Degenerate flag varieties in network coding

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Abstract. Building upon the application of flags to network coding introduced in [7], we develop a variant of this coding technique that uses degenerate flags. The information set is a metric affine space isometric to the space of upper triangular matrices endowed with the flag rank metric. This suggests the development of a theory for flag rank metric codes in analogy to the rank metric codes used in linear subspace coding.

Keywords: Network coding, degenerate flag variety, flag rank metric codes, upper triangular matrices, error correcting codes, Grassmann distance on flags

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1 Introduction

Linear subspace coding is a well established theory to transmit information in large unknown networks (see for instance [5], [4]). Here information is encoded in a subspace U of the rowspace $V := K^{n+1}$, of rows of length $(n + 1)$ over some field K. The source successively inputs vectors from U and in network nodes forward K-linear combinations of the received vectors. The success of the invention by Silva, Kötter, and Kschischang is partly based on the fact that coding for errors and erasures can be done in a linear space leading to techniques that are very similar to the classical block coding techniques. The idea here is to work with subspaces of a fixed dimension, say i, and hence in the Grassmannian

$$\mathfrak{Gr}_i(V) := \{U \leq V \mid \dim(U) = i\},$$

a full orbit under right multiplication with $\mathrm{GL}_{n+1}(K)$. The orbit $\mathfrak{Gr}_i(V)$ is partitioned further into so called *cells*, according to the pivot positions of the subspace. Each cell is a metric affine space with translations preserving the Grassmann distance. In particular the largest cell

$$\mathfrak{Gr}_i^{(0)}(V) := \{U \in \mathfrak{Gr}_i(V) \mid U = U_A = \text{ row space } (I_i|A) \text{ for some } A \in K^{i \times (n+1-i)}\}$$

is a regular orbit of the unipotent abelian subgroup

$$R_i = \{ \begin{pmatrix} I_i & A \\ 0 & I_{n+1-i} \end{pmatrix} \mid A \in K^{i \times (n+1-i)} \} \leq \mathrm{GL}_{n+1}(K).$$

To send the matrix $A \in K^{i \times (n+1-i)}$ through the network, the source inputs the rows of $(I_i|A)$. In network nodes forward linear combinations of the received vectors. So the information preserved by the network is the subspace $U_A \in \mathfrak{Gr}_i^{(0)}(V)$. The receiver gets the rows of $(G|GA)$, where $G \in K^{s \times i}$ will very likely have rank i for s big enough. Using Gaussian elimination, the receiver computes the row reduced echelon form $(I_i|A)$ of $(G|GA)$ to recover the matrix A.

To measure errors and erasures one uses the *Grassmann distance*

$$d(U, W) := d_i(U, W) := \frac{1}{2}(\dim(U + W) - \dim(U \cap W)) \text{ for all } U, W \in \mathfrak{Gr}_i(V).$$

This is a $\mathrm{GL}_{n+1}(K)$-invariant distance, so also the matrices in the subgroup R_i above preserve distances. More precisely we have $d(U_A, U_B) = \mathrm{rk}(A - B)$ for all $A, B \in K^{i \times (n+1-i)}$, so the cell above is isometric to the vector space $K^{i \times (n+1-i)}$ endowed with the rank metric. This observation motivated a very successful study of rank metric codes (see for instance [9] and the references in this paper).

In the paper [7] we suggest a new feature to network coding which we call linear flag coding. The idea origins from observing the behaviour of a commonly used network, the line network, where all nodes are aligned. In this case the linear combination of the vectors forwarded by an in network node only involves vectors that are sent before, so the network automatically preserves the flag defined by the sent vectors. Also in information transmission via unknown networks, packages need to be given a sequence number, as they usually do not arrive at the receiver in the same order as they are sent. In [7] we developed basic techniques for using flags in network coding. The idea is that the source inputs, as in linear network coding, a sequence of vectors u_i. Now the in network nodes only perform linear combinations with vectors having a lower sequence number. So the information that travels through the network is the flag

$$(U_1, \ldots, U_n) \in \mathfrak{Fl}(V)$$

where $U_i = \langle u_1, \ldots, u_i \rangle$ for all i. The flag variety

$$\mathfrak{Fl}(V) := \{(U_1, \ldots, U_n) \in \prod_{i=1}^{n} \mathfrak{Gr}_i(V) \mid U_1 \leq U_2 \leq \ldots \leq U_n\}$$

is again a full orbit under simultaneous right multiplication by $\mathrm{GL}_{n+1}(K)$. Natural $\mathrm{GL}_{n+1}(K)$-invariant distances are studied in [8] and [7], where we propose the sum of the Grassmann distance as a suitable distance function $\mathbf{d}$ on $\mathfrak{Fl}(V)$. Again $\mathfrak{Fl}(V)$ partitions into cells. These cells are regular orbits under certain unipotent algebraic subgroups of $\mathrm{GL}_{n+1}(K)$; the largest cell

$$\mathfrak{Fl}^{(0)}(V) := \mathfrak{Fl}(V) \cap \prod_{i=1}^{n} \mathfrak{Gr}_i^{(0)}(V)$$

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has dimension $n(n+1)/2$. One major disadvantage here is that the relevant subgroups are not abelian.

The present paper aims to overcome this difficulty by replacing the flag variety by a certain natural degeneration, $\mathfrak{Fl}^{(a)}(V)$ (see Definition 2.3).

In mathematical theory, the degenerate flag variety $\mathfrak{Fl}^{(a)}(V)$ has been introduced in the framework of PBW degenerations by E. Feigin [2]. In fact, the ordinary flag variety is isomorphic to the highest weight orbit of any regular, finite-dimensional $\mathrm{GL}_{n+1}(K)$-module L_λ. The PBW degeneration L_λ^a of L_λ is induced by the natural PBW filtration on the universal enveloping algebra $U(\mathfrak{gl}_{n+1}(K))$ [3]. L_λ^a is again a regular module for an (abelian) algebraic group of dimension $n(n+1)/2$. Then E. Feigin defined the PBW degenerated flag variety as the highest weight orbit (of an abelian algebraic group) on L_λ^a. Hence, by construction, $\mathfrak{Fl}^{(a)}(V)$ is a projective variety of dimension $n(n+1)/2$ over K. Again in [2], a realization of the degenerate flag variety in terms of subspace conditions is provided, the one we will be working with for our purposes.

For applications it is enough to work with the easily accessible linear algebra model given in [6]. We obtain a cell decomposition of $\mathfrak{Fl}^{(a)}(V)$ as for flags where the largest cell

$$\mathfrak{Fl}^{(a0)}(V) = \mathfrak{Fl}^{(a)}(V) \cap \prod_{i=1}^{n} \mathfrak{Gr}_i^{(0)}(V)$$

again has dimension $n(n+1)/2$. The use of degenerate flags has two advantages: On the one hand the manipulations by the in network nodes require less additions compared to the linear subspace or flag coding (see Remark 4.1). Much more important: the cells in $\mathfrak{Fl}^{(a)}(V)$ are again affine spaces yielding an even better analogy between degenerate flag coding and linear subspace coding. In particular the largest cell $\mathfrak{Fl}^{(a0)}(V)$ is a regular orbit of the vector space $U^n(K)$ of all $n \times n$ upper triangular matrices. The space $U^n(K)$ has dimension $n(n+1)/2$. As the translations preserve the Grassmann distance, this distance translates to a generalised rank metric (see Definition 2.7) on $U^n(K)$ which we call the *flag rank metric*. Flag rank metric codes are introduced in Section 3. In particular we elaborate codes in which each two distinct elements have maximal possible distance (Propositions 3.3 and 3.4). A more sophisticated analysis of flag rank metric codes is subject to further research. We give the network coding scheme using degenerate flags in Section 4 before we conclude the paper with a comparison of some network coding properties of degenerate flags and flags.

2 Degenerate flag varieties

Throughout the paper we let K be a field and V be a vector space over K of dimension $n + 1$. The **Grassmann** variety

$$\mathfrak{Gr}_i(V) := \{ U \leq V \mid \dim(U) = i \}$$

is the set of i-dimensional subspaces of V. It becomes a metric space by putting

$$d(U, W) := d_i(U, W) := \frac{1}{2}(\dim(U + W) - \dim(U \cap W)) \text{ for all } U, W \in \mathfrak{Gr}_i(V),$$

the Grassmann distance on $\mathfrak{Gr}_i(V)$.

Remark 2.1. *The maximum distance on $\mathfrak{Gr}_i(V)$ is*

$$\max\{d_i(U, W) \mid U, W \in \mathfrak{Gr}_i(V)\} = \min\{i, n + 1 - i\}.$$

<u>Proof.</u> If $2i \le n + 1$ then we can find $U, W \in \mathfrak{Gr}_i(V)$ such that $U \cap W = \{0\}$. Then $\dim(U + W) = \dim(U) + \dim(W) = 2i = 2d_i(U, W)$. If $2i > n + 1$ then we can find $U, W \in \mathfrak{Gr}_i(V)$ such that $U + W = V$. Then $\dim(U \cap W) = 2i - (n + 1)$ and hence $d_i(U, W) = n + 1 - i$. $\qquad\square$

Full flags and also the degenerate flags that we consider in the present paper are subvarieties of the direct product of Grassmannians. The sum of the Grassmann distances defines a metric on this direct product:

Remark 2.2. *The function*

$$\mathbf{d} : \prod_{i=1}^{n} \mathfrak{Gr}_i(V) \times \prod_{i=1}^{n} \mathfrak{Gr}_i(V) \to \mathbb{Z}_{\ge 0}, \quad \mathbf{d}((U_1, \dots, U_n), (V_1, \dots, V_n)) := \sum_{i=1}^{n} d_i(U_i, V_i)$$

defines a metric on the product of the Grassmannians which we call again Grassmann metric. The maximum distance between two elements in $\prod_{i=1}^{n} \mathfrak{Gr}_i(V)$ is

$$\max\{\mathbf{d}(\mathbf{U}, \mathbf{W}) \mid \mathbf{U}, \mathbf{W} \in \prod_{i=1}^{n} \mathfrak{Gr}_i(V)\} = \mathbf{d}_{max}(n + 1)$$

with

$$\mathbf{d}_{max}(n + 1) := \begin{cases} k^2 & n + 1 = 2k \text{ even} \\ k(k + 1) & n + 1 = 2k + 1 \text{ odd.} \end{cases}$$

<u>Proof.</u> It is immediate that the sum $\mathbf{d}$ is again a metric. So we only show the upper bound. First assume that $n + 1 = 2k$ is even. Then for any $U_i, V_i \in \mathfrak{Gr}_i(V)$ $(i = 1, \dots, n)$ we have $d_i(U_i, V_i) \le \min\{i, 2k - i\}$ by Remark 2.1. So

$$\sum_{i=1}^{2k-1} d_i(U_i, V_i) \le \sum_{i=1}^{k} i + \sum_{i=k+1}^{2k-1} (2k - i) = 2 \sum_{i=1}^{k-1} i + k = k(k - 1) + k = k^2.$$

Now assume that $n + 1 = 2k + 1$ is odd. Then for any $U_i, V_i \in \mathfrak{Gr}_i(V)$ $(i = 1, \dots, n)$ we have $d_i(U_i, V_i) \le \min\{i, 2k + 1 - i\}$ (again by Remark 2.1). So

$$\sum_{i=1}^{2k} d_i(U_i, V_i) \le \sum_{i=1}^{k} i + \sum_{i=k+1}^{2k} (2k + 1 - i) = 2 \sum_{i=1}^{k} i = k(k + 1).$$

$\square$

To define the degenerate flag variety we use the notation of [6]. Fix a basis $(f_1, \ldots, f_{n+1})$ of V and identify the elements of V with their coordinate rows, $V \cong K^{1\times(n+1)}$. For $1 \le i \le n+1$ we define the projection

$$\mathrm{pr}_i : V \to V, f_j \mapsto \begin{cases} f_j & \text{if } j \ne i \\ 0 & \text{if } j = i \end{cases}.$$

With respect to the chosen basis, the endomorphism pr_i acts by right multiplication with the diagonal matrix

$$D_i = \mathrm{diag}(1, \ldots, 1, 0, 1, \ldots, 1) \in K^{(n+1)\times(n+1)}$$

having the 0 on position (i, i). For $j < i$ we put $\mathrm{pr}_{j,i} := \prod_{k=j+1}^{i} \mathrm{pr}_k$ and

$$D_{j,i} = \prod_{k=j+1}^{i} D_k = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0, 1 \ldots, 1),$$

with in total $i - j$ zeros on the diagonal on positions $(j+1, j+1), \ldots, (i, i)$. In abuse of notation we also put $\mathrm{pr}_{i,i} := \mathrm{id}_V$. Then clearly

$$\mathrm{rk}(\mathrm{pr}_i) = \mathrm{rk}(D_i) = n, \ \ \mathrm{pr}_j \, \mathrm{pr}_i = \mathrm{pr}_i \, \mathrm{pr}_j, \ \ \mathrm{rk}(\mathrm{pr}_{j,i}) = \mathrm{rk}(D_{j,i}) = n - i + j.$$

Definition 2.3. *The degenerate flag variety $\mathfrak{Fl}^{(a)}(V)$ is defined as*

$$\mathfrak{Fl}^{(a)}(V) := \{(V_1, \ldots, V_n) \in \prod_{i=1}^{n} \mathfrak{Gr}_i(V) \mid \mathrm{pr}_{i+1}(V_i) \subseteq V_{i+1} \text{ for all } i = 1, \ldots, n-1\}.$$

$\mathfrak{Fl}^{(a)}(V)$ *becomes a metric space via the restriction of the Grassmann distance $\mathbf{d}$ from Remark 2.2.*

From Remark 2.2 we immediately conclude

Remark 2.4. *The maximum distance on $\mathfrak{Fl}^{(a)}(V)$ is bounded from above by $\mathbf{d}_{max}(n+1)$.*

As we will see below (Proposition 3.3 and Proposition 3.4) the maximum distance $\mathbf{d}_{max}(n + 1)$ is achieved in the subset $\mathfrak{Fl}^{(a0)}(V)$ of $\mathfrak{Fl}^{(a)}(V)$.

The main result of [6] establishes a torus invariant bijection between a slight variation of the metric space $\mathfrak{Fl}^{(a)}(V)$ and a partial flag variety. In particular [6, Theorem 1.2] elaborates an isometry between a certain Schubert cell in this partial flag variety and a cell $\mathfrak{Fl}^{(a0)}(V)$ of maximal dimension in $\mathfrak{Fl}^{(a)}(V)$. It turns out that this cell is indeed an affine space of dimension $n(n + 1)/2$ where the translations respect the Grassmann metric $\mathbf{d}$ (see Theorem 2.8).

To define this cell $\mathfrak{Fl}^{(a0)}(V)$ we fix the basis $(f_1, \ldots, f_{n+1})$ of V from above with which we identify V with the row space $K^{1\times(n+1)}$. Recall that

$$\mathfrak{Gr}_i^{(0)}(V) = \{U_A = \text{ row space } (I_i | A) \mid A \in K^{i\times(n+1-i)}\}.$$

Definition 2.5. $\mathfrak{Fl}^{(a0)}(V) := \mathfrak{Fl}^{(a)}(V) \cap \prod_{i=1}^{n} \mathfrak{Gr}_i^{(0)}(V)$.

Given $(V_1, \ldots, V_n) \in \mathfrak{Fl}^{(a0)}(V)$ then for any i the space $V_i = U_{A_i}$ is the row space of

$$\Psi_i(A_i) := \left(\; I_i \mid A_i \; \right) \in K^{i \times (n+1)}$$

for a unique matrix $A_i = A(V_i) \in K^{i \times (n+1-i)}$. The condition that $\mathrm{pr}_{i+1}(V_i) \subseteq V_{i+1}$ is equivalent to the condition that

$$A(V_{i+1}) = \left(\begin{array}{c} \widehat{A(V_i)} \\ a_i \end{array} \right) \text{ for some row } a_i \in K^{1 \times (n-i)}$$

where $\widehat{A_i}$ is the $i \times (n-i)$ matrix obtained by deleting the first column from the $i \times (n+1-i)$ matrix A_i.

For an $n \times n$-matrix $\Delta \in K^{n \times n}$ and $1 \leq i \leq n$ we denote by

$$\Delta_{[i]} := (\Delta_{k,\ell})_{k=1,\ldots,i, \ell=i,\ldots n} \in K^{i \times (n+1-i)}$$

the upper right rectangular corner of Δ. Using this notation we obtain the following parametrization of $\mathfrak{Fl}^{(a0)}(V)$ by upper triangular matrices:

Definition 2.6. *The space of upper triangular matrices is denoted by*

$$U^n(K) := \{\Delta \in K^{n \times n} \mid \Delta_{ij} = 0 \text{ for } i > j\}.$$

The function $\Delta^{(a)} : \mathfrak{Fl}^{(a0)}(V) \to U^n(K)$ *associates to* $(V_1, \ldots, V_n) \in \mathfrak{Fl}^{(a0)}(V)$ *the upper triangular matrix*

$$\Delta := \Delta^{(a)}((V_1, \ldots, V_n)) \in K^{n \times n}$$

defined by $\Delta_{[i]} := A(V_i)$. *It defines a bijection between* $\mathfrak{Fl}^{(a0)}(V)$ *and* $U^n(K)$ *where the inverse mapping is given by*

$$\mathfrak{F}^{(a)} : U^n(K) \to \mathfrak{Fl}^{(a0)}(V), \Delta \mapsto (V_1, \ldots, V_n)$$

where $V_i = U_{\Delta_{[i]}}$ *is the row space of* $\Psi_i(\Delta_{[i]})$ *for all* i.

To obtain an isometry with respect to the Grassmann distance $\mathbf{d}$ from Definition 2.3 we define the flag rank metric frk on $U^n(K)$.

Definition 2.7. *The* flag rank *of a matrix* $\Delta \in U^n(K)$ *is defined as*

$$\mathrm{frk}(\Delta) := \sum_{i=1}^{n} \mathrm{rk}(\Delta_{[i]}).$$

Theorem 2.8. *For* $(U_1, \ldots, U_n), (V_1, \ldots, V_n) \in \mathfrak{Fl}^{(a0)}(V)$ *we have*

$$\mathbf{d}((U_1, \ldots, U_n), (V_1, \ldots, V_n)) = \mathrm{frk}(\Delta^{(a)}(U_1, \ldots, U_n) - \Delta^{(a)}(V_1, \ldots, V_n)).$$

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<u>Proof.</u> Let $\Delta := \Delta^{(a)}(U_1, \ldots, U_n)$ and $\Lambda := \Delta^{(a)}(V_1, \ldots, V_n)$. Then U_i is the row space of $\psi_i(\Delta_{[i]})$ and V_i is the row space of $\psi_i(\Lambda_{[i]})$ for $1 \leq i \leq n$. So by [5, Proposition 4] we have $d_i(U_i, V_i) = \mathrm{rk}(\Delta_{[i]} - \Lambda_{[i]})$. Summing over all i yields the result. $\qquad\square$

We conclude this section with an algorithm for computing $\Delta^{(a)}((U_1, \ldots, U_n))$ for any $(U_1, \ldots, U_n) \in \mathfrak{Fl}^{(a0)}(V)$:

Algorithm 2.9. Input: $(U_1, \ldots, U_n) \in \mathfrak{Fl}^{(a0)}(V)$.
Output: $\Delta := \Delta^{(a)}((U_1, \ldots, U_n)) \in U^n(K)$.
Algorithm:

(1) *Choose $u = (u_1, \ldots, u_{n+1}) \in U_1$ such that $u_1 \neq 0$ and put $\Delta_{[1]} := (u_2/u_1, \ldots, u_{n+1}/u_1)$.*

(2) *Assume that we computed $\Delta_{[i-1]} \in K^{(i-1)\times(n-i+2)}$ and put $A := \widehat{\Delta_{[i-1]}} \in K^{(i-1)\times(n-i+1)}$ by deleting the first column of $\Delta_{[i-1]}$.*

(3) *Choose $w = (w_1, \ldots, w_{n+1}) \in U_i$ such that $w_i \neq 0$. Put $v := (w_{i+1}, \ldots, w_{n+1}) \in K^{n-i+1}$ and compute*
$$u := v - \sum_{j=1}^{i-1} w_j A_j$$
where A_j is the j-th row of the matrix A from (2). Put
$$\Delta_{[i]} := \begin{pmatrix} A \\ (u_{i+1}/w_i, \ldots, u_{n+1}/w_i) \end{pmatrix}.$$

(4) *Return the matrix $\Delta \in U^{(n)}(K)$ obtained from the computed $\Delta_{[i]}$, $i = 1, \ldots, n$.*

3 Flag rank metric codes in $U^n(K)$

Definition 3.1. *A flag rank metric code $\mathcal{C}$ is a K-subspace $\mathcal{C} \leq U^n(K)$. The minimum distance of $\mathcal{C}$ is defined as*
$$d_{\mathrm{frk}}(\mathcal{C}) := \min\{\mathrm{frk}(\Delta) \mid 0 \neq \Delta \in \mathcal{C}\}.$$

Remark 2.4 and Theorem 2.8 hence imply

Remark 3.2. $\max\{\mathrm{frk}(\Delta) \mid \Delta \in U^n(K)\} \leq \mathbf{d}_{max}(n+1)$.

Proposition 3.3. *Assume that $n + 1 = 2k$ is even.*

(a) $\max\{\mathrm{frk}(\Delta) \mid \Delta \in U^n(K)\} = k^2 = \mathbf{d}_{max}(n+1)$.

(b) *For any flag rank metric code $\mathcal{C} \leq U^n(K)$ with $d_{\mathrm{frk}}(\mathcal{C}) = k^2$ we have $\dim(\mathcal{C}) \leq k$.*

(c) *If K admits a field extension of degree k then there is a flag rank metric code $\mathcal{C} \leq U^n(K)$ with*
$$d_{\mathrm{frk}}(\mathcal{C}) = k^2 \text{ and } \dim(\mathcal{C}) = k.$$

<u>Proof.</u> To see (c) let F be a field extension of K of degree k. The regular representation $\rho : F \hookrightarrow K^{k\times k}$ defines an embedding of F into $K^{k\times k}$ such that for any $0 \neq f \in F$ the matrix $\rho(f)$ is invertible, so has full rank k. Now let

$$\mathcal{C} := \left\{ \Delta_f := \begin{pmatrix} 0 & \rho(f) \\ 0 & 0 \end{pmatrix} \mid f \in F \right\} \leq U^n(K).$$

Note that the 0s on the diagonal are $(k-1) \times k$ respectively $k \times (k-1)$-matrices. Then clearly $\dim(\mathcal{C}) = \dim(F) = k$. As any ℓ rows or columns of $\rho(f)$ are linearly independent for any $0 \neq f \in F$ we compute $d_{\text{frk}}(\Delta_f) = \mathbf{d}_{max}(2k)$.
Now (a) follows from the fact that $\text{frk}(\Delta_1) = \mathbf{d}_{max}(n+1)$ together with Theorem 2.8 and Remark 2.4.
To prove (b) let $\mathcal{C} \leq U^n(K)$ be a flag rank metric code of dimension $\geq k+1$. Then the projection of $\mathcal{C}$ onto the first k entries of the last column is not injective, so $\mathcal{C}$ contains some matrix M of the form

$$M = \begin{pmatrix} X & 0 \\ 0 & a \end{pmatrix} \quad \text{with } X \in K^{k\times(2k-2)}.$$

If $Y \in K^{k\times(k-1)}$ consists of the last $k-1$ columns of X then $M_{[k]} = \begin{pmatrix} Y \mid 0 \end{pmatrix} \in K^{k\times k}$ has at most rank $k-1$, so $\text{frk}(M) \leq k^2 - 1$. $\qquad\square$

Proposition 3.4. *Assume that $n+1 = 2k+1$ is odd.*

(a) $\max\{\text{frk}(\Delta) \mid \Delta \in U^n(K)\} = k(k+1) = \mathbf{d}_{max}(n+1)$.

(b) *For any flag rank metric code $\mathcal{C} \leq U^n(K)$ with $d_{\text{frk}}(\mathcal{C}) = k(k+1)$ we have $\dim(\mathcal{C}) \leq k+1$.*

(c) *If K admits a field extension of degree k then there is a flag rank metric code $\mathcal{C} \leq U^n(K)$ with*
$$d_{\text{frk}}(\mathcal{C}) = k(k+1) \text{ and } \dim(\mathcal{C}) = k.$$

<u>Proof.</u> To see (c) we consider the same code

$$\mathcal{C} := \left\{ \Delta_f := \begin{pmatrix} 0 & \rho(f) \\ 0 & 0 \end{pmatrix} \mid f \in F \right\} \leq U^n(K)$$

as in the proof of Proposition 3.3. Now the 0s on the diagonal are $k \times k$-matrices. As before the minimum distance of $\mathcal{C}$ is $k(k+1)$ and $\dim(\mathcal{C}) = k$.
Again (a) follows from the fact that $\text{frk}(\Delta_1) = \mathbf{d}_{max}(n+1)$ together with Theorem 2.8 and Remark 2.4.
To prove (b) let $\mathcal{C} \leq U^n(K)$ be a flag rank metric code of dimension $\geq k+2$. Then the projection of $\mathcal{C}$ onto the last $k+1$ entries of the first row is not injective, so $\mathcal{C}$ contains some matrix M of the form

$$M = \begin{pmatrix} a & 0 \\ b & X \\ 0 & y \end{pmatrix} \quad \text{with } X \in K^{(k-1)\times(k+1)}.$$

Then $M_{[k]} = \begin{pmatrix} 0 \\ X \end{pmatrix} \in K^{k \times (k+1)}$ has at most rank $k-1$, so $\mathrm{frk}(M) \le k(k+1) - 1$. $\square$

3.1 An example

To illustrate the notation we give a small example. Let $K = \mathbb{F}_3$ be the field with 3 elements and let $n = 4$. Let $\mathcal{T}$ be the 4-dimensional flag rank metric code with basis

$$\left\{ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Then $\mathcal{T} \le U_4(K)$ is a flag rank metric code of dimension 4 with minimum distance 5. This is best possible for any field K: If $\mathcal{C} \le U_4(K)$ is a flag rank metric code with minimum distance ≥ 5, then the intersection of $\mathcal{C}$ with the 6-dimensional subspace

$$\mathcal{D} := \begin{pmatrix} K & K & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}$$

is $\{0\}$ as the maximal flag rank of a matrix in $\mathcal{D}$ is 4. So the dimension of $\mathcal{C}$ cannot exceed $\dim(U_4(K)) - \dim(\mathcal{D}) = 10 - 6 = 4$.

In the next table we apply the degenerate flag coding map $\mathfrak{F}^{(a)}$ to the first two basis elements and their sum:

Δ	$\mathfrak{F}^{(a)}(\Delta) = (U_1, U_2, U_3, U_4)$
$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$U_1 = \langle f_1 + f_2 + f_4 + f_5 \rangle$ $U_2 = \langle f_1 + f_4 + f_5, f_2 + f_3 \rangle$ $U_3 = \langle f_1 + f_4 + f_5, f_2, f_3 + f_4 + f_5 \rangle$ $U_4 = \langle f_1 + f_5, f_2, f_3 + f_5, f_4 \rangle$
$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$U_1 = \langle f_1 + 2f_3 + f_4 \rangle$ $U_2 = \langle f_1 + 2f_3 + f_4, f_2 + 2f_3 + 2f_4 \rangle$ $U_3 = \langle f_1 + f_4, f_2 + 2f_4, f_3 + f_4 \rangle$ $U_4 = \langle f_1, f_2, f_3, f_4 + f_5 \rangle$
$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$U_1 = \langle f_1 + f_2 + 2f_3 + 2f_4 + f_5 \rangle$ $U_2 = \langle f_1 + 2f_3 + 2f_4 + f_5, f_2 + 2f_4 \rangle$ $U_3 = \langle f_1 + 2f_4 + f_5, f_2 + 2f_4, f_3 + 2f_4 + f_5 \rangle$ $U_4 = \langle f_1 + f_5, f_2, f_3 + f_5, f_4 + f_5 \rangle$

3.2 Duality and decoding of flag rank metric codes

The space $U^n(K)$ is equipped with a non degenerate K-bilinear form

$$A \cdot B := \text{trace}(AB^{\text{tr}}) \text{ for all } A, B \in U^n(K).$$

We hence have the notion of dual code $\mathcal{C}^\perp$ of a flag rank metric code and can use syndrome decoding to find an element of the flag rank metric code $\mathcal{C} \leq U^n(K)$ having minimum distance to any given element $\Delta \in U^n(K)$.

Definition 3.5. *For $\mathcal{C} \leq U^n(K)$ the* dual code *is*

$$\mathcal{C}^\perp := \{\Delta \in U^n(K) \mid trace(\Delta\Lambda^{\text{tr}}) = 0 \text{ for all } \Lambda \in \mathcal{C}\}.$$

From the non-degeneracy of the bilinear form we obtain that

$$\dim(\mathcal{C}) + \dim(\mathcal{C}^\perp) = \dim(U^n(K)) = n(n+1)/2 \text{ and } \mathcal{C} = (\mathcal{C}^\perp)^\perp.$$

We now fix a flag rank metric code $\mathcal{C} \leq U^n(K)$ and choose a basis $\underline{\Delta} := (\Delta_1, \ldots, \Delta_\ell)$ of $\mathcal{C}^\perp$.

Remark 3.6. *(a) For $A \in U^n(K)$ the* syndrome *of A with respect to $\underline{\Delta}$ is*

$$\text{syn}_{\underline{\Delta}}(A) := (trace(\Delta_1 A^{\text{tr}}), \ldots, trace(\Delta_\ell A^{\text{tr}})) \in K^\ell.$$

(b) For $A \in U^n(K)$ we have that $A \in \mathcal{C}$ if and only if $\text{syn}_{\underline{\Delta}}(A) = (0, \ldots, 0)$.

(c) For $f \in K^\ell$ the set

$$E_f := \{A \in U^n(K) \mid \text{syn}_{\underline{\Delta}}(A) = f\} = A_f + \mathcal{C}$$

is a coset of $\mathcal{C}$ in $U^n(K)$.

(d) We have that $E_{af} = aE_f = aA_f + \mathcal{C}$ for any $a \in K$.

(e) Let $d_f := \min\{\text{frk}(A) \mid A \in E_f\}$. Then any $A_f \in E_f$ with $\text{frk}(A_f) = d_f$ is called a minimal coset leader for f.

(f) For $0 \neq a \in K$ and $f \in K^\ell$ we have that $d_{af} = d_f$. If A_f is a minimal coset leader for f then aA_f is a minimal coset leader for af.

The usual syndrome decoding algorithm for the case that K is a finite field now reads as follows:

Algorithm 3.7. Input: *A flag rank metric code $\mathcal{C} \leq U^n(K)$ and $A \in U^n(K)$.*
Output: *One element $C \in \mathcal{C}$ such that $\text{frk}(A - C) = \min\{\text{frk}(A - C') \mid C' \in \mathcal{C}\}$.*
Precomputing:

P1 *Compute a basis $\underline{\Delta} := (\Delta_1, \ldots, \Delta_\ell)$ of $\mathcal{C}^\perp$.*

P2 *For all* $\langle f \rangle \in \mathfrak{Gr}_1(K^\ell)$ *compute a minimal coset leader A_f for f in E_f and store the set* $M_{\underline{\Delta}} := \{(f, A_f) \mid \langle f \rangle \in \mathfrak{Gr}_1(K^\ell)\}$.

Decoding:

D1 *Compute* $\text{syn}_{\underline{\Delta}}(A) =: g$. *If* $g = (0, \dots, 0)$ *then return A.*

D2 *Find the $(f, A_f) \in M_{\underline{\Delta}}$ such that $\langle f \rangle = \langle g \rangle$ and $a \in K$ with $g = af$.*

D3 *Return $A - aA_f$.*

<u>Proof.</u> We have $C := A - aA_f \in \mathcal{C}$ by Remark 3.6 (b) as $\text{syn}_{\underline{\Delta}}(A - aA_f) = g - af = 0$. Moreover

$$\text{frk}(A - C) = \text{frk}(aA_f) = \text{frk}(A_f) = d_f = d_g = \min\{\text{frk}(A - C') \mid C' \in \mathcal{C}\}.$$

$\square$

The syndrome decoding algorithm is a very general, in general not very efficient, decoding algorithm. For special types of flag rank metric codes, there are much better decoding procedures as illustrated for flag codes in [8, Section 3.4.4].

4   Network coding with degenerate flags

The network coding techniques from [7] can be easily adapted to degenerate flags: Any element $\mathbf{U} = (U_1, \dots, U_n) \in \mathfrak{Fl}^{(a0)}(V)$ is represented by a matrix $X \in K^{n \times (n+1)}$ as follows. If X_i is the i-th row of X, then

$$U_i = \langle \text{pr}_{j,i}(X_j) \mid j = 1, \dots, i \rangle$$

for all $0 \leq i < n$. The source sends the matrix X, row by row, where the i-th row has sequence number i:

$$(X_1, 1), (X_2, 2), \dots, (X_n, n).$$

In network nodes receive vectors (Y_i, i) for $i = 1, \dots, n$. For each $i = 1, \dots, n$ they form linear combinations

$$Z_i := \sum_{j=1}^{i} a_{ij} \, \text{pr}_{j,i}(Y_j) \qquad (\star)$$

where the $(a_{ij})_{1 \leq j \leq i \leq n}$ is a lower triangular matrix in $K^{n \times n}$. Then the node forwards $(Z_1, 1)$, ..., (Z_n, n). The receiver hence gets a number of rows $(R_{i,t}, i)$ for $i = 1, \dots, n$ and $t = 1, \dots, s_i$ and puts

$$W_i := \langle \text{pr}_{j,i}(R_{j,t_j}) \mid 1 \leq t_j \leq s_j, 1 \leq j \leq i \rangle$$

for $i = 0, \dots, n - 1$. Then $\mathbf{W} = (W_1, \dots, W_n)$ satisfies $\text{pr}_{i+1}(W_i) \subseteq W_{i+1}$ for all $i = 1, \dots, n - 1$. If $\dim(W_i) = i$ then we again have that $\mathbf{W} \in \mathfrak{Fl}^{(a)}(V)$. As $\mathfrak{Fl}^{(a0)}(V)$ is dense in $\mathfrak{Fl}^{(a)}(V)$ it is very likely that then $\mathbf{W} \in \mathfrak{Fl}^{(a0)}(V)$.

Remark 4.1. *Note that the operation $(\star)$ for the degenerated flags requires less additions than in the network coding scheme that uses full flags. If $Y = (y_1, \ldots, y_{n+1})$, then $\mathrm{pr}_{j,i}(Y) = (y_1, \ldots, y_{j-1}, 0, \ldots, 0, y_{i+1}, \ldots, y_{n+1})$ so we may just ignore $i - j + 1$ entries in $\mathrm{pr}_{j,i}(Y_j)$ when computing $(\star)$. Assuming that the node received exactly one vector Y_j for each sequence number j, the number of coordinate multiplications and additions in time $i + 1$ is*

$$\sum_{j=1}^{i}(n + 1 - (i - j + 1)) = i(n - i) + \frac{i(i + 1)}{2} = i(n + 1) - \frac{i^2 + i}{2}$$

compared to $i(n + 1)$ in the case of subspace coding or flag coding.

Using a flag rank metric code $\mathcal{C} \leq U^n(K)$ as input and the precomputation in Algorithm 3.7 we hence get the following algorithm

Algorithm 4.2.   • *To send an element $X \in \mathcal{C}$:*

- *the source inputs the i-th row X_i of $\psi_i(X_{[i]})$ with sequence number i*

- *in network nodes receive vectors (Y_i, i) for $i = 1, \ldots, n$*

- *compute $Z_i := \sum_{j=1}^{i} a_{ij}\,\mathrm{pr}_{j,i}(Y_j)$ with $a_{ij} \in K$ randomly chosen*

- *and forward (Z_i, i) for $= 1, \ldots, n$.*

- *if $(R_{i,t}, i)$, $t = 1, \ldots, s_i$ are the received vectors with sequence number $i = 1, \ldots, n$ then*

- *the receiver puts*

$$W_i := \langle \mathrm{pr}_{j,i}(R_{j,t_j}) \mid 1 \leq t_j \leq s_j, 1 \leq j \leq i \rangle$$

  for $i = 0, \ldots, n - 1$.

- *Check that $\mathbf{W} = (W_1, \ldots, W_n) \in \mathfrak{Fl}^{(a0)}(V)$ and compute $A := \Delta(\mathbf{W})$ using Algorithm 2.9.*

- *Use the decoding part of Algorithm 3.7 to find $C \in \mathcal{C}$ with $\mathrm{frk}(A - C)$ minimal.*

- *Decode to C.*

5   A comparison

This final section highlights the advantages to use degenerate flags instead of flags for network coding. As before we place ourselves in the situation where V is a vector space over a field K of dimension $(n + 1)$. Then

$$\mathfrak{Fl}(V) := \{(V_1, \ldots, V_n) \in \prod_{i=1}^{n} \mathfrak{Gr}_i(V) \mid V_i \subseteq V_{i+1} \text{ for all } i = 1, \ldots, n - 1\}$$

and

$$\mathfrak{Fl}^{(a)}(V) := \{(V_1, \ldots, V_n) \in \prod_{i=1}^{n} \mathfrak{Gr}_i(V) \mid \mathrm{pr}_{i+1}(V_i) \subseteq V_{i+1} \text{ for all } i = 1, \ldots, n-1\}.$$

Both subsets of $\prod_{i=1}^{n} \mathfrak{Gr}_i(V)$ are equipped with the same metric $\mathbf{d}$, the sum of the Grassmann distances. Both sets partition into cells according to the pivot sets, where a cell of maximal dimension is obtained as

$$\mathfrak{Fl}^{(0)}(V) := \mathfrak{Fl}(V) \cap \prod_{i=1}^{n} \mathfrak{Gr}_i^{(0)}(V) \text{ resp. } \mathfrak{Fl}^{(a0)}(V) := \mathfrak{Fl}^{(a)}(V) \cap \prod_{i=1}^{n} \mathfrak{Gr}_i^{(0)}(V).$$

Both cells have the same dimension and the same maximum distance $\mathbf{d}_{max}(n+1)$.

Both cells are in bijection to $U^n(K)$, where for a matrix $\Delta \in U^n(K)$ the degenerate flag is $\mathfrak{F}^{(a)}(\Delta) \in \mathfrak{Fl}^{(a0)}(V)$ defined after Definition 2.6. For the full flags the map

$$\mathfrak{F} : U^n(K) \to \mathfrak{Fl}^{(0)}(V), \Delta \mapsto (U_1, \ldots, U_n)$$

where U_i is the space spanned by the first i rows of the matrix $\Phi(\Delta) \in U^{n+1}(K)$ with ones on the diagonal and upper triangular entries $\Phi(\Delta)_{i,j+1} = \Delta_{i,j}$ for all $1 \leq i \leq n, i \leq j \leq n$.

Remark 5.1.
$$\mathbf{d}(\mathfrak{F}(0), \mathfrak{F}(\Delta)) = \mathrm{frk}(\Delta)$$

but in general
$$\mathbf{d}(\mathfrak{F}(\Gamma), \mathfrak{F}(\Delta)) = \mathrm{frk}(\Phi^{-1}(\Phi(\Gamma)\Phi(\Delta)^{-1}))$$

involves the non-commutative matrix product of $\Phi(\Delta)$ and $\Phi(\Gamma)$, whereas
$$\mathbf{d}(\mathfrak{F}^{(a)}(\Gamma), \mathfrak{F}^{(a)}(\Delta)) = \mathrm{frk}(\Gamma - \Delta).$$

Also the network coding operations in $\mathfrak{Fl}^{(a0)}(V)$ involve less computations than the ones in $\mathfrak{Fl}^{(0)}(V)$ (see Remark 4.1).

In both cases the input alphabet is $U^n(K)$, a vector space of dimension $n(n+1)/2$. Whereas retrieving the upper triangular matrix $\Delta \in U^n(K)$ associated to the flag $(U_1, \ldots, U_n) \in \mathfrak{Fl}^{(0)}(V)$ requires a (partial) Gaussian elimination, the decoding algorithm 2.9 computing $\Delta^{(a)}((U_1, \ldots, U_n))$ is much simpler.

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