Self-dual codes and invariant theory [1]

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Abstract. A formal notion of a Typ T of a self-dual linear code over a finite left R-module V is introduced which allows to give explicit generators of a finite complex matrix group, the associated Clifford-Weil group $\mathcal{C}(T) \leq \mathrm{GL}_{|V|}(\mathbb{C})$, such that the complete weight enumerators of self-dual isotropic codes of Type T span the ring of invariants of $\mathcal{C}(T)$. This generalizes Gleason's 1970 theorem to a very wide class of rings and also includes multiple weight enumerators (see Section 2.7), as these are the complete weight enumerators $\mathrm{cwe}_m(C) = \mathrm{cwe}(R^m \otimes C)$ of $R^{m\times m}$-linear self-dual codes $R^m \otimes C \leq (V^m)^N$ of Type T^m with associated Clifford-Weil group $\mathcal{C}_m(T) = \mathcal{C}(T^m)$. The finite Siegel Φ-operator mapping $\mathrm{cwe}_m(C)$ to $\mathrm{cwe}_{m-1}(C)$ hence defines a ring epimorphism $\Phi_m : \mathrm{Inv}(\mathcal{C}_m(T)) \to \mathrm{Inv}(\mathcal{C}_{m-1}(T))$ between invariant rings of complex matrix groups of different degrees. If $R = V$ is a finite field, then the structure of $\mathcal{C}_m(T)$ allows to define a commutative algebra of $\mathcal{C}_m(T)$ double cosets, called a Hecke algebra in analogy to the one in the theory of lattices and modular forms. This algebra consists of self-adjoint linear operators on $\mathrm{Inv}(\mathcal{C}_m(T))$ commuting with Φ_m. The Hecke-eigenspaces yield explicit linear relations among the cwe_m of self-dual codes $C \leq V^N$.

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1. The Type of a code

1.1. Basic notations.

Classically a linear **code** C over a finite field $\mathbb{F}$ is a subspace $C \leq \mathbb{F}^N$. N is called the **length** of the code. $C^{\perp} := \{v \in \mathbb{F}^N \mid v \cdot c = \sum_{i=1}^N v_i c_i = 0 \text{ for all } c \in C\}$ the **dual code**. C is called **self-dual**, if $C = C^{\perp}$. If $\mathbb{F}$ is of even degree over its prime field, then $\mathbb{F}$ has a unique automorphism $\overline{}$ of order 2 and one might replace the Euclidean inner product $v \cdot c$ by the Hermitian inner product $\overline{v} \cdot c = \sum_{i=1}^N \overline{v_i} c_i$ to obtain the **Hermitian dual code**.
Important for the error correcting properties of C is the **distance**

$$d(C) := \min\{d(c, c') \mid c \neq c' \in C\} = \min\{w(c) \mid 0 \neq c \in C\}$$

where

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$$w(c) := |\{1 \leq i \leq N \mid c_i \neq 0\}|$$

is the **Hamming weight** of c and $d(c, c') = w(c - c')$ the **Hamming distance**. The **Hamming weight enumerator** of a code $C \leq \mathbb{F}^N$ is the degree N homogeneous polynomial

$$\mathrm{hwe}_C(x, y) := \sum_{c \in C} x^{N-w(c)} y^{w(c)} \in \mathbb{C}[x, y]_N.$$

1.2. The Gleason-Pierce Theorem

One motivation to introduce the notion of the Type of a code is the following remarkable theorem on the divisibility of the weights of codewords in self-dual codes:
Theorem. (Gleason, Pierce (1967))
If $C = C^{\perp} \leq \mathbb{F}_q^N$ be a linear self-dual code over the field with q elements such that $w(c) \in m\mathbb{Z}$ for all $c \in C$ and some $m > 1$ then one of the following cases occurs:
I) $q = 2$ and $m = 2$ (all self-dual binary codes).
II) $q = 2$ and $m = 4$ (all doubly even self-dual binary codes).
III) $q = 3$ and $m = 3$ (all ternary codes).
IV) $q = 4$ and $m = 2$ (all Hermitian self-dual codes).
o) $q = 4$ and $m = 2$ (certain Euclidean self-dual codes).
d) q arbitrary, $m = 2$ and $\mathrm{hwe}_C(x, y) = (x^2+(q-1)y^2)^{N/2}$. In this case $C =\perp^{N/2} [1, a]$ is the orthogonal sum of self-dual codes of length 2 where either q is even and $a = 1$ or $q \equiv 1 \pmod 4$ and $a^2 = -1$ or C is Hermitian self-dual and $a\overline{a} = -1$.
The self-dual codes in the first four families are called Type I, II, III and IV codes respectively.
The Gleason-Pierce Theorem implies that for codes of Type I, II and IV the Hamming weight enumerator is a polynomial in x^2 and y^2 and for Type III codes, it is a polynomial in x and y^3.
In the following we give famous examples for codes of all four Types, where the code is given by its **generator matrix**, the lines of which form a basis of the code.

1.2.1. Binary codes.

The **repetition code** $i_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ has $\mathrm{hwe}_{i_2}(x, y) = x^2 + y^2$.
The **extended Hamming code**

$$e_8 = \begin{bmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{bmatrix}$$

has $\mathrm{hwe}_{e_8}(x, y) = x^8 + 14x^4y^4 + y^8$ and hence is a Type II code.
The **binary Golay code**

$$g_{24} = \begin{bmatrix} 110101110001100000000000 \\ 101010111000110000000000 \\ 100101011100011000000000 \\ 100010101110001100000000 \\ 100001010111000110000000 \\ 100000101011100011000000 \\ 100000010101110001100000 \\ 100000001010111000110000 \\ 100000000101011100011000 \\ 100000000010101110001100 \\ 100000000001010111000110 \\ 100000000000101011100011 \end{bmatrix}$$

is also of Type II with Hamming weight enumerator

$$\text{hwe}_{g_{24}}(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}$$

1.2.2. Ternary codes.

The **tetracode** $t_4 := \begin{bmatrix} 1\,1\,1\,0 \\ 0\,1\,2\,1 \end{bmatrix} \leq \mathbb{F}_3^4$ is a Type III code with $\text{hwe}_{t_4}(x, y) = x^4 + 8xy^3$.
The **ternary Golay code**

$$g_{12} := \begin{bmatrix} 1\,1\,1\,2\,1\,0\,2\,0\,0\,0\,0\,0 \\ 1\,0\,1\,1\,2\,1\,0\,2\,0\,0\,0\,0 \\ 1\,0\,0\,1\,1\,2\,1\,0\,2\,0\,0\,0 \\ 1\,0\,0\,0\,1\,1\,2\,1\,0\,2\,0\,0 \\ 1\,0\,0\,0\,0\,1\,1\,2\,1\,0\,2\,0 \\ 1\,0\,0\,0\,0\,0\,1\,1\,2\,1\,0\,2 \end{bmatrix} \leq \mathbb{F}_3^{12}$$

$$\text{hwe}_{g_{12}}(x, y) = x^{12} + 264x^6y^6 + 440x^3y^9 + 24y^{12}$$

1.2.3. Hermitian self-dual codes over $\mathbb{F}_4$.

The **repetition code** $i_2 \otimes \mathbb{F}_4 = \begin{bmatrix} 1\,1 \end{bmatrix}$ has $\text{hwe}_{i_2 \otimes \mathbb{F}_4}(x, y) = x^2 + 3y^2$.
The **hexacode** $h_6 = \begin{bmatrix} 1\,0\,0\,1\;\omega\;\omega \\ 0\,1\,0\;\omega\;1\;\omega \\ 0\,0\,1\;\omega\;\omega\;1 \end{bmatrix} \leq \mathbb{F}_4^6$ where $\omega^2 + \omega + 1 = 0$. The hexacode is a Type IV code and has Hamming weight enumerator $\text{hwe}_{h_6}(x, y) = x^6 + 45x^2y^4 + 18y^6$.

1.2.4. MacWilliams' theorem.

Theorem. (Jessie MacWilliams (1962))
Let $C \leq \mathbb{F}_q^N$ be a code. Then

$$\text{hwe}_{C^\perp}(x, y) = \frac{1}{|C|} \text{hwe}_C(x + (q-1)y, x - y).$$

In particular, if $C = C^\perp$, then hwe_C is invariant under the **MacWilliams transformation**

$$h_q : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

1.2.5. Gleason's theorem

Theorem. ([3])
If C is a self-dual code of Type I,II,III or IV then $\mathrm{hwe}_C \in \mathbb{C}[f, g]$ where

Type	f	g
I	$x^2 + y^2$ i_2	$x^2y^2(x^2 - y^2)^2$ Hamming code e_8
II	$x^8 + 14x^4y^4 + y^8$ Hamming code e_8	$x^4y^4(x^4 - y^4)^4$ binary Golay code g_{24}
III	$x^4 + 8xy^3$ tetracode t_4	$y^3(x^3 - y^3)^3$ ternary Golay code g_{12}
IV	$x^2 + 3y^2$ $i_2 \otimes \mathbb{F}_4$	$y^2(x^2 - y^2)^2$ hexacode h_6

Proof.
Let $C \leq \mathbb{F}_q$ be a code of Type $T =$ I, II, III, or IV. Then $C = C^{\perp}$ hence hwe_C is invariant under MacWilliams transformation h_q. Because of the Gleason-Pierce theorem, hwe_C is also invariant under the diagonal transformation $d_m := \mathrm{diag}(1, \zeta_m)) : x \mapsto x, y \mapsto \zeta_m y$ where $\zeta_m = \exp(2\pi i/m)$ denotes a **primitive** m**-th root of unity.** Hence

$$\mathrm{hwe}(C) \in \mathrm{Inv}(\langle h_q, d_m \rangle =: G_T)$$

lies in the invariant ring of the complex matrix group G_T. In all cases G_T is a complex reflection group and the invariant ring of G_T is the polynomial ring $\mathbb{C}[f, g]$ generated by the two polynomials given in the table.

Corollary. The length of a Type II code is divisible by 8.
The length of a Type III code is divisible by 4.
Proof. $\zeta_8 I_2 \in G_{\mathrm{II}}$ and $\zeta_4 I_2 \in G_{\mathrm{III}}$.

In the meantime many more Types of codes, like codes over $\mathbb{Z}/4\mathbb{Z}$ have been discovered and for all these Types a theorem like Gleason's theorem has been proven separately. In [13], Rains and Sloane distinguished nine Types of self-dual codes. Again each version of Gleason's theorem was treated separately. Our recent book [10] introduces a formal notion of a Type (see Section 1.4 below) that allows to prove a general theorem (the main theorem in Section 2.3, [10, Theorem 5.5.7, Corollary 5.7.5]) that may be applied to all known Types of codes and to many more.

1.3. Extremal codes

One main application of Gleason's theorem is to bound the minimum weight of a self-dual code of a given Type and given length. Codes with maximal possible minimum weight are called **extremal**.

Theorem.
Let C be a self-dual code of Type T and length N. Then $d(C) \leq m + m\lfloor\frac{N}{\deg(g)}\rfloor$.
I) If $T = \mathrm{I}$, then $d(C) \leq 2 + 2\lfloor\frac{N}{8}\rfloor$.
II) If $T = \mathrm{II}$, then $d(C) \leq 4 + 4\lfloor\frac{N}{24}\rfloor$.
III) If $T = \mathrm{III}$, then $d(C) \leq 3 + 3\lfloor\frac{N}{12}\rfloor$.
IV) If $T = \mathrm{IV}$, then $d(C) \leq 2 + 2\lfloor\frac{N}{6}\rfloor$.
Remark.
Using the notion of the shadow of a code, the bound for Type I codes has been improved by Eric Rains [14]

$$d(C) \leq 4 + 4\lfloor\frac{N}{24}\rfloor + a$$

where $a = 2$ if $N \pmod{24} = 22$ and $a = 0$ in all other cases.

1.4. A formal definition of a Type

In our recent book [10] we formalize the notion of a Type. The definition that is given here is slightly more restrictive, in general the square of the antiautomorphism ^{J} is conjugation by ϵ which need not be assumed to be central. Also it is not necessary to assume that the ring R and the alphabet V be finite. The presentation given here might be easier accessible and suffices for all common Types of codes.
Let R be a finite ring (with 1), $^J : R \to R$ an involution of R, so

$$(ab)^J = b^J a^J \text{ and } (a^J)^J = a \text{ for all } a, b \in R,$$

and let V be a finite left R-module.
Then $V^* = \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Q}/\mathbb{Z})$ is also a left R-module via

$$(rf)(v) = f(r^J v) \text{ for } v \in V, f \in V^*, r \in R.$$

We assume that $V \cong V^*$ as left R-modules, which means that there is an isomorphism

$$\beta^* : V \to V^*, \beta^*(v) : w \to \beta(v, w)$$

$\beta : V \times V \to \mathbb{Q}/\mathbb{Z}$ is hence biadditive and satisfies

$$\beta(rv, w) = \beta(v, r^J w) \text{ for } r \in R, v, w \in V.$$

A **code** over the alphabet V of length N is an R-submodule $C \leq V^N$.
The **dual code** (with respect to β) is

$$C^\perp := \{x \in V^N \mid \beta^N(x, c) = \sum_{i=1}^{N} \beta(x_i, c_i) = 0 \text{ for all } c \in C\}.$$

C is called **self-dual** (with respect to β) if $C = C^\perp$.
To obtain $(C^\perp)^\perp = C$ (and not having to talk about left and right dual codes) we impose the condition that β is ϵ-Hermitian for some central unit ϵ in R, satisfying $\epsilon^J \epsilon = 1$,

$$\beta(v, w) = \beta(w, \epsilon v) \text{ for } v, w \in V.$$

If $\epsilon = 1$ then β is symmetric, if $\epsilon = -1$ then β is skew-symmetric.

1.4.1. Isotropic codes.

For any **self-orthogonal** code ($C \subseteq C^{\perp}$) it automatically holds that $\beta^N(c, rc) = 0$ for all $c \in C$ and $r \in R$. The mapping $x \mapsto \beta(x, rx)$ is a **quadratic mapping** in
$\mathrm{Quad}_0(V, \mathbb{Q}/\mathbb{Z}) := \{\phi : V \to \mathbb{Q}/\mathbb{Z} \mid \phi(0) = 0$ and
$\phi(x + y + z) - \phi(x + y) - \phi(x + z) - \phi(y + z) + \phi(x) + \phi(y) + \phi(z) = 0\}$.
This is the set of all mappings $\varphi : V \to \mathbb{Q}/\mathbb{Z}$ for which

$$\lambda(\varphi) : V \times V \to \mathbb{Q}/\mathbb{Z}, (v, w) \mapsto \varphi(v + w) - \varphi(v) - \varphi(w)$$

is biadditive. Let $\Phi \subset \mathrm{Quad}_0(V, \mathbb{Q}/\mathbb{Z})$ and let $C \leq V^N$ be a code. Then C is called **isotropic** (with respect to Φ) if

$$\phi^N(c) := \sum_{i=1}^{N} \phi(c_i) = 0 \text{ for all } c \in C \text{ and } \phi \in \Phi.$$

1.4.2. The definition of a Type.

The quadruple (R, V, β, Φ) is called a **Type** if
a) $\Phi \leq \mathrm{Quad}_0(V, \mathbb{Q}/\mathbb{Z})$ is a subgroup and for all $r \in R$, $\phi \in \Phi$ the mapping $\phi[r] : x \mapsto \phi(rx)$ is again in Φ. Then Φ is an **R-qmodule.**
b) For all $\phi \in \Phi$ there is some $r_\phi \in R$ such that $\lambda(\phi)(v, w) = \beta(v, r_\phi w)$ for all $v, w\, in V$.
c) For all $r \in R$ the mapping $\phi_r : V \to \mathbb{Q}/\mathbb{Z}, v \mapsto \beta(v, rv)$ lies in Φ.

1.4.3. Examples of Types.

Type I codes (2_{I}).
$R = \mathbb{F}_2 = V,\ \beta(x, y) = \frac{1}{2}xy,\ \Phi = \{\varphi : x \mapsto \frac{1}{2}x^2 = \beta(x, x), 0\}.$
Type II codes (2_{II}).
$R = \mathbb{F}_2 = V,\ \beta(x, y) = \frac{1}{2}xy,\ \Phi = \{\phi : x \mapsto \frac{1}{4}x^2, 2\phi = \varphi, 3\phi, 0\}.$
Type III codes (3).
$R = \mathbb{F}_3 = V,\ \beta(x, y) = \frac{1}{3}xy,\ \Phi = \{\varphi : x \mapsto \frac{1}{3}x^2 = \beta(x, x), 2\varphi, 0\}.$
Type IV codes (4^H).
$R = \mathbb{F}_4 = V,\ \beta(x, y) = \frac{1}{2}\,\mathrm{trace}(x\overline{y}),\ \Phi = \{\varphi : x \mapsto \frac{1}{2}x\overline{x}, 0\}$ where $\overline{x} = x^2$.
Additive codes over $\mathbb{F}_4$ (4^{H+}).
$R = \mathbb{F}_2,\ V = \mathbb{F}_4,\ \beta(x, y) = \frac{1}{2}\,\mathrm{trace}(x\overline{y}),\ \Phi = \{\varphi : x \mapsto \frac{1}{2}x\overline{x}, 0\}$
Generalized doubly-even codes over $\mathbb{F}_q$, $q = 2^f$ (q^E_{II}).
$R = \mathbb{F}_q = V,\ \beta(x, y) = \frac{1}{2}\,\mathrm{trace}(xy),\ \Phi = \{x \mapsto \frac{1}{4}\,\mathrm{trace}(ax^2) : a \in \mathbb{F}_q\}.$
Euclidean self-dual codes over $\mathbb{F}_q$, $q = p^f$ odd, (q^E).
$R = \mathbb{F}_q = V,\ \beta(x, y) = \frac{1}{p}\,\mathrm{trace}(xy),\ \Phi = \{\varphi_a : x \mapsto \frac{1}{p}\,\mathrm{trace}(ax^2) : a \in \mathbb{F}_q\}.$
Euclidean self-dual codes over $\mathbb{F}_q$ containing the all ones vector, $q = p^f$ odd, (q^E_1).
$R = \mathbb{F}_q = V,\ \beta(x, y) = \frac{1}{p}\,\mathrm{trace}(xy),$
$\Phi = \{\varphi_{a,b} : x \mapsto \frac{1}{p}(\mathrm{trace}(ax^2 + bx)) : a, b \in \mathbb{F}_q\}.$

Self-dual codes over $\mathbb{Z}/m\mathbb{Z}$ ($m^{\mathbb{Z}}$).
$R = \mathbb{Z}/m\mathbb{Z} = V,\ \beta(x, y) = \frac{1}{m}xy,\ \Phi = \{x \mapsto \frac{1}{m}(ax^2) : a \in \mathbb{Z}/m\mathbb{Z}\}$.
Even self-dual codes over $\mathbb{Z}/m\mathbb{Z}$ ($m^{\mathbb{Z}}_{\mathrm{II}}$) ($m$ even).
$R = \mathbb{Z}/m\mathbb{Z} = V,\ \beta(x, y) = \frac{1}{m}xy,\ \Phi = \{x \mapsto \frac{1}{2m}(ax^2) : a \in \mathbb{Z}/m\mathbb{Z}\}$.

1.5. Equivalence of codes.

Let $T := (R, V, \beta, \Phi)$ be a Type. Then $\mathrm{Aut}(T) :=$

$$\{\varphi \in \mathrm{End}_R(V) \mid \beta(\varphi(v), \varphi(w)) = \beta(v, w), \phi(\varphi(v)) = \phi(v) \text{ for all } v, w \in V, \phi \in \Phi\}$$

is the **automorphism group** of the Type T.
The group

$$\mathrm{Aut}_N(T) := \mathrm{Aut}(T) \wr S_N = \{(\varphi_1, \ldots, \varphi_N)\pi \mid \pi \in S_N, \varphi_i \in \mathrm{Aut}(T)\}$$

acts on the set $M_N(T)$ of codes of Type T and length N.
Two codes $C, D \leq V^N$ of Type T are called **T-equivalent**, if there is $\sigma \in \mathrm{Aut}_N(T)$ such that $\sigma(C) = D$.
The **automorphism group** of C is

$$\mathrm{Aut}_T(C) := \{\sigma \in \mathrm{Aut}(T) \wr S_N \mid \sigma(C) = C\}$$

For example for Hermitian codes over $\mathbb{F}_4$ the automorphism group is $\mathrm{Aut}(4^H) = \mathbb{F}_4^* = \{1, \omega, \omega^2\}$ whereas for Euclidean codes over $\mathbb{F}_4$ the automorphism group is $\mathrm{Aut}(4^E) = \{1\}$. So the $\mathbb{F}_4$-codes with generator matrix $[1, 1]$ respectively $[1, \omega]$ are equivalent as Hermitian codes over $\mathbb{F}_4$ but not as Euclidean codes.
So equivalence is not a property of the codes alone but a property of the Type.

1.6. A method to classify all codes of a given Type.

This method is based on an algorithm originally formulated by Martin Kneser [7] to enumerate unimodular lattices (up to equivalence).
For a Type T let $M_N(T) := \{C \leq V^N \mid C \text{ of Type } T\}$.
For $C \in M_N(T)$, the equivalence class

$$[C] := \{D \leq V^N \text{ of Type } T \mid D = \pi(C) \text{ for some } \pi \in \mathrm{Aut}_N(T)\}.$$

Then $M_N(T) = \bigcup_{j=1}^{h} [C_j]$ is the disjoint union of equivalence classes.
Now Kneser's method is roughly as follows: We start with some code $C \in M_N(T)$ (usually an orthogonally decomposable code) and then successively calculate the **neighbours** D of C, which are these codes $D \in M_N(T)$ such that $C/C \cap D$ is a simple R-module (if R is a field, this means that $\dim(C \cap D) = \dim(C) - 1$). Test whether D is equivalent to a known code and continue with all new D.

1.6.1. Number of equivalence classes of codes of Type T

N	I	II	III	IV
2	1(1)	–	–	1(1)
4	1(1)	–	1(1)	1(1)
6	1(1)	–	–	2(1)
8	2(1)	1(1)	1(1)	3(1)
10	2	–	–	5(2)
12	3(1)	–	3(1)	10
14	4(1)	–	–	21(1)
16	7	2(2)	7(1)	55(4)
18	9	–	–	244(1)
20	16	–	24(6)	(2)
22	25(1)	–	–	
24	55	9(1)	338(2)	
26	103	–	–	
28	261	–	(6931)	
30	731	–	–	
32	3295	85(5)		
34	24147	–	–	

The number of extremal codes is given in brackets and empty spaces left to be filled out later by the reader, since this classification is a still ongoing process (see also [6]). [5] and [4] use the classification of unimodular lattices to obtain the ternary codes of length 24 and the extremal ones of length 28. The binary codes of length 34 are obtained in [1]. The other results were obtained by the Kneser-neighbouring method with [2].

1.7. The mass formula

The mass formula is a helpful tool to verify the completeness of a list of self-dual codes. We put $m_N(T) := |M_N(T)|$ and $a_N(T) := |\operatorname{Aut}_N(T)|$.
Theorem. (mass formula)

$$\sum_{j=1}^{h} \frac{1}{|\operatorname{Aut}(C_j)|} = \frac{m_N(T)}{a_N(T)}.$$

Proof. $\operatorname{Aut}_N(T)$ acts on $M_N(T)$ and the equivalence classes are precisely the $\operatorname{Aut}_N(T)$-orbits. So

$$|[C_j]| = \frac{|\operatorname{Aut}_N(T)|}{|\operatorname{Aut}(C_j)|}$$

is the index of the stabilizer and

$$|M_N(T)| = \sum_{j=1}^{h} |[C_j]| = \sum_{j=1}^{h} \frac{|\operatorname{Aut}_N(T)|}{|\operatorname{Aut}(C_j)|}.$$

Type	$m_N(T)$	$a_N(T)$
I	$\prod_{i=1}^{N/2-1}(2^i+1)$	$N!$
II	$2\prod_{i=1}^{N/2-2}(2^i+1)$	$N!$
III	$2\prod_{i=1}^{N/2-1}(3^i+1)$	$2^N N!$
IV	$\prod_{i=0}^{N/2-1}(2^{2i+1}+1)$	$3^N N!$

2. The Clifford-Weil group

2.1. Complete weight enumerators

For $c=(c_1,\ldots,c_N)\in V^N$ and $v\in V$ put

$$a_v(c) := |\{i\in\{1,\ldots,N\} \mid c_i = v\}|.$$

Then

$$\mathrm{cwe}_C := \sum_{c\in C}\prod_{v\in V} x_v^{a_v(c)} \in \mathbb{C}[x_v : v\in V]$$

is called the **complete weight enumerator of the code** C.
The tetracode t_4 has complete weight enumerator $\mathrm{cwe}_{t_4}(x_0,x_1,x_2)=x_0^4+x_0x_1^3+x_0x_2^3+3x_0x_1^2x_2+3x_0x_1x_2^2$ and hence
$\mathrm{hwe}_{t_4}(x,y)=\mathrm{cwe}_{t_4}(x,y,y)=x^4+8xy^3$.

2.2. The Clifford-Weil group

Let $T:=(R,V,\beta,\Phi)$ be a Type. Then the **associated Clifford-Weil group** $\mathcal{C}(T)$ is a subgroup of $\mathrm{GL}_{|V|}(\mathbb{C})$

$$\mathcal{C}(T)=\langle m_r, d_\phi, h_{e,u_e,v_e} \mid r\in R^*, \phi\in\Phi, e=u_ev_e\in R \text{ symmetric idempotent } \rangle$$

Let $(e_v|v\in V)$ denote a basis of $\mathbb{C}^{|V|}$. Then

$$m_r : e_v \mapsto e_{rv},\ \ d_\phi : e_v \mapsto \exp(2\pi i\phi(v))e_v$$

$$h_{e,u_e,v_e} : e_v \mapsto |eV|^{-1/2}\sum_{w\in eV}\exp(2\pi i\beta(w,v_ev))e_{w+(1-e)v}$$

Using the notation of Section 1.4.3 one computes the following Clifford-Weil groups:
$\mathcal{C}(\mathrm{I})=\langle d_\varphi=\mathrm{diag}(1,-1), h_{1,1,1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=h_2\rangle=G_{\mathrm{I}}$
isomorphic to the dihedral group of order 16.
$\mathcal{C}(\mathrm{II})=\langle d_\phi=\mathrm{diag}(1,i), h_{1,1,1}\rangle=G_{\mathrm{II}}$ a complex reflection group of order 192.

$\mathcal{C}(\mathrm{III}) = \langle m_2 = \begin{pmatrix} 100 \\ 001 \\ 010 \end{pmatrix}, d_\varphi = \mathrm{diag}(1, \zeta_3, \zeta_3), h_{1,1,1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix} \rangle$

isomorphic to $Z_4 \times \mathrm{SL}_2(3)$ of order 96.

$\mathcal{C}(\mathrm{IV}) = \langle m_\omega = \begin{pmatrix} 1000 \\ 0001 \\ 0100 \\ 0010 \end{pmatrix}, d_\varphi = \mathrm{diag}(1, -1, -1, -1), h_{1,1,1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \rangle$

isomorphic to $D_{12} \times Z_3$ of order 36.

2.3. A general Gleason theorem.

Theorem.
Let $C \leq V^N$ be a self-dual isotropic code of Type T. Then cwe_C is invariant under $\mathcal{C}(T)$.
Proof.
Invariance under m_r ($r \in R^*$) because C is a code.
Invariance under d_ϕ ($\phi \in \Phi$) because C is isotropic.
Invariance under h_{e,u_e,v_e} because C is self-dual.
So it is obvious that the weight enumerators lie in the **ring of invariant polynomials** $\mathrm{Inv}(\mathcal{C}(T))$ of the associated Clifford-Weil group. In fact in many cases this invariant ring is spanned as a $\mathbb{C}$-vector-space by the complete weight enumerators. We conjecture that this holds for arbitrary finite rings see [10, Conjecture 5.7.2]. Note that it is in general not possible to obtain a similar theorem for the Hamming weight enumerators (see Section 2.4).
The main theorem.(N,, Rains, Sloane (1999-2006) [10])
If R is a direct product of matrix rings over chain rings, then

$$\mathrm{Inv}(\mathcal{C}(T)) = \langle \mathrm{cwe}_C \mid C \text{ of Type } T \rangle.$$

The proof of this theorem is quite involved and led us to write the book [10].

2.4. Symmetrizations

Let (R, J) be a ring with involution. Then the **central unitary group** is

$$\mathrm{ZU}(R, J) := \{g \in Z(R) \mid gg^J = g^J g = 1\}.$$

Theorem. Let $T = (R, V, \beta, \Phi)$ be a Type and

$$U := \{u \in \mathrm{ZU}(R, J) \mid \phi(uv) = \phi(v) \text{ for all } \phi \in \Phi, v \in V\}.$$

Then $m(U) := \{m_u \mid u \in U\}$ is in the center of $\mathcal{C}(T)$.
Let $X_0, \ldots, X_n$ be the U-orbits on V. The **U-symmetrized Clifford-Weil group** is $\mathcal{C}^{(U)}(T) = \{g^{(U)} \mid g \in \mathcal{C}(T)\} \leq \mathrm{GL}_{n+1}(\mathbb{C})$.
If $g(\frac{1}{|X_i|}\sum_{v\in X_i} e_v) = \sum_{j=0}^n a_{ij}(\frac{1}{|X_j|}\sum_{w\in X_j} e_w)$ then $g^{(U)}(x_i) = \sum_{j=0}^n a_{ij}x_j$.
Remark. The invariant ring of $\mathcal{C}^{(U)}(T)$ consists of the U-symmetrized invariants of $\mathcal{C}(T)$. In particular, if the invariant ring of $\mathcal{C}(T)$ is spanned by the complete weight enu-

merators of self-dual codes in T, then the invariant ring of $\mathcal{C}^{(U)}(T)$ is spanned by the U-symmetrized weight-enumerators of self-dual codes in T.
Let $X_0, \dots, X_n$ denote the orbits on U on V and for $c = (c_1, \dots, c_N) \in C$ and $0 \le j \le n$ define

$$a_j(c) = |\{1 \le i \le N \mid c_i \in X_j\}$$

Then the **U-symmetrized weight-enumerator** of C is

$$\mathrm{cwe}_C^{(U)} = \sum_{c \in C} \prod_{j=0}^{n} x_j^{a_j(c)} \in \mathbb{C}[x_0, \dots, x_n].$$

2.5. Gleason's Theorem revisited.

For Type I,II,III,IV the central unitary group $\mathrm{ZU}(R, J)$ is transitive on $V \setminus \{0\}$, so there are only two orbits:

$$x \leftrightarrow \{0\}, \; y \leftrightarrow V \setminus \{0\}$$

and the symmetrized weight enumerators are the Hamming weight enumerators.
The symmetrized Clifford-Weil groups are precisely Gleason's groups:
$G_{\mathrm{I}} = \mathcal{C}(\mathrm{I})$, $G_{\mathrm{II}} = \mathcal{C}(\mathrm{II})$, $G_{\mathrm{III}} = \mathcal{C}^{(U)}(\mathrm{III})$, and $G_{\mathrm{IV}} = \mathcal{C}^{(U)}(\mathrm{IV})$.

2.6. Hermitian codes over $\mathbb{F}_9$. [10, Section 5.8]

$(9^H) : R = V = \mathbb{F}_9$, $\beta(x, y) = \frac{1}{3}\,\mathrm{trace}(x\overline{y})$, $\Phi = \{\varphi : x \mapsto \frac{1}{3}x\overline{x}, 2\varphi, 0\}$.
Let α be a primitive element of $\mathbb{F}_9$ and put $\zeta = \zeta_3 \in \mathbb{C}$. Then with respect to the $\mathbb{C}$-basis $(0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7)$ of $\mathbb{C}[V]$, the associated Clifford-Weil group $\mathcal{C}(9^H)$ is generated by
$d_\varphi := \mathrm{diag}(1, \zeta, \zeta^2, \zeta, \zeta^2, \zeta, \zeta^2, \zeta, \zeta^2)$,

$$m_\alpha := \begin{pmatrix} 100000000 \\ 000000001 \\ 010000000 \\ 001000000 \\ 000100000 \\ 000010000 \\ 000001000 \\ 000000100 \\ 000000010 \end{pmatrix}, \; h := \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta & 1 & \zeta & \zeta & \zeta^2 & 1 & \zeta^2 \\ 1 & \zeta & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta^2 & \zeta & 1 \\ 1 & 1 & \zeta^2 & \zeta^2 & \zeta & 1 & \zeta & \zeta & \zeta^2 \\ 1 & \zeta & 1 & \zeta & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta^2 \\ 1 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta^2 & \zeta & 1 & \zeta \\ 1 & \zeta^2 & \zeta^2 & \zeta & 1 & \zeta & \zeta & \zeta^2 & 1 \\ 1 & 1 & \zeta & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta^2 & \zeta \\ 1 & \zeta^2 & 1 & \zeta^2 & \zeta^2 & \zeta & 1 & \zeta & \zeta \end{pmatrix}$$

$\mathcal{C}(9^H)$ is a group of order 192 with Molien series

$$\frac{\theta(t)}{(1-t^2)^2(1-t^4)^2(1-t^6)^3(1-t^8)(1-t^{12})}$$

where

$$\begin{aligned}\theta(t) := 1 &+ 3t^4 + 24t^6 + 74t^8 + 156t^{10} + 321t^{12} + 525t^{14} + 705t^{16} \\ &+ 905t^{18} + 989t^{20} + 931t^{22} + 837t^{24} + 640t^{26} + 406t^{28} \\ &+ 243t^{30} + 111t^{32} + 31t^{34} + 9t^{36} + t^{38} \,,\end{aligned}$$

So the invariant ring of $\mathcal{C}(9^H)$ has at least

$$\theta(1) + 9 = 6912 + 9 = 6921$$

generators and the maximal degree (=length of the code) is 38.
We cannot symmetrize directly to obtain Hamming weight enumerators but we can only symmetrize by $(\mathbb{F}_9^*)^2 = \mathrm{ZU}(9^H)$. This group has 3 orbits on $V = \mathbb{F}_9$:

$$\{0\} = X_0,\ \{1, \alpha^2, \alpha^4, \alpha^6\} =: X_1,\ \{\alpha, \alpha^3, \alpha^5, \alpha^7\} =: X_2$$

and the symmetrized Clifford-Weil group is

$$\mathcal{C}^{(U)}(9^H) = \langle d_\varphi^{(U)} = \mathrm{diag}(1, \zeta, \zeta^2),\ m_\alpha^{(U)} = \begin{pmatrix} 100 \\ 001 \\ 010 \end{pmatrix},\ h^{(U)} = \frac{1}{3}\begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & \text{-}2 \\ 1 & \text{-}2 & 1 \end{pmatrix} \rangle$$

of order $\frac{192}{4} = 48$. The invariant ring $\mathrm{Inv}(\mathcal{C}^{(U)}(9^H))$ is a polynomial ring spanned by the U-symmetrized weight enumerators

$$q_2 = x_0^2 + 8x_1x_2,\ \ q_4 = x_0^4 + 16(x_0x_1^3 + x_0x_2^3 + 3x_1^2x_2^2)$$
$$q_6 = x_0^6 + 8(x_0^3x_1^3 + x_0^3x_2^3 + 2x_1^6 + 2x_2^6) + 72(x_0^2x_1^2x_2^2 + 2x_0x_1^4x_2 + 2x_0x_1x_2^4) + 320x_1^3x_2^3$$

of the three codes with generator matrices

$$\begin{bmatrix} 1 & \alpha \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \alpha & 2\alpha & 0 & 1 & 2 \end{bmatrix}.$$

Their Hamming weight enumerators are

$$\begin{aligned} r_2 &= q_2(x, y, y) := x^2 + 8y^2 \,, \\ r_4 &= q_4(x, y, y) := x^4 + 32xy^3 + 48y^4 \,, \\ r_6 &= q_6(x, y, y) := x^6 + 16x^3y^3 + 72x^2y^4 + 288xy^5 + 352y^6 \,. \end{aligned}$$

The polynomials r_2, r_4 and r_6 generate the ring $\mathrm{Ham}(9^H)$ spanned by the Hamming weight enumerators of the codes of Type 9^H.

$\mathrm{Ham}(9^H) = \mathbb{C}[r_2, r_4] \oplus r_6\mathbb{C}[r_2, r_4]$ with the syzygy

$$r_6^2 = \frac{3}{4}r_2^4r_4 - \frac{3}{2}r_2^2r_4^2 - \frac{1}{4}r_4^3 - r_2^3r_6 + 3r_2r_4r_6 \,.$$

Note that $\mathrm{Ham}(9^H)$ is **not** the invariant ring of a finite group.

2.7. Higher genus complete weight enumerators.

Let $c^{(i)} := (c_1^{(i)}, \dots, c_N^{(i)}) \in V^N$, $i = 1, \dots, m$, be m not necessarily distinct codewords. For $v := (v_1, \dots, v_m) \in V^m$, let

$$a_v(c^{(1)}, \dots, c^{(m)}) := |\{j \in \{1, \dots, N\} \mid c_j^{(i)} = v_i \text{ for all } i \in \{1, \dots, m\}\}| .$$

The **genus-m complete weight enumerator** of C is

$$\mathrm{cwe}_m(C) := \sum_{(c^{(1)},\dots,c^{(m)}) \in C^m} \prod_{v \in V^m} x_v^{a_v(c^{(1)},\dots,c^{(m)})} \in \mathbb{C}[x_v : v \in V^m] .$$

$$\begin{array}{cccccc} c_1^{(1)} & c_2^{(1)} & \dots & c_j^{(1)} & \dots & c_N^{(1)} \\ c_1^{(2)} & c_2^{(2)} & \dots & c_j^{(2)} & \dots & c_N^{(2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ c_1^{(m)} & c_2^{(m)} & \dots & c_j^{(m)} & \dots & c_N^{(m)} \\ & & & \uparrow & & \\ & & & v \in V^m & & \end{array}$$

$$\mathrm{cwe}_2(i_2) = x_{00}^2 + x_{11}^2 + x_{01}^2 + x_{10}^2.$$

$$\mathrm{cwe}_2(e_8) = x_{00}^8 + x_{01}^8 + x_{10}^8 + x_{11}^8 + 168 x_{00}^2 x_{01}^2 x_{10}^2 x_{11}^2 +$$
$$14(x_{00}^4 x_{01}^4 + x_{00}^4 x_{10}^4 + x_{00}^4 x_{11}^4 + x_{01}^4 x_{10}^4 + x_{01}^4 x_{11}^4 + x_{10}^4 x_{11}^4)$$

2.8. The genus-m Clifford-Weil group.

For $C \le V^N$ and $m \in \mathbb{N}$ let

$$C(m) := R^{m \times 1} \otimes C = \{(c^{(1)}, \dots, c^{(m)})^{\mathrm{Tr}} \mid c^{(1)}, \dots, c^{(m)} \in C\} \le (V^m)^N$$

Then

$$\mathrm{cwe}_m(C) = \mathrm{cwe}(C(m)).$$

Moreover if C is a self-dual isotropic code of Type $T = (R, V, \beta, \Phi)$, then $C(m)$ is a self-dual isotropic code of Type

$$T^m = (R^{m \times m}, V^m, \beta^{(m)}, \Phi^{(m)})$$

and hence $\mathrm{cwe}_m(C)$ is invariant under $\mathcal{C}_m(T) := \mathcal{C}(T^m)$, **the genus-m Clifford-Weil group.**

This is the main reason why we also allow non commutative rings R in our main theorem. Even for codes over a finite field $\mathbb{F}$, the underlying ring $R = \mathbb{F}^{m \times m}$ for the genus-m Clifford-Weil group is not commutative. Our main theorem from Section 2.3 also applies to this situation and in particular to higher genus weight enumerators of codes.

2.8.1. $\mathcal{C}_2(\mathrm{I})$

$$R = \mathbb{F}_2^{2\times 2},\ R^* = \mathrm{GL}_2(\mathbb{F}_2) = \langle a := \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix},\ b := \begin{pmatrix} 0\,1 \\ 1\,1 \end{pmatrix} \rangle$$

$$V = \mathbb{F}_2^2 = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}, \text{ symmetric idempotent } e = \mathrm{diag}(1,0)$$

$$\mathcal{C}_2(\mathrm{I}) = \langle m_a = \begin{pmatrix} 1000 \\ 0010 \\ 0100 \\ 0001 \end{pmatrix},\ m_b = \begin{pmatrix} 1000 \\ 0001 \\ 0100 \\ 0010 \end{pmatrix},\ h_{e,e,e} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\,1\,0\,0 \\ 1\text{-}1\,0\,0 \\ 0\,0\,1\,1 \\ 0\,0\,1\text{-}1 \end{pmatrix},\ d_{\varphi e} = \mathrm{diag}(1,-1,1,-1) \rangle$$

of order 2304 and Molien series $\frac{1+t^{18}}{(1-t^2)(1-t^8)(1-t^{12})(1-t^{24})}$. As a minimal set of generators for the invariant ring of $\mathcal{C}(\mathrm{I})$ we may take the genus-2 weight enumerators of the codes $i_2, e_8, d_{12}^+, g_{24}$ and $(d_{10}e_7f_1)^+$.

2.8.2. $\mathcal{C}_2(\mathrm{II})$

$\mathcal{C}_2(\mathrm{II}) = \langle m_a, m_b, h_{e,e,e}, d_{\phi e} = \mathrm{diag}(1,i,1,i) \rangle$ has order 92160 and Molien series $\frac{1+t^{32}}{(1-t^8)(1-t^{24})^2(1-t^{40})}$ where the generators correspond to the genus 2 complete weight enumerators of the codes $e_8, g_{24}, d_{24}^+, d_{40}^+$, and d_{32}^+. $\mathcal{C}_2(\mathrm{II})$ has a reflection subgroup of index 2, No. 31 on the Shephard-Todd list.

2.8.3. Higher genus Clifford-Weil groups for the classical Types of codes over finite fields.

The higher genus Clifford-Weil groups of the classical Types T of codes over fields have the structure

$$\mathcal{C}_m(T) = S.(\ker(\lambda) \times \ker(\lambda)).\mathcal{G}_m(T)$$

where $S = \mathcal{C}_m(T) \cap \mathbb{C}^*\,\mathrm{id}$ is the scalar subgroup (of order $|S| = \min\{N \mid$ there is a code of Type T and length $N\}$), $\ker(\lambda) \times \ker(\lambda)$ is a linear $\mathrm{GL}_{2m}(R)$-module and $\mathcal{G}_m(T) \le \mathrm{GL}_{2m}(R)$ is one of the following classical groups:

R	J	ϵ	$\mathcal{G}_m(T)$
$\mathbb{F}_q \oplus F_q$	$(r,s)^J = (s,r)$	1	$\mathrm{GL}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_{q^2}$	$r^J = r^q$	1	$U_{2m}(\mathbb{F}_{q^2})$
$\mathbb{F}_q,\ q$ odd	$r^J = r$	1	$\mathrm{Sp}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q,\ q$ odd	$r^J = r$	-1	$O_{2m}^+(\mathbb{F}_q)$
$\mathbb{F}_q,\ q$ even	doubly even		$\mathrm{Sp}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q,\ q$ even	singly even		$O_{2m}^+(\mathbb{F}_q)$

For Type I, II, III, IV one gets:
$\mathcal{C}_m(\mathrm{I}) = 2_+^{1+2m}.O_{2m}^+(\mathbb{F}_2)$, $\mathcal{C}_m(\mathrm{II}) = Z_8 Y 2^{1+2m}.\mathrm{Sp}_{2m}(\mathbb{F}_2)$, $\mathcal{C}_m(\mathrm{III}) = Z_4.\mathrm{Sp}_{2m}(\mathbb{F}_3)$, and $\mathcal{C}_m(\mathrm{IV}) = Z_2.U_{2m}(\mathbb{F}_4)$.

3. Hecke operators for codes.

This Section introduces Hecke operators for codes and therewith answers a question raised in 1977 by Michel Broué. A general reference for this section is [11].

3.1. Motivation.

Determine linear relations between $\mathrm{cwe}_m(C)$ for $C \in M_N(T) = \{C \le V^N \mid C \text{ of Type } T\}$.
$M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$ and these two codes have the same genus 1 and 2 weight enumerator, but $\mathrm{cwe}_3(e_8 \perp e_8)$ and $\mathrm{cwe}_3(d_{16}^+)$ are linearly independent.
$h(M_{24}(\mathrm{II})) = 9$ and only the genus 6 weight enumerators are linearly independent, there is one relation for the genus 5 weight enumerators.
$h(M_{32}(\mathrm{II})) = 85$ and here the genus 10 weight enumerators are linearly independent, whereas there is a unique relation for the genus 9 weight enumerators.
There are three different approaches:
1) Determine all the codes and their weight enumerators.
If $\dim(C) = n = N/2$ there are $\prod_{i=0}^{d-1}(2^n - 2^i)/(2^d - 2^i)$ subspaces of dimension d in C.
Problem: $N = 32$, $d = 10$ yields more than 10^{18} subspaces, so it is impossible to calculate the genus 10 weight enumerator of a code of length 32.
2) Use Molien's theorem:
$\mathrm{Inv}_N(\mathcal{C}_m(\mathrm{II})) = \langle \mathrm{cwe}_m(C) \mid C \in M_N(\mathrm{II})\rangle$ and if $a_N := \dim(\mathrm{Inv}_N(\mathcal{C}_m(\mathrm{II})))$ then

$$\sum_{N=0}^{\infty} a_N t^N = \frac{1}{|\mathcal{C}_m(\mathrm{II})|} \sum_{g \in \mathcal{C}_m(\mathrm{II})} (\det(1-g))^{-1}$$

Problem: $\mathcal{C}_{10}(\mathrm{II}) \le \mathrm{GL}_{1024}(\mathbb{C})$ has order $> 10^{69}$. Even with the use normal subgroups of $\mathcal{C}_m(\mathrm{II})$, we can only calculate the Molien series up to $m = 4$.
3) Use Hecke operators. In the following I will comment on this approach.

3.2. The Kneser-Hecke operator.

Fix a Type $T = (\mathbb{F}_q, \mathbb{F}_q, \beta, \Phi)$ of self-dual codes over a finite **field** with q elements.

$$M_N(T) = \{C \le \mathbb{F}_q^N \mid C \text{ of Type } T\} = [C_1] \,\dot\cup \ldots \dot\cup\, [C_h]$$

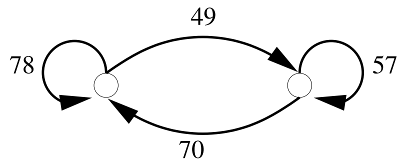
where $[C]$ denotes the **permutation equivalence** class of the code C. Clearly permutation equivalent codes have the same complete weight enumerator and - on the other hand - if $\mathrm{cwe}_n(D) = \mathrm{cwe}_n(C)$ for $n := \frac{N}{2} = \dim(C)$ then C and D are permutation equivalent.
$C, D \in M_N(T)$ are called **neighbours**, if $\dim(C) - \dim(C \cap D) = 1$, $C \sim D$.

$$\mathcal{V} = \mathbb{C}[C_1] \oplus \ldots \oplus \mathbb{C}[C_h] \cong \mathbb{C}^h$$

$$K_N(T) \in \mathrm{End}(\mathcal{V}),\ \ K_N(T) : [C] \mapsto \sum_{D \in M_N(T), D \sim C} [D].$$

Kneser-Hecke operator. (adjacency matrix of neighbouring graph)

Example. $M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$



$$K_{16}(\mathrm{II}) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$$

3.3. The Kneser-Hecke operator is self-adjoint.

$\mathcal{V}$ has a Hermitian positive definite inner product defined by

$$\langle [C_i], [C_j] \rangle := |\,\mathrm{Aut}(C_i)|\delta_{ij}.$$

Theorem. (N. 2006)

The Kneser-Hecke operator K is a self-adjoint linear operator.

$$\langle v, Kw \rangle = \langle Kv, w \rangle \text{ for all } v, w \in \mathcal{V}.$$

Example. $\frac{7}{10} = \frac{|\,\mathrm{Aut}(e_8 \perp e_8)|}{|\,\mathrm{Aut}(d_{16}^+)|}$ hence $\mathrm{diag}(7, 10) K_{16}(\mathrm{II})^{\mathrm{Tr}} = K_{16}(\mathrm{II})\, \mathrm{diag}(7, 10)$.

3.4. The eigenspaces of the Kneser-Hecke operator.

$$\mathrm{cwe}_m : \mathcal{V} \to \mathbb{C}[X], \sum_{i=1}^{h} a_i [C_i] \mapsto \sum_{i=1}^{h} a_i\, \mathrm{cwe}_m(C_i)$$

is a linear mapping with kernel

$$\mathcal{V}_m := \ker(\mathrm{cwe}_m).$$

Then

$$\mathcal{V} =: \mathcal{V}_{-1} \geq \mathcal{V}_0 \geq \mathcal{V}_1 \geq \ldots \geq \mathcal{V}_n = \{0\}.$$

is a filtration of $\mathcal{V}$ yielding the orthogonal decomposition

$$\mathcal{V} = \bigoplus_{m=0}^{n} \mathcal{Y}_m \text{ where } \mathcal{Y}_m = \mathcal{V}_{m-1} \cap \mathcal{V}_m^{\perp}.$$

$$\mathcal{V}_0 = \{\sum_{i=1}^{h} a_i [C_i] \mid \sum a_i = 0\} \text{ and } \mathcal{V}_0^{\perp} = \mathcal{Y}_0 = \langle \sum_{i=1}^{h} \frac{1}{|\operatorname{Aut}(C_i)|} [C_i] \rangle.$$

Theorem. (N. 2006)
The space $\mathcal{Y}_m = \mathcal{Y}_m(N)$ is the $K_N(T)$-eigenspace to the eigenvalue $\nu_N^{(m)}(T)$ with $\nu_N^{(m)}(T) > \nu_N^{(m+1)}(T)$ for all m.

Type	$\nu_N^{(m)}(T)$
q_{I}^E	$(q^{n-m} - q - q^m + 1)/(q-1)$
q_{II}^E	$(q^{n-m-1} - q^m)/(q-1)$
q^E	$(q^{n-m} - q^m)/(q-1)$
q_1^E	$(q^{n-m-1} - q^m)/(q-1)$
q^H	$(q^{n-m+1/2} - q^m - q^{1/2} + 1)/(q-1)$
q_1^H	$(q^{n-m-1/2} - q^m - q^{1/2} + 1)/(q-1)$

Corollary. The neighbouring graph is connected.
Proof. The maximal eigenvalue ν_0 of the adjacency matrix is simple with eigenspace $\mathcal{Y}_0$.

3.4.1. Doubly even codes of length 16.

$M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$ and the possible eigenvalues are $(2^{8-m-1} - 2^m : m = 0, 1, 2, 3) = (127, 62, 28, 8)$
$K_{16}(\mathrm{II}) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$ has eigenvalues 127 and 8 with eigenvectors $(7, 10)$ and $(1, -1)$. Hence

$$\begin{aligned} \mathcal{Y}_0 &= \langle 7[e_8 \perp e_8] + 10[d_{16}^+] \rangle \\ \mathcal{Y}_1 &= \mathcal{Y}_2 = 0 \\ \mathcal{Y}_3 &= \langle [e_8 \perp e_8] - [d_{16}^+] \rangle. \end{aligned}$$

3.4.2. Doubly even codes of length 24.

$M_{24}(\mathrm{II}) = [e_8^3] \cup [e_8 d_{16}] \cup [e_7^2 d_{10}] \cup [d_8^3] \cup [d_{24}] \cup [d_{12}^2] \cup [d_6^4] \cup [d_4^6] \cup [g_{24}]$

$$K_{24}(\mathrm{II}) = \begin{pmatrix} 213 & 147 & 344 & 343 & 0 & 0 & 0 & 0 & 0 \\ 70 & 192 & 896 & 490 & 7 & 392 & 0 & 0 & 0 \\ 10 & 14 & 504 & 490 & 0 & 49 & 980 & 0 & 0 \\ 1 & 3 & 192 & 447 & 0 & 36 & 1152 & 216 & 0 \\ 0 & 990 & 0 & 0 & 133 & 924 & 0 & 0 & 0 \\ 0 & 60 & 480 & 900 & 1 & 206 & 400 & 0 & 0 \\ 0 & 0 & 72 & 216 & 0 & 3 & 1108 & 648 & 0 \\ 0 & 0 & 0 & 45 & 0 & 0 & 720 & 1218 & 64 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1771 & 276 \end{pmatrix}$$

m	0	1	2	3	4	5	6
ν_m	2047	1022	508	248	112	32	-32
$\dim(\mathcal{Y}_m)$	1	1	1	2	2	1	1

$\langle 99[e_8^3] - 297[e_8 d_{16}] - 3465[d_8^3] + 7[d_{24}] + 924[d_{12}^2] + 4928[d_6^4] - 2772[d_4^6] + 576[g_{24}] \rangle =$ $\ker(\mathrm{cwe}_5) = \mathcal{V}_5$.

3.5. The Dimension of $\mathcal{Y}_m(N)$ for doubly-even binary self-dual codes.

N, m	0	1	2	3	4	5	6	7	8	9	≥ 10
8	1										
16	1	0	0	1							
24	1	1	1	2	2	1	1				
32	1	1	2	5	10	15	21	18	8	3	1

The Molien series of $\mathcal{C}_m$(II) is

$$1 + t^8 + a(m)t^{16} + b(m)t^{24} + c(m)t^{32} + \dots$$

where

m	1	2	3	4	5	6	7	8	9	≥ 10
a	1	1	2	2	2	2	2	2	2	2
b	2	3	5	7	8	9	9	9	9	9
c	2	4	9	19	34	55	73	81	84	85

3.6. The Dimension of $\mathcal{Y}_m(N)$ for singly-even binary self-dual codes.

N, m	0	1	2	3	4	5	6	7	8	9	10	11
2	1											
4	1											
6	1											
8	1	1										
10	1	1										
12	1	1	1									
14	1	1	1	1								
16	1	2	1	2	1							
18	1	2	2	2	2							
20	1	2	3	4	4	2						
22	1	2	3	6	7	4	2					
24	1	3	5	9	15	13	7	2				
26	1	3	6	12	23	29	20	8	1			
28	1	3	7	18	40	67	75	39	10	1		
30	1	3	8	23	65	142	228	189	61	10	1	
32	1	4	10	33	111	341	825	1176	651	127	15	1

The Molien series of $\mathcal{C}_m$(I) is

$$1 + t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + \sum_{N=12}^{\infty} a_N(m)t^N$$

where $a_N(m) := \dim\langle \operatorname{cwe}_m(C) \mid C = C^\perp \leq \mathbb{F}_2^N \rangle$ is given in the following table:

m, N	12	14	16	18	20	22	24	26	28	30	32
2	3	3	4	5	6	6	9	10	11	12	15
3	3	4	6	7	10	12	18	22	29	35	48
4	3	4	7	9	14	19	33	45	69	100	159
5	3	4	7	9	16	23	46	74	136	242	500
6	3	4	7	9	16	25	53	94	211	470	1325
7	3	4	7	9	16	25	55	102	250	659	2501
8	3	4	7	9	16	25	55	103	260	720	3152
9	3	4	7	9	16	25	55	103	261	730	3279
10	3	4	7	9	16	25	55	103	261	731	3294
≥ 11	3	4	7	9	16	25	55	103	261	731	3295

References

[1] R. T. Bilous, Enumeration of the binary self-dual codes of length 34. J. Combin. Math. Combin. Comput. 59 (2006), 173–211.

[2] J. Cannon et al., *The Magma Computational Algebra System for Algebra, Number Theory and Geometry*, published electronically at http://magma.maths.usyd.edu.au/magma/.

[3] A. M. Gleason, Weight polynomials of self-dual codes and the MacWilliams identities, in *Actes, Congrés International de Mathématiques (Nice, 1970)*, Gauthiers-Villars, Paris, 1971, Vol. 3, pp. 211–215.

[4] M. Harada, A. Munemasa, B. Venkov, Classification of ternary extremal self-dual codes of length 28. (preprint)

[5] M. Harada, A. Munemasa, A complete classification of ternary self-dual codes of length 24. (preprint)

[6] W. C. Huffman, On the classification and enumeration of self-dual codes, *Finite Fields Applic.* **11** (2005), 451–490.

[7] M. Kneser, Klassenzahlen definiter quadratischer Formen, *Archiv der Math.* **8** (1957), 241-250.

[8] G. Nebe, E. M. Rains and N. J. A. Sloane, The invariants of the Clifford groups, *Designs, Codes, and Cryptography* **24** (2001), 99–121.

[9] G. Nebe, E. M. Rains and N. J. A. Sloane, Codes and invariant theory, *Math. Nachrichten*, **274–275** (2004), 104–116.

[10] G. Nebe, E. M. Rains and N. J. A. Sloane, *S*elf-dual codes and invariant theory. Springer-Verlag (2006).

[11] G. Nebe, Kneser-Hecke-operators in coding theory. Abh. Math. Sem. Univ. Hamburg 76 (2006) 79-90

[12] G. Nebe, Finite Weil-representations and associated Hecke-algebras. (preprint)

[13] E. M. Rains and N. J. A. Sloane, Self-dual codes, in *Handbook of Coding Theory*, ed. V. S. Pless and W. C. Huffman, Elsevier, Amsterdam, 1998, pp. 177–294.

[14] E. Rains, Shadow bounds for self-dual codes. IEEE Trans. Inform. Theory 44 (1998), no. 1, 134–139.

[15] B. Runge, Codes and Siegel modular forms, *Discrete Math.* **148** (1996), 175–204.

[16] A. Weil, Sur certaines groupes d'opérateurs unitaires, *Acta Math.* **111** (1964), 143–211. *Oeuvres Scientifiques* III, Springer-Verlag, 1979, pp. 1–69.

[17] H. Yoshida, *The Action of Hecke Operators on Theta Series.* Algebraic and topological theories (Kinosaki, 1984), 197–238, Kinokuniya, Tokyo, 1986.