On extremal even unimodular 72-dimensional lattices.

Gabriele Nebe[1] and Richard Parker[2]

Abstract. By computer search we show that the lattice Γ from [7] is the unique extremal even unimodular 72-dimensional lattices that can be contructed as proposed in [4].

Keywords: extremal even unimodular lattice.
MSC: primary: 11H06, secondary: 11H31, 11H50, 11H55, 11H56, 11H71

1 Introduction.

In this paper a lattice (L, Q) is always an even unimodular positive definite lattice, i.e. a free $\mathbb{Z}$-module L equipped with an integral positive definite quadratic form $Q : L \to \mathbb{Z}$ of determinant 1. The minimum of L is the minimum of the quadratic form on the non-zero vectors of L

$$\min(L) = \min(L, Q) = \min\{Q(\ell) \mid 0 \neq \ell \in L\}.$$

From the theory of modular forms it is known that the minimum of an even unimodular lattice of dimension n is always $\leq \lfloor \frac{n}{24} \rfloor + 1$. Lattices achieving equality are called extremal. Of particular interest are extremal unimodular lattices in the "jump dimensions" - the multiples of 24. There are five extremal even unimodular lattices known in the jump dimensions, the Leech lattice Λ_{24}, the unique even unimodular lattice in dimension 24 without roots, three lattices called P_{48p}, P_{48q}, P_{48n}, of dimension 48 which have minimum 6 [2], [6] and one lattice Γ in dimension 72 [7].

If (L, Q) is an even unimodular lattice, then $L/2L$ becomes a non-degenerate quadratic space over $\mathbb{F}_2$ with quadratic form $q(\ell + 2L) := Q(\ell) + 2\mathbb{Z}$. This has Witt defect 0, so there are totally isotropic subspaces $U, V \leq (L/2L, q)$ such that $L/2L = U \oplus V$. Let $2L \leq M, N \leq L$ denote the preimages of U, V, respectively. Then $(M, \frac{1}{2}Q)$ and $(N, \frac{1}{2}Q)$ are again even unimodular lattices. We call (M, N) a **polarisation** of L.

Definition 1.1. *([4], [8, Construction I], [7]) Given such a polarisation (M, N) of the even unimodular lattice (L, Q) let*

$$\begin{aligned} \mathcal{L}(M,N) &:= \{(a,b,c) \in L \perp L \perp L \mid a+b+c \in M, a+b \in N, a+c \in N\} \\ &= \{(x+m, y+m, z+m) \in L \perp L \perp L \mid m \in M, x,y,z \in N, x+y+z \in 2L\}. \end{aligned}$$

Then the lattice $(\mathcal{L}(M,N), \tilde{Q})$ is an even unimodular lattice where

$$\tilde{Q}(a,b,c) := \frac{1}{2}(Q(a) + Q(b) + Q(c)).$$

---

[1] Corresponding author, Gabriele Nebe, Lehrstuhl D für Mathematik, RWTH Aachen university, 52056 Aachen, Germany, nebe@math.rwth-aachen.de

[2] Parker's visit to Aachen was financed by the DFG (GRK 1632)

Lemma 1.2. *(see [7]) Let* (M,N) *be a polarisation of* L *and assume that* $d = \min(L,Q) = \min(N, \frac{1}{2}Q) = \min(M, \frac{1}{2}Q)$*. Then*

$$\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M,N), \tilde{Q}) \leq 2d.$$

The vectors of norm $\frac{3d}{2}$ *in* $\mathcal{L}(M,N)$ *are exactly those triples* (a,b,c) *where* $a,b,c \in \mathrm{Min}(L)$ *satisfy* $a+N = b+N = c+N$ *and* $a+b+c \in M$*.*

Proof. Let $\lambda = (a,b,c) \in \mathcal{L}(M,N)$. According to the number of non-zero components one gets up to permutation:
1) One non-zero component: Then $\lambda = (a,0,0)$ with $a = 2\ell \in 2L$ so

$$\tilde{Q}(\lambda) = \frac{1}{2}Q(2\ell) = 2Q(\ell) \geq 2d.$$

2) Two non-zero components: Then $\lambda = (a,b,0)$ with $a,b \in N$ so $\tilde{Q}(\lambda) = \frac{1}{2}(Q(a)+Q(b)) \geq 2d$.
3) Three non-zero components: Then $\tilde{Q}(\lambda) = \frac{1}{2}(Q(a)+Q(b)+Q(c)) \geq \frac{3}{2}d$.
If $\tilde{Q}(\lambda) = \frac{3d}{2}$ then $\lambda = (a,b,c)$ has three non-zero components and $Q(a) = Q(b) = Q(c) = \frac{d}{2}$. By construction all components of λ lie in the same coset of N and their sum is in M. □

Example 1.3. *Let* $L = \mathbb{E}_8$ *be the unique even unimodular lattice of dimension* 8*. Then* $\mathrm{Aut}(L)$ *has a unique orbit on the polarisations* (M,N) *of* L*, so there is up to isometry just one lattice* $\mathcal{L}(M,N)$ *with* $L = \mathbb{E}_8$*. This lattice is an even unimodular lattice of dimension* 24 *with minimum* 2*, so it is isometric to the Leech lattice* Λ_{24}*. We use this construction of* Λ_{24} *to fix a Gram matrix* F *of the Leech lattice. Let* $\alpha \in \mathrm{End}(\mathbb{E}_8)$ *be such that* $\alpha^2 - \alpha + 2 = 0$ *(there is a unique* $\mathrm{Aut}(\mathbb{E}_8)$ *conjugacy class of such endomorphisms). Then* $\mathbb{Z}[\alpha] \cong \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ *and* $\beta := (1-\alpha)$ *satisfies* $\alpha\beta = 2$*. Put* $M := \alpha L$ *and* $N := \beta L$*. Then a Gram matrix of* $\Lambda_{24} = \mathcal{L}(M,N)$ *is given by*

$$\mathcal{F} := \frac{1}{2}\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & 0 \\ 0 & \beta & \beta \end{pmatrix} \mathrm{diag}(F,F,F) \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & 0 \\ 0 & \beta & \beta \end{pmatrix}^{tr} = \begin{pmatrix} 3F & X & X \\ X^{tr} & 2F & F \\ X^{tr} & F & 2F \end{pmatrix}$$

where F *is a Gram matrix of* $\mathbb{E}_8$ *and* $X = \alpha F (1-\alpha)^{tr}$*:*

$$F = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} -3 & 0 & 2 & -1 & 1 & 1 & -2 & 0 \\ 0 & -3 & 0 & 1 & 0 & 1 & -1 & 2 \\ 1 & 0 & -3 & 1 & 1 & -1 & 2 & -1 \\ 1 & 2 & 2 & -3 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 2 & -3 & 2 & 0 & 0 \\ -1 & -1 & 1 & 1 & 1 & -3 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 2 & -3 & 1 \\ 0 & -2 & 1 & 0 & 0 & 0 & 2 & -3 \end{pmatrix}$$

Starting with a suitable polarisation (M_0, N_0) of $L = \Lambda_{24}$ the extremal even unimodular lattice $\Gamma = \mathcal{L}(M_0, N_0)$ of dimension 72 was constructed in [7]. This paper reports on a computer demonstration that Γ is the unique extremal even unimodular lattice of dimension 72 that can be constructed by this tripling construction. The main result is

Theorem 1.4. *Let (M, N) be a polarisation of $L = \Lambda_{24}$ such that $\mathcal{L}(M, N)$ has minimum 4. Then there is some $g \in \mathrm{Aut}(\Lambda_{24})$ such that $M = gM_0$, $N = gN_0$ and hence $\mathcal{L}(M, N) \cong \Gamma$.*

As already remarked in [4] if (M, N) is a polarisation of Λ_{24} such that $\mathcal{L}(M, N)$ is extremal, then $(M, \frac{1}{2}Q)$ and $(N, \frac{1}{2}Q)$ are both isometric to the Leech lattice. We call such sublattices **good**. A polarisation (M, N) of Λ_{24} is called **good**, if both sublattices N and M are good. A good polarisation (M, N) is called **extremely good**, if $\mathcal{L}(M, N)$ is extremal. The strategy to find orbit representatives of all extremely good polarisations of Λ_{24} starts by constructing representatives of all orbits of $\mathrm{Aut}(\Lambda_{24})$ on the set of good sublattices. It turns out that there are 16 orbits, so there are 16 candidates for the second entry N in the (extremely) good polarisations (M, N).

2 Orbit representatives of the subspaces.

In this section we list the orbits of the automorphism group of the Leech lattice Λ_{24} on the good sublattices N of Λ_{24} Such a sublattice N necessarily contains $2\Lambda_{24}$ and hence corresponds to a totally singular subspace $E = N/2\Lambda_{24} \leq \Lambda_{24}/2\Lambda_{24}$.

Definition 2.1. *Let N be a good sublattice of Λ_{24}. Then any nonzero class $0 \neq f + N \in \Lambda_{24}/N$ contains exactly 24 pairs $\{\pm v_1, \dots, \pm v_{24}\}$ of minimal vectors in Λ_{24} (so $Q(v_i) = 2$ for all i). The set*

$$B(N, f) := \{(v_i + v_j + v_k) + 2\Lambda_{24} \mid 1 \leq i, j, k \leq 24\} \subset \Lambda_{24}/2\Lambda_{24}$$

is called the set of **bad vectors** *for N and f. Their union*

$$B(N) := \bigcup_{0 \neq f + N \in \Lambda_{24}/N} B(N, f)$$

is called the set of **bad vectors** *for N. The* **profile** *of N is the multiset*

$$\mathit{prof}(N) := \{|B(N, f)| \mid 0 \neq f + N \in \Lambda_{24}/N\}.$$

Theorem 2.2. *The automorphism group $\mathrm{Aut}(\Lambda_{24}) = 2.Co_1$ has exactly 16 orbits on the good sublattices as given in the table below.*

Proof. The number of good sublattices N is given in [3]. With partly random search we found enough totally isotropic subspaces $E = N/2\Lambda_{24}$ of $\Lambda_{24}/2\Lambda_{24}$ such that the full preminage N of E is similar to the Leech lattice (again by [3] the proportion of such subspaces in the set of all maximal totally singular subspaces is about 1/68107). We then compute the stabiliser and profile of these subspaces which turn out to distinguish the orbits. □

The stabilisers of orbit representatives of good sublattices

	$\mathrm{Stab}_{Co_1}(E)$	$\lvert\mathrm{Stab}_{2.Co_1}(E)\rvert$	$\mathrm{prof}(N)$
1	$PSL_2(25):2$	$2^5 3 \cdot 5^2 13$	$64^{65}, 256^{650}, 1024^{3380}$
2	$A_7 \times PSL_2(7)$	$2^7 3^3 5 \cdot 7^2$	$256^{2625}, 1024^{1470}$
3	$S_3 \times PSL_2(13)$	$2^4 3^2 7 \cdot 13$	$256^{1365}, 1024^{2730}$
4	$3.A_6 \times A_5$	$2^7 3^4 5^2$	$256^{225}, 1024^{3870}$
5	$PSL_2(7) \times PSL_2(7)$	$2^7 3^2 7^2$	$32^7, 128^{196}, 512^{2548}, 2048^{1344}$
6	$A_5 \times$ soluble	$2^{16} 3^3 5$	$64^{63}, 256^{960}, 1024^{3072}$
7	$G_2(4) \times A_4$	$2^{16} 3^4 5^2 7 \cdot 13$	64^{4095}
8	$PSL_2(23)$	$2^4 3 \cdot 11 \cdot 23$	$128^{253}, 512^{2530}, 2048^{1312}$
9	soluble	$2^{12} 3$	$32, 128^{30}, 512^{2784}, 2048^{1280}$
10	soluble	$2^{13} 3^2$	$64^{15}, 256^{240}, 1024^{3840}$
11	soluble	$2^9 3 \cdot 7$	$64^7, 256^{80}, 1024^{3808}$
12	soluble	$2^{12} 3^2$	$32, 128^{286}, 512^{1504}, 2048^{2304}$
13	$3.A_7.2$	$2^5 3^3 5 \cdot 7$	$64^{63}, 256^{1260}, 1024^{2772}$
14	soluble	$2^{10} 3 \cdot 5$	$64^{15}, 256^{1200}, 1024^{2880}$
15	soluble	$2^9 3 \cdot 7$	$32, 128^{14}, 512^{1904}, 2048^{2176}$
16	soluble	$2^{15} 3^3$	$64^{159}, 256^{2400}, 1024^{1536}$

To encode a basis of representatives of these 16 orbits we work with respect to a basis B with Gram matrix $\mathcal{F}$ of Λ_{24} given in Example 1.3. Any $v \in \Lambda_{24}/2\Lambda_{24}$ has a unique expression $v = \sum_{i=1}^{24} a_i \overline{B_i}$ with $a_i \in \{0,1\}$. Then $\mathrm{num}(v) := \sum_{i=1}^{24} a_i 2^{24-i} \in \{0, \ldots, 2^{24}-1\}$ A basis of the subspace $E = \langle e_1, \ldots, e_{12} \rangle_{\mathbb{F}_2} \leq \Lambda_{24}/2\Lambda_{24}$ is encoded by giving the numbers $\mathrm{num}(e_i) - 2^{24-i}$. Apart from this renormalisation we did not pay any attention to find small numbers (by choosing either a better basis or a different orbit representative). Explicit generator matrices are available from the first author's homepage.

1	1465	938	3283	3558	1133	2623	2648	802	1901	2171	539	2029
2	4447	2579	2509	2265	4760	45	569	868	483	6407	6695	-2747
3	717	2761	10347	2206	10348	10730	8271	725	9189	2800	-1617	-2718
4	762	67421	66339	2025	67054	779	66906	-30808	-16145	-7871	-4075	-1954
5	279159	278691	16921	279303	16470	-114417	-65128	-16052	-24146	-11304	-5897	-2512
6	213	450	82	6484	2863	2555	5117	961	6601	4432	2779	-2314
7	4541	4541	1075	5383	1275	381	256	1333	6139	1018	4422	-690
8	1014	352	1657	1830	2504	608	2081	3373	4	2144	3720	761
9	4485	4155	599	5910	6113	1336	4098	193	638	6021	5071	-102
10	1107	110	4206	4439	1115	164	1929	221	10	5229	1960	-1614
11	11139	10899	10760	10713	985	890	2178	939	2179	2120	-1375	-2929
12	1589	2157	548	2012	870	3451	3827	327	817	1972	1172	3533
13	1094	434	1173	9609	12557	313	5360	5550	13418	442	-6028	-3571
14	12835	12896	498	12619	8300	526	5024	8293	4173	531	-6027	-2135
15	1051	10197	8547	9983	1559	990	8868	9428	9727	9747	-2665	-980
16	34128	38064	1413	32913	32832	33981	4438	33013	-16351	-8085	-4733	-3109

3 The extremely good polarisations.

The key observation to find all extremely good polarisations by an exhaustive computer search is the following easy lemma.

Lemma 3.1. *Let N be a good sublattice of Λ_{24}. The polarisation (M, N) of Λ_{24} is extremely good if and only if $(M/2\Lambda_{24}) \cap B(N) = \emptyset$.*

Proof. The polarisation (M, N) is good, if $\min(M, Q) = 4$. By assumption $(M, \frac{1}{2}Q)$ is an even unimodular lattice so the condition that $\min(M, Q) = 4$ is equivalent to the fact that M does not contain any minimal vectors of Λ_{24}. Each of these $4095 \cdot 48$ minimal vectors belongs to exactly one frame $\{\pm v_1, \dots, \pm v_{24}\} \subset f + N$ and by construction their classes mod $2\Lambda_{24}$ belong to $B(N, f)$. By Lemma 1.2 the vectors of norm 3 in $\mathcal{L}(M, N)$ are triples (a, b, c) where $a, b, c \in \mathrm{Min}(\Lambda_{24})$ belong to the same class modulo N. The sum $a + b + c$ of all such triples is included in the set $B(N)$ of bad vectors of N. Therefore a necessary and sufficient condition for (M, N) to be an extremely good polarisation is that $B(N) \cap M/2\Lambda_{24} = \emptyset$.
□

Corollary 3.2. *Let $N \leq \Lambda_{24}$ be good sublattice of Λ_{24}. If $2048 \in$ prof(N), then there is no extremely good polarisation (M, N) containing N as second entry.*

Proof. Assume that there is an extremely good polarisation (M, N). Then $F := M/2\Lambda_{24}$ is a complement of $E := N/2\Lambda_{24}$ in $\Lambda_{24}/2\Lambda_{24}$. In particular F consists of a system of isotropic representatives of all classes $f + N \in \Lambda_{24}/N$. Any nonzero class $f + E$ contains exactly $2^{11} = 2048$ isotropic vectors (if f is isotropic, then all isotropic vectors in $f + E$ are $\{f + e \mid e \in E \cap f^{\perp}\}$). By construction all elements of $B(N, f)$ are isotropic elements in $f + E$. If there is one class $f + E$ in which all isotropic elements are bad, then $F \cap B(N, f) \neq \emptyset$, contradicting the assumption that (M, N) is extremely good. □

Now the procedure to find all extremely good polarisations (M, N) is as follows: We fix one of the 16 orbit representatives of good sublattices as the second entry N, compute $B(N)$, put $E := N/2\Lambda_{24}$ and work in $\mathbb{F}_2^{24} \cong \Lambda_{24}/2\Lambda_{24}$. We fix some basis $(e_1, \dots, e_{12})$ of E and the dual basis $(f_1, \dots, f_{12})$ of some totally singular complement F_0 of E. Any M corresponds to a totally singular complement F of E and this complement has a unique basis $(b_1, \dots, b_{12})$ such that $(b_i, e_j) = \delta_{ij}$. Then $b_i = f_i + \sum_{j=1}^{12} x_{ij} e_j$ where $(x_{ij}) = (x_{ij})^{tr} \in \mathbb{F}_2^{12\times 12}$ satisfies $x_{ii} = 0$ for all i. Now (M, N) is an extremely good polarisation, if and only if $F \cap B(N) = \emptyset$. We recursively build the basis $(b_1, \dots, b_{12})$ of F by running through

$$\{b_1 \in f_1 + E \mid b_1 \text{ isotropic } \} \setminus B(N, f_1).$$

If $(b_1, \dots, b_k)$ are chosen put $U := \langle b_1, \dots, b_k \rangle$. Then the candidates for b_{k+1} are those isotropic elements in $f_{k+1} + E$ that are perpendicular to U and for which $(b_{k+1} + U) \cap B(N) = \emptyset$. It is very helpful to order the basis vectors $(e_1, \dots, e_{12})$ such that $|B(N, f_i)|$ is as big as possible for the first few values of i.

There is still an action of $S := \mathrm{Stab}_{Co_1}(E)$ on the complements F of E which we use in the hard cases: If the program has proven that there is no extremely good complement F that contains the vector b_1, we can exclude the full orbit Sb_1 and replace $B(N)$ by $B(N) \cup Sb_1$. For the case $S \cong G_2(4) \times A_4$ we even had to use the action of the stabiliser S_1 of b_1 in S on the candidates for b_2.

We let this program run for all 11 orbit representatives N of good sublattices where $2048 \notin \text{prof}(N)$. For the first possibility of N, where $S = \text{PSL}_2(25) : 2$, we found two extremely good polarisations (M, N) which belong to the same orbit under S. They both give rise to the lattice Γ constructed in [7]. For the other ten orbit representatives N, no extremely good polarisation (M, N) was found.

References

[1] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4):235-265, 1997

[2] J.H. Conway, N.J.A. Sloane, *Sphere packings, lattices and groups.* Springer Grundlehren 290, 1993.

[3] C. Dong, H. Li, G. Mason, S. P. Norton, *Associative subalgebras of the Griess algebra and related topics* in: Ferrar, J. (ed.) et al., The Monster and Lie algebras. Proceedings of a special research quarter at the Ohio State University, Columbus, OH, USA, May 1996. Berlin: de Gruyter. Ohio State Univ. Math. Res. Inst. Publ. **7**, 27-42

[4] Robert L. Griess Jr., Rank 72 high minimum norm lattices. J. Number Theory 130 (2010) 1512-1519.

[5] J. Lepowsky, A. Meurman, An $\mathbb{E}_8$-approach to the Leech lattice and the Conway group. J. Algebra 77 (1982) 484-504.

[6] G. Nebe, Some cyclo-quaternionic lattices, J. Alg. 199 (1998) 472-498.

[7] G. Nebe, An even unimodular 72-dimensional lattice of minimum 8. J. Reine und Angew. Math. (to appear)

[8] H.-G. Quebbemann, A construction of integral lattices. Mathematika, 31 (1984) 137-140.