Automorphisms of extremal unimodular lattices in dimension 72.

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ABSTRACT. The paper narrows down the possible automorphisms of extremal even unimodular lattices of dimension 72. With extensive computations in MAGMA using the very sophisticated algorithm for computing class groups of algebraic number fields written by Steve Donnelly it is shown that the extremal even unimodular lattice Γ_{72} from [17] is the unique extremal even unimodular lattice of dimension 72 that admits a large automorphism, where a $d \times d$ matrix is called large, if its minimal polynomial has an irreducible factor of degree $> d/2$.

1 Introduction

This paper is part of a series of papers classifying extremal lattices with given automorphism.

A *lattice* L is a finitely generated $\mathbb{Z}$-submodule of full rank in Euclidean space $(\mathbb{R}^d, (,))$, so there is some basis $(b_1, \ldots, b_d)$ such that $L = \{\sum_{i=1}^{d} a_i b_i \mid a_i \in \mathbb{Z}\}$. The *automorphism group* of L is its stabilizer in the orthogonal group

$$\text{Aut}(L) = \{g \in \text{GL}_d(\mathbb{R}) \mid g(L) = L \text{ and } (g(x), g(y)) = (x, y) \text{ for all } x, y \in \mathbb{R}^d\}.$$

With respect to a basis of L, $\text{Aut}(L)$ is a finite integral matrix group, $\text{Aut}(L) \leq \text{GL}_d(\mathbb{Z})$.

The *dual lattice* is $L^{\#} := \{x \in \mathbb{R}^d \mid (x, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}$. We call L *integral* if $L \subseteq L^{\#}$ and *even* if $(\ell, \ell) \in 2\mathbb{Z}$ for all $\ell \in L$. The *minimum* of L is $\min(L) := \min\{(\ell, \ell) \mid \ell \in L, \ell \neq 0\}$.

Even unimodular lattices are even lattices L with $L = L^{\#}$. They only exist if the dimension is a multiple of 8. The theory of modular forms allows to bound the minimum of a d-dimensional even unimodular lattice by $\min(L) \leq 2\lfloor \frac{d}{24} \rfloor + 2$ (see for instance [11]) and lattices achieving equality are called *extremal*. Very particular lattices are extremal lattice whose dimension is a multiple of 24: Their sphere packing density realises a local maximum among all lattice sphere packings in this dimension and the vectors of given length in such lattices form spherical 11-designs (see e.g. [3]). In dimension 24 there is a unique extremal lattice, the Leech lattice Λ_{24}. The Leech lattice is the densest lattice in dimension 24, its automorphism group is a covering group of the sporadic simple Conway group. In dimension 48 one knows 4 extremal lattices (see [19]). The paper [18] shows that there are no other 48-dimensional extremal lattices that admit a large automorphism. The aim of the present paper is to establish a similar classification in dimension 72. More precisely we prove the following theorem.

Theorem 1.1. *Let Γ be an extremal lattice of dimension 72 that admits an automorphism σ such that the minimal polynomial of σ has an irreducible factor of degree > 36. Then $\Gamma = \Gamma_{72}$ is the unique known extremal lattice of dimension 72 constructed in [17].*

2 Some important lattices

In this paper we denote the n-th *cyclotomic polynomial* by Φ_n and its degree, the value of the *Euler phi function* at n, by $\varphi(n)$. To start with let us introduce some important lattices. The best studied family of lattices are certainly the *root lattices*. These are even lattices that are generated by vectors ℓ of norm $(\ell, \ell) = 2$. All root lattices are orthogonal sums of the indecomposable lattices $\mathbb{A}_d$, $\mathbb{D}_d$ $(d \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (see for instance [11]). The next result is certainly well known.

Proposition 2.1. *Let R be a root lattice admitting an automorphism $g \in \mathrm{Aut}(R)$ of prime order $p > 2$ with irreducible minimal polynomial Φ_p. Then g stabilises each orthogonal summand of R. If $p = 3$ then R is an orthogonal sum of copies of $\mathbb{A}_2$, $\mathbb{D}_4$, $\mathbb{E}_6$ or $\mathbb{E}_8$. For $p = 5$ the orthogonal summands are $\mathbb{A}_4$ and $\mathbb{E}_8$ and for $p \geq 7$ the lattice R is an orthogonal sum of copies of $\mathbb{A}_{p-1}$.*

<u>Proof.</u> Write $R = R_1^{n_1} \oplus \ldots \oplus R_s^{n_s}$ with pairwise non isometric indecomposable root lattices R_i. Then $\mathrm{Aut}(R) \cong \mathrm{Aut}(R_1) \wr S_{n_1} \times \ldots \times \mathrm{Aut}(R_s) \wr S_{n_s}$. Let $\pi : \mathrm{Aut}(R) \to S_{n_1} \times \ldots \times S_{n_s}$ denote the corresponding group homomorphism. Then $g \in \ker(\pi)$, because $\Phi_p(g) = 0$, so g stabilizes all orthogonal summands of R. As Φ_p is irreducible, this shows that g acts on each of the summands R_i with minimal polynomial Φ_p. Now $\mathrm{Aut}(\mathbb{A}_d) = \langle -1 \rangle \times S_{d+1}$ contains an element with minimal polynomial Φ_p if and only if S_{d+1} has a transitive permutation of order p, so if and only if $d + 1 = p$. If $d > 4$ then $\mathrm{Aut}(\mathbb{D}_d) \cong C_2 \wr S_d$ contains no element with minimal polynomial Φ_p and for $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ the result follows by inspection of the conjugacy classes of elements in the automorphism group. $\qquad\square$

Closely related to the root lattices are the so called *Niemeier lattices*, by which we mean the even unimodular lattices of dimension 24. B. Venkov ([10, Chapter 18]) has shown that if L is an even unimodular lattice of dimension 24, then its *root sublattice R*, which is the sublattice of L generated by the vectors of norm 2 in L, is either $\{0\}$ (and then $L = \Lambda_{24}$ is the Leech lattice, the unique extremal lattice of dimension 24) or R has full rank. Moreover an explicit classification shows that there are 23 possible $R \neq \{0\}$ and that L is then uniquely determined by R. We will denote the Niemeier lattice L with root sublattice R by $N(R)$. If R is non empty then $\mathrm{Aut}(N(R)) \leq \mathrm{Aut}(R)$. As the Leech lattice admits an automorphism of order 7 with irreducible minimal polynomial this shows the following corollary.

Corollary 2.2. *Let L be some even unimodular lattice of dimension 24 admitting an automorphism σ with minimal polynomial Φ_7. Then either $L \cong \Lambda_{24}$ or $L \cong N(\mathbb{A}_6^4)$.*

The fixed lattice L of an automorphism of prime order p of some even unimodular lattice is an even p-elementary lattice, i.e. an even lattice L such that $pL^{\#} \subseteq L$. The possible genera of such lattices are described in [10, Theorem 13, Chapter 15]. The following result holds in greater generality but we only need it for $p = 3$.

Lemma 2.3. *Let L be some even 3-elementary lattice of dimension $n \leq 24$ such that $\min(L^\#) > 2$. Then $L = L^\# = \Lambda_{24}$.*

Proof. As L is an even lattice of odd determinant, the dimension $n = 2m$ must be even. By [10, Theorem 13, Chapter 15] there is a genus of even 3-elementary lattices of determinant 3^k and dimension $2m$ if and only if $k \equiv m \pmod 2$ and this genus is unique. In particular we may always add some orthogonal sum R of root lattices $\mathbb{A}_2, \mathbb{E}_6$, and $\mathbb{E}_8$ so that $M := L \oplus R$ is of dimension 24. By the uniqueness of the 3-elementary genus, the lattice M is contained in some even unimodular lattice Λ of dimension 24, so

$$M = L \oplus R \leq \Lambda \leq L^\# \oplus R^\# = M^\#.$$

If $\min(L^\#) > 2$, then the root system of Λ is contained in $R^\#$, in particular it cannot have full rank. Venkov's classification of the 24-dimensional even unimodular lattices shows that this implies that the root system of Λ is empty, so $\Lambda = \Lambda_{24}$ is the Leech lattice, $R = \{0\}$ and $\Lambda_{24} \subseteq L^\# \subseteq \frac{1}{3}\Lambda_{24}$. By [14, Theorem 4.1] all non-zero classes of $\frac{1}{3}\Lambda_{24}/\Lambda_{24}$ contain vectors of norm $\leq \frac{18}{9} = 2$. So $L^\# = \Lambda_{24} = L$. $\square$

Among the possible fixed lattices of automorphisms of order 5 there is one famous lattice that was already studied by J. Tits [24] and which we call $\mathcal{T}_{24}$, the *Tits lattice*. It is a 5-elementary lattice of dimension 24, minimum 8, and automorphism group $(2.J_2 \circ SL_2(5)) : 2$, that is isometric to its rescaled dual lattice, $\mathcal{T}_{24} \cong \sqrt{5}\mathcal{T}_{24}^\#$. It is hence an extremal 5-modular lattice in the sense of [21].

Lemma 2.4. *Let L be a 24-dimensional 5-elementary even lattice with determinant 5^{12} and minimum 8.*

(a) *If L has an automorphism g of order 21 with minimal polynomial Φ_{21}, such that the minimal polynomial of the action of g on the discriminant group $L^\#/L \cong \mathbb{F}_5^{12}$ is also Φ_{21}, then $L \cong \mathcal{T}_{24}$.*

(b) *If L has an automorphism g of order 28 with minimal polynomial Φ_{28}, such that the minimal polynomial of the action of g on the discriminant group $L^\#/L \cong \mathbb{F}_5^{12}$ is also Φ_{28}, then $L \cong \mathcal{T}_{24}$.*

Proof. In both cases the minimal polynomial μ_g is a cyclotomic polynomial and so L can be viewed as some 2-dimensional Hermitian lattice over the ring of integers in a cyclotomic number field. Both parts of the lemma follow by computing the full genus of these Hermitian lattices but the techniques are different, due to the fact that in part (a) the lattices are maximal $\mathbb{Z}[\zeta_{21}]$-lattices whereas in part (b), the $\mathbb{Z}[\zeta_{28}]$-lattice L embeds into two even unimodular $\mathbb{Z}[\zeta_{28}]$-lattices and we may use the classification of even unimodular 24-dimensional $\mathbb{Z}$-lattices.

(a) Here $\mu_g \equiv f_1 f_2 \in \mathbb{F}_5[x]$ is a product of two irreducible polynomials modulo 5, where the inverses of the roots of f_1 are again roots of f_1. So as a $\mathbb{F}_5[g]$-module

$$L^\#/L \cong \mathbb{F}_5[x]/f_1 \oplus \mathbb{F}_5[x]/f_2 \cong S_1 \oplus S_2$$

is a direct sum of two self-dual simple modules. This shows that L is an $\mathcal{A}$-maximal Hermitian lattice, where $\mathcal{A}$ is the discriminant of $\mathbb{Z}[\zeta_{21}]$. Using the mass-formula

[12, Proposition 4.5] one computes the mass of the full genus of Hermitian lattices L with $L^{\#}/L \cong S_1 \oplus S_2$ to be

$$m_{21} := 2^{-2\cdot 6} \cdot \frac{48}{63} \cdot \frac{16}{3} \cdot 2 \cdot (5^3 - 1)^2 = \frac{1922}{63}.$$

Starting with the lattice $L := \mathcal{T}_{24}$ we use the Kneser neighboring method for both ideals dividing 7 as described in [22] to enumerate the full genus of L. The class number is 1286, there are 223 lattice of minimum 4, 1062 lattice of minimum 6 and a unique lattice of minimum 8, namely $\mathcal{T}_{24}$.

(b) Again $\mu_g \equiv f_1 f_2 \in \mathbb{F}_5[x]$ is a product of two irreducible polynomials modulo 5, but the inverses of the roots of f_1 are roots of f_2. So as a $\mathbb{F}_5[g]$-module

$$L^{\#}/L \cong \mathbb{F}_5[x]/f_1 \oplus \mathbb{F}_5[x]/f_2 \cong S \oplus S^*$$

is a direct sum of a simple module S and its dual. So both invariant sublattices of $L^{\#}$ that contain L are unimodular. In particular for any given lattice L there is a unique even unimodular g-invariant lattice M containing L such that $M/L \cong S$. Let $\wp := (f_1(\zeta_{28}), 5) \trianglelefteq \mathbb{Z}[\zeta_{28}]$ be the corresponding prime ideal dividing 5 and view L and M as $\mathbb{Z}[\zeta_{28}]$-lattices via the action of g. Then $M/\wp M \cong S \oplus S$ and $L/\wp M$ is one of the $5^6 + 1 = 15626$ non-trivial submodules of this module.

There are 2 even unimodular lattices M whose automorphism group contains an element a of order 7 with an irreducible minimal polynomial, $M_1 = \Lambda_{24}$, the Leech lattice and $M_2 = N(\mathbb{A}_6^4)$ the Niemeier lattice with root system $\mathbb{A}_6^4$. The automorphism group of each of these lattices contains a unique element of order 28 with centraliser of order 168 resp. 196. This again may be verified with the mass formula in [12], which yields

$$2^{-2\cdot 6} \cdot \frac{8}{7} \cdot \frac{416}{21} \cdot 2 = 13/1176 = 1/168 + 1/196$$

as the mass of the genus of M. So the mass of the genus of L is $(5^6 + 1)\frac{13}{1176} \sim$ 172.736... and one could again use the Kneser neighboring method to enumerate the genus. But here it is much faster to compute the lattices L as sublattices of M_1 resp. M_2. There are 2606 orbits of invariant lattices L of M_1 under the action of the centraliser of g in $\mathrm{Aut}(M_1)$, 703 of them have minimum 4, 1902 have minimum 6 and a unique sublattice has minimum 8. Again this lattice is isometric to $\mathcal{T}_{24}$. For M_2 one finds 2234 orbits, 2 of minimum 2, 606 of minimum 4, and 1626 of minimum 6.

$\square$

To exclude automorphisms of order 25 we classified a certain genus of 2-dimensional hermitian lattices over $\mathbb{Z}[\zeta_{25}]$:

Lemma 2.5. *Let Z be an even lattice of dimension* 40 *that admits an automorphism $\sigma \in$* $\mathrm{Aut}(Z)$ *with minimal polynomial Φ_{25} and such that $(1 - \sigma^5)Z^{\#} = Z$. Then $\min(Z) \leq 6$.*

<u>Proof.</u> To compute the mass of the genus of lattices Z let $L := (1 - \sigma)^{-2}Z$. Then $L^{\#} = (1 - \sigma)^2 Z^{\#} = (1 - \sigma)^{-1}L$. As there is a bijection between the lattices L and Z the

mass of the genus of Hermitian $\mathbb{Z}[\zeta_{25}]$-lattices L and Z coincide. Also L is not a maximal lattice, $L^\#/L$ is isomorphic to the hyperbolic plane over $\mathbb{F}_5$ and there are two unimodular σ-invariant overlattices M of L. On the other hand, $M \supseteq L \supseteq (1-\sigma)M$, so L corresponds to one of the 6 one-dimensional subspaces of $M/(1-\sigma)M \cong \mathbb{F}_5^2$. Using the Venkov trick (as explained in [1, Proposition 2.4]) we hence get for the masses of the genera of Hermitian $\mathbb{Z}[\zeta_{25}]$-lattices

$$\mathrm{mass}(Z) = \mathrm{mass}(L) = \frac{6}{2}\mathrm{mass}(M) = \frac{6}{2} \cdot 2^{-2 \cdot 10} \cdot \frac{512}{25} \cdot \frac{5825408}{75} \cdot 2 = \frac{45511}{5000}$$

where we used the mass formula in [12] to compute the mass of the genus of M.

Starting with the lattice $\mathbb{A}_4^{10}$ we enumerate the lattices in the genus of Z using Kneser neighboring method until we found enough lattices Z. They fall into 50 $\mathbb{Z}$-isometry classes of lattices. Summing up the reciprocals of the orders of the centralizers $C_G(z)$, for representatives of all conjugacy classes of automorphisms $z \in G = \mathrm{Aut}_{\mathbb{Z}}(Z)$ such that the minimal polynomial of z is Φ_{25} we find almost the full mass up to a summand $3/250$. The remaining lattices are constructed "by hand": There is an obvious even unimodular lattice $M = \mathbb{E}_8^5$ that has an automorphism of order 25 with irreducible minimal polynomial. Starting with this lattice M we constructed a sublattice Z as illustrated above. $G = \mathrm{Aut}_{\mathbb{Z}}(Z)$ has 3 conjugacy classes of elements of order 25 with irreducible minimal polynomial and centralisers of order 500, 250, 250, respectively. The last lattice was found as a sublattice of index 25 of Z invariant under this group $C_G(z)$ of order 500 by rescaling the form by $(1-z)(1-z^{-1})$. This lattice has the same Hermitian automorphism group but already the $\mathbb{Z}$-lattices are not isometric to each other. So in total there are 52 $\mathbb{Z}$-isometry classes of lattices Z, which split into 482 isometry classes of Hermitian $\mathbb{Z}[\zeta_{25}]$-lattices. None of the $\mathbb{Z}$-lattices Z has minimum ≥ 8. $\qquad\square$

A remark seems to be in place how to handle high dimensional lattices with the computer. There is no fast algorithm known that can compute the minimum of a, say, 72-dimensional lattice from its Gram matrix. But there are algorithms to find small vectors in these lattices. We never needed to prove extremality of any of the computed lattices, the only task was to find vectors of norm ≤ 6. This was done by a combination of the fast available reduction algorithms in MAGMA.

Remark 2.6. *To prove that an even lattice L given by its Gram matrix F has minimum < 8 we performed the following reduction steps in* MAGMA:
F:=LLLGram(F:Delta:=0.99,Eta:=0.501);
F:=SeysenGram(F);
F:=PairReduceGram(F);
until the minimum diagonal entry of F becomes ≤ 6. If this minimum is ≥ 8 after 1000 *loops then we replaced F by some equivalent matrix by transforming it by some small random element in* $\mathrm{GL}_n(\mathbb{Z})$. *If L is unimodular then $F \in \mathrm{GL}_n(\mathbb{Z})$ and we may take the equivalent matrix to be $F^{-1} = F^{-1}FF^{-tr}$. We then repeated* 1000 *loops of the reduction steps once more. If during these procedures no diagonal entry became smaller than 8, there is a fair chance that the minimum of L is ≥ 8.*

3 Automorphisms of prime order

The notion of the type of an automorphism of a lattice L was introduced in [18]. It is motivated by the analogous notion of a type of an automorphism of a code.

Let $\sigma \in \mathrm{GL}_d(\mathbb{Q})$ be an element of prime order p. Let $\tilde{F} := \ker(\sigma - 1)$ and $\tilde{Z} := \mathrm{im}(\sigma - 1)$. Then $\tilde{F}$ is the fixed space of σ and the action of σ on $\tilde{Z}$ gives rise to a vector space structure on $\tilde{Z}$ over the p-th cyclotomic number field $\mathbb{Q}[\zeta_p]$. In particular $d = f + z(p-1)$, where $f := \dim_{\mathbb{Q}}(\tilde{F})$ and $z = \dim_{\mathbb{Q}[\zeta_p]}(\tilde{Z})$.

If L is a σ-invariant $\mathbb{Z}$-lattice, then L contains a sublattice M with

$$L \geq M = (L \cap \tilde{F}) \oplus (L \cap \tilde{Z}) =: F(\sigma) \oplus Z(\sigma) = F \oplus Z \geq pL$$

of finite index $[L : M] = p^s$ where $s \leq \min(f, z)$.

Definition 3.1. *The triple* $p - (z, f) - s$ *is called the* type *of the element* $\sigma \in \mathrm{GL}(L)$.

This section is devoted to the study of the possible types of automorphisms of prime order of extremal even unimodular lattices of dimension $24m$. The arguments are similar to the ones in [18, Section 4.1], where we obtained contradictions by considering the lattices F and Z. Just for further reference we recall some results from [18] and [19] that will be used in the following arguments without further notice.

Remark 3.2. *Let L be an even unimodular lattice and $\sigma \in \mathrm{Aut}(L)$ of prime order p and type $p - (z, f) - s$.*

(a) $s \leq \min(z, f)$ *and* $s \equiv z \pmod{2}$.

(b) *If $s = 0$ then z is even.*

(c) *If $s = z$ then $\sqrt{p}Z^{\#}$ is the trace lattice of an Hermitian unimodular $\mathbb{Z}[\zeta_p]$ lattice of rank z.*

(d) *If $s = f$ and p is odd, then $\frac{1}{\sqrt{p}}F$ is an even unimodular lattice. In particular f is a multiple of 8.*

The arguments below will be based on the fact that both lattices F and Z have minimum $\geq \min(L)$ and determinant $\leq p^z$. So the Hermite function is

$$\gamma(F) \geq (2m + 2)/(p^{z/f}) \text{ respectively } \gamma(Z) \geq (2m + 2)/(p^{1/(p-1)}).$$

Already Blichfeldt [6] obtained a good upper bound on the *Hermite constant* γ_d, the maximum value of the Hermite function on d-dimensional lattices:

$$\gamma_d \leq \frac{2}{\pi}\Gamma(2 + \frac{d}{2})^{2/d} =: B(d).$$

The currently best upper bounds for $d \leq 36$ are given in [8].

Theorem 3.3. *Let L be an extremal even unimodular lattice of dimension $24m$ and $\sigma \in \mathrm{Aut}(L)$ of prime order p and type $p - (z, f) - s$ with $z = 1$. Then $p = 23$, $m = 1$, and $L = \Lambda_{24}$ or $p = 47$, $m = 2$, and $L = P_{48q}$.*

<u>Proof.</u> By Remark 3.2 we get $s \leq \min(z, f)$ and $s \equiv z \pmod{2}$. So if $z = 1$ then $s = 1$ and in particular $f \geq 1$. This excludes that $p = 24m + 1$ (this also follows from [4], as the characteristic polynomial of σ is then the irreducible polynomial Φ_p).

Assume now that $p = 24m - 1$, so $f = 2$. Then F is a 2-dimensional lattice of minimum $2m + 2$ and determinant $p = 24m - 1$, so

$$\gamma(F)^2 = \frac{(2m + 2)^2}{24m - 1}.$$

As the hexagonal lattice is the densest 2-dimensional lattice we have $\gamma(F)^2 \leq 4/3$ from which we immediately compute that $m \leq 6$. If $m = 4, 5, 6$ then $24m - 1$ is not a prime. The case $m = 1$ is clear because the Leech lattice has an automorphism of order 23, for $m = 2$ the result is [18, Theorem 5.6]. It remains to consider the case $m = 3$, so $p = 71$. This case has been already treated by Skoruppa [23]. Then Z is an ideal lattice in $\mathbb{Q}[\zeta_{71}]$ of determinant 71 and minimum 8. With Magma we checked that all such ideal lattices have minimum ≤ 6.

So from now on we may assume that $p \leq 24m - 3$.

First assume that $p \leq 12m - 1$. Then Z is $(p - 1)$-dimensional lattice of determinant p and minimum $\geq 2m + 2$. By Blichfeldt's bound we get

$$\frac{2m + 2}{p^{1/(p-1)}} \leq \gamma(Z) \leq B(p - 1) = \frac{2}{\pi}((\frac{p - 1}{2} + 1)!)^{2/(p-1)}.$$

Put $a := \frac{p-1}{2}$ and take the a-th power to obtain

$$(\star) \quad (\pi(m + 1))^a \leq \sqrt{2a + 1}(a + 1)a! \leq \sqrt{2a + 1}(a + 1)(1 + \frac{1}{11a})\sqrt{2\pi a}\left(\frac{a}{e}\right)^a$$

by Stirling's formula. Now we use the assumption that $p \leq 12m - 1$, so $m \geq \frac{a+1}{6} > \frac{a}{6}$ to conclude that

$$\left(\frac{e\pi}{6}\right)^a \leq \sqrt{\pi}(a + 1)(2a + 1)(1 + \frac{1}{11a}) < 6a^2$$

for $a > 3$. As $\frac{e\pi}{6} > \sqrt{2}$ we find $2^{a/2} \leq 6a^2$ which implies that $a \leq 23$. As we know all automorphisms of the Leech lattice and the case $m = 2$ is already treated in [18] we may assume that $m \geq 3$. But no $1 \leq a \leq 23$ satisfies the first inequality in $(\star)$ for $m = 3$.

Now assume that $12m + 1 \leq p \leq 24m - 3$. Then we consider the lattice F of determinant p, minimum $\geq 2m + 2$, and dimension $2a$ with

$$2 \leq a = (24m - p + 1)/2 \leq (24m - (12m + 1) + 1)/2 = 6m.$$

Blichfeldt's bound gives us

$$(\star\star) \quad \frac{2m + 2}{p^{1/(2a)}} \leq \gamma(F) \leq \frac{2}{\pi}((a + 1)!)^{1/a}$$

and as above we take the a-th power and apply Stirling's formula to simplify the right hand side and obtain

$$\left(\frac{e\pi(m + 1)}{a}\right)^a \leq \sqrt{p}\sqrt{2\pi a}(a + 1)(1 + \frac{1}{11a}).$$

To bound $\sqrt{p}$ on the right hand side, we divide by $\left(\frac{e\pi(m+1)}{a}\right)^{1/2}$ to obtain

$$\left(\frac{e\pi(m + 1)}{a}\right)^{a - \frac{1}{2}} \leq \sqrt{\frac{p}{m + 1}}\sqrt{\frac{2}{e}}a(a + 1)(1 + \frac{1}{11a}).$$

As $a \leq 6m \leq 6m + 6$ the left hand side is again lower bounded by $(e\pi/6)^{a-\frac{1}{2}}$ and hence by $\sqrt{2}^{a-\frac{1}{2}}$. For the right hand side (rhs) we use that $p \leq 24m \leq 25(m+1)$ and $2/e \leq 1$ to obtain (rhs) $\leq 5a(a+1)(11a+1)/(11a) \leq 6(a-\frac{1}{2})^2$ whenever $a \geq 12$. As before we conclude

7

that $a \le 23$. For each value of $a \in \{2, \ldots, 23\}$ the inequality $(\star\star)$ can be expressed as the negativity of a polynomial in m of degree $2a$ (after substituting $p = 24m - 2a + 1$ and rising to the $2a$-power) for which explicit computations show that the unique negative coefficient is the one of m^1. This gives an easy bound on the maximal m for which the polynomial could possibly be positive. For $a > 1$ this m is always < 4 and it is immediately checked that these polynomials are always positive for $2 \le a \le 6m + 1$. $\qquad\square$

Similar arguments can also be applied to treat other series of parameters, but as these are usually immediately ruled out by direct computations in any concrete dimension, we omit the heavy calculations. Note also that one might not expect such a general theorem for even z, because here $24m = z(p - 1)$ and $L = Z$ is always one possibility.

3.1 Dimension 72

Theorem 3.4. *Let Γ be some extremal even unimodular lattice of dimension 72 and $-I_{72} \ne \sigma \in \mathrm{Aut}(\Gamma)$ be some element of prime order p. Then either $p \in \{37, 19, 13, 7\}$ and the minimal polynomial of σ is irreducible (so σ has type $p - (72/(p-1), 0) - 0$) or $p \le 5$ where the possible types are*

$p=5$ $5 - (z, 72 - 4z) - z$ *with* $9 \le z \le 12$,
    $5 - (13, 20) - a$ *with* $a = 13, 11$,
    $5 - (14, 16) - a$ *with* $a = 14, 12, 10$,
    $5 - (16, 8) - 8$, *or* $5 - (18, 0) - 0$.

$p=3$ $3 - (z, 72 - 2z) - s$ *with* $18 \le z \le 23$ *or*
    $3 - (24, 24) - 24$ *or* $3 - (36, 0) - 0$.

$p=2$ $2 - (z, 72 - z) - s$ *with* $z = 24, 32, 36, 40, 48$ *where* $z = s$ *if* $z = 24$ *or* $z = 48$.

Proof. Let $\sigma \in \mathrm{Aut}(\Gamma)$ be an element of prime order p. By Theorem 3.3 we may assume that $z \ge 2$, in particular $p \le 37$. We denote the fixed lattice $F(\sigma)$ by F and $Z(\sigma)$ by Z. Then Z is an ideal lattice in the p-th cyclotomic number field.

$p = 37$ By Theorem 3.3 we obtain that $z = 2$.

$p = 23, 29, 31$ Then $z = 2$ and in all cases $\gamma(F) > B(72 - 2(p - 1))$, or $z = 3$ and $p = 23$. Then F is a 6-dimensional lattice of determinant 23^3 and minimum 8. So the density of F exceeds the one of the root lattice $\mathbb{E}_6$ which is known to be the densest lattice of dimension 6.

$p = 19$ Here Blichfeldt's bound leads to the conclusion that $z = 4$ is the only possibility.

$p = 17$ Using Blichfeldt's bound we are left with the possibility that $z = 4$. Then the fixed lattice has dimension 8, determinant 17^4 and minimum 8. There is a unique genus of 8-dimensional 17-elementary even lattices of determinant 17^4, which we enumerated completely using the standard procedure implemented in Magma. Its class number is 118 and the maximal possible minimum is 6.

$p = 13$ Here Blichfeldt's bound implies that $z = 6$.

$p = 11$ Using Blichfeldt's bound we are left with the possibility that $z = 6$. Then the fixed lattice is in the genus of the 11-modular lattices of dimension 12. By [16] this genus contains no lattice of minimum 8.

8

$p = 7$ Blichfeldt's bound allows us to conclude that $z \in \{9, 10, 11, 12\}$. If $z = 11$, then the fixed lattice is a 7-elementary even lattice of dimension 6 and determinant 7^5. There is a unique genus, this genus has class number 1, and the unique lattice in this genus is $\sqrt{7}\mathbb{A}_6^{\#}$ with minimum 6.

If $z = 10$, then the fixed lattice is a 12-dimensional, 7-elementary even lattice of determinant 7^{10}. There is a unique genus, this has class number 12 and the maximal possible minimum is 6.

If $z = 9$ then the fixed lattice lies in the genus of the 7-modular lattices of dimension 18. By [3, Theorem 6.1] this genus contains no lattice of minimum 8.

$p = 5$ Here we may use Blichfeldt's bound to obtain $z \in \{9, 10, 11, 12, 13, 14, 15, 16, 18\}$. To exclude $z = 15$ note that in this case the fixed lattice of σ has dimension 12 and determinant 5^{11}, more precisely it is in the genus of $\sqrt{5}(\mathbb{E}_8 \oplus \mathbb{A}_4)^{\#}$. This genus has class number 2 and both lattices have minimum 4. All the other restrictions on the values of s are obtained by comparing Blichfeldt's bound with $\gamma(F)$. Only to exclude the type $5 - (12, 24) - 10$ we need the slightly stronger bounds in [8].

$p = 3$ The cases $z \leq 14$ are excluded by Blichfeldt's bound. To exclude the cases $z = 15, 16, 17$ we look at the lattice Z which is (or contains) a unimodular $\mathbb{Z}[\zeta_3]$-lattice. So viewed as a $\mathbb{Z}$-lattice this is a 3-modular $\mathbb{Z}$-lattice of dimension $2z = 30, 32, 34$. The bound in [21] shows that the maximal possible minimum of such a lattice is 6. The cases $25 \leq z \leq 35$ are not possible by Lemma 2.3 applied to $L = \sqrt{3}F^{\#}$ whence $L^{\#} = \frac{1}{\sqrt{3}}F$ of minimum $\geq \frac{8}{3} > 2$. The same lemma also implies that $z = s$ if $z = 24$ and then $F = \sqrt{3}\Lambda_{24}$.

$p = 2$ If σ is an automorphism of type $2 - (z, f) - s$, then $-\sigma$ has type $2 - (f, z) - s$, so we may assume that $f \leq z$. The fixed lattice F of σ is hence of dimension $f \leq 36$. By [18, Lemma 4.9] the lattice F contains $\sqrt{2}U$ for some unimodular lattice U. As $\min(F) \geq 8$, we conclude that $\min(U) \geq 4$ and hence $f \in \{24, 32, 36\}$ by the results in [9, Theorem 2] and [2].

$\square$

Corollary 3.5. *If p^2 divides $|\operatorname{Aut}(\Gamma)|$ then $p \leq 5$.*

Proof. Assume that $p > 5$. Then by Theorem 3.4 the automorphisms of order p act with an irreducible minimal polynomial. This shows that $C_p \times C_p$ is not a subgroup of $\operatorname{Aut}(\Gamma)$. Also as p does not divide 72 there is no element of order p^2 in $\operatorname{Aut}(\Gamma)$, which shows the corollary. $\square$

Proposition 3.6. $\operatorname{Aut}(\Gamma)$ *does not contain an element of order* 25.

Proof. Assume that $\sigma \in \operatorname{Aut}(\Gamma)$ has order 5^2. Then σ^5 has type $5 - (z, f) - s$ so that z is a multiple of 5. By Theorem 3.4 the type of σ^5 is $5 - (10, 32) - 10$. So $Z(\sigma^5)$ is a lattice as in Lemma 2.5 and hence $\min(Z(\sigma^5)) \leq 6$. $\square$

Remark 3.7. *Let Γ_{72} be the extremal even unimodular lattice of dimension 72 constructed in [17]. Then $\operatorname{Aut}(\Gamma_{72})$ has order $2^8 3^2 5^2 7 \cdot 13$ and contains elements of type $13 - (6, 0) - 0$, $7 - (12, 0) - 0$, $5 - (18, 0) - 0$, $5 - (12, 24) - 12$ where the fixed lattice is $\mathcal{T}_{24}$, $3 - (24, 24) - 24$ where the fixed lattice is $\sqrt{3}\Lambda_{24}$, $2 - (48, 24) - 24$ where the fixed lattice is $\sqrt{2}\Lambda_{24}$, $2 - (36, 36) - 36$, and $2 - (24, 48) - 24$.*

3.2 Some remarks on dimension 96, 120, 144.

The bounds on the density of the fixed lattice F or its orthogonal complement Z seem to become more and more restrictive in higher dimensions. To illustrate this I give some preliminary results in the next three dimensions.

Theorem 3.8. *Let L be an extremal even unimodular lattice of dimension 96, 120, or 144 and $\sigma \in \mathrm{Aut}(L)$ some automorphism of prime order $p > 7$. Then one of*

- *$\dim(L) = 96$, $p \in \{13, 17\}$ and $\mu_\sigma = \Phi_p$, or the type of σ is $11 - (8, 16) - 8$ or $13 - (7, 12) - 7$.*

- *$\dim(L) = 120$, $p \in \{11, 13, 31, 61\}$ and $\mu_\sigma = \Phi_p$, or the type of σ is $13 - (9, 12) - 9$.*

- *$\dim(L) = 144$, $p \in \{13, 19, 37\}$ and $\mu_\sigma = \Phi_p$.*

<u>Proof.</u> First assume that $\dim(L) = 96$ and $\min(L) = 10$. Combining Blichfeldt's bound with the known values for γ_d for $d \leq 8$, excludes all possible types except the ones in the Theorem and $11 - (9, 6) - 5$ and $19 - (5, 6) - 5$. In these cases a complete enumeration of the genus of $F(\sigma)$ shows that there is no lattice of minimum ≥ 10.
If $\dim(L) = 120$ then Blichfeldt's bound (we need the stronger bounds in [8] to exclude the type $11 - (10, 20) - 10$) leaves us with the possibilities in the Theorem, type $11 - (9, 10) - 9$, and type $17 - (7, 8) - 7$. A complete enumeration of the genus of $F(\sigma)$ in the lattes two cases (class number 3) shows that there is no such lattice $F(\sigma)$ of minimum ≥ 12.
Now assume that $\dim(L) = 144$. Then Blichfeldt's bound give us with the possibilities in the theorem or type $29 - (5, 4) - 3$, $13 - (11, 12) - 11$, $11 - (13, 14) - 13$. If σ has type $29 - (5, 4) - 3$ then the $F(\sigma)$ is one of the two even 29-elementary lattices of dimension 4 and determinant 29^3. They have minimum 6 and 4 respectively. If σ has type $13 - (11, 12) - 11$ then the $F(\sigma)$ is in the genus of $\sqrt{13}\mathbb{A}_{12}^{\#}$. This genus has class number 6 and the minima are $4, 4, 10, 10, 12, 12$. In particular no lattice in this genus has minimum $\geq 14 = \min(L)$. To exclude type $11 - (13, 14) - 13$ we enumerated the genus of the even 11-elementary lattices of dimension 14 and determinant 11^{13}. The class number is 8 and no lattice in this genus has minimum ≥ 14. $\qquad\square$

4 Ideal lattices

This section describes the computations to classify all 72-dimensional extremal even unimodular lattices L that have an automorphism σ of order m such that the characteristic polynomial of σ is the m-th cyclotomic polynomial. In other words the lattice L is an ideal lattice in the m-th cyclotomic number field $K = \mathbb{Q}[\zeta_m]$. By [5] for all these fields there is some positive definite even unimodular ideal lattice.

Remark 4.1. *Let $\sigma \in \mathrm{Aut}(L)$ be an automorphism of the lattice L with characteristic polynomial Φ_m, the m-th cyclotomic polynomial. Then the action of σ on $\mathbb{Q}L$ turns the vector space $\mathbb{Q}L$ into a one-dimensional vector space over the m-th cyclotomic number field $K = \mathbb{Q}[\zeta_m]$ which we identify with K. Then the lattice L is a $\mathbb{Z}[\zeta_m]$-submodule, hence isomorphic to a fractional ideal J in K. The symmetric positive definite bilinear form $B : L \times L \to \mathbb{Q}$ is ζ_m-invariant, since $B(\sigma(x), \sigma(y)) = B(x, y)$ for all $x, y \in L$. In particular*

$$(L, B) \cong (J, b_\alpha), \text{ where } b_\alpha(x, y) = trace_{K/\mathbb{Q}}(\alpha x \overline{y}), \text{ for all } x, y \in J.$$

Here $^-$ is the complex conjugation on K, the involution with fixed field $K^+ := \mathbb{Q}[\zeta_m + \zeta_m^{-1}]$, and $\alpha \in K^+$ is totally positive, i.e. $\iota(\alpha) > 0$ for all embeddings $\iota : K^+ \to \mathbb{R}$.

Definition 4.2. *Let $K = \mathbb{Q}[\zeta_m]$ with maximal real subfield $K^+ := \mathbb{Q}[\zeta_m + \zeta_m^{-1}]$. Let $\mathrm{Cl}(K)$ denote the ideal class group of K and $\mathrm{Cl}^+(K^+)$ the ray class group of K^+, where two ideals I, J are equal in $\mathrm{Cl}^+(K^+)$ if there is some totally positive $\alpha \in K^+$ with $I = \alpha J$. Since $x\overline{x}$ is totally positive for any $0 \neq x \in K$, the norm induces a group homomorphism*

$$\mathrm{Cl}(K) \to \mathrm{Cl}^+(K^+), \ [J] \mapsto [J\overline{J} \cap K^+].$$

The positive class group $\mathrm{Cl}^+(K)$ is the kernel of this homomorphism and $h^+(K) := |\mathrm{Cl}^+(K)|$.

The proof of the next lemma is direct elementary computation following the lines of [5] and [18].

Lemma 4.3. *Let J be a fractional ideal of K, $\alpha \in K^+$ totally positive, and $\Gamma := \mathrm{Gal}(K/\mathbb{Q})$ the Galois group of K over $\mathbb{Q}$.*

(a) Let

$$\Delta := \{x \in K \mid \mathrm{trace}_{K/\mathbb{Q}}(x\overline{y}) \in \mathbb{Z} \text{ for all } y \in \mathbb{Z}[\zeta_m]\}$$

denote the different of the ring of integers $\mathbb{Z}_K = \mathbb{Z}[\zeta_m]$ of K. Then the dual lattice of (J, b_α) is

$$(J, b_\alpha)^{\#} = (\overline{J}^{-1}\Delta\alpha^{-1}, b_\alpha).$$

So (J, b_α) is unimodular, if and only if $(J\overline{J})^{-1}\Delta\alpha^{-1} = \mathbb{Z}_K$.

(b) $(J, b_\alpha) \cong (g(J), b_{g(\alpha)})$ for all $g \in \Gamma$.

(c) $(J, b_\alpha) \cong (J, b_{u\overline{u}\alpha})$ for all $u \in \mathbb{Z}[\zeta_m]^*$.

(d) For $0 \neq a \in K$ we have $(J, b_\alpha) \cong (aJ, b_{\alpha(a\overline{a})^{-1}})$.

From this lemma we get the following strategy for finding representatives of all isometry classes of even unimodular ideal lattices:

Remark 4.4. *Let J be a fractional ideal of K, $\alpha \in K^+$ totally positive, so that (J, b_α) is unimodular. Then all isometry classes of even unimodular ideal lattices in K are represented by*

$$(IJ, b_{v\alpha a_I^{-1}}), \text{ where}$$

(a) $[IJ]$ runs through a system of representatives of the Γ-orbits on $\mathrm{Cl}^+(K)[J]$,

(b) $a_I \in K^+$ is a totally positive generator of $I\overline{I} \cap K^+ = \mathbb{Z}_{K^+}a_I$, and

(c) v runs through a set of representatives of U^+/N, where

$$U^+ := (\mathbb{Z}_{K^+}^*)_{>>0} = \{v \in \mathbb{Z}_{K^+}^* \mid v \text{ is totally positive }\}$$

is the group of totally positive units in $\mathbb{Z}_{K^+}$ and

$$N := N(\mathbb{Z}_K^*) := \{u\overline{u} \mid u \in \mathbb{Z}_K^*\}$$

the subgroup of norms of units in $\mathbb{Z}_K$.

- To compute a system of representatives as in Remark 4.4 (c) we use [25, Theorem 8.3], that gives a system $\underline{u} := (u_1, \ldots, u_{r-1})$ of multiplicatively independent units in $\mathbb{Z}_{K^+}^*$ and the index $i = [\mathbb{Z}_{K^+}^* : \langle \underline{u} \rangle]$. To find the totally positive units U^+ (modulo the squares) we put $u_r := -1$ and compute a matrix $R(\underline{u}) \in \mathbb{F}_2^{r \times r}$ with

$$R(\underline{u})_{i,j} = \begin{cases} 1 & \text{if } \epsilon_j(u_i) < 0 \\ 0 & \text{if } \epsilon_j(u_i) > 0 \end{cases}$$

for $i = 1, \ldots, r$ and all real embeddings $\epsilon_1, \ldots, \epsilon_r$ of K^+. Any s in the kernel of $R(\underline{u})$ corresponds to a totally positive unit $\prod_{i=1}^r u_i^{s_i}$ which is a square in $\langle u_1, \ldots, u_r \rangle$, if and only if $s = 0$. If $i = [\mathbb{Z}_{K^+}^* : \langle u_1, \ldots, u_r \rangle]$ is odd, then the kernel of $R(\underline{u})$ gives a system of representatives of the totally positive units modulo $(\mathbb{Z}_{K^+}^*)^2$. If not, then one of these totally positive elements $u = \prod_{i=1}^r u_i^{s_i}$ is a square in K^+ and we find its square root $v \in \mathbb{Z}_{K^+}^*$ by factoring $X^2 - u$ in $K^+[X]$ and replace one of the u_i with $s_i = 1$ by v until we find a subgroup $V := \langle u_1, \ldots, u_r \rangle$ with $[\mathbb{Z}_{K^+}^* : V]$ odd. Then $U^+/(\mathbb{Z}_{K^+}^*)^2 \cong V_{>>0}/V^2$.
- By [13, Satz 14 and Satz 27] $[N : (\mathbb{Z}_{K^+}^*)^2] = 2$ whenever m is not a prime power. To find N without computing $\mathbb{Z}_K^*$ we use the Galois action on $U^+/(\mathbb{Z}_{K^+}^*)^2 \cong V_{>>0}/V^2$. This group is an $\mathbb{F}_2\Gamma$ module containing the trivial $\mathbb{F}_2\Gamma$ submodule $N/(\mathbb{Z}_{K^+}^*)^2$. In all cases this $\mathbb{F}_2\Gamma$ module turned out to have a unique trivial submodule which is hence the image of N.
- To obtain suitable representatives of the Galois orbits on the ideal classes as in Remark 4.4 (a) we usually choose a totally decomposed small prime p and compute the permutation action of Γ on the set of prime ideals $\{P_1, \ldots, P_{72}\}$ of $\mathbb{Z}_K$ that contain p. In most cases it was easy to find a small prime p such that

$$\langle [P_1], \ldots, [P_{72}] \rangle = \mathrm{Cl}^+(K)$$

and to compute the explicit $\mathbb{Z}\Gamma$-module epimorphism

$$f : \mathbb{Z}^{72} \to \mathrm{Cl}^+(K), v := (v_1, \ldots, v_{72}) \mapsto \prod_{i=1}^{72} [P_i]^{v_i}$$

If not we take several primes p and consider the direct product of permutation modules. We successively compute small elements $v \in \mathbb{Z}^{72}$ such that $\bigcup_v \{f(w) \mid w \in v^\Gamma\} = \mathrm{Cl}^+(K)$. Of course, if $J \neq 1$, then we need to compute the action of Γ on the elements $[J \prod_{i=1}^{72} P_i^{v_i}]$. In our cases we could choose $[J]$ to be a class of order 4, so $[J]^\gamma = [J]^{\pm 1}$ for all γ in Γ. As $[J \prod_{i=1}^{72} P_i^{v_i}]^\gamma = [J]^\gamma [\prod_{i=1}^{72} P_i^{v_i}]^\gamma$ this can be controlled by adding an additional block diagonal entry ± 1 to the matrices representing the Γ-action on $\mathbb{Z}^{72}$ and replacing f accordingly.
- The following table summarizes the computations that led to the theorem below. Here $K = \mathbb{Q}[\zeta_m]$ with $m \not\equiv 2 \pmod 4$ and $\varphi(m) = 72$. Then h_K displays the structure of the class group as computed with MAGMA. The class number coincides with the one given in [25]. In all cases $\mathrm{Cl}^+(K) = \mathrm{Cl}(K)^2$. The third row gives the index of the norms in the group of totally positive units and line four an ideal class $[J]$ that supports even unimodular positive definite lattices. We then list the primes p for which the prime ideals

of $\mathbb{Z}_K$ that divide p generate $\mathrm{Cl}^+(K)$ followed by the number of Galois orbits on $\mathrm{Cl}^+(K)$.

m	91	95	111	117	135
h_K	4×13468	107692	19×25308	$3 \times 3 \times 9 \times 1638$	75961
$|U^+/N|$	4	2	2	2	1
J	1	Q_{11}	Q_{1999}	1	1
p	547^2	191	223	1171	271
$\#orbs$	202	751	3433	942	1086
m	148	152	216	228	252
h_K	4827501	$19 \times 171 \times 513$	1714617	238203	$7 \times 28 \times 364$
$|U^+/N|$	1	1	1	1	4
J	1	1	1	1	1
p	593	457	433	229	$2, 5, 71$
$\#orbs$	68891^*	24568^*	25337^*	3878	338

Here Q_{11} resp. Q_{1999} denotes a suitable product of three prime ideals in $\mathbb{Z}_K$ that divide 11 resp. 1999 so that the classes $[Q_{11}]$ and $[Q_{1999}]$ have order 4.
For $m = 91$ the entry $p = 547^2$ means that we took the squares of the prime ideals dividing 547, as these generate $\mathrm{Cl}^+(K) = \mathrm{Cl}(K)^2$.
Some of the numbers of orbits are marked with a $*$:
For $m = 148$ our random procedure generating small elements $v \in \mathbb{Z}^{72}$ such that $\bigcup_v f(v^\Gamma) = \mathrm{Cl}^+(K)$ did not find certain elements of order 9 and 3. So additionally to the 68891 orbit representatives we considered all elements in $C_9 \cong \langle [P_{37}] \rangle \leq \mathrm{Cl}^+(K)$.
Similarly for $m = 152$ the 24568 Galois orbits did not represent some elements of order 9 in the class group. We additionally considered all elements in the Sylow 3-subgroup ($\cong C_9 \times C_{27}$) of the class group.
Finally for $m = 216$ we also failed to find the six elements of order 9 in the class group using the random search for small representatives. These are found in the subgroup $\langle [P_3] \rangle \cong C_9$ generated by the prime ideals dividing 3.

Using the strategy explained before we find the following result.

Theorem 4.5. *Let L be an extremal even unimodular lattice such that $\mathrm{Aut}(L)$ contains some element σ with irreducible characteristic polynomial. Then the order of σ is 91 or 182 and $L = (I, b_\alpha) \cong \Gamma_{72}$ for all six ideal classes $[I] \in \mathrm{Cl}(\mathbb{Q}[\zeta_{91}])$ of order 7 and a unique α modulo $N(\mathbb{Z}[\zeta_{91}]^*)$.*

To prove this theorem we computed representatives of all isomorphism classes of ideal lattices in the m-th cyclotomic number field as explained above using the class group algorithm available in MAGMA. We then performed the reduction procedure described in Remark 2.6. The lattices that survived with minimum ≥ 8 correspond to ideal lattices (I, b_α) for the six ideal classes $[I]$ of order 7 in $\mathrm{Cl}(\mathbb{Q}[\zeta_{91}])$ with a unique α modulo squares, so these are all in the same Galois orbit and hence isometric. As the lattice Γ_{72} from [17] has an automorphism of order 91 with irreducible characteristic polynomial, this ideal lattice has to be the extremal even unimodular lattice Γ_{72}.

5 Large automorphisms.

Similar as ideal lattices we can classify lattices that admit a large automorphism.

Definition 5.1. *Let $\sigma \in \mathrm{GL}_d(\mathbb{Z})$. Then σ is called* large, *if the minimal polynomial of σ has an irreducible factor of degree $> d/2$.*

If $\sigma \in \mathrm{Aut}(L)$ is a large automorphism then the irreducible factor of the minimal polynomial is some cyclotomic polynomial Φ_n with $\varphi(n) > d/2$ and the lattice L contains a sublattice $Z \oplus F$ of finite index where Z is an ideal lattice in the n-th cyclotomic number field. The lattice L is an ideal lattice, if $F = \{0\}$ or equivalently $\varphi(n) = d$.

Remark 5.2. *If $\sigma \in \mathrm{Aut}(L)$ is a large automorphism then the order n of σ satisfies $\varphi(n) > d/2$.*

This section describes the computations that lead to the following result:

Theorem 5.3. *Let Γ be an extremal even unimodular lattice of dimension 72 and assume that there is some large automorphism $\sigma \in \mathrm{Aut}(\Gamma)$. Then the order of σ is either 91, 182 or 168 and $\Gamma = \Gamma_{72}$.*

Note that $\mathrm{Aut}(\Gamma_{72})$ has four conjugacy classes of elements of order 168. All these elements g have minimal polynomial $\Phi_{168}\Phi_{56}$ of degree 72. So if F is the fixed lattice of g^{56} then $F \cong \sqrt{3}\Lambda_{24}$ is a rescaled Leech lattice. For each of the two conjugacy classes $[h]$ of elements of order 56 in $\mathrm{Aut}(\Lambda_{24})$ exactly two conjugacy classes of elements $g \in \mathrm{Aut}(\Gamma_{72})$ satisfy $g_{|F} \in [h]$.

During the whole section we keep the assumptions of Theorem 5.3. As we already dealt with ideal lattices in the previous section we assume that the characteristic polynomial of σ is not irreducible. In particular Γ contains a sublattice $Z \oplus F$, so that Z is an ideal lattice in the n-th cyclotomic number field and F is the fixed lattice of $\sigma^{n/p}$ for some prime divisor p of n.

From Theorem 3.4 we hence obtain the following list of possible element orders n such that $n \not\equiv 2 \pmod 4$ and $36 < \varphi(n) < 72$:

$\varphi(n) = 64$. Then $n = 2^7$, $n = 2^5 5$, $n = 2^6 3$, or $n = 2^4 3 \cdot 5$.

$\varphi(n) = 54$ and $n = 3^4$.

$\varphi(n) = 48$. Then the following table displays the possible values for n together with the structure h_K of the class group of $K = \mathbb{Q}[\zeta_n]$ and the positive class group h_k^+:

n	65	104	105	112	140	144	156	168	180
h_K	$2 \times 2 \times 4 \times 4$	3×117	13	3×156	39	13×39	156	84	5×15
h_K^+	2×2	3×117	13	6×39	39	13×39	78	42	5×15
$|U_+/N|$	16	1	1	2	1	1	2	2	1

U_+/N is the group of totally positive units in $\mathbb{Z}[\zeta_n + \zeta_n^{-1}]^*$ modulo the group of norms of units in $\mathbb{Z}[\zeta_n]$

Lemma 5.4. $n \neq 3^4$.

<u>Proof.</u> Otherwise let $\sigma \in \mathrm{Aut}(\Gamma)$ be some element of order 3^4. Then $\alpha := \sigma^{27}$ has type $3 - (27, 18) - s$ which contradicts Theorem 3.4. $\qquad \square$

Lemma 5.5. $\varphi(n) \neq 64$.

<u>Proof.</u> In this case F has dimension 8. By Theorem 3.4 the only possibility is that F is the fixed lattice of some element of order 5, and then $F = \sqrt{5}E_8$. This lattice has no automorphism of order 2^5 or $2^4 3$. $\qquad \square$

Lemma 5.6. $n \neq 65 = 5 \cdot 13$.

<u>Proof.</u> Assume that $n = 65$, then F is the fixed lattice of an element of order 5, so $\dim(F) = 24$, $\det(F) = 5^{12}$ and Z is an ideal lattice of determinant 5^{12}. There is no such positive definite ideal lattice Z. $\hfill\square$

Lemma 5.7. $n \neq 104 = 2^3 13$.

<u>Proof.</u> If $n = 104$, then F is the fixed lattice of an element of order 2 and hence $F \cong \sqrt{2}\Lambda_{24}$ and Z is a positive definite ideal lattice of determinant 2^{24}. All ideal classes support a unique such lattice Z, but only for those classes in a certain cyclic subgroup of order 9 this lattice Z has minimum ≥ 8. These 9 classes fall into 3 Galois orbits. In all cases ζ_{104} acts like a primitive 52nd root of unity on $Z^{\#}/Z \cong \mathbb{F}_2^{24}$. The automorphism group $\mathrm{Aut}(\Lambda_{24})$ of the Leech lattice has a unique conjugacy class of elements g of order 52, so the extremal even unimodular 72-dimensional lattices Γ with an automorphism σ of order 104 are maximal isotropic (ζ_{104}, g)-invariant submodules of

$$X := (Z^{\#} \oplus \frac{1}{\sqrt{2}}\Lambda_{24})/(Z \oplus \sqrt{2}\Lambda_{24}) \cong \mathbb{F}_2^{48}$$

The module X has $2^{12} - 1$ invariant minimal submodules M that fall into 317 orbits under the action of the centralizer $\langle (1, g) \rangle$. For representatives M of these orbits we computed the maximal isotropic invariant submodules of $M^{\perp}/M$ and the corresponding 72-dimensional unimodular lattices. In all these lattices our reduction algorithm 2.6 finds vectors of norm < 8, so none of these lattices is extremal. $\hfill\square$

Lemma 5.8. $n \neq 105 = 3 \cdot 5 \cdot 7$.

<u>Proof.</u> If $n = 105$, then F is the fixed lattice of an element of order p for $p \in \{3, 5\}$. Moreover σ acts on $F^{\#}/F$ as an element of order $\frac{105}{p} \in \{35, 21\}$. Hence $F \cong \sqrt{3}\Lambda_{24}$ if $p = 3$ and by Lemma 2.4 (a) $F \cong \mathcal{T}_{24}$ if $p = 5$. So Z is a positive definite ideal lattice of determinant 3^{24} or 5^{12} and minimum ≥ 8. There are two Galois orbits on the ideal classes of $\mathbb{Z}[\zeta_{105}]$, both support a unique positive definite lattice of determinant 3^{24} and 5^{12}. For $p = 3$ both lattices have minimum 8, for $p = 5$ only the principal ideals support lattices of minimum 8.

<u>p=3:</u> The automorphism group $\mathrm{Aut}(\Lambda_{24})$ of the Leech lattice has a unique conjugacy class of elements g of order 35, so the extremal even unimodular 72-dimensional lattices Γ with an automorphism σ of order 105 correspond to maximal isotropic (ζ_{105}, g)-invariant submodules M of

$$X := (Z^{\#} \oplus \frac{1}{\sqrt{3}}\Lambda_{24})/(Z \oplus \sqrt{3}\Lambda_{24}) \cong \mathbb{F}_3^{48} \cong S^2 \oplus (S^*)^2$$

as a (ζ_{105}, g)-module, where S^* is the dual module of the simple module $S \cong \mathbb{F}_3^{12}$. So $M = Y \oplus Y^{\perp}$ where $Y \leq S^2$ and $Y^{\perp}$ is its annihilator in $(S^*)^2$. There are $3^{12} + 3 = 531444$ submodules $Y \leq S^2$ which fall into 15187 orbits under the action of the centralizer $\langle (1, g) \rangle$. For representatives Y of these orbits we compute the unimodular lattice Γ as the full preimage of $Y \oplus Y^{\perp}$. For both ideals no such lattice has minimum ≥ 8.

<u>p=5:</u> The automorphism group $\mathrm{Aut}(\mathcal{T}_{24})$ of the Tits lattice has a unique conjugacy class of elements g of order 21, so the extremal even unimodular 72-dimensional lattices Γ

with an automorphism σ of order 105 correspond to maximal isotropic (ζ_{105}, g)-invariant submodules M of
$$X := (Z^\# \oplus \mathcal{T}_{24}^\#)/(Z \oplus \mathcal{T}_{24}) \cong \mathbb{F}_5^{24}.$$

The module X has $2 \cdot (5^6 + 1) = 31252$ minimal submodules falling into 1492 orbits under the action of the centralizer $\langle (1, g) \rangle$. For all representatives we compute the maximal isotropic invariant overmodules and the unimodular lattices as full preimages. None of them has minimum ≥ 8. $\qquad\square$

Lemma 5.9. $n \neq 112 = 2^4 7$.

<u>Proof.</u> If $n = 112$, then F is the fixed lattice of an element of order 2, so $F \cong \sqrt{2}\Lambda_{24}$ and Z is an ideal lattice of determinant 2^{24}. The Galois group acts on the ideal class group with 16 orbits and all ideals support a unique positive definite lattice of determinant 2^{24}. None of these lattices has minimum ≥ 8. $\qquad\square$

Lemma 5.10. $n \neq 140 = 2^2 \cdot 5 \cdot 7$.

<u>Proof.</u> If $n = 140$, then F is the fixed lattice of an element of order p for $p \in \{2, 5\}$. Moreover σ acts on $F^\#/F$ as an element of order 35 resp. 28. Hence $F \cong \sqrt{2}\Lambda_{24}$ if $p = 2$ and by Lemma 2.4 (b) $F \cong \mathcal{T}_{24}$ if $p = 5$. So Z is a positive definite ideal lattice of determinant 2^{24} or 5^{12} and minimum ≥ 8. There are 4 Galois orbits on the ideal classes of $\mathbb{Z}[\zeta_{140}]$, all four support a unique positive definite lattice of determinant 2^{24} and 5^{12}. For $p = 2$ the lattices that have minimum 8 correspond to the ideal classes of order 3 and 1, for $p = 5$ only the principal ideals support lattices of minimum 8.

<u>p=2:</u> The automorphism group $\mathrm{Aut}(\Lambda_{24})$ of the Leech lattice has a unique conjugacy class of elements g of order 35, so the extremal even unimodular 72-dimensional lattices Γ with an automorphism σ of order 140 correspond to maximal isotropic (ζ_{140}, g)-invariant submodules M of
$$X := (Z^\# \oplus \frac{1}{\sqrt{2}}\Lambda_{24})/(Z \oplus \sqrt{2}\Lambda_{24}) \cong \mathbb{F}_2^{48} \cong S^2 \oplus (S^*)^2$$

as a (ζ_{140}, g)-module, where S^* is the dual module of the simple module $S \cong \mathbb{F}_2^{12}$. So $M = Y \oplus Y^\perp$ where $Y \leq S^2$ and $Y^\perp$ is its annihilator in $(S^*)^2$. There are $2^{12} + 3 = 4099$ submodules $Y \leq S^2$ which fall into 121 orbits under the action of the centralizer $\langle (1, g) \rangle$. For representatives Y of these orbits we compute the unimodular lattice Γ as the full preimage of $Y \oplus Y^\perp$. For both ideals no such lattice has minimum ≥ 8.

<u>p=5:</u> The automorphism group $\mathrm{Aut}(\mathcal{T}_{24})$ of the Tits lattice has a unique conjugacy class of elements g of order 28, so the extremal even unimodular 72-dimensional lattices Γ with an automorphism σ of order 140 correspond to maximal isotropic (ζ_{140}, g)-invariant submodules M of
$$X := (Z^\# \oplus \mathcal{T}_{24}^\#)/(Z \oplus \mathcal{T}_{24}) \cong \mathbb{F}_5^{24} \cong S^2 \oplus (S^*)^2$$

as a (ζ_{140}, g)-module, where S^* is the dual module of the simple module $S \cong \mathbb{F}_2^{12}$. So $M = Y \oplus Y^\perp$ where $Y \leq S^2$ and $Y^\perp$ is its annihilator in $(S^*)^2$. There are $5^6 + 3 = 15628$ submodules $Y \leq S^2$ which fall into 562 orbits under the action of the centralizer $\langle (1, g) \rangle$. For representatives Y of these orbits we compute the unimodular lattice Γ as the full preimage of $Y \oplus Y^\perp$. No such lattice has minimum ≥ 8. $\qquad\square$

Lemma 5.11. $n \neq 144 = 2^4 3$.

Proof. If $n = 144$, then F is the fixed lattice of an element of order 2 or 3. In particular Z is an ideal lattice of determinant 2^{24} or 3^{24} in $\mathbb{Q}[\zeta_{144}]$. Explicit computations show that ζ_{144} acts as an element of order 36 (with minimal polynomial $(x^6 + x^3 + 1)^4 \in \mathbb{F}_2[x]$) (for $p = 2$) respectively an element of order 48 (with minimal polynomial $(x^4 + x^2 + 2)^3(x^4 + 2x^2 + 2)^3 \in \mathbb{F}_3[x]$) (for $p = 3$) on $Z^\#/Z$. This leads to a contradiction, as $\text{Aut}(\Lambda_{24})$ has no such elements acting this way on $\Lambda_{24}/p\Lambda_{24}$. $\qquad \square$

Lemma 5.12. $n \neq 156 = 2^2 3 \cdot 13$.

Proof. Assume that $n = 156$. Then F is the fixed lattice of an element of order 2 or 3, so Z is an ideal lattice of determinant 2^{24} resp. 3^{24}. The Galois group of $\mathbb{Q}[\zeta_{156}]$ has eight orbits on $\text{Cl}^+(\mathbb{Q}[\zeta_{156}]) \cong C_{78}$ according to the eight different element orders in this group. On all ideals I there is some positive definite even unimodular form B_{α_I}.

p=3: In $\mathbb{Z}[\zeta_{156} + \zeta_{156}^{-1}]$ the ideal $(3) = (p_3)^2(p_3')^2$ with $p_3, p_3' \in \mathbb{Z}[\zeta_{156} + \zeta_{156}^{-1}]$ so that $p_3 p_3'$ is totally positive. So the possible ideal lattices Z are $Z_1 := (I, B_{p_3^2 \alpha_I})$, $Z_2 := (I, B_{(p_3')^2 \alpha_I})$, and $Z_3 := (I, B_{p_3 p_3' \alpha_I})$. On $Z_1^\#/Z_1$ and also on $Z_2^\#/Z_2$ the element ζ_{156} acts as an element of order 156. Such an element does not exist in $\text{Aut}(\Lambda_{24})$. So we need to consider the ideal lattices $(I, B_{p_3 p_3' \alpha_I})$ and $(I, B_{u p_3 p_3' \alpha_I})$ for all 8 representatives $[I]$ of the Galois orbits and some unit u that is not a norm. We find 5 such ideal lattices that have minimum ≥ 8: For the elements $[I]$ of order 39 in the class group and one unit, the elements of order 26 and all units and the elements of order 1, again both units. The Leech lattice has a unique conjugacy classes of automorphisms of order 52. As already described before we compute the invariant unimodular overlattices of $Z \perp \sqrt{3}\Lambda_{24}$ for all five lattices Z. None of them has minimum 8.

p=2: Here we find on each ideal class $2 = |U^+/N|$ lattices Z of determinant 2^{24}, only for the principal ideals these lattices have minimum ≥ 8. The Leech lattice has two conjugacy classes of automorphisms of order 39, represented by, say, g and h. As already described before we compute the unimodular overlattices of $Z \perp \sqrt{2}\Lambda_{24}$ for both lattices Z invariant under (ζ_{156}, g) respectively (ζ_{156}, h). None of them has minimum 8. $\qquad \square$

Lemma 5.13. If $n = 168 = 2^3 3 \cdot 7$ then $\Gamma = \Gamma_{72}$.

Proof. Assume that $n = 168$. Then F is the fixed lattice of an element of order 2 or 3, so Z is an ideal lattice of determinant 2^{24} resp. 3^{24}. The Galois group of $\mathbb{Q}[\zeta_{168}]$ has eight orbits on $\text{Cl}^+(\mathbb{Q}[\zeta_{168}]) \cong C_{42}$ according to the eight different element orders in this group. On all ideals I there is some positive definite even unimodular form B_{α_I}.

p=3: In $\mathbb{Z}[\zeta_{168} + \zeta_{168}^{-1}]$ the ideal $(3) = (p_3)^2(p_3')^2$ with $p_3, p_3' \in \mathbb{Z}[\zeta_{168} + \zeta_{168}^{-1}]$ so that $p_3 p_3'$ is totally positive. So the possible ideal lattices Z are $Z_1 := (I, B_{p_3^2 \alpha_I})$, $Z_2 := (I, B_{(p_3')^2 \alpha_I})$, and $Z_3 := (I, B_{p_3 p_3' \alpha_I})$. On $Z_1^\#/Z_1$ and also on $Z_2^\#/Z_2$ the element ζ_{168} acts as an element of order 168. Such an element does not exist in $\text{Aut}(\Lambda_{24})$. So we need to consider the ideal lattices $(I, B_{p_3 p_3' \alpha_I})$ and $(I, B_{u p_3 p_3' \alpha_I})$ for all 8 representatives $[I]$ of the Galois orbits and some unit $u \in U^+ \setminus N$. We find 9 such ideal lattices that have minimum ≥ 8: For the elements $[I]$ of order 2, 6, and 14 in the class group and both units, the elements of order 21 and the unit 1, and for the elements of order 7 and 1 and the unit u. The Leech lattice has two conjugacy classes of automorphisms of order 56. As already described before we compute the invariant unimodular overlattices of $Z \perp \sqrt{3}\Lambda_{24}$ for all nine lattices Z and representatives of both conjugacy classes. Only for the lattices Z where the underlying

ideal is principal the algorithm 2.6 failed to find vectors of norm < 8 in exactly 2 even unimodular overlattices of $Z \perp \sqrt{3}\Lambda_{24}$ (for both conjugacy classes of elements of order 56 in $\mathrm{Aut}(\Lambda_{24})$). As $\mathrm{Aut}(\Gamma_{72})$ has four conjugacy classes of elements of order 168 and these have minimal polynomial $\Phi_{168}\Phi_{56}$ these four lattices are indeed isometric to Γ_{72}.

p=2: Here we find on each ideal class $2 = |U^+/N|$ lattices Z of determinant 2^{24}, only for the classes of order 1 and 7 and the unit 1, and the classes of order 2 and 6 and both units these lattices have minimum ≥ 8. The Leech lattice has a unique conjugacy class $[g]$ of automorphisms of order 84, which act as elements of order 42 on $\Lambda_{24}/2\Lambda_{24}$. For all six lattices Z invariant under (ζ_{168}, g) we compute the unimodular overlattices of $Z \perp \sqrt{2}\Lambda_{24}$. None of them has minimum 8. $\qquad\square$

Lemma 5.14. $n \neq 180 = 2^2 3^2 5$.

<u>Proof.</u> If $n = 180$, then F is the fixed lattice of an element of order 2, 3, or 5. In particular Z is an ideal lattice of determinant 2^{24}, 3^{24}, or 5^{12} in $\mathbb{Q}[\zeta_{180}]$. Explicit computations show that ζ_{180} acts as an element of order 45 (for $p = 2$) respectively an element of order 60 with minimal polynomial $(x^4 + x^3 + 2x + 1)^3(x^4 + 2x^3 + x + 1)^3 \in \mathbb{F}_3[x]$ (for $p = 3$) on $Z^\#/Z$. This leads to a contradiction, as $\mathrm{Aut}(\Lambda_{24})$ has no such elements acting this way on $\Lambda_{24}/p\Lambda_{24}$. For $p = 5$ we compute the minima of the suitable ideal lattices of determinant 5^{12} to see that there is no such ideal lattice with minimum ≥ 8. $\qquad\square$

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