On the classification of lattices over $\mathbb{Q}(\sqrt{-3})$ which are even unimodular $\mathbb{Z}$-lattices of rank 32

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Abstract

We classify the lattices of rank 16 over the Eisenstein integers which are even unimodular $\mathbb{Z}$-lattices (of dimension 32). There are exactly 80 unitary isometry classes.

1 Introduction

Let $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ be the ring of integers in the imaginary quadratic field $K = \mathbb{Q}[\sqrt{-3}]$. An *Eisenstein lattice* is a positive definite Hermitian $\mathcal{O}$-lattice (Λ, h) such that the *trace lattice* (Λ, q), with $q(x,y) := trace_{K/\mathbb{Q}} h(x,y) = h(x,y) + \overline{h(x,y)}$ is an even unimodular $\mathbb{Z}$-lattice. The rank of the free $\mathcal{O}$-lattice Λ is $r = \frac{n}{2}$ where $n = \dim_{\mathbb{Z}}(\Lambda)$. Eisenstein lattices (or the more general theta-lattices introduced in [HKN2]) are of interest in the theory of modular forms, as their theta series is a modular form of weight r for the full Hermitian modular group with respect to $\mathcal{O}$ (cf. [HKN1].). The paper [HKN1] contains a classification of the Eisenstein lattices for $n = 8$, 16, and 24. In these cases one can use the classifications of even unimodular $\mathbb{Z}$-lattices by Kneser and Niemeier and look for automorphisms with minimal polynomial $X^2 - X + 1$.

For $n = 32$ this approach does not work as there are more than 10^9 isometry classes of even unimodular $\mathbb{Z}$-lattices (cf. [Ki, Corollary 17]). In this case we apply a generalisation of Kneser's neighbor method (compare [Sc]) over $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ to construct enough representatives of Eisenstein lattices and then use the mass formula developed in [HKN1] (and in a more general setting in [HKN2]) to check that the list of lattices is complete.

Given some ring R that contains $\mathcal{O}$, any R-module is clearly also an $\mathcal{O}$-module. In particular the classification of Eisenstein lattices can be used to obtain a classification of even unimodular $\mathbb{Z}$-lattices that are R-modules for the maximal order

$$R = \mathfrak{M}_{2,\infty} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+ij}{2}$$

respectively

$$R = \mathfrak{M}_{3,\infty} = \mathbb{Z} + \mathbb{Z}\frac{1+i\sqrt{3}}{2} + \mathbb{Z}j + \mathbb{Z}\frac{j+ij\sqrt{3}}{2},$$

where $i^2 = j^2 = -1$, $ij = -ji$, in the rational definite quaternion algebra of discriminant 2^2 respectively 3^2. For the Hurwitz order $\mathfrak{M}_{2,\infty}$ these lattices have been determined in [BN], the classification over $\mathfrak{M}_{3,\infty}$ is new (cf. [He]).

2 Statement of Results

Theorem. *The mass of the genus of Eisenstein lattices of rank* 16 *is*

$$\mu_{16} = \sum_{i=1}^{h} \frac{1}{|U(\Lambda_i)|} = \frac{16519 \cdot 3617 \cdot 1847 \cdot 809 \cdot 691 \cdot 419 \cdot 47 \cdot 13}{2^{31} \cdot 3^{22} \cdot 5^4 \cdot 11 \cdot 17} \sim 0.002.$$

There are exactly $h = 80$ *isometry classes* $[\Lambda_i]$ *of Eisenstein lattices of rank* 16.

Proof. The mass was computed in [HKN1]. The 80 Eisenstein lattices of rank 16 are listed in the following table with the order of their unitary automorphism group. These groups have been computed with MAGMA. We also checked that these lattices are pairwise not isometric. Using the mass formula one verifies that the list is complete. □

To obtain the complete list of Eisenstein lattices of rank 16 we first constructed some lattices as orthogonal sums of Eisenstein lattices of rank 12 and 4 and from known 32-dimensional even unimodular lattices. We also applied coding constructions from ternary and quaternary codes in the same spirit as described in [Ba]. To this list of lattices we applied Kneser's neighbor method. For this we made use of the following facts (cf. [Sc]): Let Γ be an integral $\mathcal{O}$-lattice and $\mathfrak{p}$ a prime ideal of $\mathcal{O}$ that does not divide the discriminant of Γ. An integral $\mathcal{O}$-lattice Λ is called a $\mathfrak{p}$-neighbor of Γ if

$$\Lambda/(\Gamma \cap \Lambda) \cong \mathcal{O}/\mathfrak{p} \text{ and } \Gamma/(\Gamma \cap \Lambda) \cong \mathcal{O}/\overline{\mathfrak{p}}.$$

All $\mathfrak{p}$-neighbors of a given $\mathcal{O}$-lattice Γ can be constructed as

$$\Gamma(\mathfrak{p}, x) := \mathfrak{p}^{-1}x + \Gamma_x, \quad \Gamma_x := \{y \in \Gamma \mid h(x,y) \in \mathfrak{p}\},$$

where $x \in \Gamma \setminus \mathfrak{p}\Gamma$ with $h(x,x) \in \mathfrak{p}\overline{\mathfrak{p}}$ (such a vector is called *admissible)*. We computed (almost random) neighbors (after rescaling the already computed lattices to make them integral) for the prime elements 2, $2-\sqrt{-3}$, and $4-\sqrt{-3}$ by randomly choosing admissible vectors x from a set of representatives and constructing $\Gamma(\mathfrak{p}, x)$ or all integral overlattices of Γ_x of suitable index. For details of the construction we refer to [Sc].

Corollary. *There are exactly* 83 $\mathfrak{M}_{3,\infty}$*-lattices of rank* 8 *that yield even unimodular* $\mathbb{Z}$*-lattices of rank* 32.

Proof. Since $\mathfrak{M}_{3,\infty}$ is generated by its unit group $\mathfrak{M}_{3,\infty}^* \cong C_3 : C_4$ one may determine the structures over $\mathfrak{M}_{3,\infty}$ of an Eisenstein lattice Γ as follows. Let $\frac{-1+\sqrt{-3}}{2} =: \sigma \in U(\Gamma)$ be a third root of unity. If the $\mathcal{O}$-module structure of Γ can be extended to a $\mathfrak{M}_{3,\infty}$ module structure, the $\mathcal{O}$-lattice Γ needs to be isometric to its complex conjugate lattice $\overline{\Gamma}$. Let τ_0 be such an isometry, so

$$\tau_0 \in \mathrm{GL}_{\mathbb{Z}}(\Gamma),\ \tau_0\sigma = \sigma^{-1}\tau_0 \text{ and } h(\tau_0 x, \tau_0 y) = \overline{h(x,y)} \text{ for all } x, y \in \Gamma.$$

Let

$$U'(\Gamma) := \langle U(\Gamma), \tau_0 \rangle \cong U(\Gamma).C_2.$$

Then we need to find representatives of all conjugacy classes of elements $\tau \in U'(\Gamma)$ such that

$$\tau^2 = -1 \text{ and } \tau\sigma = -\sigma^2\tau.$$

This can be shown as in [KM] in the case of the Gaussian integers. □

Alternatively, one can classify these lattices directly using the neighbor method and a mass formula, which can be derived from the mass formula in [Ha] as in [BN]. The results are contained in [He]. For details on the neighbor method in a quaternionic setting we refer to [Co].

The Eisenstein lattices of rank up to 16 are listed in the following tables ordered by the number of roots. For the sake of completeness we have included the results from [HKN1] in rank 4, 8 and 12. R denotes the root system of the corresponding even unimodular $\mathbb{Z}$-lattice (cf. [CS, Ch. 4]). In the column $\sharp$Aut the order of the unitary automorphism group is given. The next column contains the number of structures of the lattice over $\mathfrak{M}_{3,\infty}$. For lattices with a structure over the Hurwitz quaternions $\mathfrak{M}_{2,\infty}$ (note that $(i+j+ij)^2 = -3$, so all lattices with a structure over $\mathfrak{M}_{2,\infty}$ have a structure over $\mathcal{O}$), the name of the corresponding Hurwitz lattice used in [BN] is given in the last column.

Table 1: The lattice of rank 4

no.	R	$\sharp$Aut	$\mathfrak{M}_{3,\infty}$	$\mathfrak{M}_{2,\infty}$
1	E_8	155520	1	E_8

Table 2: The lattice of rank 8

no.	R	$\sharp$Aut	$\mathfrak{M}_{3,\infty}$	$\mathfrak{M}_{2,\infty}$
1	$2E_8$	48372940800	2	$2E_8$

Table 3: The lattices of rank 12

no.	R	$\sharp$Aut	$\mathfrak{M}_{3,\infty}$	$\mathfrak{M}_{2,\infty}$
1	$3E_8$	22568879259648000	2	$3E_8$
2	$4E_6$	8463329722368	1	
3	$6D_4$	206391214080	1	$L_6(\mathfrak{P}^6)$
4	$12A_2$	101016305280	1	
5	$\emptyset$	2690072985600	1	Λ_{24}

Table 4: The lattices of rank 16

no.	R	$\sharp$Aut	$\mathfrak{M}_{3,\infty}$	$\mathfrak{M}_{2,\infty}$
1	$4E_8$	14039648409841827840000	3	$4E_8$
2	$4E_6+E_8$	1316217038422671360	1	
3	$6D_4+E_8$	32097961613721600	1	$E_8 \perp L_6(\mathfrak{P}^6)$
4	$12A_2+E_8$	15710055797145600	1	
5	$4A_2+4E_6$	2742118830047232	1	
6	$4D_4+2E_6$	40122452017152	1	
7	E_8	418360150720512000	1	$E_8 \perp \Lambda_{24}$
8	$10A_2+2E_6$	71409344532480	1	
9	$8D_4$	443823666757632	2	$L_8(\mathfrak{P}^8)$
10	$4A_2+3D_4+E_6$	313456656384		
11	$13A_2+E_6$	11604018486528		
12	$6D_4$	825564856320	1	
13	$6A_2+D_4+E_6$	48977602560		
14	$4A_2+4D_4$	15479341056	1	
15	$7A_2+E_6$	21427701120		
16	$16A_2$	1851353376768	3	
17	$8A_2+2D_4$	8707129344	1	
18	$4A_2+3D_4$	1451188224		
19	$4A_2+E_6$	9795520512		
20	$4D_4$	82556485632	1	$L_8(\mathfrak{P}^4)$
21	D_4+E_6	1277045637120		
22	$6A_2+2D_4$	302330880	2	
23	$9A_2+D_4$	1836660096		
24	A_2+E_6	22448067840		
25	$4A_2+2D_4$	107495424	1	
26	$7A_2+D_4$	52907904		
27	$10A_2$	408146688	1	
28	$6A_2+D_4$	22674816		
29	$2A_2+2D_4$	134369280	1	
30	$5A_2+D_4$	8398080		
31	$8A_2$	423263232	2	
32	$8A_2$	7558272	4	
33	$4A_2+D_4$	4478976		
34	$2D_4$	7644119040	1	$L_8(\mathfrak{P}^2)$
35	$2D_4$	656916480	1	
36	$7A_2$	1530550080		
37	$7A_2$	2834352		
38	$3A_2+D_4$	113374080		
39	$3A_2+D_4$	2519424		
40	$6A_2$	1679616	1	
41	$6A_2$	629856	2	
42	$2A_2+D_4$	1710720		

	R	$\sharp$Aut	$\mathfrak{M}_{3,\infty}$	$\mathfrak{M}_{2,\infty}$
43	$5A_2$	139968		
44	A_2+D_4	3265920		
45	A_2+D_4	2426112		
46	$4A_2$	161243136	2	
47	$4A_2$	68024448	1	
48	$4A_2$	4199040	2	
49	$4A_2$	1399680	1	
50	$4A_2$	314928		
51	$4A_2$	139968	1	
52	$4A_2$	69984	3	
53	D_4	660290641920		
54	D_4	1813985280		
55	D_4	87091200		$L_8(\mathfrak{P})$
56	D_4	1990656		
57	$3A_2$	58320		
58	$3A_2$	15552		
59	$2A_2$	606528		
60	$2A_2$	186624	1	
61	$2A_2$	41472	1	
62	$2A_2$	25920		
63	$2A_2$	18144	2	
64	$2A_2$	18144	2	
65	$2A_2$	16200	4	
66	A_2	2204496		
67	A_2	108864		
68	A_2	3888		
69	A_2	2916		
70	$\emptyset$	303216721920	2	BW_{32}, Λ''_{32}
71	$\emptyset$	15552000	5	Λ'_{32}
72	$\emptyset$	9289728	3	Λ_{32}
73	$\emptyset$	1658880	1	
74	$\emptyset$	387072	3	
75	$\emptyset$	29376	2	
76	$\emptyset$	10368	1	
77	$\emptyset$	8064	2	
78	$\emptyset$	5760	4	
79	$\emptyset$	4608	2	
80	$\emptyset$	2592	3	

A list of the Gram matrices of the lattices is given in [Ht].

Remark. *a) The* 80 *corresponding* $\mathbb{Z}$*-lattices belong to mutually different* $\mathbb{Z}$*-isometry classes.*

b) Each of the lattices listed above is isometric to its conjugate. Hence the associated Hermitian theta series are symmetric Hermitian modular forms (cf. [HKN2]).

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