On the classification of even unimodular lattices with a complex structure

by

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Abstract. This paper classifies the even unimodular lattices that have a structure as a Hermitian $\mathcal{O}_K$-lattice of rank $r \leq 12$ for rings of integers in imaginary quadratic number fields K of class number 1. The Hermitian theta series of such a lattice is a Hermitian modular form of weight r for the full modular group, therefore we call them theta lattices. For arbitrary imaginary quadratic fields we derive a mass formula for the principal genus of theta lattices which is applied to show completeness of the classifications.

Keywords: Hermitian lattices, Hermitian theta series, mass formulas, even unimodular lattices
11E41; 11F03, 11E39, 11H06

1 Introduction

The classification of even unimodular $\mathbb{Z}$-lattices is explicitly known only in the cases of rank $8, 16$ and 24 (cf. [CS]). Given an imaginary quadratic number field K with discriminant d_K and ring of integers $\mathcal{O}_K$, Cohen and Resnikoff [CR] showed that there exists a free $\mathcal{O}_K$-module M of rank r, which is even and satisfies $\det M = (2/\sqrt{d_K})^r$ if and only if $r \equiv 0 \bmod 4$, where explicit examples were described in [DK]. Each such $\mathcal{O}_K$-module is an even unimodular $\mathbb{Z}$-lattice of rank $2r$ and the associated Hermitian theta series is a Hermitian modular form of weight r for the full modular group. An explicit description of the isometry classes of these lattices has so far only been obtained for $r = 4, 8$ and 12 for the Gaussian number field $K = \mathbb{Q}(\sqrt{-1})$ in [I], [S] and [KM] and for the Eisenstein number field $K = \mathbb{Q}(\sqrt{-3})$ in [HKN]. In these cases one can basically use the Niemeier classification and has to look for the exceptional automorphisms with minimal polynomial $x^2 + 1$ resp. $x^2 - x + 1$ due to the exceptional units in K.

In this paper we derive analogous results for $\mathcal{O}_K$-modules whenever K is an arbitrary imaginary quadratic number field. At first we derive a mass formula for the appropriate genus of lattices. In the case of class number 1, either an explicit classification of the isometry classes is given or a large lower bound on the number of isometry classes is derived. We apply the neighboring method to the explicit example given in [DK] in order to obtain a classification. There are two notable effects. At first there are even unimodular $\mathbb{Z}$-lattices which do not have the structure of an $\mathcal{O}_K$-module. On the other hand there are lattices which are not isometric as $\mathcal{O}_K$-modules although the underlying $\mathbb{Z}$-lattices are isometric (cf. [SSS], [KM], [HKN]). Finally the results are applied to the associated Hermitian theta series in order to obtain a filtration similar to [NV] and [HKN].

2 Mass of even and odd unimodular lattices

Throughout this paper let $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$ squarefree, be an imaginary quadratic number field with discriminant d_K, Dirichlet character χ_K and ring of integers

$$\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega_K, \quad \omega_K = \begin{cases} (1+\sqrt{-d})/2 & \text{if } d \equiv 3 \bmod 4, \\ \sqrt{-d} & \text{if } d \equiv 1, 2 \bmod 4. \end{cases}$$

An $\mathcal{O}_K$-*lattice* M in a finite dimensional K vector space $\mathcal{V}$ is a finitely generated projective $\mathcal{O}_K$-submodule M of $\mathcal{V}$ that contains a K-basis of $\mathcal{V}$.

Now let $(\mathcal{V}, h)$ be the r-dimensional standard Hermitian vector space over K, so there exists a basis $(e_1, \ldots, e_r)$ of $\mathcal{V}$ such that $h(e_i, e_j) = \delta_{ij}$. In this section we want to use the method from [BN] (for a more structural approach using the theory of Bruhat-Tits buildings see [CNP]) to derive the mass of the genus of all positive definite even unimodular $\mathcal{O}_K$-lattices that have a Hermitian structure over $\mathcal{O}_K$, from the one of the so called *principal genus*, the genus $\mathcal{N}_r$ of the Hermitian lattice

$$I_r = \mathcal{O}_K e_1 + \ldots + \mathcal{O}_K e_r.$$

This genus consists of all odd $\mathcal{O}_K$-lattices $N \subset \mathcal{V}$ that are Hermitian unimodular, i.e. that are equal to their *Hermitian dual* lattice

$$N^* := \{x \in \mathcal{V};\ h(x, N) \subset \mathcal{O}_K\}.$$

We only treat dimensions $r = 2k$ that are multiples of 4, so k is even. Let

$$\mathcal{N} := \{N;\ N\, \mathcal{O}_K\text{-lattice in } \mathcal{V},\ N = N^*,\ \exists\, x \in N \text{ such that } h(x, x) \notin 2\mathbb{Z}\}$$

be the genus of all odd Hermitian unimodular lattices in $\mathcal{V}$. Since

$$h(x + y, x + y) = h(x, x) + h(y, y) + \mathrm{Trace}_{K/\mathbb{Q}}(h(x, y))$$

a unimodular lattice is automatically odd if 2 is not ramified in K. In the case $d \not\equiv 3 \bmod 4$ (when 2 is ramified) there are two genera of unimodular lattices in $\mathcal{V}$ (see for instance [J]), $\mathcal{N}_r$ and the genus

$$\mathcal{M}_r := \{M;\ M\, \mathcal{O}_K\text{-lattice in } \mathcal{V},\ M = M^*, h(x, x) \in 2\mathbb{Z} \text{ for all } x \in M\}$$

of even unimodular lattices in $\mathcal{V}$.

To avoid clumsy notation we put $\mathcal{M}_r := \mathcal{N}_r$ if $d \equiv 3 \bmod 4$.

Remark 1. Let p be a prime divisor of d_K. Let R be the completion of $\mathcal{O}_K$ at the prime ideal $\wp \trianglelefteq \mathcal{O}_K$ that divides $p\mathcal{O}_K$ and π a prime element of R. For $M \in \mathcal{M}_r$ we define the non degenerate bilinear $\mathbb{F}_p$-space $V = \pi^{-1}RM/RM$,

$$\beta : V \times V \to \mathbb{F}_p = R/\wp, \quad \beta(x + RM, y + RM) := ph(x, y) + \wp.$$

If $p = 2$, then $\mathrm{trace}(\frac{1}{\pi}R) \subseteq 2\mathbb{Z}_2$ if and only if $2 \mid d$. In that case

$$q : V \to \mathbb{F}_2 = R/\wp, \quad q(y + RM) := h(y, y) + \wp = \frac{1}{2}\beta(y, y)$$

gives a well-defined quadratic form on the space V with associated bilinear form β.

Two $\mathcal{O}_K$-lattices M, N are called *isometric* if there exists an *isometry* $U \in \mathcal{U}(\mathcal{V})$ such that

$$N = UM.$$

The (unitary) *automorphism group* of M is defined by

$$\mathrm{Aut}(M) := \{U \in \mathcal{U}(\mathcal{V});\ UM = M\}.$$

The *mass* of a genus $\mathcal{G}$ is

$$\mu(\mathcal{G}) := \sum_{[M] \subset \mathcal{G}} \frac{1}{\sharp \,\mathrm{Aut}(M)},$$

where we sum over representatives M of the isometry classes $[M]$ in $\mathcal{G}$.

We quote the particular case $r \equiv 0 \bmod 4$ from [HK, Theorem 5.6] as

Theorem 1. *Let $r = 2k \equiv 0 \bmod 4$. Then the mass*

$$\mu_r^{(odd)} := \mu(\mathcal{N}_r) = \sum_{[N] \subset \mathcal{N}_r} \frac{1}{\sharp \,\mathrm{Aut}(N)}$$

is equal to

$$\frac{1}{2^{t+r-1} \cdot r!} \prod_{j=1}^{k} |B_{2j} \cdot B_{2j-1}, \chi_K| \prod_{2 \neq p | d} (p^k + 1) \cdot \begin{cases} (2^r - 1) & \text{if } d \equiv 1 \bmod 4, \\ 2^k(2^r - 1) & \text{if } d \equiv 2 \bmod 4, \\ 1 & \text{if } d \equiv 3 \bmod 4, \end{cases}$$

where t is the number of distinct prime divisors of the discriminant d_K and B_j (resp. B_j, χ_K) are the (generalized) Bernoulli numbers.

For the definition of the (generalized) Bernoulli numbers confer [M, p.89 resp. p.94].

Next we want to compute the mass of the genus $\mathcal{M}_r$.

Lemma 1. *Let* $r = 2k \equiv 0 \bmod 4$ *and* $d \equiv 2 \bmod 4$*. Then one has*

$$\mu_r^{(even)} := \mu(\mathcal{M}_r) = (2^k(2^k - 1))^{-1}\mu_r^{(odd)}.$$

Proof. The idea of the proof is that any odd unimodular lattice is a 2-neighbor of some even unimodular lattice and vice versa. Since the two genera $\mathcal{N}_r$ and $\mathcal{M}_r$ are only different on the completion R of $\mathcal{O}_K$ at the prime ideal over 2, we extend scalars to R. Remark 1 implies that the situation is exactly the same as for $\mathbb{Z}$-lattices (see for instance [Bo]): For $N \in \mathcal{N}_r$ the R-lattice RN is isometric to RI_r with orthonormal basis $(e_1, \ldots, e_r)$. Its even sublattice is

$$N_0 := \{x \in RN;\ h(x,x) \in 2\mathbb{Z}_2\} = \langle \sqrt{-d}e_1, e_1 + e_2, \ldots, e_1 + e_r\rangle_R$$

and

$$N_0^* = \langle N_0, \frac{1}{\sqrt{-d}}(e_1 + \ldots + e_r), e_1\rangle_R.$$

So the three unimodular lattices that contain N_0 are

$$N = \langle N_0, e_1\rangle,\ N_1 = \langle N_0, z\rangle,\ M = \langle N_0, y\rangle$$

where $y := \frac{1}{\sqrt{-d}}(e_1 + \ldots + e_r)$ and $z := y + e_1$. One computes

$$h(z,z) = \frac{1}{d}(r - 1 + (1 + d)) = \frac{r}{d} + 1 \in R^*$$

and

$$h(y,y) = \frac{r}{d} \in 2R.$$

By assumption r is a multiple of 4 so N and N_1 are odd unimodular lattices and M is the unique even neighbor of N.

Now let $M \in \mathcal{M}_r$ be an even unimodular lattice and $X \subset M$ a sublattice of index 2 such that X^* contains an odd unimodular lattice. Then

$$RX^*/RM \subset R\frac{1}{\sqrt{-d}}M/RM$$

is a one-dimensional anisotropic subspace of the quadratic space from Remark 1. Again by [T], Exercise 11.3, the number of such anisotropic subspaces is

$$(2^{2k} - 1) - (2^k - 1)(2^{k-1} - 1) = 2^{k-1}(2^k - 1).$$

Each such X defines 2 odd neighbors of M, so

$$\mu(\mathcal{N}_r) = 2^k(2^k - 1)\mu(\mathcal{M}_r).$$ □

In the case where $d \equiv 1 \bmod 4$ one may apply the same strategy to compare the mass of $\mathcal{N}_r$ and $\mathcal{M}_r$.

Lemma 2. *Let $r = 2k \equiv 0 \bmod 4$ and $d \equiv 1 \bmod 4$. Then one has*

$$\mu_r^{(odd)} = \frac{2^r - 1}{2} \mu_r^{(even)}.$$

Proof. The proof is similar to the one of Lemma 1 but there are two major differences:
(1) The prime element $\pi := (1 + \sqrt{-d}) \in R$ in the 2-adic completion R now has trace 2. Therefore the dual of the even sublattice of the standard lattice is

$$N_0^* = \langle N_0, y := \frac{1}{\pi}(e_1 + \ldots + e_r), e_1 \rangle_R$$

where

$$h(y, y) = \frac{r}{1+d} \quad \text{and} \quad h(y + e_1, y + e_1) = \frac{r+2}{1+d} + 1$$

are both even. So any odd unimodular lattice N has 2 even neighbors.
(2) If $M \in \mathcal{M}_r$ is an even unimodular lattice, then the space

$$\left(\frac{1}{\pi} RM/RM, \beta \right) \quad \text{(defined as in Remark 1)}$$

is a symplectic vector space and any sublattice $X \subset M$ of index 2 defines a unique odd neighbor of M. So the number of odd neighbors of M is $2^r - 1$. □

3 Theta lattices.

Any Hermitian $\mathcal{O}_K$-lattice (L, h) defines a positive definite $\mathbb{Z}$-lattice (L, F_h) where

$$F_h(x, y) := \begin{cases} \frac{1}{2} \operatorname{Trace}_{K/\mathbb{Q}}(h(x, y)) & \text{if } d \equiv 1, 2 \bmod 4 \ \text{(so if 2 is ramified)}, \\ \operatorname{Trace}_{K/\mathbb{Q}}(h(x, y)) & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

The *integral dual lattice* of L is

$$\begin{aligned} (L, F_h)^{\#} &:= \{x \in \mathbb{Q}L;\ F_h(x, L) \subset \mathbb{Z}\} \\ &= \{x \in KL;\ h(x, L) \subset \frac{1}{\sqrt{-d}} \mathcal{O}_K\} = \frac{1}{\sqrt{-d}} L^*. \end{aligned}$$

Definition 1. The $\mathcal{O}_K$-lattice (L, h) is called a *theta lattice*, if (L, F_h) is an even unimodular lattice, so if $L = (L, F_h)^{\#}$ and $F_h(x, x) \in 2\mathbb{Z}$ for all $x \in L$.

Since even unimodular $\mathbb{Z}$-lattices only exist if the $\mathbb{Q}$-dimension $2r$ is a multiple of 8, we are only interested in the case where r is a multiple of 4. Let

$$\mathcal{L}_r := \{L;\ L\ \mathcal{O}_K\text{-lattice in } \mathcal{V}, (L, F_h) = (L, F_h)^{\#}, F_h(x, x) \in 2\mathbb{Z} \text{ for all } x \in L\}$$

be the set of all theta lattices in $\mathcal{V}$, where $\mathcal{V}$ is the r-dimensional standard Hermitian K vector space from the previous section.

Remark 2. The genus $\mathcal{L}_r$ of theta lattices in $\mathcal{V}$ always contains a free $\mathcal{O}_K$-lattice.

Proof. We give an example of a free theta lattice L of rank 4 just as in [DK]. Then the orthogonal sum of $r/4$ copies of L is a free theta lattice in $\mathcal{L}_r$. We work with coordinates with respect to an orthonormal basis (e_1, e_2, e_3, e_4) of $\mathcal{V}$.

We choose $\alpha, \beta \in \mathbb{Z}$ such that

$$d + 1 + \alpha^2 + \beta^2 \equiv 0 \bmod d_K$$

and set $u = \alpha + \beta + \sqrt{-d}$, $v = \alpha - \beta + \sqrt{-d} \in \mathcal{O}_K$ as well as

$$L = \mathcal{O}_K \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathcal{O}_K \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathcal{O}_K \frac{1}{\sqrt{d_K}} \begin{pmatrix} u \\ v \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}_K \frac{1}{\sqrt{d_K}} \begin{pmatrix} -\overline{v} \\ \overline{u} \\ -1 \\ 1 \end{pmatrix}.$$

Remark 3. a) Let $L \in \mathcal{L}_r$. If $d \equiv 1, 2 \bmod 4$ then $h(x, x) \in 2\mathbb{Z}$ for any $x \in L$, so any sublattice of L is automatically even as a Hermitian lattice.
b) Let p be a prime divisor of d and R be the completion of $\mathcal{O}_K$ at the prime ideal $(p, \sqrt{-d}) =: \wp$. For a theta lattice $L \in \mathcal{L}_r$ we define the non degenerate symplectic $\mathbb{F}_p$-space $W = RL/\sqrt{-d}RL$,

$$f : W \times W \to \mathbb{F}_p = R/\wp, f(x + RL, y + RL) := \sqrt{-d}h(x, y) + \wp.$$

If $M = M^*$ is a Hermitian unimodular sublattice of L, then $RM/\sqrt{-d}RL$ is a maximal totally isotropic subspace of (W, f). On the other hand if $M \in \mathcal{M}_r$

is an (even) unimodular Hermitian lattice and (V, q) the quadratic space as in Remark 1 then the maximal totally singular subspaces of (V, q) are in bijection with the R-lattices RL

$$M \subset L \subset \frac{1}{\sqrt{-d}} M \text{ and } L \in \mathcal{L}$$

The number c_V resp. c_W of maximal totally singular subspaces of (V, q) resp. (W, f) is given in [T]. For $r = 2k$ we find

$$c_V = \prod_{j=0}^{k-1} (p^j + 1), \quad c_W = \prod_{j=1}^{k} (p^j + 1) \text{ and hence } \frac{c_W}{c_V} = \frac{p^k + 1}{2}.$$

Applying the argument from [BN] for all prime divisors of d we hence obtain the following formula for the mass $\mu_r := \mu(\mathcal{L}_r)$ of the theta lattices:

Theorem 2. *Let $r \equiv 0 \bmod 4$. The mass of the genus of theta lattices in $\mathcal{V}$ is given by*

$$\mu_r := \mu(\mathcal{L}_r) = \frac{1}{2^{r-1} \cdot r!} \cdot \prod_{j=1}^{r/2} |B_{2j} \cdot B_{2j-1,\chi_K}|,$$

where B_j (resp. B_{j,χ_K}) are the (generalized) Bernoulli numbers.

4 Classification of theta lattices for class number 1

In this section we assume that the class number of K is equal to 1, i.e.

$$d_K = -3, -4, -7, -8, -11, -19, -43, -67, -163.$$

In this case any $\mathcal{O}_K$-lattice is a free $\mathcal{O}_K$-module

$$M = \mathcal{O}_K b_1 + \cdots + \mathcal{O}_K b_r.$$

If M is even then M is a theta lattice if and only if

$$\det(h(b_\nu, b_\mu)) = (2/\sqrt{d_K})^r$$

(cf. [CR]).

Now we evaluate the formulas from Theorem 2 and start with $r = 4$. We obtain the theta lattice given in Remark 2. Then we apply the neighboring method described in [S] to this lattice at the ideal $(1 + \sqrt{-d}) \subset \mathcal{O}_K$ if d is odd resp. $(\sqrt{-2}) \subset \mathcal{O}_K$ if $d = 2$. The result is

Corollary 1. *If $r = 4$ we obtain the following table of masses and numbers of isometry classes of theta lattices.*

d_K	-3	-4	-7	-8	-11	-19	-43	-67	-163
μ_4	$\frac{1}{155.520}$	$\frac{1}{46.080}$	$\frac{1}{5.040}$	$\frac{1}{3.840}$	$\frac{1}{1.920}$	$\frac{11}{5.760}$	$\frac{83}{5.760}$	$\frac{251}{5.760}$	$\frac{463}{1.152}$
$\sharp$	1	1	1	1	1	2	4	6	16

The underlying $\mathbb{Z}$-lattice is in any case of course the E_8-lattice. A list of representatives of the isometry classes can be found in [H2].

We proceed in the same way for $r = 8$. The result is

Corollary 2. *If $r = 8$ we obtain the following numbers of isometry classes of theta lattices.*

d_K	-3	-4	-7	-8	-11	-19	-43	-67	-163
$\sharp$	1	3	3	6	7	83	> 480.000	$> 2 \cdot 10^7$	$> 3 \cdot 10^{13}$

The estimates are obtained trivially from the value of μ_8 and $\sharp \operatorname{Aut} M \geq 2$.

The analogous procedure yields

Corollary 3. *If $r = 12$ we obtain the following numbers of isometry classes of theta lattices*

d_K	-3	-4	-7	-8	-11	-19	-43	-67	-163
$\sharp$	5	28	464	> 3.882	$> 2 \cdot 10^8$	$> 10^{16}$	$> 3 \cdot 10^{27}$	$> 6 \cdot 10^{33}$	$> 2 \cdot 10^{46}$

The results on $d_K = -3$ are contained in [HKN], on $d_K = -4$ in [I], [S] and [KM]. Representatives are given in [H1].

Remark 4. a) The orders of the automorphism groups of the theta lattices for $r = 4$ are given by the mass if $d_K = -3, -4, -7, -8, -11$, resp. In the other cases they are

$$\begin{aligned} 720 \ \text{ or } \ 1.920 \ \text{ if } d_K &= -19, \\ 120, 240, 720 \ \text{ or } \ 1.920 \ \text{ if } d_K &= -43, \\ 48, 120, 120, 240, 720 \ \text{ or } \ 1.920 \ \text{ if } d_K &= -67. \end{aligned}$$

The group of order 1920 is isomorphic to $2_{-}^{1+4}.A_5$, the automorphism group of the quaternionic structure of the E_8 lattice over the Hurwitz order (see [BN]).
b) If $r = 8$ the lattices $E_8 \oplus E_8$ and D_{16}^+ have got the structure of an $\mathcal{O}_K$-lattice for $d_K = -7, -8, -11, -19$, but D_{16}^+ does not occur for $d_K = -3, -4$.
c) If $r = 12$ and $d_K = -7$ there are exactly 9 isometry classes of $\mathcal{O}_K$-lattices whose $\mathbb{Z}$-lattice is isometric to the Leech lattice. They were used in [N] in order to construct a 72-dimensional extremal even unimodular lattice.
d) Not all 24 Niemeier lattices have got a structure of an $\mathcal{O}_K$-module. The number is given by the following table

d_K	-3	-4	-7
$\sharp$	5	19	21

5 The Hermitian theta series

We consider the Hermitian half-space of degree n

$$\mathcal{H}_n = \left\{ Z \in \mathbb{C}^{n \times n};\ \frac{1}{2i}(Z - \overline{Z}^{tr}) > 0 \right\}.$$

The Hermitian modular group

$$\Gamma_n(\mathcal{O}_K) := \{U \in \mathrm{SL}(2n; \mathcal{O}_K);\ UJ\overline{U}^{tr} = J\}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

acts on $\mathcal{H}_n$ by the usual fractional linear transformation. The space $[\Gamma_n(\mathcal{O}_K), r]$ of Hermitian modular forms of degree n and weight r consists of all holomorphic functions $f : \mathcal{H}_n \to \mathbb{C}$ satisfying

$$f(U\langle Z \rangle) = \det(CZ + D)^r \cdot f(Z) \quad \text{for all} \quad U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(\mathcal{O}_K)$$

with the additional condition of boundedness for $n = 1$ (cf. [B3]).

It was shown in [CR] and [HN] that for a theta lattice M of rank r, the associated Hermitian theta series

$$\begin{aligned}\Theta^{(n)}(Z, M) := & \sum_{(\lambda_1,\ldots,\lambda_n)\in M^n} e^{\pi i \operatorname{trace}((h(\lambda_\nu,\lambda_\mu))\cdot Z)}, \quad Z \in \mathcal{H}_n, \\ & = \sum_{T\geq 0} \sharp(H,T) e^{\pi i \operatorname{trace}(TZ)},\end{aligned}$$

with the Fourier coefficients

$$\sharp(H,T) := \sharp\{(\lambda_1, \ldots, \lambda_n) \in M^n;\ (h(\lambda_\nu, \lambda_\mu))_{\nu,\mu} = T\} \in \mathbb{N}_0$$

belongs to

$$[\Gamma_n(\mathcal{O}_K), r].$$

Just as in the case of Siegel modular forms (cf. [E]) we obtain the analytic version of Siegel's main theorem involving the Siegel Eisenstein series in $[\Gamma_n(\mathcal{O}_K), r]$, because cusp forms can be defined as the kernel of the Siegel ϕ-operator in the case of class number 1 (cf. [K]).

Corollary 4. *Let $M_1, \ldots, M_s$ be representatives of the isometry classes of theta lattices of* rank r *and let the class number of K be equal to* 1. *If $r > 2n$ one has*

$$\frac{1}{\mu_r}\sum_{j=1}^{s}\frac{1}{\sharp \operatorname{Aut}(M_j)}\Theta^{(n)}(Z, M_j) = E_r^{(n)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right):\left(\begin{smallmatrix} * & * \\ 0 & *\end{smallmatrix}\right)\backslash\Gamma_n} \det(CZ + D)^{-r}.$$

Hel Braun [B2, Theorem] proved a more general version, without the assumption that the class number be 1, where the analytic part is not as explicit as in Corollary 4.

Computing a few Fourier coefficients for $n = 2$ yields

Corollary 5. *The Hermitian theta series $\Theta^{(2)}(\,\cdot\,, M_j)$ of the representatives of the isometry classes of theta lattices of* rank 4 *are linearly independent, whenever*

$$d_K = -19, -43, -67.$$

If $d_K = -163$ we conjecture the following filtration (cf. [NV]) of the space spanned by Hermitian theta series of weight 4, i.e. the dimension of spaces of cusp forms spanned by theta series of theta lattices:

n	0	1	2	3
dim	1	0	13	2

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