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Low dimensional strongly perfect lattices. II: Dual strongly perfect lattices of dimension 13 and 15.

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Résumé. Un réseau est dual fortement parfait si le réseau et son dual sont fortement parfaits. On demontre qu'il n'y a pas des réseaux dual fortement parfaits en dimension 13 et 15.

Abstract. A lattice is called dual strongly perfect if both, the lattice and its dual, are strongly perfect. We show that there are no dual strongly perfect lattices of dimension 13 and 15.
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1. Introduction.

This paper continues the classification of strongly perfect lattices in [10], [6], [7], [8]. A lattice L in Euclidean space $(\mathbb{R}^n, (,))$ is called strongly perfect, if the set of its minimal vectors forms a spherical 4-design. Strongly perfect lattices realise a local maximum of the sphere packing density function.

All strongly perfect lattices are known up to dimension 12. They all share the property of being dual strongly perfect, which means that both lattices L and its dual lattice L^* are strongly perfect (see Definition 2.1). The only known strongly perfect lattice for which the dual is not strongly perfect is K'_{21} (see [10, Tableau 19.2]) in dimension 21. The dual strongly perfect lattices of dimension 14 have been classified in [8]. The present paper deals with dimension 13 and 15. We show that there are no dual strongly perfect lattices of dimension 13 and 15. In fact we conjecture that in these dimensions there are no strongly perfect lattices. Some cases of the non-existence proof of strongly perfect lattices in dimension ≥ 13 seem to be rather hard. They become much easier if we restrict to those lattices, for which also the dual lattice is strongly perfect, since this allows us to apply the strategy described in the introduction of [8] to obtain bounds on the exponent of the discriminant group L^*/L which either allow the direct classification of all candidates for L or the use of modular forms to prove their non-existence. A new method (see Theorem 2.9) is described that is particularly helpful to narrow down the kissing number of a dual strongly perfect lattice of minimal type. For a more detailed description of the proofs and in particular of the computations and the used MAGMA and SAGE programs we refer to Elisabeth Nossek's PhD thesis [9]. This thesis also shows that there are no universally perfect lattices in dimension 17. A lattice L is called universally perfect, if all non-empty layers $L_a := \{\ell \in L \mid (\ell, \ell) = a\}$ form spherical 4-designs (see [8]). By the theta-transformation formula also the dual lattice L^* then is universally perfect, so this is a stronger notion than being dual strongly perfect. Universally perfect lattices also play a role in Riemannian geometry, as the torus $\mathbb{R}^n/L^*$ defined by the dual lattice L^* provides a strict local minimum of the height function on the set of all n-dimensional flat tori of volume 1 ([4, Theorem 1.2]).

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2. Some general equations

2.1. Designs and strongly perfect lattices. Let $L \subset \mathbb{R}^n$ be an n-dimensional lattice. Then $\min(L) := \min\{(x,x) \mid 0 \neq x \in L\}$ is called the minimum of L and $\mathrm{Min}(L) := \{x \in L \mid (x,x) = \min(L)\}$ the set of minmal vectors. $\mathrm{Min}(L)$ is a finite antipodal spherical code and we let $S(L)$ be some fixed set such that

$$\mathrm{Min}(L) = S(L) \,\dot\cup\, -S(L) \text{ and put } s(L) := |S(L)| = \frac{|\mathrm{Min}(L)|}{2}.$$

The Hermite function is

$$\gamma(L) := \frac{\min(L)}{\det(L)^{1/n}}$$

where $\det(L)$ is the determinant of L, the determinant of the Gram matrix $((b_i,b_j))_{i,j=1}^n$ of any lattice basis $B := (b_1,\ldots,b_n)$ of L. The dual lattice is

$$L^* := \{v \in \mathbb{R}^n \mid (v,\ell) \in \mathbb{Z} \text{ for all } \ell \in L\}.$$

As L^* is spanned by the dual basis of B, $\det(L^*)\det(L) = 1$.

The geometry of the minimal vectors $\mathrm{Min}(L)$ has been used by Korkine, Zolotareff, and Voronoi to characterise the local maxima of the Hermite function (see [5]).

Definition 2.1. *([10]) A lattice $L \subset \mathbb{R}^n$ is called* strongly perfect*, if its minimal vectors form a spherical* 4*-design, which means that for all* $\alpha \in \mathbb{R}^n$

$$(D4)(\alpha): \quad \sum_{x \in S(L)} (x,\alpha)^4 = \frac{3sm^2}{n(n+2)}(\alpha,\alpha)^2.$$

where $m := \min(L)$ and $s = s(L) = |S(L)|$ is half the kissing number *of L. We call L* dual strongly perfect*, if both lattices, L and L^* are strongly perfect.*

Strongly perfect lattices have been introduced by the third author in [10]. They provide examples of local maxima of γ. In particular they are similar to rational lattices and hence for any strongly perfect lattice L the quantity

$$r(L) := \min(L)\min(L^*) \in \mathbb{Q}$$

is a rational number.

From $(D4)(\alpha)$ one obtains the following equations $(Di) = (Di)(\alpha)$ and $(Dij) = (Dij)(\alpha,\beta)$.

$$\begin{array}{lrr}
(D2): & \sum_{x\in S(L)}(x,\alpha)^2 = & \frac{sm}{n}(\alpha,\alpha) \\
(D11): & \sum_{x\in S(L)}(x,\alpha)(x,\beta) = & \frac{sm}{n}(\alpha,\beta) \\
(D22): & \sum_{x\in S(L)}(x,\alpha)^2(x,\beta)^2 = & \frac{sm^2}{n(n+2)}(2(\alpha,\beta)^2+(\alpha,\alpha)(\beta,\beta)) \\
(D13): & \sum_{x\in X}(x,\alpha)(x,\beta)^3 = & \frac{3sm^2}{n(n+2)}(\alpha,\beta)(\beta,\beta) \\
n_2(\alpha) := & \frac{1}{12}\sum_{x\in S(L)}(x,\alpha)^4-(x,\alpha)^2 = & \frac{sm}{12n}(\alpha,\alpha)(\frac{3m}{n+2}(\alpha,\alpha)-1)
\end{array}$$

Note that $(D2)(\alpha)$, $(D22)(\alpha,\beta)$, $(D4)(\alpha)$, $n_2(\alpha)$ as well as $\frac{1}{6}(D13-D11)(\alpha,\beta) = \frac{sm}{6n}(\alpha,\beta)(\frac{3m}{n+2}(\beta,\beta)-1)$ are non negative integers for all $\alpha,\beta \in L^*$.

Since $n_2(\alpha) \geq 0$ for $\alpha \in S(L^*)$, we see that any strongly perfect lattice L of dimension n satisfies

$$r(L) = \min(L)\min(L^*) \geq \frac{n+2}{3} \quad ([10, \text{Theorem } 10.4]).$$

A strongly perfect lattice L is called of minimal type, if $r(L) = \frac{n+2}{3}$ and of general type otherwise. Lattices of minimal type satisfy $n_2(\alpha) = 0$ for all $\alpha \in \mathrm{Min}(L^*)$ and hence $(\alpha,x) \in \{0,\pm 1\}$ for all $x \in \mathrm{Min}(L), \alpha \in \mathrm{Min}(L^*)$.

Remark 2.2. *For $\alpha \in L^*$ such that $|(x,\alpha)| \leq 2$ for all $x \in S(L)$ we obtain*

$$n_2(\alpha) = |N_2(\alpha)| \text{ where } N_2(\alpha) = \{x \in \mathrm{Min}(L) \mid (x,\alpha) = 2\}$$

In this case $\sum_{x\in N_2(\alpha)} x = c\alpha$ by [6, Lemma 2.1] *where $c = \frac{2n_2(\alpha)}{(\alpha,\alpha)}$.*

The last two lemmata allow the construction of sublattices of small index. We let $\mathbb{Z}_{(p)} := \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$ denote the localization of $\mathbb{Z}$ in $\mathbb{Q}$.

Lemma 2.3. *([7, Lemma 2.8]) Let Γ be a lattice such that $(\gamma,\gamma) \in \mathbb{Z}_{(2)}$ for all $\gamma \in \Gamma$. Let $\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha,\alpha) \in 2\mathbb{Z}_{(2)}\}$. If $\Gamma^{(e)}$ is a sublattice of Γ, then $[\Gamma : \Gamma^{(e)}] \in \{1,2,4\}$.*

Lemma 2.4. *([7, Lemma 2.9]) Let Γ be a lattice such that $(\gamma,\gamma) \in \mathbb{Z}_{(3)}$ for all $\gamma \in \Gamma$. Let $\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha,\alpha) \in 3\mathbb{Z}_{(3)}\}$. Assume that*

$$(\alpha,\beta)^2 - (\alpha,\alpha)(\beta,\beta) \in 3\mathbb{Z}_{(3)} \text{ for all } \alpha,\beta \in \Gamma.$$

Then $\Gamma^{(t)}$ is a sublattice of Γ and $[\Gamma : \Gamma^{(t)}] \in \{1,3\}$.

2.2. Bounds on kissing numbers and Hermite constants. The best known upper bounds on the Hermite constant

$$\gamma_n := \max\{\gamma(L) \mid L \text{ is an } n-\text{dimensional lattice }\}$$

are given in [3]. In particular we obtain

$$\gamma_{13} \leq 2.65 \text{ and } \gamma_{15} \leq 2.91.$$

There are also well known upper bounds $\mathrm{kis}(n)$ for the half kissing number $s(L)$ of n-dimensional lattice L (see [2]). For $n = 13$ and 15 we obtain

$$\mathrm{kis}(13) \leq 933 \text{ and } \mathrm{kis}(15) \leq 2481.$$

If L is strongly perfect, then the lattice is also perfect and therefore the projections along its minimal vectors span the space of all symmetric endomorphisms (see [5]). Therefore $s(L) \geq \frac{n(n+1)}{2}$.

Our starting point for the classification of all strongly perfect lattices L of dimension n is to find a finite list of possible parameters $(s(L), r(L)) \in \mathbb{N} \times \mathbb{Q}$.

Remark 2.5. *Let L be a strongly perfect lattice. Then $s := s(L) \in [\frac{n(n+1)}{2}, \mathrm{kis}(n)]$ and $r := r(L) \in [\frac{n+2}{3}, \gamma_n^2] \cap \mathbb{Q}$ is a rational zero of the polynomial*

$$n_2(r) - n_2 := \frac{sr}{12n}(\frac{3r}{n+2} - 1) - n_2$$

for some $n_2 \in \{0,\ldots,s(L)\}$. Moreover $\frac{s\cdot r}{n} \in \mathbb{Z}$ and $\frac{3sr^2}{n(n+2)} \in \mathbb{Z}$. If additionally $r < 8$ then $n_2 \leq \frac{r}{8-r}$ (see [7, Lemma 2.4]). From the proof of [7, Lemma 2.4] we also find that $n_2 = \frac{r}{8-r}$ if and only if $(x,x') = \frac{1}{2}(x,x)$ for all $x \neq x' \in N_2(\alpha)$ and then $N_2(\alpha)$ spans a rescaled root lattice A_{n_2}. In particular $n_2(r) \leq n$ in this case.
By [8, Lemma 2.8] we get the bound $n_2(8) \leq 2(n-1)$ also for $(\alpha,\alpha) = 8$. If equality holds, then $N_2(\alpha)$ spans a rescaled root lattice D_n.

Proposition 2.6. *Applying the strategy from Remark 2.5 and using the fact that $n_2(r) \neq 1$ ([7, Lemma 2.6]) we find the following list of possibilities*

$(r(L), s(L), n_2(r)) = (r, s, n_2)$ for a strongly perfect lattice of dimension 13 *resp.* 15*:*

$n = 13$

r	$39/7$	6	$13/2$	$20/3$	7
s	490	$130a$	$80a$	351	390
a	$-$	$2,3$	$2,3,4$	$-$	$-$
n_2	2	a	a	5	7

$n = 15$

r	6	$45/7$	$20/3$	7	$15/2$	$68/9$	8	$153/19$	$170/21$	$25/3$
s	$510a$	833	$153a$	765	816	1215	$765a$	1805	$882a$	$459a$
a	$2,3$	$-$	$2..5$	$-$	$-$	$-$	$1,2$	$-$	$1,2$	$1..5$
n_2	a	4	a	7	11	17	$14a$	34	$17a$	$10a$

or L is of minimal type.

2.3. Dual strongly perfect lattices. For dual strongly perfect lattices L we can exclude certain values of $(r(L), s(L))$ with the following lemma.

Lemma 2.7. *Let $p \geq 5$ be a prime dividing $n(n+2)$ and let L be a dual strongly perfect lattice of dimension n. Assume that p does not divide $s(L)s(L^*)$. Then $r(L) \in p^2\mathbb{Z}_{(p)}$.*

Proof. We use the equality $D4$ to see that

$$\begin{array}{lll} \frac{3s}{n(n+2)} \min(L)^2(\alpha,\alpha)^2 & \in \mathbb{Z} & \text{for all } \alpha \in L^* \\ \frac{3s}{n(n+2)} \min(L^*)^2(x,x)^2 & \in \mathbb{Z} & \text{for all } x \in L. \end{array}$$

Rescaling L so that $\min(L) = \frac{p}{6s(L)}$, then $\Gamma := L^*$ is an even lattice with minimum

$$\min(\Gamma) = \frac{6r(L)s(L)}{p}.$$

Interchanging the role of L and L^* we see that

$$\sqrt{\frac{6^2 r(L)s(L^*)s(L)}{p^2}}\Gamma^*$$

is also an even lattice If p^2 does not divide $r(L)$, then all inner products of elements of Γ^* are p-adic multiples of p, which is impossible for the dual of some integral lattice. □

Corollary 2.8. *Let L be a dual strongly perfect lattice of dimension* 15*. Then $r(L) \notin \{\frac{45}{7}, \frac{15}{2}, \frac{68}{9}, \frac{153}{19}, \frac{170}{21}\}$ and if $r(L) = \frac{20}{3}$, then $s(L) = 153 \cdot 5$ or $s(L^*) = 153 \cdot 5$.*

This strategy does not give many restrictions for lattices of minimal type, where $r = \frac{n+2}{3}$ and $s \in [\frac{n(n+1)}{2}, \mathrm{kis}(n)]$ is any number such that $\frac{s \cdot r}{n} \in \mathbb{Z}$. Here, and more generally for small values of $r(L)$, the following method is very helpful to exclude certain cases.

Theorem 2.9. *Let L be a dual strongly perfect lattice of dimension n and let $r := r(L) = r(L^*)$, $s := s(L)$ and $t := s(L^*)$. Assume that $(\alpha, x) \in \{0, \pm 1, \pm 2\}$ for all $\alpha \in S(L^*), x \in S(L)$. Put*

$$\begin{array}{rl} n_2 := & \frac{tsr}{12n}(\frac{3r}{n+2} - 1) \\ n_1 := & \frac{tsr}{n} - 4n_2 \\ n_0 := & st - n_1 - n_2 \end{array}$$

Then n_2, n_1, n_0 are non-negative integers divisible by s and t. Moreover for any $b \in \mathbb{R}$

$$\begin{array}{rl} P(b) := & (s+t)^2(\frac{15}{(n(n+2)(n+4)} + (2b - \frac{1}{4})\frac{3}{(n(n+2)} + (b^2 - \frac{b}{2})/n - \frac{b^2}{4}) \\ & -2(n_1(\frac{1}{r} - \frac{1}{4})(\frac{1}{r} + b)^2 + n_2(\frac{4}{r} - \frac{1}{4})(\frac{4}{r} + b)^2 - n_0\frac{b^2}{4}) \\ & -\frac{3}{4}(s+t)(1+b)^2 \leq 0. \end{array}$$

Proof. We first remark that $n_i = |\{(x, \alpha) \in S(L) \times S(L^*) \mid (x, \alpha) = \pm i\}|$ for $i = 0, 1, 2$. Rescale L so that $\min(L) = 1$ and $\min(L^*) = r$ and put $M := \frac{1}{\sqrt{r}} L^*$, $S := \mathrm{Min}(L)$ and $T := \mathrm{Min}(M)$. Then $S \cap T = \emptyset$ and $X := S \dot\cup T$ is a spherical 5-design. Moreover for all $x \neq x' \in S$, $y \neq y' \in T$ we have

$$(x, x) = (y, y) = 1, \ (x, x')^2 \leq \frac{1}{4}, \ (y, y')^2 \leq \frac{1}{4}, \ (x, y)^2 \in \{0, \frac{1}{r}, \frac{4}{r}\}$$

For $b \in \mathbb{R}$ define

$$f_b(x) := (x^2 - 1/4)(x^2 + b)^2 = x^6 + (2b - 1/4)x^4 + (b^2 - b/2)x^2 - 1/4b^2.$$

Then clearly $f_b((x, x')) \leq 0$ for all $x \neq x' \in S$ and similarly $f_b((y, y')) \leq 0$ for all $y \neq y' \in T$. Using the fact that T and S are spherical 4-designs and that

$$\sum_{x,y \in S \cup T} (x, y)^6 \geq \frac{15}{n(n+2)(n+4)}(|S| + |T|)^2$$

we compute

$$0 \geq \sum_{x \neq \pm x' \in S} f_b((x, x')) + \sum_{y \neq y' \in T} f_b((y, y')) \geq P(b). \tag{1}$$

Observe that $s = \frac{1}{2}|S|$ and $t = \frac{1}{2}|T|$. □

2.4. Dual strongly perfect lattices of minimal type.

Corollary 2.10. *The polynomial $P(b)$ from Theorem 2.9 is a quadratic polynomial in b and we can test the non positivity of $P(b)$ by checking that $P(0) \leq 0$ and that the discriminant of $P(b)$ is ≤ 0. In particular for lattices* *of minimal type this leads to*

$$\begin{aligned}&(-8n^3+152n^2-208n+64)(s+t)^4+\\&(-12n^5-48n^4+108n^3+1272n^2+1920n)(s+t)^3+\\&(-36n^5-288n^4-720n^3-576n^2)(s+t)^2+\\&(32n^3-416n^2-128n+512)(s+t)^2st+\\&(24n^5+96n^4-216n^3-1968n^2-4416n)(s+t)st+\\&(-32n^3+224n^2+1088n-1280)(st)^2\leq 0\end{aligned}$$

and

$$(16-n)(s+t)^2-(n^3+6n^2+8n)(s+t)+(2n^2-12n-80)/(n+2)st\leq 0$$

Remark 2.11. *Let L be a dual strongly perfect lattice of minimal type scaled such that* $\min(L)=1$, $\min(L^*)=\frac{n+2}{3}=: r$.

(a) *If $n>8$ then $\mathrm{Min}(L^*)$ is not a rescaled root system, so there are $\alpha,\beta\in\mathrm{Min}(L^*)$ such that $-r/2<(\alpha,\beta)<0$. Then $\gamma:=\alpha+\beta\in L^*$ satisfies $r<(\gamma,\gamma)<2r$ and $(x,\gamma)\in\{0,\pm1,\pm2\}$ for all $x\in\mathrm{Min}(L)$. Since $(\gamma,\gamma)<2r$ we get*

$$n_2(\gamma)<\frac{sr}{6n}\left(\frac{6r}{n+2}-1\right)=\frac{s(n+2)}{18n}.$$

(b) *From Equation $D2$ one obtains $\frac{s(L)(n+2)}{3n}\in\mathbb{Z}$.*

3. Dimension 13

3.1. Strongly perfect lattices of general type. In this section we exclude some of the cases listed in Proposition 2.6. Here is is not necessary to assume that the dual lattice is also strongly perfect, more precisely we will prove the following theorem.

Theorem 3.1. *Let L be a strongly perfect lattice of dimension 13. Then we have the following 4 possibilities*

(a) $s(L)=260$ *and* $r(L)=6$.
(b) $s(L)=390$ *and* $r(L)=6$.
(c) $s(L)=390$ *and* $r(L)=7$.
(d) $r(L)=5$, *which means that L is of minimal type.*

For the proof we put $s:=s(L)$ and scale L, such that $\min(L)=1$. Then $r:=r(L)=\min(L^*)$ For $\alpha\in L^*$ write $(\alpha,\alpha)=\frac{p}{q}$ with coprime integers p and q. Then

$$(\star)\quad \frac{1}{12}(D4-D2)(\alpha)=\frac{s}{12\cdot 13}\frac{p}{q}\frac{p-5q}{5q}\in\mathbb{Z}.$$

Moreover $D2(\alpha)=\frac{s}{13}\frac{p}{q}$ and $D4(\alpha)=\frac{s}{5\cdot 13}\frac{p^2}{q^2}\in\mathbb{Z}$ which yields that $s\cdot p$ is divisible by $5\cdot 13$ and q^2 divides s.

Lemma 3.2. $(s, r) \neq (490, 39/7)$.

Proof. Since s is not divisible by 13 we have $p = 13p_1$ for all $\alpha \in \Lambda^*$. Moreover q^2 divides s hence $q \in \{1, 7\}$. Let $\Gamma := \sqrt{\frac{7}{13}}\Lambda^*$. Then $\Gamma^* = \sqrt{\frac{13}{7}}\Lambda$ is a strongly perfect lattice of minimum $\frac{13}{7}$. For $\alpha, \beta \in \Gamma$ the equality $\frac{1}{6}(D13 - D11)(\alpha, \beta)$ yields

$$\frac{(\alpha, \beta)}{3}(13(\beta, \beta) - 35) \in \mathbb{Z}.$$

In particular $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in \Gamma$ for which $(\beta, \beta) \in 2\mathbb{Z}$ and $(\alpha, \beta) \in 3\mathbb{Z}$ for all $\alpha, \beta \in \Gamma$ for which $(\beta, \beta) \in 3\mathbb{Z}$. Therefore we can apply Lemma 2.4 and Lemma 2.3 to construct the sublattice $\Gamma^{(6)} = \Gamma^{(e)} \cap \Gamma^{(t)}$ of index ≤ 12. Then $\frac{1}{\sqrt{3}}\Gamma^{(6)}$ is an even lattice of determinant

$$\det(\frac{1}{\sqrt{3}}\Gamma^{(6)}) \leq 12^2(\frac{7}{39})^{13}\det(\Lambda^*) \leq 0.01$$

which is a contradiction. □

Lemma 3.3. $r \neq 13/2$.

Proof. Assume that $r = 13/2$, then $s = 80a$ for $a = 2, 3$, or 4. Since s is not divisible by 13 we have $p = 13p_1$ for all $\alpha \in L^*$. $(\star)$ now reads as $\frac{2^2 a p_1(13p_1 - 5q)}{3q^2} \in \mathbb{Z}$. Hence $q \in \{1, 2, 4\}$ and $\Gamma := \sqrt{8/13}L^*$ is an even lattice. If $a \neq 3$ then [7, Lemma 2.9] (using the equality $(D22(\alpha))$) yields that $\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 3\mathbb{Z}\}$ is a sublattice of Γ of index ≤ 3. Hence $\frac{1}{\sqrt{3}}\Gamma^{(t)}$ is an even lattice of determinant

$$\det(\frac{1}{\sqrt{3}}\Gamma^{(t)}) \leq 9 \cdot (\frac{8}{39})^{13}\det(L^*) \leq 9 \cdot \left(\frac{8}{39}\right)^{13}\left(\frac{\gamma_{13}}{\min(L)}\right)^{13} \leq 0.004$$

which is a contradiction.
For $a = 3$ the equality $(\star)$ directly implies that $q \in \{1, 2\}$ and $\Gamma := \sqrt{4/13}L^*$ is an even lattice of minimum 2. But $\Gamma^* = \sqrt{\frac{13}{4}}L$ has minimum $\frac{13}{4} > 2$ which is a contradiction. □

Lemma 3.4. $(s, r) \neq (351, 20/3)$.

Proof. Now $s = 3^3 \cdot 13$ and $(\star)$ yields $\frac{9p}{2^2 \cdot 5 \cdot q^2}(p - 5q) \in \mathbb{Z}$, so $p = 5p_1$ is divisible by 5 and $q \in \{1, 3\}$. Hence the norms of the elements in $\Gamma := \sqrt{\frac{3}{5}}L^*$ are integers and $D22$ implies that also the scalar products in Γ are integers. Therefore the even sublattice $\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 2\mathbb{Z}\}$ is a sublattice of Γ of index 1 or 2. By $(\star)$, the norms in $\Gamma^{(e)}$ are all divisible by 4, hence $\frac{1}{\sqrt{2}}\Gamma^{(e)}$ is an even lattice of determinant

$$\det(\frac{1}{\sqrt{2}}\Gamma^{(e)}) \leq 2^{-11}(\frac{3}{5})^{13}\det(\Gamma) \leq 0.21$$

which is a contradiction. □

3.2. Dual strongly perfect lattices of general type. In this section we will show that there is no 13-dimensional dual strongly perfect lattice of general type. By Theorem 3.1 for any such lattice L either $r(L) = 7$ or $r(L) = 6$.

Lemma 3.5. *$r(L) \neq 7$.*

Proof. Let L be a dual strongly perfect lattice of minimum $\min(L) = 7$ such that $\min(L^*) = 1$. Then $s(L) = s(L^*) = 390$. For any $x \in S(L^*)$ we obtain that

$$N_2(x) := \{\alpha \in \mathrm{Min}(L) \mid (\alpha, x) = 2\} = \{\alpha_1, \ldots, \alpha_7\}$$

with $(\alpha_i, \alpha_j) = 7/2$ for all $i \neq j$. Put $\alpha := \alpha_1 + \alpha_2 + \alpha_3 \in L$ and $\beta := \alpha_4$. Then $(\alpha, \alpha) = 42$, $(\beta, \beta) = 7$ and $(\alpha, \beta) = \frac{21}{2}$ and so $\frac{1}{6}(D13 - D11) =$

$$\frac{1}{6}\sum_{x\in S(L)} (x, \alpha)^3(x, \beta) - (x, \alpha)(x, \beta) = (\alpha, \beta)((\alpha, \alpha) - 5) = \frac{21 \cdot 37}{2} \notin \mathbb{Z}$$

which is a contradiction. □

Lemma 3.6. *If $r(L) = 6$ then $(s(L), s(L^*)) \neq (260, 260)$.*

Proof. Assume that $r(L) = 6$ and $s(L) = 260$. Rescale L so that $\min(L) = \frac{1}{2}$. Then $\Gamma := L^*$ is an even lattice of minimum 12. Moreover by $D22$ the inner product $(\alpha, \beta) \in \{0, \pm 3, \pm 6, \pm 12\}$ for all $\alpha, \beta \in \mathrm{Min}(\Gamma)$. Since $\mathrm{Min}(\Gamma)$ is a 4-design of cardinality $2s(L^*) = 520$, we can compute $n_i(\alpha) := |\{\beta \in \mathrm{Min}(\Gamma) \mid (\alpha, \beta) = i\}|$ and obtain a contradiction since $n_3(\alpha)$ is not integral. □

Lemma 3.7. *If $r(L) = 6$ then $s(L) \neq 390$.*

Proof. Rescale L so that $\min(L) = 1$, put $\Gamma := L^*$, so $\min(\Gamma) = 6$. Then

$$41093 \leq \det(\Gamma) \leq 317832.$$

Then $\frac{1}{6}(D13 - D11)(\alpha, \beta) = (\alpha, \beta)((\beta, \beta) - 5) \in \mathbb{Z}$ for all $\alpha, \beta \in \Gamma$ so $\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 2\mathbb{Z}\}$ is an even lattice of index $1, 2$, or 4 in Γ. In particular $\det(\Gamma^{(e)}) \in 2\mathbb{Z}$.
If $s(L^*) = 390$ then similar considerations apply to $\sqrt{6}L$ so $\det(\Gamma) = 2^a 3^b$ for some $a \in \{-3, \ldots, 16\}$, $b \in \{0, \ldots, 13\}$. Interchanging the roles of L and L^* we may assume that $b \geq 7$ whence $a \in \{-3, \ldots, 7\}$ by the

bound on $\det(\Gamma)$ above. Moreover $\min(\Gamma^{(e)}) = 6$, $\Gamma^{(e)}$ is strongly perfect, so $\min(\Gamma^{(e),*}) \geq \frac{5}{6}$. Therefore $\Gamma^{(e)}$ is a sublattice of some maximal even lattice M of determinant $2^x 3^y$ such that $\min(M^*) \geq \frac{5}{6}$. Magma computations show that there are 19 such lattices M, there is only one sublattice T of 3-power index in those lattices such that $3^7 \mid \det(T)$ and $\min(T^*) \geq \frac{5}{6}$. The minimum of T is 4 and no sublattice of index 2 or 3 of T has dual minimum $\geq \frac{5}{6}$.

If $s(L^*) = 260$ then $\Delta := \sqrt{4}L$ contains an even sublattice $\Delta^{(t)} := \{\alpha \in \Delta \mid (\alpha, \alpha) \in \mathbb{Z}\}$ of index 1 or 3. with $S(\Delta) = S(\Delta^{(t)})$ of cardinality $s(L) = 390$. So also $\Delta^{(t)}$ is strongly perfect of minimum 4 and hence $\min(\Delta^{(t)})^* \geq 5/4$. Therefore $\Delta^{(t)}$ is contained in some maximal even lattice M of determinant only involving the primes 2 and 3. One computes that there is only one such maximal lattice, $M \cong E_7 \perp E_6$, such that $\min(M^*) \geq \frac{5}{4}$. Then M^* has minimum $\frac{4}{3}$, in particular there is some vector $\beta \in (\Delta^{(t)})^*$ such that $(\beta, \beta) = \frac{4}{3}$. One computes

$$\frac{1}{12}(D4 - D2)(\beta) = \frac{4s(\Delta^{(t)})}{12 \cdot 13} \frac{4}{3} (\frac{12 \cdot 4}{15 \cdot 3} - 1) = \frac{8}{9}$$

which is not integral, a contradiction. □

3.3. Dual strongly perfect lattices of minimal type. Let L be a dual strongly perfect lattice of minimal type, $\min(L) = 1$, $\min(L^*) = 5$. Then by Remark 2.11 (b) $s(L) = 13 s_1$ and by Remark 2.11 (a) the lattice L^* contains some vector γ of norm $5 < r := (\gamma, \gamma) < 10$. So $n_2 := n_2(\gamma) \in \mathbb{N}$, $n_2 < \frac{s(n+2)}{18n} = \frac{5}{6} s_1$ and r is a rational zero of $n_2(r) - n_2$. With this strategy and using the bounds from Remark 2.5 we find the following possibilities for s_1 and r:

Remark 3.8. *Let $\gamma \in L^*$ of norm $5 < r := (\gamma, \gamma) < 10$ such that $|(x, \gamma)| \leq 2$ for all $x \in S(L)$. Then (s_1, r) are listed in the following table:*

s_1	10	15	16	20	25	27	30	32	35	
r	8, 9	8, 9	15/2	6, 8, 9	8, 9	20/3	6, 7, 8, 9	15/2	8, 9	
s_1	40	45	49	50	54	55	60	64	65	70
r	8, 9	8, 9	60/7	8, 9	25/3	8, 9	8, 9	35/4	8, 9	8, 9

We first exclude some small values of s_1 and then apply Theorem 2.9.

Lemma 3.9. *If s_1 is odd, then L^* contains a vector γ of norm $(\gamma, \gamma) \in 4\mathbb{Z}_{(2)} \cap [5, 7.82]$.*

<u>Proof.</u> If s_1 is odd, then $\Gamma := L^*$ is a 2-integral lattice. By $n_2(\alpha)$ all norms in $\Gamma^{(e)}$ are in $4\mathbb{Z}_{(2)}$. Since $\Gamma^{(e)}$ is a sublattice of index 2 in L^*, we get

$$\min(\Gamma^{(e)}) \leq \gamma_{13} \det(\Gamma^{(e)})^{1/13} \leq 2.65(4 \cdot 317831.1)^{1/13} \leq 7.82 .$$

□

Since all vectors of norm < 9 in L^* satisfy the conditions of Remark 2.2, Remark 3.8 lists all possible norms < 9 of vectors in Γ. Looking at the norms in $4\mathbb{Z}_{(2)}$ that are ≤ 7.82 we can immediately exclude all cases where s_1 is odd, except for the case $s_1 = 27$.

Lemma 3.10. *s_1 is even.*

Proof. It remains to exclude the case that $s_1 \neq 27$. Let $\Delta := \sqrt{\frac{3}{5}} L^*$. Then all norms in Δ are integers. Evaluating $\frac{1}{6}((D13)-(D11))(\alpha,\beta)$ we conclude $(\alpha,\beta) \in 2\mathbb{Z}$ for all $\alpha,\beta \in \Delta$ with $(\alpha,\alpha) \in 2\mathbb{Z}$. Moreover

$$\det(\Delta) \leq (\gamma_{13}/\min(\Delta^*))^{13} \leq (2.65\frac{3}{5})^{13} \leq 415.2 \ .$$

Let $\Delta^{(e)} := \{\alpha \in \Delta \mid (\alpha,\alpha) \in 2\mathbb{Z}\}$. Then $\Delta^{(e)}$ is a sublattice of Δ of index ≤ 4. By $(D4)-(D2)$ all norms in $\Delta^{(e)}$ are divisible by 4 and hence $\frac{1}{\sqrt{2}}\Delta^{(e)}$ is still even. Therefore

$$8192 = 2^{13} \text{ divides } \det(\Delta^{(e))}) \leq 4^2 \det(\Delta) \leq 4^2 \cdot 415.2 = 6643.2$$

which yields a contradicition. □

We now treat the cases for which s_1 is not divisible by 3 similarly by looking at the sublattice $\Gamma^{(t)}$.

Remark 3.11. *Assume that s_1 is not divisible by 3. $\Gamma^{(t)}$ is a sublattice of $\Gamma := L^*$ of index 3 and*

$$\min(\Gamma^{(t)}) \leq \gamma_{13} \det(\Gamma^{(t)})^{1/13} \leq 2.65(9 \cdot 317831.1)^{1/13} \leq 8.32.$$

Remark 3.8 lists all possible norms < 9 of elements in L^*, so this allows to exclude most of the cases where s_1 is not divisible by 3:

Corollary 3.12. *$s_1 \neq 10$, $s_1 \neq 40$, $s_1 \neq 50$, $s_1 \neq 64$, $s_1 \neq 70$.*

Lemma 3.13. *$s_1 \neq 16$ and $s_1 \neq 32$.*

Proof. Assume that $s_1 = 16a$ for $a \in \{1,2\}$ and put $\Delta := \sqrt{\frac{4}{5}} L^*$. Then Δ is classical integral, $\min(\Delta) = 4$, $\min(\Delta^*) = \frac{5}{4}$ and $\det(\Delta) \leq 17472.95$. Equality $(D4)-(D2)$ yields that

$$5a(\alpha,\alpha)((\alpha,\alpha)-4) \in 12\mathbb{Z} \text{ for all } \alpha \in \Delta \ .$$

Hence Δ is an even lattice and $\Delta^{(t)}$ is a sublattice of Δ of index 3 such that $\frac{1}{\sqrt{3}}\Delta^{(t)}$ is still integral. Hence

$$1594323 = 3^{13} \text{ divides } \det(\Delta^{(t)}) = 9\det(\Delta) \leq 157256$$

yields a contradiction. □

Lemma 3.14. *If L is a dual strongly perfect lattice of minimal type and of dimension 13 then $s(L) = s(L^*) = 260$.*

Proof. Let $s := s(L)$ and $t := s(L^*)$. Then by the discussion above $s/13, t/13 \in \{20, 30, 54, 60\}$. Only $(s,t) = (20\cdot 13, 20\cdot 13)$ satisfies Corollary 2.10. □

Theorem 3.15. *There is no dual-strongly perfect lattice of dimension 13.*

Proof. It suffices to treat such lattices L of minimal type. Then we know that $s(L) = s(L^*) = 260$. Rescaled so that $\min(L) = 1/2$, $\min(L^*) = 10$ the lattice $\Gamma := L^*$ is an even lattice. Moreover for all $\alpha, \beta \in \Gamma$

$$\begin{aligned} D22(\alpha,\beta) &= \frac{20}{15\cdot 2^2}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \\ &= \frac{1}{3}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z}. \end{aligned}$$

Therefore the lattice $\Gamma^{(t)} := \{\alpha \in \Gamma | (\alpha,\alpha) \in 3\mathbb{Z}\}$ is a sublattice of index 3 in Γ and $3^{13} \mid \det(\Gamma^{(t)})$ so $3^{11} \mid \det(\Gamma)$. But $\Lambda := \sqrt{20}\Gamma^*$ is an even (dual strongly perfect) lattice. So

$$3^{11} \mid \det(\Gamma) \mid 20^{13}$$

yields a contradiction. □

4. Dimension 15

4.1. Dual strongly perfect lattices of general type. Throughout this section let L be a 15-dimensional dual strongly perfect lattice. We go through the possibilities listed in Proposition 2.6, five of which where already excluded in Corollary 2.8.

Lemma 4.1. $r(L) \neq \frac{20}{3}$.

Proof. Assume that $r(L) = \frac{20}{3}$. Interchanging L and L^* if necessary and using Corollary 2.8, we may assume that $s(L) = 3^2 5 \cdot 17$. Rescale L such that $\min(L) = \frac{1}{3}$. From Equation $D4$ and $D22$ we find that $\Gamma := L^*$ is an integral lattice of minimum 20. Now $D22$ shows that $\Gamma^{(t)}$ is a sublattice of Γ of index 3, such that $\frac{1}{\sqrt{3}}\Gamma^{(t)}$ is integral. Hence $3^{13} \mid \det(\Gamma)$. If $s(L^*) \neq 3^3 17$, then a similar argument shows that $3^{13} \mid \det(\sqrt{240}\Gamma^*)$ hence $3^{26} \mid 240^{15}$, a contradiction. So $s(L^*) = 3^3 17$ and rescaled so that $\min(L) = 4$ the lattice L is integral (by (D4) and (D22)). Hence the even sublattice $L^{(e)}$ of L is of index 1 or 2. Moreover $\frac{1}{6}(D13 - D11)$ shows that $\frac{1}{\sqrt{2}}L^{(e)}$ is integral, so

$$2^{13} \mid \det(L) \leq \frac{\gamma_{15}}{\min(L^*)}^{15} < 4273 < 2^{13}$$

a contradiction. □

Lemma 4.2. $r(L) \neq 7$.

Proof. This is exactly the same argument as in Lemma 3.5. If $\min(L) = \frac{1}{3}$, then $\Gamma = L^*$ is an integral lattice of minimum 21. Moreover for any $x \in \mathrm{Min}(L)$ the set $N_2(x) = \{\alpha_1, \ldots, \alpha_7\}$ generates a rescaled root lattice A_7 and $\alpha := \alpha_1 + \alpha_2 + \alpha_3$, $\beta := \alpha_4$ leads to $\frac{1}{3}(D13 - D11)(\alpha, \beta) \notin \mathbb{Z}$. □

Lemma 4.3. $r(L) \neq 6$.

Proof. Assume that $r(L) = 6$ and put $s := |S(L)|$ and $t := |S(L^*)|$. Then $s, t \in \{510a \mid a \in \{2, 3\}\}$. All these possibilities can be excluded using Theorem 2.9. □

Lemma 4.4. *If L is a strongly perfect lattice of dimension 15 with $r(L) = 25/3$ then $s(L) = 459a$ with $a = 2$ or $a = 4$.*

Proof. Otherwise $s(L) = 459a$ for $a = 1, 3, 5$. Rescale L so that $\min(L) = \frac{1}{3}$ and put $\Gamma := L^*$. Then $\min(\Gamma) = 25$ and from $n_2(\alpha)$ and $D22$ we see that Γ is an integral lattice and $(\alpha, \alpha) \in 4\mathbb{Z}$ if $\alpha \in \Gamma$ has even norm. In particular the even sublattice $\Gamma^{(e)}$ has index ≤ 2 in Γ and $\min(\Gamma^{(e)}) \geq 28$. For the determinant of $\Gamma^{(e)}$ we get

$$\det(\Gamma^{(e)}) \leq 2^2 \det(\Gamma) \leq 2^2 \left(\frac{\gamma_{15}}{\min(L)}\right)^{15} < 4 \cdot 125106519575005.$$

From this one computes the value of the Hermite function on $\Gamma^{(e)}$ as $\gamma(\Gamma^{(e)}) > 2.93 > \gamma_{15}$ which is absurd. □

Lemma 4.5. $r(L) \neq 25/3$.

Proof. Rescale L so that $\min(L) = 10$. Then L and $\Gamma := \sqrt{12}L^*$ are even lattices with $\det(L)\det(\Gamma) = 2^{30}3^{15}$. From $(10/\gamma_{15})^{15} \leq \det(L) \leq (6\gamma_{15}/5)^{15}$ we obtain only the possibility

$$\det(L) = 2^{19}3^5, \det(\Gamma) = 2^{11}3^{10}$$

or vice versa. Without loss of generality assume that $\det(L) = 2^{19}3^5$. Since the maximal elementary divisor of a Gram matrix of L is 12 we obtain six possible genera of lattices. We construct lattices L_i in each of these genera and compute the first few coefficients of their theta series. If L is in the genus of L_i, then $S := \theta_{L\perp(6)} - \theta_{L_i\perp(6)}$ is a cusp form of weight 8 for $\Gamma_0(12)$. With MAGMA we compute a basis of this 11-dimensional space. Applying the Atkin-Lehner operator and using the fact that we know the first coefficients of $\theta_{L\perp(6)}$ and $\theta_{\Gamma\perp(2)}$, we end with a one parametric family of solutions for these theta series. This family does not contain an element with non-negative coefficients. □

Lemma 4.6. $r(L) \neq 8$.

Proof. Assume that $r(L) = 8$ and scale L so that $\min(L) = 2$, $\min(L^*) = 4$ and put $\Gamma := L^*$. Then $s(L) = 765a$, $s(L^*) = 765a'$ with $a, a' = 1, 2$. For $\alpha \in \mathrm{Min}(\Gamma)$

$$N_2(\alpha) = \{x \in \mathrm{Min}(L) \mid (x, \alpha) = 2\} = \{x_1, \ldots, x_k\}$$

with $k = 14a = a(n-1)$. For $a = 2$ we can rearrange $N_2(\alpha)$ so that $x_{2i-1} + x_{2i} = \alpha$ for $i = 1, .., 14$. Then $N_2(\alpha)$ spans a sublattice of L that is isometric to D_{15}. In particular $\det(L)$ is a square. If $a' = 2$ then we may interchange the role of L and L^* to obtain that also $2^{-15}\det(L^*)$ is a square, which is a contradiction. So we may assume that $a = 1$. Then

$$\Gamma^{(e,t)} = \{\alpha \in \Gamma \mid (\alpha, \alpha) \in \mathbb{Z}\}$$

is an even sublattice of Γ of index $1, 2, 3, 6$ with $\det(\Gamma^{(e,t)}) = 2^x 3^y$ with $1 \le x \le 15$, $0 \le y \le 2$. Moreover $\mathrm{Min}(\Gamma) = \mathrm{Min}(\Gamma^{(e,t)})$ so $\Gamma^{(e,t)}$ is strongly perfect of minimum 4, whence its dual minimum is $\ge \frac{17}{12}$. We compute all candidates of maximal even overlattices of $\Gamma^{(e,t)}$ and their sublattices of index 3. Proceeding to compute sublattices of 2-power index one does not find a lattice of minimum 4 whose dual has minimum $\ge \frac{17}{12}$. □

4.2. Dual strongly perfect lattices of minimal type. Let L be a dual strongly perfect lattice of minimal type, $\min(L) = 1$, $\min(L^*) = \frac{17}{3}$. Then by Remark 2.11 (b) $s(L) = 45s_1$ is divisible by 45. Moreover by Remark 2.11 (a) the lattice L^* contains some vector γ of norm $\frac{17}{3} < r := (\gamma, \gamma) < \frac{34}{3}$. So $n_2 := n_2(\gamma) \in \mathbb{N}$, $n_2 < \frac{s(n+2)}{18n}$ and r is a rational zero of $n_2(r) - n_2$. With this strategy and using the bounds from Remark 2.5 we find the following possibilities for s_1 and r:

s_1	16	17	25	
r	17/2	20/3, 7, 8, 29/3, 32/3, 11	136/15, 51/5	
s_1	32	34	48	49
r	17/2	6, 8, 26/3, 9, 29/3, 10, 32/3, 11	17/2	68/7
s_1	50	51	54	
r	136/15, 51/5	25/3, 28/3, 29/3, 32/3, 11	85/9	

Applying Theorem 2.9 to the possible pairs

$$(s(L), s(L^*)) \in \{16, 17, 25, 32, 34, 48, 49, 50, 51, 54\}^2$$

we always arrive at a contradiction. Combining this with the results of the previous section we have shown the following theorem.

Theorem 4.7. *There is no dual strongly perfect lattice in dimension* 15.

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Boris Venkov

Boris Venkov died in November 2011 before we could finish the paper