_{ij} = \frac{1}{|o_i|} |\{x \in o_i \mid xg \in o_j\}|.$$

As $e^\circ = e$, the algebra $eKGe$ inherits the natural involution $\circ : ege \mapsto eg^{-1}e = eg^{tr}e$. The dimensions of the composition factors of Ve and their multiplicities can be predicted by character theoretic methods.

In our applications we took 5-10 random group elements g_i, and computed the K-algebra $A := \langle eg_ie, eg_i^{-1}e = (eg_ie)^\circ \rangle$. The composition factors of the A-module Ve are obtained using meataxe methods. We check, whether these do have the predicted dimension and then compute an element $a = -a^\circ$ in A acting as a unit X on such a composition factor Se. Then Lemma 1 together with Section 3.1.1 allow us to deduce the orthogonal discriminant of S as

$$\mathrm{disc}(S) = (-1)^{\dim(Se)/2} \det(X) \, \mathrm{disc}(S(1-e)_{|P}).$$

To obtain the orthogonal discriminant for number fields K it is essential to use Corollary 1 to compile a finite list of possible orthogonal discriminants, as meataxe methods only perform well for finite fields. Given this list of possible discriminants we compute enough p-modular reductions (usually for small primes p not dividing the group order) of $\mathrm{disc}(S)$ to conclude the exact value in $K^\times/(K^\times)^2$.

The largest permutation module V handled so far is the one of degree $108,345,600$ of the Harada Norton group. Using fixed point condensation with the Sylow 5-subgroup of HN, we obtain a module Ve of dimension 7008. As Ve is an $e\mathbb{Z}[\frac{1}{5}]HNe$-module, we are free to reduce this module modulo all primes $\neq 5$ to compute and analyse the composition factors.

A more sophisticated implementation of the meataxe should be able to handle even larger examples.

3.3.3 Summary

Direct methods in characteristic $\neq 2$ usually compute the discriminant of the natural involution to deduce the orthogonal discriminant of χ. In characteristic 2 these do not work and, in particular, we do not have a provable method to use condensation techniques for computing orthogonal discriminants. Here, we compute the Gram matrix of the invariant quadratic form in the original representation, and use it to compute the discriminant. (The implementation in `GAP` uses an algorithm due to Jon Thackray.)

- Many matrix representations are publicly available via the ATLAS of Group Representations [Wil+]. The data file marks these entries with "AGR".
- We can reduce the permutation representations that are available via the ATLAS of Group Representations [Wil+] modulo primes dividing the group order, compute their absolutely irreducible constituents, and determine the orthogonal discriminants of those that are orthogonal and have even degree. The data file marks these entries with "const(desc)" where desc is the identifier of the permutation representation.
- Many representations have been constructed by Richard Parker in order to compute the orthogonal discriminant. The data file marks these entries with "RP".
- The orthogonal discriminants that have been obtained by Gabriele Nebe using condensation methods as described in Section 3.3.2 are marked by "GNcond".
- In certain cases decomposition matrices allow us to conclude orthogonal discriminants using Theorem 3. Entries obtained in such a way are marked by "GN".

3.4 Character Theoretic Methods

Here the idea is to use only the character table of the given character χ plus information from the character table library, concerning (character tables of) subgroups and overgroups. This information, for example known orthogonal discriminants of related characters, may suffice to deduce the orthogonal discriminant of χ. The advantage of this approach is that checking these criteria is cheap, but the disadvantage is that they need not yield the answer.

The following criteria are used. (The string in brackets is used to mark those entries in the data file for which the criterion in question yields the value.)

Group order ("order"): In positive characteristic, if the orthogonal discriminant of χ with character field F is $O+$ ($O-$) then the order of G divides that of $\mathrm{GO}^+(\chi(1), F)$ ($\mathrm{GO}^-(\chi(1), F)$). This condition determines the orthogonal discriminant in some cases.

```
julia> ch = character_table("Co2", 2)[2];

julia> degree(ch)
22

julia> Oscar.OrthogonalDiscriminants.od_from_order(ch)
(true, "O+")
```

Group automorphisms ("grpaut(n)"): For a character χ of the group G and a group automorphism σ of G, the character χ^σ is defined by $\chi^\sigma(g) =$

$\chi(g^\sigma)$, for $g \in G$. If χ has an orthogonal discriminant then χ^σ has the same orthogonal discriminant.

Galois action (`"galaut(n)"`): For a character χ of the group G, and a field automorphism σ of the character field of χ, the character χ^σ is defined by $\chi^\sigma(g) = \chi(g)^\sigma$, for $g \in G$. In characteristic zero, if χ has orthogonal discriminant d then χ^σ has orthogonal discriminant d^σ. In positive characteristic, if χ has an orthogonal discriminant then χ^σ has the same orthogonal discriminant.

Transitive permutation characters (`"permchar"`): If π is a transitive permutation character of G, i. e., there is a subgroup H of G such that π is the induced character 1_H^G, then $\chi = \pi - 1_G$ is the character of a rational representation that fixes a symmetric bilinear form of determinant $\pi(1)$. If χ is orthogonally stable then its orthogonal discriminant is $(-1)^{\chi(1)/2}\pi(1)$ (modulo squares). If χ is absolutely irreducible then this yields the value, otherwise it yields a condition on the orthogonal discriminants of the constituents of χ.

Eigenvalues (`"ev"`): Assume that χ is either an ordinary character, or a p-modular Brauer character for an odd prime p. If χ is orthogonal, and if there is $g \in G$ such that a representation ρ affording χ map g to a matrix that does not have an eigenvalue ± 1, then the restriction of χ to the subgroup $\langle g \rangle$ is orthogonally stable, and has determinant $\det(\rho(g) - \rho(g^{-1}))$, modulo squares, see [Neb22b, Cor. 4.2]. (This is a special case of the criterion from Section 3.3.1.) Note that the eigenvalues of $\rho(g)$, and hence, the determinant can be computed from the power map information that belongs to the character table of G.

```
julia> ch = character_table("Co3", 3)[2];

julia> degree(ch)
22

julia> Oscar.OrthogonalDiscriminants.od_from_eigenvalues(ch)
(true, "O+")
```

Jantzen-Schaper formula (`"specht"`): The ordinary irreducible representations of the symmetric group on n points are parameterized by the partitions of n, and the determinant of the bilinear form that is fixed by the representing matrices for the partition λ can be expressed in terms of λ, via the Jantzen-Schaper formula [Mat99, p. 5.33]. This yields the orthogonal discriminants of those characters of the alternating group on n points that extend to the symmetric group. We are interested in the cases $5 \leq n \leq 13$.

```
julia> ch = character_table("A12")[26];

julia> degree(ch)
1728
```

```
julia> Oscar.OrthogonalDiscriminants.od_for_specht_module(ch)
(true, "1")
```

Restriction to p-subgroups (`"syl(p)"`): Let p be an odd prime, and let χ be a character in characteristic different from p. The restriction χ_P of χ to a p-subgroup P of G is orthogonally stable if and only if the trivial character of P is not a constituent of χ_P, and the orthogonal discriminant of χ_P can be computed in terms of $\chi(1)$ and the character field of χ_P (see [Neb22a, Section 4.1] and Section 3.1.1). Note that in order to check whether χ_P is orthogonally stable, it is sufficient to know the permutation character 1_P^G, we do not need the character table of P.

```
julia> ch = character_table("R(27)")[16];

julia> degree(ch)
18278

julia> Oscar.OrthogonalDiscriminants.od_from_p_subgroup(ch, 3)
(true, "-3")
```

Restriction to subgroups (`"rest(...)"`, `"ext(...)"`): If H is a subgroup of G whose character table is known, and if the restriction χ_H is orthogonally stable then we can argue as follows. If the orthogonal discriminants of the constituents of χ_H are known, then we can deduce that of χ; in this case, the data file contains the label `"ext(...)"`. If the orthogonal discriminant of χ is known, then we get a condition on the orthogonal discriminants of the constituents of χ_H; for example, if all of them except one are already known, then we can deduce the missing one; in this case, the data file contains the label `"rest(...)"`.

Regard ordinary characters as Brauer characters (`"lift(+...)"`): Let χ be a p-modular Brauer character. If χ is the restriction of an ordinary character whose orthogonal discriminant is known, then reducing this value modulo p often yields the orthogonal discriminant of χ. If χ is a constituent of the restriction of an ordinary character whose orthogonal discriminant is known, then reducing this value modulo p often yields the orthogonal discriminant of χ if the discriminants of the other constituents are known.

Tensor products (`"tensor(...)"`): [Neb99, Section 3.1.3] lists formulae for the determinants of the invariant bilinear forms of tensor products $\chi \cdot \psi$ and of symmetric squares $\chi^{2+} - 1_G$ and antisymmetric squares χ^{2-}. In those cases where these tensor products and symmetrizations are orthogonally stable, this yields conditions on the orthogonal discriminants of their constituents, as in the above criteria.

Consistency checks: Often an orthogonal discriminant can be computed with several criteria, and the results must be consistent. A posteriori, also those conditions about constituents of restrictions, tensor products, p-modular reductions that were not sufficient to deduce the orthogonal discriminants can be used for consistency checks.

4 Examples

In total, we compiled data for ATLAS groups containing almost 20 000 orthogonal discriminants, that can be displayed in more than 1 000 ordinary and Brauer character tables that are available in the `OSCAR` character table library. To illustrate the output and some of the methods, we give a few examples in this section.

4.1 The Orthogonal Discriminants of $G_2(3)$

The 2-modular Brauer character table of the simple group $G_2(3)$ together with the stored orthogonal discriminants can be displayed as follows.

```
julia> Oscar.OrthogonalDiscriminants.show_with_ODs(
       character_table("G2(3)", 2))
G2(3)mod2

            2   6   3   3   .  1  1  .  .  .  .   .   .
            3   6   6   6   6  4  4  .  3  3  3   .   .
            7   1   .   .   .  .  .  1  .  .  .   .   .
           13   1   .   .   .  .  .  .  .  .  .   1   1

               1a  3a  3b  3c 3d 3e 7a 9a 9b 9c 13a 13b
           2P  1a  3a  3b  3c 3d 3e 7a 9a 9c 9b 13b 13a
           3P  1a  1a  1a  1a 1a 1a 7a 3c 3c 3c 13a 13b
           7P  1a  3a  3b  3c 3d 3e 1a 9a 9b 9c 13b 13a
          13P  1a  3a  3b  3c 3d 3e 7a 9a 9b 9c  1a  1a
     d OD   2
 X_1 1      +   1   1   1   1  1  1  1  1  1  1   1   1
 X_2 1 O-   +  14   5   5  -4  2 -1  .  2 -1 -1   1   1
 X_3 2      o  64  -8  -8   1  4 -2  1  1  A /A  -1  -1
 X_4 2      o  64  -8  -8   1  4 -2  1  1 /A  A  -1  -1
 X_5 1 O-   +  78  -3  -3  -3 -3  6  1  .  .  .   .   .
 X_6 1 O+   +  90   9   9   9  .  . -1  .  .  .  -1  -1
 X_7 1 O-   +  90  -9  18   .  3 -3 -1 -3  .  .  -1  -1
 X_8 1 O-   +  90  18  -9   .  3 -3 -1 -3  .  .  -1  -1
 X_9 1 O-   + 378  -9  -9   9 -3 -6  .  3  .  .   1   1
X_10 2 O+   + 448  16  16 -11 -2 -2  .  1  1  1   B  B*
X_11 2 O+   + 448  16  16 -11 -2 -2  .  1  1  1  B*   B
X_12 1 O+   + 832 -32 -32  -5  4  4 -1  1  1  1   .   .

A = 3z_3 + 1
/A = -3z_3 - 2
B = -z_13^11 - z_13^8 - z_13^7 - z_13^6 - z_13^5 - z_13^2 - 1
B* = z_13^11 + z_13^8 + z_13^7 + z_13^6 + z_13^5 + z_13^2
```

The new data are contained in the column headed by `OD`. Here we give the type of the invariant quadratic form as described in Definition 1. For instance this allows us to read off that the image of the 14-dimensional absolutely irreducible 2-modular representation of $G_2(3)$ is contained in $O_{14}^-(\mathbb{F}_2)$.

The group $G_2(3)$ is one of the interesting examples where all ordinary and modular orthogonal discriminants can be obtained directly from the known decomposition matrix.

```
julia> show_OD_info("G2(3)")
G2(3):  2^6*3^6*7*13
--------------------

 i| chi|     K|disc|           2|3|            7|      13
--+----+------+----+------------+-+-------------+--------
 2| 14a|     Q|  -3|         14a| |          14a|     14a
  |    |      |    |          O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
 5| 78a|     Q|  -3|         78a| |          78a|     78a
  |    |      |    |          O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
 9|104a|     Q|  21|     14a+90a| |     (def. 1)|    104a
  |    |      |    |      O-, O+| |             |      O-
--+----+------+----+------------+-+-------------+--------
10|168a|     Q|  13|     78a+90a| |         168a|(def. 1)
  |    |      |    |      O-, O+| |           O-|
--+----+------+----+------------+-+-------------+--------
11|182a|     Q|  -3| 14a+78a+90c| |         182a|    182a
  |    |      |    |  O-, O-, O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
12|182b|     Q|  -3| 14a+78a+90b| |         182b|    182b
  |    |      |    |  O-, O-, O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
15|448a|Q(b13)|   1|        448a| |         448a|14a+434a
  |    |      |    |          O+| |           O+| O+,  O+
--+----+------+----+------------+-+-------------+--------
16|448b|Q(b13)|   1|        448b| |         448b|14a+434a
  |    |      |    |          O+| |           O+| O+,  O+
--+----+------+----+------------+-+-------------+--------
17|546a|     Q|  -3|78a+90b+378a| |         546a|    546a
  |    |      |    | O-, O-,  O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
18|546b|     Q|  -3|78a+90c+378a| |         546b|    546b
  |    |      |    | O-, O-,  O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
19|728a|     Q|   1|    14a+378a| |         728a|    728a
  |    |      |    |     O-,  O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
20|728b|     Q|   1|    14a+378a| |         728b|    728b
  |    |      |    |     O-,  O-| |           O+|      O+
--+----+------+----+------------+-+-------------+--------
23|832a|     Q|   1|        832a| |64ab+78a+626a|    832a
  |    |      |    |          O+| |  O+, O+,  O+|      O
```

The function `show_OD_info` collects the information about ordinary and modular orthogonal discriminants that are stored in our data. The rows of the table correspond to the ordinary indicator + characters χ of even degree. The first column lists the ATLAS number of χ followed by the degree. Then we give the character field $\mathbb{Q}(\chi)$, and column four displays a representative of $\mathrm{disc}(\chi)$. The following columns are headed by the prime divisors p of the group order. If $\chi \pmod p$ is orthogonally stable, then we give the corresponding character degrees of the p-modular constituents of χ and their corresponding orthogonal discriminants. The entry "(def. 1)" means that $\chi \pmod p$ is not orthogonally stable but has defect 1, from which we know that p is ramified in the discriminant field extension by Remark 4.

For the case of $G_2(3)$ all orthogonal discriminants can be obtained from the decomposition matrices: For the ordinary characters we know from Theorem 3 that the discriminant field extension is unramified at all primes but possibly those that divide 3, except for the characters number 9 and 10 where we know that 7 respectively 13 are ramified. In all cases, this yields a unique possibility for the respective quadratic extension $\mathrm{Disc}(\chi)$ of the character field. Let us illustrate the consideration for the two non-rational characters number 15 and 16. Here the character field is $\mathbb{Q}(\sqrt{13})$ and we know that the discriminant field is either the character field or a totally real quadratic extension of $\mathbb{Q}(\sqrt{13})$ that is unramified at all primes but possibly those dividing 3. There are no such quadratic extensions, as can be computed with the commands

```
julia> K, _ = quadratic_field(13)
(Real quadratic field defined by x^2 - 13, sqrt(13))

julia> ray_class_field(3*maximal_order(K))
Class field defined mod (<3, 3>, InfPlc{AbsSimpleNumField,
↪  AbsSimpleNumFieldEmbedding}[]) of structure Z/1
```

This way we get all the ordinary orthogonal discriminants. The p-modular reductions allow us to find all the modular discriminants from the ordinary ones as we illustrate for the prime $p = 2$:

As -3 is not a 2-adic square, the prime 2 is inert in the quadratic extension $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ and hence the orthogonal discriminant of the 2-modular reductions of χ_i are $O-$ for $i \in \{2, 5, 11, 12, 17, 18\}$. A similar argument yields discriminant $O-$ for the 2-modular reduction of χ_9 and χ_{10}. This gives the orthogonal discriminants for the 2-modular characters `14a`, `78a`, `90a`, `90c`, `90b`, `378a`, and two checks coming from the 2-modular reduction of χ_{10} and χ_{18}. The 2-modular reduction of the ordinary characters of discriminant 1 have orthogonal discriminant $O+$, from which we get the orthogonal discriminants of `448a`, `448b`, and `832a` as well as a check coming from the characters χ_{19} and χ_{20}.

4.2 The Ordinary Orthogonal Discriminants of J_2

This example illustrates how we can obtain the ordinary orthogonal discriminants of a group by only using representations over finite (prime) fields. These representations can be constructed with Richard Parker's C-meataxe by reducing permutation representations or tensor products of known representations. This way we obtain the following table.

	11	19	29	31	41	59	disc	13	17
$14a$	$O-$	$O+$	$O-$	$O+$	$O-$	$O-$	-3		
$14b$	$O-$	$O+$	$O-$	$O+$	$O-$	$O-$	-3		
$36a$	$O+$	$O+$	$O+$	$O+$	$O+$	$O+$	5	$O-$	$O-$
$70a$	$O-$	$O+$	$O-$	$O+$	$O-$	$O-$	-3		
$70b$	$O-$	$O+$	$O-$	$O+$	$O-$	$O-$	-3		
$90a$	$O+$	$O-$	$O+$	$O-$	$O-$	$O-$	-7	$O-$	$O-$
$126a$	$O-$	$O-$	$O+$	$O-$	$O+$	$O-$	-5	$O-$	$O-$
$160a$	$O+$	$O+$	$O+$	$O+$	$O+$	$O+$	1	$O+$	$O+$
$224a$	$O+$	$O+$	$O+$	$O+$	$O+$	$O+$	1		
$224b$	$O+$	$O+$	$O+$	$O+$	$O+$	$O+$	1		
$288a$	$O-$	$O-$	$O-$	$O-$	$O+$	$O+$	105	$O+$	$O-$
$300a$	$O-$	$O-$	$O-$	$O-$	$O+$	$O+$	21	$O-$	$O+$
$336a$	$O+$	$O+$	$O+$	$O+$	$O+$	$O+$	1	$O+$	$O+$

The rows of this table are named by the degrees of the ordinary even degree indicator + irreducible characters of the sporadic simple group J_2. The entries of all the columns headed by a prime are computed and in total allow us to deduce the orthogonal discriminant of the character as given in column disc. We kept the ordering of the columns as it was given in Parker's handwritten table. The character fields of the irreducible ordinary characters of J_2 are either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. This is why we first chose primes not dividing the group order for which 5 is a square, allowing us to construct the corresponding absolutely irreducible representation over the prime field. But of course this information is not enough, for instance, to decide whether the discriminant of the rational character `36a` is 5 or 1. So we constructed the representations with rational characters also over $\mathbb{F}_{13}$ and $\mathbb{F}_{17}$, and computed the discriminant of an invariant quadratic form there.

As an example for our arguments, we treat the character `228a`. As the character is rational of degree a multiple of 4, we know that the discriminant field is a real quadratic number field L that is unramified outside $\{2, 3, 5, 7\}$, the set of prime divisors of the group order. Our computed information yields that the primes $11, 19, 29, 31$, and 17 are inert in L and the primes $41, 59, 13$ are split in L. In other words $L = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{N}$ is squarefree and divides $2 \cdot 3 \cdot 5 \cdot 7$. Moreover d is a square mod $41, 59$, and 13, and a non-square modulo $11, 19, 29, 31$, and 17. This yields the unique solution $d = 105 = 3 \cdot 5 \cdot 7$.

By computer, we need to solve a system of linear equations over the field $\mathbb{F}_2$: The entry (p, q) of the matrix below tells us whether p is a square modulo q (entry 0) or not (entry 1).

	11	19	29	31	41	59	13	17
-1	1	1	0	1	0	1	0	0
2	1	1	1	0	0	1	1	0
3	0	1	1	1	1	0	0	1
5	0	0	0	0	0	0	1	1
7	1	0	0	0	1	0	1	1

To compute the discriminant of the character `300a`, for example, we compute the unique linear combination of the rows of this matrix that yields $(1, 1, 1, 1, 0, 0, 1, 0)$. It is easy to see that this row is the sum of the rows of 3 and 7, so the discriminant of `300a` is 21.

5 Applications

This section lists some aspects of the computations, and implications of the results.

5.1 Which Discriminant Fields are Galois Extensions of the Rationals?

The number fields that do occur in representation theory of finite groups are usually abelian extensions of the rationals, i.e. contained in some cyclotomic fields. Also discriminant fields are very often abelian extensions of the rationals:

Theorem 6 *Let χ be an orthogonally simple ordinary character of a finite group G, and put $L := \mathrm{Disc}(\chi)$ to denote the discriminant field.*

- *If χ is not absolutely irreducible (i.e. of type $\circ$ or $-$ in Remark 3), then L is an abelian extension of $\mathbb{Q}$.*
- *If G is solvable, then L is an abelian extension of $\mathbb{Q}$ (see [Neb22a] and [Rot22])*
- *For G of type L_2, all discriminant fields are abelian extensions of the rationals (see [BN17]).*

Proposition 2 *The discriminant field is Galois over* $\mathbb{Q}$ *if and only if the discriminant, a square class of the character field, is stable under all Galois automorphisms of the character field.*

For the proof we need the following easy lemma in Galois theory:

Lemma 2 *Given a tower* $A \subseteq B \subseteq C$ *of fields such that* B/A *is Galois and* C/B *is Galois and* $[C : A] < \infty$*, then* C/A *is Galois if and only if for all* $g \in \mathrm{Gal}(B/A)$*, there is* $f \in \mathrm{Aut}(C)$ *such that* $f_{|B} = g$*.*

Proof Under the conditions of the lemma the sequence

$$1 \to \mathrm{Gal}(C/B) \to \mathrm{Aut}_A(C) \to \mathrm{Aut}_A(B) \to 1$$

is exact and hence $|\mathrm{Aut}_A(C)| = [C : A]$, which implies that C/A is Galois.□

Proof (of Proposition 2) Now we apply this to our situation where $F = F(\chi)$ is the character field of an ordinary orthogonally stable character χ, and $K = F[\sqrt{\delta}]$ is the discriminant field.

To prove Proposition 2, we need to show that $K/\mathbb{Q}$ is Galois if and only if $\delta(F^{\times})^2$ is stable under the full Galois group of $F/\mathbb{Q}$, i.e., for all $g \in \mathrm{Gal}(F/\mathbb{Q})$ there is $k_g \in F$ such that $g(\delta) = k_g^2\delta$.

For the proof let $\alpha := \sqrt{\delta} \in K$.

Assume that $K/\mathbb{Q}$ is Galois.

Then $\langle\sigma\rangle := \mathrm{Gal}(K/F)$ is a normal subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ of order 2, and hence central.

The minimal polynomial of α over F is $X^2 - \delta$ and any automorphism $f \in \mathrm{Aut}(K)$ that extends $g \in \mathrm{Gal}(F/\mathbb{Q})$ satisfies $f(\alpha)^2 = g(\delta)$ and $f(F) \subseteq F$. Now f commutes with σ so $k_g := f(\alpha)/\alpha \in \mathrm{Fix}_{\sigma}(K) = F$ and $k_g^2 = f(\alpha)^2/\alpha^2 = g(\delta)/\delta$, so $g(\delta) = k_g^2\delta$.

To see the opposite direction, we extend $g \in \mathrm{Gal}(F/\mathbb{Q})$ to an automorphism f of K by putting $f(a\alpha + b) := g(a)k_g\alpha + g(b)$ for all $a, b \in F$. It is easy to see that f is a field automorphism of K extending g. So Proposition 2 follows from Lemma 2. □

Remark 5 In the notation of the proof we get that the discriminant field is an abelian extension of $\mathbb{Q}$ if and only if $f(k_g)k_f = g(k_f)k_g$ for all $f, g \in \mathrm{Gal}(F/\mathbb{Q})$.

Corollary 2 *Let* χ *be an orthogonally stable ordinary character of* G *and* $K := F(\chi)$ *its character field. Assume that* $\mathrm{Aut}(G)$ *acts transitively on the Galois orbit* $\chi^{\mathrm{Gal}(K/\mathbb{Q})}$*. Then,* $\mathrm{Disc}(\chi)$ *is Galois over* $\mathbb{Q}$*.*

In particular all discriminant fields of the orthogonally stable characters of the alternating groups are Galois over $\mathbb{Q}$.

Example 1 Conjecture 3.9 in [Cra22] states that any absolutely irreducible character with indicator + and degree congruent to 2 (mod 4) is expected to have an orthogonal discriminant α such that $\sqrt{\alpha}$ lies in a cyclotomic field.

A counterexample is provided by the two irreducible characters of degree 169290 of the sporadic simple O'Nan group. Their orthogonal discriminants are $-53 \pm 36\sqrt{2}$, see [NP23, Remark 7.3].

So far, all non Galois discriminant fields that we are aware of occur for sporadic simple groups and their automorphism groups.

Example 2 During our computations we only found the following ordinary orthogonally simple (see Remark 3) characters of finite simple groups for which the discriminant fields $\mathbb{Q}(\sqrt{\delta})$ are not Galois over $\mathbb{Q}$:

G	χ	δ	$\mathrm{Gal}(\mathbb{Q}(\sqrt{\delta})/\mathbb{Q})$
J_1	$56ab$	$(31+5\sqrt{5})/2$	D_8
J_1	$120abc$	$29-18c_{19}-9c_{19}^{*2}$	$C_2 \times A_4$
J_3	$1920abc$	$63-30y_9-7y_9^{*2}$	A_4
He	$21504ab$	$357+68\sqrt{21}$	D_8
Ru	$27000abc$	$119y_7+49y_7^{*2}+170$	A_4
Ru	$34944ab$	$41-16\sqrt{6}$	D_8
Ru	$110592ab$	$(1015-185\sqrt{29})/2$	D_8
ON	$169290ab$	$-36\sqrt{2}-53$	D_8
ON	$175616ab$	$225+84\sqrt{5}$	D_8
ON	$207360abc$	$-496c_{19}+1767c_{19}^{*4}+3472$	$C_2 \times A_4$
HN	$5103000ab$	$17+4\sqrt{5}$	D_8

The table lists the groups, the characters χ (full Galois orbit) in the form $\chi(1)ab...$, the orthogonal discriminant of $\chi(1)a$ in ATLAS notation (see Section 2.1.1), and the Galois group of the normal closure of the discriminant field. The characters of $G = J_3$ and $G = He$ extend to characters of $G.2$ with the same degree, character field, and orthogonal discriminant.

We can select the entries in question from the known data as follows, using the criterion from Proposition 2.

```
julia> function is_galois_discriminant_field(data)
         chi = Oscar.OrthogonalDiscriminants.character_of_entry(data)
         F, emb = character_field(chi)
         c = conductor(emb(gen(F)))
         galgens = Oscar.AbelianClosure.generators_galois_group_cyclotom ⌋
         ↪  ic_field(c)
         delta = atlas_irrationality(data[:valuestring])
         return all(x -> is_square(preimage(emb, delta * x(delta))),
```

```
                 galgens)
       end;

julia> info = all_od_infos(characteristic => 0, is_simple);

julia> filter!(r -> r[:valuestring] != "?" &&
                    conductor(atlas_irrationality(r[:valuestring])) > 1,
               info);

julia> length(info)
58

julia> filter!(!is_galois_discriminant_field, info);

julia> length(info)
26

julia> println(sort!(collect(Set([r[:groupname] for r in info]))))
["HN", "He", "J1", "J3", "ON", "Ru"]
```

5.2 No even Discriminants?

Richard Parker conjectured that orthogonal discriminants in characteristic zero are always odd (see [Neb22a, Conjecture 1.3]). This conjecture is true for characters of solvable groups (see [Neb22a, Theorem 1.5]), and it holds also for all characters of Atlas groups which we have computed so far.

```
julia> info = all_od_infos(characteristic => 0, degree => 1);

julia> all(x -> x[:valuestring] == "?" ||
                is_odd(parse(Int, x[:valuestring])),
           info)
true
```

Note that the sketch of a proof of this conjecture over the rationals given in [Cra22, p. 7] is not correct, as it assumes that there is always an even lattice of square-free discriminant.

5.3 Groups Embedding in both Orthogonal Groups of same Degree

The final remark in [SW91] asks whether there is a group G with irreducible orthogonal representations of the same even degree and over the same character field in characteristic two, such that one of them has orthogonal discriminant $O+$ and the other has orthogonal discriminant $O-$.

The data about Atlas groups provide exactly one such example: The simple group $G_2(3)$ has three 90-dimensional absolutely irreducible representations over the field with two elements, `"90a"` (the one which is invariant under the outer automorphism) has orthogonal discriminant $O+$, whereas `"90b"` and `"90c"` (which are conjugate under the outer automorphism) have orthogonal discriminant $O-$, cf. Section 4.1.

```
julia> plus = [];  minus = [];

julia> for d in all_od_infos()
         if d[:valuestring] == "O+"
           push!(plus, (d[:groupname], d[:characteristic], d[:degree],
                        parse(Int, filter(isdigit, d[:charname]))))
         elseif d[:valuestring] == "O-"
           push!(minus, (d[:groupname], d[:characteristic], d[:degree],
                         parse(Int, filter(isdigit, d[:charname]))))
         end
       end

julia> both = intersect!(plus, minus);

julia> filter(x -> x[2] == 2, both)
1-element Vector{Any}:
 ("G2(3)", 2, 1, 90)

julia> length(both)
103
```

(We see that there are many examples in odd characteristic.)

5.4 Accessing the Atlas of Orthogonal Discriminants

The information about orthogonal discriminants of Atlas groups can be used in `GAP` and `OSCAR`, as follows.

The `GAP` function `Display` and the `OSCAR` function `show`, respectively, can be called with the option to extend the shown character table by a column for orthogonal discriminants. One can also access the list of known orthogonal discriminants for an ATLAS character table, via the functions `OrthogonalDiscriminants` (in `GAP`) and `orthogonal_discriminants` (in `OSCAR`), respectively.

5.5 New Findings for the Old Character Tables

The following new information has been obtained as a by-product of the computation of orthogonal discriminants.

- Listing the orthogonal discriminants of the orthogonal absolutely irreducible characters of a group requires the knowledge of the Frobenius Schur indicators of these characters (see Section 2.3). In characteristic two, this information is not known for all character tables we are interested in. Several 2-modular Frobenius Schur indicators that had been missing are now known. They have been either computed explicitly once we had the representation in question, or determined using [GW95, Lemma 1.2].
- The Brauer character tables of $L_2(49)$ mod 7, $L_2(81)$ mod 3, and $L_6(2)$ mod 2 had been missing.
- Several class fusions between Atlas character tables, which turned out to be useful for restrictions of characters to subgroups, have been added to the character table library.
- A so-called generality problem for the sporadic simple group HN and its automorphism group $HN.2$ has been solved. This problem concerns the consistency between the 11- and 19-modular character tables of these groups, as follows.

  In the ordinary character table of HN, the conjugacy classes `20A` and `20B` are distinguished only by the two algebraic conjugate irreducible characters χ_{51}, χ_{52} of degree 5 103 000. Their values on `20A` and `20B` are $1 \pm 2\sqrt{5}$.

  According to the Brauer character tables in the library of character tables up to version 1.3.4, the conjugacy class `20A` of HN was the class for which both the unique irreducible 11-modular Brauer character of degree 628 426 and the unique irreducible 19-modular Brauer character of degree 1 074 075 have the value $1 - 2r_5$. The orthogonal discriminant of χ_{51} is either $4\sqrt{5} + 17$ or $-4\sqrt{5} + 17$. In the former case, the 11-modular reduction of χ_{51} is orthogonally stable, and the 19-modular reduction is not; in the latter case, it is the other way round. However, with the above choice of the class `20A`, both the 11- and 19-modular reductions of χ_{51} are orthogonally stable (and the 11- and 19-modular reductions of χ_{52} are not). Thus we have shown that the choice of `20A` in the two character tables is not consistent. In order to make the two character tables consistent, we have changed the 11-modular table in version 1.3.5 of the table library, by swapping the columns of `20A` and `20B`.

  (As a consequence, also the 11-modular table of the automorphism group $HN.2$ of HN had to be adjusted. There are still open questions about the consistency of other conjugacy classes in Brauer character tables of HN. They are independent of the question about `20A` and `20B`, and they cannot be answered by considering orthogonal discriminants.)

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