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# Golden lattices

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*This paper is dedicated to Boris Venkov.*

Abstract. A golden lattice is an even unimodular $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$-lattice of which the Hilbert theta series is an extremal Hilbert modular form. We construct golden lattices from extremal even unimodular lattices and obtain families of dense modular lattices.

## 1. Introduction

**1.1. Even unimodular $R$-lattices.** Let $R$ be the ring of integers in a totally real number field $K$. Let $\Lambda$ be a full $R$-lattice in Euclidean $n$-space $(K^n, Q)$, so $\Lambda$ is a finitely generated $R$-submodule of $K^n$ that spans $K^n$ over $K$ and $Q : K^n \to K$ is a totally positive definite quadratic form. The *polar form* of $Q$ is the positive definite symmetric $K$-bilinear form $B$ defined by $B(x, y) := Q(x+y)-Q(x)-Q(y)$. The lattice $(\Lambda, Q)$ is called *even* if $Q : \Lambda \to R$ is an $R$-valued quadratic form and *unimodular* over $R$, if

$$\Lambda = \Lambda^{\#} := \{x \in K^n \mid B(x, \lambda) \in R \text{ for all } \lambda \in \Lambda\}.$$

For small dimension and small fields even unimodular $R$-lattices have been classified in [**2**], [**8**], [**9**].

**1.2. Trace lattices.** Any $R$-lattice $(\Lambda, Q)$ and any totally positive $\alpha \in K_+$ give rise to a positive definite $\mathbb{Z}$-lattice

$$L_\alpha := (\Lambda, \mathrm{tr}_{K/\mathbb{Q}}(\alpha Q))$$

of dimension $n[K : \mathbb{Q}]$. $L_\alpha$ will be called a *trace lattice* of $(\Lambda, Q)$, since the quadratic form

$$q := q_\alpha : L_\alpha \to \mathbb{Q}, x \mapsto \mathrm{tr}_{K/\mathbb{Q}}(\alpha Q(x))$$

is obtained as a trace. In this way an $R$-lattice $(\Lambda, Q)$ defines family of positive definite $\mathbb{Z}$-lattices $\{L_\alpha \mid \alpha \in K_+\}$ where $\alpha$ varies over the $[K : \mathbb{Q}]$-dimensional rational cone $K_+$ of totally positive elements in $K$. The most important invariants of all the lattices $L_\alpha$ in this family, like minimum and determinant, and also the

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theta series may be read off from the corresponding invariants of the lattice $(\Lambda, Q)$ and its Hilbert theta series. For instance the dual lattice

$$(1.1) \qquad L_\alpha^* := \{x \in K^n \mid \mathrm{tr}_{K/\mathbb{Q}}(\alpha B(x,\lambda)) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\} = \alpha^{-1} R^* \Lambda^{\#}$$

where $R^* := \{s \in K \mid \mathrm{tr}_{K/\mathbb{Q}}(sR) \subseteq \mathbb{Z}\}$ is the *inverse different* of $R$.

**1.3. Extremal even unimodular lattices.** In particular if $(\Lambda, Q)$ is an even unimodular $R$-lattice and $\alpha$ is a totally positive generator of $R^* = \alpha R$ then the trace lattice $L_\alpha$ is an even unimodular integral lattice of dimension $N = n[K : \mathbb{Q}]$. It is well known that the minimum

$$\min(L, q) = \min\{q(x) \mid 0 \neq x \in L\}$$

of an even unimodular $\mathbb{Z}$-lattice $L$ of dimension $N$ is bounded by $\min(L,q) \leq 1 + \lfloor \frac{N}{24} \rfloor$ (see [**17**, end of proof of Satz 2]). Lattices that achieve equality are called *extremal*. If the dimension $N$ is a multiple of 24 then the known extremal even unimodular lattices are densest known lattices (see [**15**]). There are 4 such lattices known, the Leech lattice $\Lambda_{24}$ in dimension 24, three lattices $P_{48p}$, $P_{48q}$, and $P_{48n}$ of dimension 48 and one lattice $\Gamma_{72}$ of dimension 72.

**1.4. Golden lattices.** This article considers the situation where $R = \mathbb{Z}[\vartheta]$ and $1 + \vartheta = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Then $K$ is a real quadratic number field of minimal possible discriminant 5, $R$ is a principal ideal domain and

$$R^* = \eta^{-1} R \text{ where } \eta = 3 + \vartheta = \frac{5+\sqrt{5}}{2}$$

is a totally positive generator of the prime ideal of norm 5. In Section 3 the extremal even unimodular lattices $\Lambda_{24}$, $P_{48n}$ and $\Gamma_{72}$ are obtained as trace lattices $L_{\eta^{-1}}$ of even unimodular $\mathbb{Z}[\vartheta]$-lattices. This structure allows to construct interesting families of dense modular lattices (Section 2.4).

## 2. Hilbert theta series of golden lattices

**2.1. Symmetric Hilbert modular forms.** Let $R := \mathbb{Z}[\vartheta]$ be the ring of integers in the real quadratic number field $K := \mathbb{Q}[\sqrt{5}]$. Then the Hilbert theta series (Definition 2.1) of an $n$-dimensional even unimodular $R$-lattice $(\Lambda, Q)$ is a Hilbert modular form of weight $n/2$ (see [**5**, Section 5.7]). If $(\Lambda, Q)$ is Galois invariant, then so is its theta series and hence this Hilbert modular form is *symmetric*. Hilbert modular forms for $R$ are holomorphic functions on the direct product $\mathbb{H}_K := \mathbb{H} \times \mathbb{H}$ of 2 copies of the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

If $\sigma_1, \sigma_2$ denote the two embeddings of $K$ into $\mathbb{R} \subset \mathbb{C}$ then $\mathrm{SL}_2(R)$ acts on $\mathbb{H}_K$ by

$$(z_1, z_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \frac{\sigma_2(a)z_2 + \sigma_2(b)}{\sigma_2(c)z_2 + \sigma_2(d)}\right)$$

and the Galois automorphism just interchanges the two copies of $\mathbb{H}$. The ring of symmetric Hilbert modular forms for the group $\mathrm{SL}_2(R)$ is a polynomial ring

$$\mathcal{H} := \mathbb{C}[A_2, B_6, C_{10}]$$

graded by the weight. The explicit generators of weight 2, 6, and 10 have been obtained in [**6**] (see [**5**, Corollary 5.4]). We denote by

$$\mathcal{H}_w := \{f \in \mathcal{H} \mid \text{ weight of } f = w\}$$

the space of symmetric Hilbert modular forms of weight $w$.

DEFINITION 2.1. Let $(\Lambda, Q)$ be an even $R$-lattice. Then the *Hilbert theta series* of $\Lambda$ is

$$\Theta(\Lambda, Q) := \sum_{\lambda\in\Lambda} \exp(2\pi i\, \mathrm{tr}_{K/\mathbb{Q}}(zQ(\lambda))) = 1 + \sum_{X\in R_+} A_X \exp(2\pi i\, \mathrm{tr}_{K/\mathbb{Q}}(zX))$$

where $A_X := |\{\lambda \in \Lambda \mid Q(\lambda) = X\}|$ and $\mathrm{tr}_{K/\mathbb{Q}}(zX) := z_1\sigma_1(X) + z_2\sigma_2(X)$ for $z = (z_1, z_2) \in \mathbb{H}_K$.

Since $(1, \eta^{-1})$ is a $\mathbb{Q}$-basis of $K$ the traces of $Q(\lambda)$ and $\eta^{-1}Q(\lambda)$ uniquely determine the value $Q(\lambda) \in K$. So $\Theta(\Lambda, Q)$ is determined by the $(q_0, q_1)$-*expansion*

$$\Theta(\Lambda, Q) := \sum_{\lambda\in\Lambda} q_0^{\mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda))} q_1^{\mathrm{tr}_{K/\mathbb{Q}}(Q(\lambda))} \in \mathbb{C}[[q_0, q_1]]$$

which is very convenient for computations. Replacing $q_1$ by 1 yields the usual theta series of the trace lattice $L_{\eta^{-1}}$ and substituting $q_0$ by 1 gives the theta series of $L_1$. Explicit computations with Magma [**1**] show that $A_2(q_0, 1) = \Theta(E_8)$ the Eisenstein series of weight 4, $B_6(q_0, 1) = \Delta$, the cusp form of weight 12, and $C_{10}(q_0, 1) = 0$. In particular replacing $q_1$ by 1 yields a surjective ring homomorphism from $\mathcal{H}$ onto the ring of elliptic modular forms of weight divisible by 4 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

EXAMPLE 2.2. The $(q_0, q_1)$-expansion of the three generators of $\mathcal{H}$ start with

$$A_2 = 1 + (120q_1^2 + 120q_1^3)q_0 + (120q_1^3 + 600q_1^4 + 720q_1^5 + 600q_1^6 + 120q_1^7)q_0^2 + O(q_0^3)$$
$$B_6 = (q_1^2 + q_1^3)q_0 + (q_1^3 + 20q_1^4 - 90q_1^5 + 20q_1^6 + q_1^7)q_0^2 + O(q_0^3)$$
$$C_{10} = (q_1^4 - 2q_1^5 + q_1^6)q_0^2 + (-2q_1^5 - 18q_1^6 + 20q_1^7 + 20q_1^8 - 18q_1^9 - 2q_1^{10})q_0^3 + O(q_0^4)$$

### 2.2. Extremal Hilbert modular forms.

DEFINITION 2.3. Define a valuation on the field of fractions of $\mathbb{C}[[q_0, q_1]]$ by

$$\nu : \mathbb{C}((q_0, q_1)) \setminus \{0\} \to \mathbb{Z} \times \mathbb{Z}, \nu(\sum_{(i,j)=(s,t)}^{(\infty,\infty)} A_{(i,j)} q_0^i q_1^j) := \min\{(i, j) \mid A_{(i,j)} \neq 0\}$$

where the total ordering on $\mathbb{Z} \times \mathbb{Z}$ is lexicographic, so

$$(s, t) \leq (s', t') \text{ if } s < s' \text{ or } s = s' \text{ and } t \leq t'.$$

This gives rise to a valuation on the ring of Hilbert modular forms via the $(q_0, q_1)$-expansion. A symmetric Hilbert modular form $f \in \mathcal{H}_w$ is called an *extremal Hilbert modular form of weight* $w$, if

$$\nu(f - 1) \geq \nu(f' - 1) \text{ for all } f' \in \mathcal{H}_w.$$

With Magma [**1**] one computes

$$\nu(A_2 - 1) = (1, 2),\ \nu(B_6) = (1, 2),\ \nu(C_{10}) = (2, 4),\ \nu(X_{12}) = (2, 5)$$

where $X_{12} = \frac{1}{4}(A_2C_{10} - B_6^2)$. The valuations of the first few extremal Hilbert modular forms are given in the table in Example 2.7 below.

### 2.3. Golden lattices.

DEFINITION 2.4. An even unimodular $R$-lattice $(\Lambda, Q)$ is called a *golden lattice*, if its Hilbert theta series is an extremal symmetric Hilbert modular form.

PROPOSITION 2.5. *Let $(L, q)$ be an even unimodular $\mathbb{Z}$-lattice of dimension $N$ and $\vartheta \in \mathrm{End}_{\mathbb{Z}}(L)$ be a self-adjoint endomorphism of $L$ with minimal polynomial $X^2 + X - 1$. Then $L$ is a $\mathbb{Z}[\vartheta]$-lattice $\Lambda$ and $L = L_{\eta^{-1}}$ for the even unimodular $\mathbb{Z}[\vartheta]$-lattice $(\Lambda, Q)$ with quadratic form*

$$Q : \Lambda \to R, Q(\lambda) := \frac{1}{2}(q(\lambda) + q(\vartheta\lambda)) + \frac{1}{2}(q(\vartheta\lambda) - q(\lambda))\sqrt{5}.$$

*Assume that there is some automorphism $\sigma \in \mathrm{Aut}(L, q)$ such that $\sigma\vartheta = (-1 - \vartheta)\sigma$. Then for $N = 8, 16, 24, 32, 48, 56, 72$ the lattice $(\Lambda, Q)$ is a golden lattice, if and only if $(L, q)$ is an extremal even unimodular lattice.*

PROOF. Clearly any such endomorphism $\vartheta$ defines a $\mathbb{Z}[\vartheta]$-structure on the $\mathbb{Z}$-lattice $L$. Since $\vartheta$ is a self-adjoint with respect to the polar form of $q$, the form $q$ is a trace form, $q(\lambda) = \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda))$ for some $K$-valued quadratic form $Q$ for all $\lambda \in L$. If $Q(\lambda) = a + b\sqrt{5}$, then

$$q(\lambda) = \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda)) = a - b \text{ and } q(\vartheta\lambda) = \mathrm{tr}_{K/Q}(\vartheta^2\eta^{-1}Q(\lambda)) = a + b$$

and hence $Q(\lambda) = \frac{1}{2}(q(\lambda) + q(\vartheta\lambda)) + \frac{1}{2}(q(\vartheta\lambda) - q(\lambda))\sqrt{5}$. Clearly if $q(\lambda) \in \mathbb{Z}$ and $q(\vartheta\lambda) \in \mathbb{Z}$ then also $Q(\lambda) \in R$, therefore $(\Lambda, Q)$ is even. By Equation (1.1) the lattice $(\Lambda, Q)$ is unimodular.

The Galois invariance of $\Theta(\Lambda, Q)$ follows from the fact that $Q(\vartheta\sigma(\lambda)) = \overline{Q(\lambda)}$ for all $\lambda \in \Lambda$:

To see this write $Q(\lambda) = a + b\sqrt{5}$ and $Q(\sigma(\lambda)) = a' + b'\sqrt{5}$, so

$$Q(\vartheta\sigma(\lambda)) = \vartheta^2 Q(\sigma(\lambda)) = \frac{3a' - 5b'}{2} + \frac{3b' - a'}{2}\sqrt{5}.$$

Then

$$\begin{array}{lll} \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda)) = \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\sigma(\lambda))) & \text{yields} & a - b = a' - b' \\ \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\vartheta\lambda)) = \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\sigma(\vartheta\lambda))) \Leftrightarrow & & \\ \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}\vartheta^2 Q(\lambda)) = \mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}\overline{\vartheta}^2 Q(\sigma(\lambda))) & \text{yields} & 2a - 4b = a' + b' \end{array}$$

so in total $a' = \frac{3a-5b}{2}, b' = \frac{a-3b}{2}$ which gives

$$Q(\sigma(\lambda)) = \overline{\vartheta^2 Q(\lambda)} \text{ for all } \lambda \in \Lambda.$$

Therefore the mapping $\lambda \mapsto \vartheta\sigma(\lambda)$ gives a bijection between

$$\{\lambda \in \Lambda \mid Q(\lambda) = \alpha\} \text{ and } \{\lambda \in \Lambda \mid Q(\lambda) = \overline{\alpha}\}$$

so the Hilbert theta series of $(\Lambda, Q)$ is symmetric.

The last statement follows from explicit computations in the ring of Hilbert modular forms. These show that for weight $2, 4, 6, 8, 12, 14$, and $18$ the condition that $f(q_0, 1)$ be an extremal elliptic modular form and that $f(q_0, q_1)$ has non negative coefficients imply that $f$ is an extremal Hilbert modular form. The opposite direction follows from the table in Example 2.7. ∎

**2.4. Associated modular lattices.** Generalising unimodular lattices Quebbemann [**16**] introduced the notion of $p$-modular lattices.

DEFINITION 2.6. An even $\mathbb{Z}$-lattice $L$ in euclidean space is called *$p$-modular*, if there is a similarity $\sigma$ of norm $N$ (so $(\sigma(v),\sigma(w)) = p(v,w)$ for all $v,w \in L$) such that $\sigma(L^*) = L$.

If $p$ is one of the 6 primes for which $p+1$ divides 24, then the theta series of $p$-modular lattices generate a polynomial ring with 2 generators from which one obtains a similar notion of extremality as for unimodular lattices: Let $k := \frac{24}{p+1}$. Then any $p$-modular lattice $L$ of dimension $N$ satisfies $\min(L) \leq 1 + \lfloor \frac{N}{2k} \rfloor$. $p$-modular lattices achieving this bound are called *extremal*.

EXAMPLE 2.7. For the first few weights the extremal Hilbert modular forms $f$ turn out to be unique and start with non-negative integral coefficients. The valuation $\nu(f-1) = (s,t)$ can be read off from the following table. Any golden lattice $\Lambda$ with theta series $f$ defines an even unimodular $\mathbb{Z}$-lattice $L_{\eta^{-1}}$ of minimum $s$ and a 5-modular lattice $L_1$ of minimum $t$. The Hilbert modular form $f$ also gives us information about the minimal vectors $\mathrm{Min}(L,q) := \{v \in L \mid q(v) = \min(L)\}$. The kissing number $s_{\eta^{-1}} = |\mathrm{Min}(L_{\eta^{-1}})|$ and $s_1 = |\mathrm{Min}(L_1)|$ of these two lattices can be read off from column 3 and 4 of the table. That all minimal vectors of $L_1$ are also minimal vectors of $L_{\eta^{-1}}$ is indicated by a $+$ in the last column. A $-$ means that only half of the minimal vectors of $L_1$ are also contained in $\mathrm{Min}(L_{\eta^{-1}})$. The table summarizes the results of explicit computations in Magma (V2.18-8) using $(q_0,q_1)$-series expansions of $A_2$, $B_6$ and $C_{10}$. Only for the extremal Hilbert modular forms where the weight is marked as gold we could also construct golden lattices (see Section 3).

| weight | $\nu(f-1)$ | $s_{\eta^{-1}}$ | $s_1$ | $\subset$ |
|---|---|---|---|---|
| 2 (gold) | $(1,2)$ | 240 | 120 | $+$ |
| 4 (gold) | $(1,2)$ | 480 | 240 | $+$ |
| 6 (gold) | $(2,4)$ | 196560 | 37800 | $+$ |
| 8 (gold) | $(2,4)$ | 146880 | 21600 | $+$ |
| 10 | $(2,5)$ | 39600 | 79200 | $-$ |
| 12 (gold) | $(3,6)$ | 52416000 | 2620800 | $+$ |
| 14 | $(3,6)$ | 15590400 | 537600 | $+$ |
| 16 | $(3,7)$ | 2611200 | 2611200 | $-$ |
| 18 (gold) | $(4,8)$ | 6218175600 | 75411000 | $+$ |
| 20 | $(4,9)$ | 1250172000 | 609840000 | $-$ |
| 24 | $(5,10)$ | 565866362880 | 1655821440 | $+$ |
| 30 | $(6,13)$ | 45792819072000 | 3217294080000 | $-$ |

Note that [**13**, Proposition 3.3] applied to an even unimodular $R$-lattice $(\Lambda, Q)$ gives that

$$\frac{5}{2}\min(L_{\eta^{-1}}) \geq \min(L_1) \geq 2\min(L_{\eta^{-1}}).$$

The argument is that $\eta = 1 + \overline{\vartheta}^2$ and $5\eta^{-1} = 1 + \vartheta^2$ so for all $\lambda \in \Lambda$

$$(2.1)\qquad \begin{aligned} Q(\lambda) &= (1+\overline{\vartheta}^2)\eta^{-1}Q(\lambda) = \eta^{-1}Q(\lambda) + \eta^{-1}Q(\overline{\vartheta}\lambda) \quad \text{and} \\ 5\eta^{-1}Q(\lambda) &= (1+\vartheta^2)Q(\lambda) = Q(\lambda) + Q(\vartheta\lambda) \end{aligned}$$

Imitating the proof of this proposition we find the following bound.

PROPOSITION 2.8. *Let $f$ be a symmetric Hilbert modular form of weight $w$, $s := 1 + \lfloor \frac{w}{6} \rfloor$ and $t := \lfloor \frac{5s}{2} \rfloor$. Then $\nu(f-1) \leq (s,t)$. More precisely put $\nu(f-1) = (s',t')$. Then $s' \leq s$ and $\lfloor \frac{5s'}{2} \rfloor \geq t' \geq 2s'$.*

PROOF. Let $\nu(f-1) = (s',t')$. Since $f(q_0, 1)$ is an elliptic modular form of weight $2w$ we get that $s' \leq s$. To obtain the other two inequalities write

$$f = 1 + \sum_{X \in R_+} A_X q_0^{\mathrm{tr}(\eta^{-1}X)} q_1^{\mathrm{tr}(X)}.$$

Then $f$ is invariant under the transformation by $\mathrm{diag}(u, u^{-1}) \in \mathrm{SL}_2(R)$ for any $u \in R^*$. This shows that

$$A_X = A_{\vartheta^2 X} = A_{\overline{\vartheta}^2 X}.$$

To see that $t' \geq 2s'$ choose some totally positive $X = a + b\sqrt{5} \in R$ such that $t' = 2a$ and $A_X = A_{\overline{\vartheta}^2 X} \neq 0$. Then

$$a - b = \mathrm{tr}(\eta^{-1}X) \geq s' \text{ and } a + b = \mathrm{tr}(\eta^{-1}\overline{\vartheta}^2 X) \geq s'$$

and so $t' = 2a = (a-b)+(a+b) \geq 2s'$. To see that $t' \leq \frac{5s'}{2}$ we let $X := a+b\sqrt{5} \in R$ be a totally positive element with $\mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}X) = a - b = s'$ and $A_X \neq 0$. Then

$$2a = \mathrm{tr}_{K/\mathbb{Q}}(X) \geq t' \text{ and } 3a - 5b = \mathrm{tr}_{K/\mathbb{Q}}(\vartheta^2 X) \geq t'$$

and hence $5s' = 5a - 5b \geq 2t'$. □

The minimum of an extremal 5-modular lattice is $1 + \lfloor \frac{N}{8} \rfloor$ and for $N \geq 80$ this is strictly bigger than $\frac{5}{2}(1 + \lfloor \frac{N}{24} \rfloor)$ which yields:

COROLLARY 2.9. *Let $L$ be a golden lattice of dimension $N$. Then $L_1$ is an extremal 5-modular lattice if and only if $N = 8$ or $N = 24$.*

Any golden lattice defines a family of modular lattices:

THEOREM 2.10. *Let $(\Lambda, Q)$ a golden lattice of dimension $n$ and $(s,t) := \nu(\Theta(\Lambda, Q) - 1)$. For $a \in \mathbb{N}_0$ the trace lattice $L_{1+a\eta^{-1}}$ is an $(a^2 + 5a + 5)$-modular lattice of minimum $\geq t + as$.*

PROOF. Recall that $L_{1+a\eta^{-1}} = (\Lambda, \mathrm{tr}_{K/\mathbb{Q}}((1 + a\eta^{-1})Q))$. Since $\mathrm{tr}_{K/\mathbb{Q}}(\eta^{-1}Q)$ and $\mathrm{tr}_{K/\mathbb{Q}}(Q)$ are positive definite and take integral values on $\Lambda$, the lattice $L_{1+a\eta^{-1}}$ is even and positive definite for any $a \in \mathbb{Z}_{\geq 0}$. Clearly $\min(L_{1+a\eta^{-1}}) \geq \min(L_1) + a\min(L_{\eta^{-1}}) = t + as$. Now $(\Lambda, Q)$ is unimodular, so Equation (1.1) yields that the $\mathbb{Z}$-dual of $L_{1+a\eta^{-1}}$ is

$$L^*_{1+a\eta^{-1}} = \eta^{-1}(1 + a\eta^{-1})^{-1}\Lambda^{\#} = \eta^{-1}(1 + a\eta^{-1})^{-1}L_{1+a\eta^{-1}}.$$

The element $\eta(1 + a\eta^{-1}) \in K$ hence defines a similarity between $L^*_{1+a\eta^{-1}}$ and $L_{1+a\eta^{-1}}$ or norm $N(\eta + a) = a^2 + 5a + 5$. □

## 3. Examples

All even unimodular $\mathbb{Z}[\vartheta]$-lattices are classified in dimension 4, 8, and 12 [**7**], [**2**]. In each of these dimensions there is a unique golden lattice. For the other dimensions 16 to 36 we inspect automorphisms of some known extremal even unimodular lattice to find a golden lattice with the method from Proposition 2.5. The tensor symbol $\otimes$ denotes the Kronecker product of matrix groups, which is group theoretically the central product.

**3.1. Dimension 4.** Already Maass [**11**] has shown that there is a unique golden lattice, $F_4$, of dimension 4. It can be constructed from the maximal order $\mathcal{M}$ in the definite quaternion algebra with center $K$ that is only ramified at the two infinite places. Its $R$-automorphism group is

$$\mathrm{Aut}_R(F_4) \cong (\mathrm{SL}_2(5) \otimes \mathrm{SL}_2(5)) : 2.$$

**3.2. Dimension 8.** Maass also showed that the only 8-dimensional even unimodular $R$-lattice is the orthogonal sum $F_4 \perp F_4$.

**3.3. Dimension 12.** The golden lattices of dimension 12 are exactly the $\mathbb{Z}[\vartheta]$-structures of the unique extremal even unimodular $\mathbb{Z}$-lattice of dimension 24, the Leech lattice. [**2**] shows that there is a unique such golden lattice $\Lambda$. Its automorphism group is

$$\mathrm{Aut}_R(\Lambda) \cong 2.J_2 \otimes \mathrm{SL}_2(5).$$

**3.4. Dimension 16.** By [**7**, Table (1.2)] the mass of all even unimodular $R$-lattices of rank 16 is $> 10^6$, so a complete classification seems to be out of reach. Here it would be desirable to have a mass formula for the lattices without roots in analogy to the classical case of even unimodular $\mathbb{Z}$-lattices [**10**].

There are several known extremal even unimodular lattices in dimension 32 which have a fairly big automorphism group. In particular there are two golden lattices $\Lambda_1$ and $\Lambda_2$ that have are $\mathcal{M}$-lattices for $\mathcal{M}$ as in 3.1 (see [**3**] and [**13**, Table 2] for the automorphism group). The automorphism groups $G_i = \mathrm{Aut}_{\mathbb{Z}[\vartheta]}(\Lambda_i)$ are

$$G_1 \cong (\otimes^4 \mathrm{SL}_2(5)) : S_4,\ G_2 \cong \mathrm{SL}_2(5) \otimes 2_-^{1+6}.O_6^-(2).$$

**3.5. Dimension 20.** No golden lattice of dimension 20 is known. It is an interesting problem to construct such a golden lattice or to prove its non-existence, since this is the smallest dimension for which $\min(L_1) > 2\min(L_{\eta^{-1}})$.

From the extremal even unimodular lattice $L$ in [**15**] with automorphism group $(U_5(2) \times 2_-^{1+4}.Alt_5).2$ and an automorphism $z \in L$ of order 5 with irreducible minimal polynomial one obtains a Galois invariant $\mathbb{Z}[z + z^{-1}]$-lattice $(\Lambda, Q)$ for which the 5-modular trace lattice $L_1$ has 19800 minimal vectors of norm 4 and no vectors of norm 5.

**3.6. Dimension 24.** The extremal even unimodular lattice $P_{48n}$ constructed in [**12**, Theorem 5.3] has an obvious structure as a $\mathbb{Z}[\vartheta]$-lattice with automorphism group $\mathrm{SL}_2(13).2 \otimes \mathrm{SL}_2(5)$. This provides one example of a golden lattice of $\mathbb{Z}[\vartheta]$-dimension 24.

**3.7. Dimension 28.** Any extremal even unimodular 56-dimensional $\mathbb{Z}$-lattice that has a Galois invariant structure over $\mathbb{Z}[\vartheta]$ gives rise to a golden lattice. It should not be too difficult to construct such a lattice.

**3.8. Dimension 32.** No golden lattice of dimension 32 is known. There is one extremal even unimodular lattice $L$ of dimension 64 constructed in [**12**, p. 496] of which the extremality is proven in [**13**, Proposition 4.2]. The subgroup $G := \mathrm{SL}_2(17) \otimes \mathrm{SL}_2(5)$ of the automorphism group of $L$ has endomorphism ring $\mathbb{Z}[(1+\sqrt{17})/2, \vartheta]$ and with Proposition 2.5 the structure over $\mathbb{Z}[\vartheta]$ yields a lattice $(\Lambda, Q)$ whose Hilbert theta series is symmetric, $L = L_{\eta^{-1}}$ is extremal, but the minimum of the 5-modular lattice $L_1$ is only 6 (and not 7, as required for a golden lattice).

**3.9. Dimension 36.** In [**14**] an extremal even unimodular lattice $\Gamma_{72}$ of dimension 72 is constructed. Any Galois invariant $\mathbb{Z}[\vartheta]$-structure on $\Gamma_{72}$ will give rise to a golden lattice of rank 36. The lattice $\Gamma_{72}$ can be obtained has a Hermitian tensor product [**4**]

$$\Gamma_{72} = P_b \otimes_{\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]} P$$

where $P$ is the $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$-structure of the Leech lattice with automorphism group $\mathrm{SL}_2(25)$. The group $\mathrm{SL}_2(25) \leq \mathrm{GL}_{24}(\mathbb{Z})$ contains an element $\zeta$ of order 5 with irreducible minimal polynomial. Put $\vartheta := \zeta + \zeta^{-1}$. Then $P$ is a $\mathbb{Z}[\vartheta, \frac{1+\sqrt{-7}}{2}]$-lattice with automorphism group $U := (C_5 \times C_5) : C_4$. This yields a $\mathbb{Z}[\vartheta]$-structure on the Hermitian tensor product $\Gamma_{72}$ which defines a golden lattice of rank 36 whose automorphism group contains $U \times PSL_2(7)$.

## References


[1] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993). MR1484478

[2] Patrick J. Costello and John S. Hsia, *Even unimodular 12-dimensional quadratic forms over* $\mathbf{Q}(\sqrt{5})$, Adv. in Math. **64** (1987), no. 3, 241–278, DOI 10.1016/0001-8708(87)90009-0. MR888629 (88h:11023)

[3] Renaud Coulangeon, *Réseaux unimodulaires quaternioniens en dimension* $\leq 32$, Acta Arith. **70** (1995), no. 1, 9–24 (French). MR1318759 (96c:11040)

[4] R. Coulangeon, G. Nebe, Dense lattices as Hermitian tensor products. this issue.

[5] Wolfgang Ebeling, *Lattices and codes*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1994. A course partially based on lectures by F. Hirzebruch. MR1280458 (95c:11084)

[6] Karl-Bernhard Gundlach, *Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers* $Q(\sqrt{5})$, Math. Ann. **152** (1963), 226–256 (German). MR0163887 (29 #1186)

[7] J. S. Hsia, *Even positive definite unimodular quadratic forms over real quadratic fields*, Rocky Mountain J. Math. **19** (1989), no. 3, 725–733, DOI 10.1216/RMJ-1989-19-3-725. Quadratic forms and real algebraic geometry (Corvallis, OR, 1986). MR1043244 (91d:11034)

[8] J. S. Hsia and D. C. Hung, *Even unimodular 8-dimensional quadratic forms over* $\mathbf{Q}(\sqrt{2})$, Math. Ann. **283** (1989), no. 3, 367–374, DOI 10.1007/BF01442734. MR985237 (90c:11023)

[9] David C. Hung, *Even positive definite unimodular quadratic forms over* $\mathbf{Q}(\sqrt{3})$, Math. Comp. **57** (1991), no. 195, 351–368, DOI 10.2307/2938679. MR1079022 (92a:11043)

[10] Oliver D. King, *A mass formula for unimodular lattices with no roots*, Math. Comp. **72** (2003), no. 242, 839–863 (electronic), DOI 10.1090/S0025-5718-02-01455-2. MR1954971 (2003m:11101)

[11] Hans Maass, *Modulformen und quadratische Formen über dem quadratischen Zahlkörper* $R(\sqrt{5})$, Math. Ann. **118** (1941), 65–84 (German). MR0006209 (3,272b)

[12] Gabriele Nebe, *Some cyclo-quaternionic lattices*, J. Algebra **199** (1998), no. 2, 472–498, DOI 10.1006/jabr.1997.7163. MR1489922 (99b:11075)

[13] Gabriele Nebe, *Construction and investigation of lattices with matrix groups*, Integral quadratic forms and lattices (Seoul, 1998), Contemp. Math., vol. 249, Amer. Math. Soc., Providence, RI, 1999, pp. 205–219, DOI 10.1090/conm/249/03759. MR1732361 (2000m:11054)

[14] G. Nebe, An even unimodular 72-dimensional lattice of minimum 8. J. Reine und Angew. Math. (DOI: 10.1515/crelle.2011.175)

[15] G. Nebe, N.J.A. Sloane, A catalogue of lattices. http://www.math.rwth-aachen.de/∼Gabriele.Nebe/LATTICES/

[16] H.-G. Quebbemann, *Modular lattices in Euclidean spaces*, J. Number Theory **54** (1995), no. 2, 190–202, DOI 10.1006/jnth.1995.1111. MR1354045 (96i:11072)

[17] Carl Ludwig Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1969** (1969), 87–102 (German). MR0252349 (40 #5570)

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