On the classification of lattices over $\mathbb{Q}(\sqrt{-3})$, which are even unimodular $\mathbb{Z}$-lattices

by

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1 Introduction

The classification of even unimodular $\mathbb{Z}$-lattices is explicitly known only in the cases of rank 8, 16 and 24 (cf. [CS]). Given an imaginary quadratic number field K with discriminant d_K and ring of integers $\mathcal{O}_K$, Cohen and Resnikoff [CR] showed that there exists a free $\mathcal{O}_K$-module Λ of rank r, which is even and satisfies $\det \Lambda = (2/\sqrt{d_K})^r$ if and only if $r \equiv 0 \pmod 4$, where explicit examples can also be found in [DK]. Each such $\mathcal{O}_K$-module is an even unimodular $\mathbb{Z}$-lattice of rank $2r$ and the associated Hermitian theta series is a Hermitian modular form of weight r with respect to the full modular group. An explicit description of the isometry classes of these lattices has so far only been obtained for the Gaussian number field $K = \mathbb{Q}(i)$ and $r = 4, 8$ by Schiemann [S], where 1 resp. 3 isometry classes exist, and for $r = 12$ by Kitazume and Munemasa [KM], where 28 isometry classes exist. It should be noted that similar results over the Hurwitz order can be found in [Q] and [BN].

In this paper we derive analogous results for $\mathcal{O}_K$-modules whenever $K = \mathbb{Q}(i\sqrt{3})$ is Eisenstein's number field. There is exactly one isometry class for $r = 4$ and $r = 8$ and surprisingly there exist just 5 isometry classes if $r = 12$. In the proof we make use of the Niemeier classification and verify which of these lattices have got the structure of an $\mathcal{O}_K$-module. The main idea is that the orthogonal automorphism group of such a lattice must contain an element with minimal polynomial $p(X) = X^2 - X + 1$, which corresponds to $\frac{1}{2}(1 + i\sqrt{3})I$. Moreover we investigate the filtration of cusp forms spanned by the associated Hermitian theta series.

2 The mass of the Eisenstein lattices

Throughout the paper let $K = \mathbb{Q}(i\sqrt{3})$ be Eisenstein's number field of discriminant -3 with attached Dirichlet character

$$\chi_K(n) = \left(\frac{n}{3}\right), \quad n \in \mathbb{Z},$$

and ring of integers

$$\mathcal{O} := \mathbb{Z} + \mathbb{Z}\omega, \quad \omega := (1 + i\sqrt{3})/2,$$

which is Euclidean with respect to the norm $N(a) = a\overline{a}$. We consider K^r, $r \in \mathbb{N}$, with the standard Hermitian scalar product

$$K^r \times K^r \to K, \quad (x, y) \mapsto \langle x, y\rangle := \overline{x}^{tr} y.$$

Let $\mathcal{U}(r)$ denote the unitary group of rank r.

Definition 1. $\Lambda \subset K^r$ is called an *Eisenstein lattice* of rank r if there exist linearly independent vectors $b_1, \ldots, b_r \in K^r$ such that

(i) $\Lambda = \mathcal{O}b_1 + \ldots + \mathcal{O}b_r$,

(ii) $\det(\langle b_j, b_k\rangle) = (2/i\sqrt{3})^r$,

(iii) $\langle \lambda, \lambda\rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$.

Two such lattices Λ, Λ' are called *isometric* if there exists an *isometry* $U \in \mathcal{U}(r)$ such that

$$\Lambda' = U\Lambda.$$

The (unitary) *automorphism group* of Λ is defined by

$$\mathrm{Aut}(\Lambda) := \{U \in \mathcal{U}(r);\ U\Lambda = \Lambda\}.$$

The *mass* of the Eisenstein lattices of rank r is

$$\mu_r := \sum_{\Lambda} \frac{1}{\sharp \,\mathrm{Aut}(\Lambda)},$$

where we sum over representatives of the isometry classes of Eisenstein lattices of rank r.

Considering the symmetric bilinear form

$$K^r \times K^r \to \mathbb{Q}, \quad (x, y) \mapsto \mathrm{Re}\langle x, y\rangle,$$

we may also consider Λ as an even unimodular $\mathbb{Z}$-lattice of rank $2r$

$$\Lambda_{\mathbb{Z}} := \mathbb{Z}b_1 + \mathbb{Z}\omega b_1 + \ldots + \mathbb{Z}b_r + \mathbb{Z}\omega b_r.$$

This yields an embedding of $\mathrm{Aut}\,\Lambda$ into the (orthogonal) automorphism group $\mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ of $\Lambda_{\mathbb{Z}}$. Hence it is clear that Eisenstein lattices exist only if r is a multiple of 4.

Remark 1. Let $\mathcal{O}_\wp$ be the completion of $\mathcal{O}$ at some prime ideal $\wp$. For primes $\wp \neq (2), (i\sqrt{3})$ the completion $\Lambda \otimes \mathcal{O}_\wp$ of any Eisenstein lattice has an orthonormal basis. Since 2 is inert in $\mathcal{O}$ and $1/2 < \cdot, \cdot >$ is a unimodular Hermitian form over $\mathcal{O}_{(2)}$ the lattice $\Lambda \otimes \mathcal{O}_{(2)}$ has an orthogonal basis of vectors of norm 2. It is easy to see that the lattice $\Lambda \otimes \mathcal{O}_{(i\sqrt{3})}$ is equivalent to an orthogonal sum of 2-dimensional lattices with Gram matrix $\begin{pmatrix} 0 & 1/(i\sqrt{3}) \\ -1/(i\sqrt{3}) & 0 \end{pmatrix}$. In particular any two Eisenstein lattices of rank r are locally equivalent for all primes $\wp$ and hence these lattices form a genus (see also [F, Lemma 9]).

Theorem 1. *The mass of the genus of Eisenstein lattices of rank* r, $r \in \mathbb{N}$, $4 \mid r$, *is given by*

$$\mu_r = \frac{1}{2^{r-1} \cdot r!} \cdot \prod_{j=1}^{r/2} |B_{2j} \cdot B_{2j-1,\chi_K}|,$$

where B_j *(resp.* B_{j,χ_K}*) are the (generalized) Bernoulli numbers. In particular one has*

$$\mu_4 = \frac{1}{2^7 \cdot 3^5 \cdot 5}, \quad \mu_8 = \frac{1}{2^{15} \cdot 3^{10} \cdot 5^2}, \quad \mu_{12} = \frac{691 \cdot 809 \cdot 1847}{2^{22} \cdot 3^{17} \cdot 5^3 \cdot 7 \cdot 11 \cdot 13}.$$

Proof. Hashimoto and Koseki [HaK] computed the mass μ_r^* of the genus of unimodular Hermitian $\mathcal{O}$-modules of rank r, $4 \mid r$, to be

$$\mu_r^* = \frac{3^{r/2}+1}{2^r \cdot r!} \cdot \prod_{j=1}^{r/2} |B_{2j} \cdot B_{2j-1,\chi_K}|,$$

where B_j (resp. B_{j,χ_K}) are the (generalized) Bernoulli numbers (cf. [Z], resp. [F]). Using the same counting argument as in [BN], Proposition 2.4, we obtain

$$\mu_r = \mu_r^* \cdot \frac{c_1}{c_2},$$

where c_1 is the number of maximal isotropic subspaces of the orthogonal $\mathbb{F}_3$-vector space $\mathbb{F}_3^r$ and where c_2 is the number of maximal isotropic subspaces of the symplectic $\mathbb{F}_3$-vector space $\mathbb{F}_3^r$. These numbers can be found in [T], p. 78 resp. p. 174, and are equal to

$$c_1 = \prod_{j=1}^{r/2} (3^{j-1}+1), \quad c_2 = \prod_{j=1}^{r/2} (3^j+1).$$

This yields the first claim. The special values are obtained from an explicit calculation of the Bernoulli numbers exploiting the formulas in [Z]. □

3 Classification of Eisenstein lattices

In order to obtain a classification of the Eisenstein lattices we make use of Theorem 1. First of all we recall the fact from [DK] that there exists an Eisenstein lattice Λ_4 of rank 4 given by its Gram matrix

$$\begin{pmatrix} 2I & C \\ -C & 2I \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \frac{2i}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

An easy computer calculation yields

$$\sharp \operatorname{Aut}(\Lambda_4) = 2^7 \cdot 3^5 \cdot 5 = \frac{1}{\mu_4}.$$

In view of

$$\sharp \operatorname{Aut}(\Lambda_4 \oplus \Lambda_4) = 2 \cdot (\sharp \operatorname{Aut}(\Lambda_4))^2 = \frac{1}{\mu_8}$$

Theorem 1 implies

Corollary 1. *There is exactly one isometry class of Eisenstein lattices of* rank 4 *and of* rank 8*. Representatives are* Λ_4 *and* $\Lambda_4 \oplus \Lambda_4$.

In the case of rank 12 we apply the Niemeier classification of even unimodular 24-dimensional $\mathbb{Z}$-lattices (cf. [CS]). We use the fact that ωI is always contained in the automorphism group of an Eisenstein lattice. Its minimal polynomial over $\mathbb{Q}$ is $p(X) = X^2 - X + 1$.

Proposition 1. *Each matrix* $B \in \mathbb{Z}^{2r \times 2r}$ *with minimal polynomial* $p(X) = X^2 - X + 1$ *is in* $\mathrm{GL}(2r; \mathbb{Z})$ *conjugate to*

$$\operatorname{diag}(A, \ldots, A), \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Proof. Following Newman [N], p. 54 and Theorem III.12, the claim basically follows from the fact that $\mathcal{O} = \mathbb{Z}[\omega]$ has class number 1. □

Now we are going to investigate which Niemeier lattices possess an automorphism with minimal polynomial $p(X) = X^2 - X + 1$.

Lemma 1. *Let L be a $\mathbb{Z}$-lattice, which has a decomposition into orthogonally indecomposable non-isometric lattices*

$$L = R_1^{n_1} \oplus \ldots \oplus R_s^{n_s}.$$

Then one has

a) $\mathrm{Aut}(L) = \times_{j=1}^{s} \mathrm{Aut}(R_j^{n_j})$.

b) $\mathrm{Aut}(R_j^{n_j}) = \{\mathrm{diag}(\phi_1, \ldots, \phi_{n_j})\sigma_j;\ \phi_k \in \mathrm{Aut}(R_j), \sigma_j \in S_{n_j}\}$.

c) *If* $\phi \in \mathrm{Aut}(L)$ *exists with minimal polynomial* $p(X) = X^2 - X + 1$*, then* ϕ *belongs to the kernel of the group epimorphism* $\mathrm{Aut}(L) \to S_{n_1} \times \ldots \times S_{n_s}$ *in* a) *and* b)*.*

Proof. a), b) The results are well-known (cf. [Kn]).
c) Let P_σ be the integral orthogonal $n \times n$ matrix associated to $\sigma \in S_n$ in the j-component. Then $p(\phi) = 0$ shows that $P_\sigma^2 - P_\sigma$ is diagonal. Hence $\sigma = id$ follows. □

Thus we have reduced the problem to orthogonally indecomposable $\mathbb{Z}$-lattices. In the case of root lattices (cf. [CS], Chapter 4) we obtain the well-known

Lemma 2. *a) If $n \geq 2$ one has*

$$\mathrm{Aut}(A_n) \cong S_{n+1} \times \{\pm 1\}.$$

b) If $n \neq 4$ one has

$$\mathrm{Aut}(D_n) \cong S_n \ltimes \{\pm 1\}^n.$$

Considering our situation we obtain

Corollary 2. *If*

$$\Lambda = A_n,\ \ n \neq 2 \quad \text{or} \quad \Lambda = D_n,\ \ n \neq 4 \quad \text{or} \quad \Lambda = E_7$$

then $\mathrm{Aut}(\Lambda)$ *does not contain an automorphism with minimal polynomial* $p(X) = X^2 - X + 1$.

Proof. The claim is clear for $A_1 = \mathbb{Z}$. Suppose that $\phi \in \mathrm{Aut}(\Lambda)$ with minimal polynomial $p(X) = X^2 - X + 1$ exists. Hence $\phi^3 = -id$ follows. Considering A_n, $n \geq 3$, and the action of S_{n+1} on A_n we conclude that A_n contains a

sublattice of rank 3 which is ϕ-invariant. As 3 is odd ϕ must possess a real eigenvalue which contradicts $p(X) = X^2 - X + 1$.
$\mathrm{Aut}(D_n)$, $n = 1, 2$, does not contain an element of order 3. Hence let $n = 3$ or $n \geq 5$ and $\Lambda = D_n$. Again we can conclude that D_n contains a sublattice of rank 3 which ist ϕ-invariant. Hence the same argument as above yields the contradiction.
In view of rank $E_7 = 7$ we may proceed in the same way as in the last case. □

A verification yields

Lemma 3. *The lattices A_2, D_4, E_6, E_8 and the Leech lattice L_0 possess an automorphism with minimal polynomial $p(X) = X^2 - X + 1$. This automorphism is unique up to conjugacy.*

Proof. The existence and the uniqueness are verified by a direct computation for A_2 and D_4. Considering E_6, E_8 and L_0 the result follows from [CC]. □

Hence we get

Theorem 2. *There are exactly* 5 *isometry classes of Eisenstein lattices of* rank 12. *The underlying $\mathbb{Z}$-lattices, the orders of the unitary automorphism groups and their index in the orthogonal automorphism groups are given by the following table*

root system of Λ	$\sharp\ \mathrm{Aut}(\Lambda)$	$[\mathrm{Aut}_{\mathbb{Z}}(\Lambda) : \mathrm{Aut}(\Lambda)]$
$3E_8$	$2^{22} \cdot 3^{16} \cdot 5^3$	$2^{21} \cdot 5^3 \cdot 7^3$
$4E_6$	$2^{16} \cdot 3^{17}$	$2^{16} \cdot 5^4$
$6D_4$	$2^{21} \cdot 3^9 \cdot 5$	2^{19}
$12A_2$	$2^7 \cdot 3^{15} \cdot 5 \cdot 11$	2^{12}
$\emptyset$	$2^{14} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$2^8 \cdot 3 \cdot 5^2 \cdot 7 \cdot 23$

Proof. Apply Lemma 3 and the Niemeier classification in [CS]. After conjugation we may assume that

$$B = \mathrm{diag}(A, \ldots, A), \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

belongs to $\mathrm{Aut}(\Lambda)$ due to Proposition 1. Using

$$A\mathbb{Z}^2 \cong \mathcal{O}$$

we have got an $\mathcal{O}$-structure on Λ. Moreover we have

$$\mathrm{Aut}(\Lambda) \cong \{U \in \mathrm{Aut}_{\mathbb{Z}}(\Lambda);\ UB = BU\}.$$

Now $\sharp\,\mathrm{Aut}(\Lambda)$ and $[\mathrm{Aut}_{\mathbb{Z}}(\Lambda) : \mathrm{Aut}(\Lambda)]$ are computed explicitly using [PS]. By means of Theorem 1 we conclude that we have already found all the isometry classes. □

Remark 2. a) Explicit examples of the lattices in Theorem 2 given by their Gram matrices can be found in [H1].

b) Theorem 1 yields

$$\mu_{16} = \frac{13 \cdot 47 \cdot 419 \cdot 691 \cdot 809 \cdot 1847 \cdot 3617 \cdot 16519}{2^{31} \cdot 3^{22} \cdot 5^4 \cdot 11 \cdot 17} \approx\ 0,002.$$

So far we have constructed 5 isometry classes of Eisenstein lattices Λ'_j of rank 16, namely the orthogonal sum from those in Theorem 2 with Λ_4. But they have got only a small proportion of the whole mass

$$\sum_{j=1}^{5} \frac{1}{\sharp\,\mathrm{Aut}(\Lambda'_j)} \Big/ \mu_{16} < 5 \cdot 10^{-14}.$$

c) The Eisenstein lattices associated with $3E_8$, $6D_4$ and L_0 have also got the structure of a module over the Hurwitz order (cf. [BN]). The quaternionic automorphism group has the index 3^{12} resp. 3^6 resp. $2 \cdot 3^5 \cdot 11$ in the unitary automorphism group $\mathrm{Aut}(\Lambda)$.

d) Analogous results for other imaginary quadratic number fields and $r = 4, 8$ are achieved in [H2].

4 The Hermitian theta series

We consider the Hermitian half-space of degree n

$$\mathcal{H}_n = \{Z \in \mathbb{C}^{n\times n};\ \frac{1}{2i}(Z - \overline{Z}^{tr}) > 0\}.$$

The Hermitian modular group

$$\Gamma_n := \{M \in \mathrm{SL}(2n;\mathcal{O});\ MJ\overline{M}^{tr} = J\},\quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

acts on $\mathcal{H}_n$ by the usual fractional linear transformation. The space $[\Gamma_n, r]$ of Hermitian modular forms of degree n and weight r consists of all holomorphic functions $f : \mathcal{H}_n \to \mathbb{C}$ satisfying

$$f(M\langle Z\rangle) = \det(CZ + D)^r \cdot f(Z) \quad \text{for all} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$$

with the additional condition of boundedness for $n = 1$ (cf. [B]). The subspace $[\Gamma_n, r]_0$ of cusp forms is defined by the kernel of the Siegel ϕ-operator

$$f \mid \phi \equiv 0.$$

Proposition 2. ([CR]) *Let Λ be an Eisenstein lattice of* rank r *with Gram matrix H. Then the associated Hermitian theta series*

$$\begin{aligned} \Theta^{(n)}(Z, H) : &= \sum_{G \in \mathcal{O}^{r\times n}} e^{\pi i \,\mathrm{trace}(\overline{G}^{tr} HG\cdot Z)}, \quad Z \in \mathcal{H}_n, \\ &= \sum_{T\geq 0} \sharp(H, T) e^{\pi i \,\mathrm{trace}(TZ)}, \end{aligned}$$

with the Fourier coefficients

$$\sharp(H, T) := \sharp\{G \in \mathcal{O}^{r\times n};\ \overline{G}^{tr} HG = T\},$$

belongs to

$$[\Gamma_n, r].$$

Just as in the case of Siegel modular forms we obtain the analytic version of Siegel's main theorem involving the Siegel Eisenstein series in $[\Gamma_n, r]$ (cf. [B], [K]).

Corollary 3. *Let $H_1, \ldots, H_s$ be Gram matrices of the lattices $\Lambda_1, \ldots, \Lambda_s$, which are representatives of the isometry classes of Eisenstein lattices of* rank r. *If $r > 2n$ one has*

$$\frac{1}{\mu_r} \sum_{j=1}^{s} \frac{1}{\sharp \,\mathrm{Aut}(\Lambda_j)} \Theta^{(n)}(Z, H_j) = E_r^{(n)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \backslash \Gamma_n} \det(CZ + D)^{-r}.$$

If $r = 4$ or $r = 8$ the Eisenstein series therefore coincides with a Hermitian theta series. Let $H_1, \ldots, H_5$ be fixed Gram matrices of the Eisenstein lattices associated with $3E_8, 4E_6, 6D_4, 12A_2$ resp. L_0. Then we can describe the filtration analogous to [NV].

Theorem 3. *a) Let*

$$F^{(n)}(Z) := \Theta^{(n)}(Z,H_1) - 30\Theta^{(n)}(Z,H_2) + 135\Theta^{(n)}(Z,H_3) - 160\Theta^{(n)}(Z,H_4) + 54\Theta^{(n)}(Z,H_5) \in [\Gamma_n, 12].$$

Then $F^{(4)} \not\equiv 0$ and either $F^{(3)} \equiv 0$ or $F^{(3)}$ is a non-trivial cusp form.
b) The form

$$G^{(3)}(Z) = \Theta^{(3)}(Z,H_2) - 6\Theta^{(3)}(Z,H_3) + 8\Theta^{(3)}(Z,H_4) - 3\Theta^{(3)}(Z,H_5)$$

is a non-trivial cusp form of degree 3.
c) The form

$$H^{(2)}(Z) = \Theta^{(2)}(Z,H_3) - 2\Theta^{(2)}(Z,H_4) + \Theta^{(2)}(Z,H_5)$$

is a non-trivial cusp form of degree 2.
d) The form

$$J^{(1)}(Z) = \Theta^{(1)}(Z,H_4) - \Theta^{(1)}(Z,H_5)$$

is a non-trivial cusp form of degree 1.

Proof. The functions above are modular forms of weight 12 due to Proposition 2.
a) The modular form $F^{(4)}$ does not vanish identically, since this is already true for the restriction to the Siegel half-space, where $F^{(4)}$ is the analogous linear combination of the Siegel theta series. In this case we may use the table of Fourier coefficients in [BFW].
We know from [DK] that

$$[\Gamma_2, 12] = \mathbb{C}E_4^{(2)^3} + \mathbb{C}E_6^{(2)^2} + \mathbb{C}E_{12}^{(2)}.$$

Calculating the Fourier coefficients of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

we obtain $F^{(2)} \equiv 0$.
b) The analogous computations show that $G^{(2)} \equiv 0$. If $n = 3$ the Fourier coefficient of

$$\begin{pmatrix} 4 & 4\omega/3 & 1+\omega/3 \\ 4\overline{\omega}/3 & 4 & -2\omega/3 \\ 1+\overline{\omega}/3 & -2\overline{\omega}/3 & 4 \end{pmatrix}$$

is 5184, hence $G^{(3)} \not\equiv 0$.
c)We have

$$H^{(2)} \in \mathbb{C} \cdot \left(E_{12}^{(2)} - \frac{441}{691} \left(E_4^{(2)} \right)^3 - \frac{250}{691} \left(E_6^{(2)} \right)^2 \right) = [\Gamma_2, 12]_0,$$

because $H^{(1)} \equiv 0$ follows from calculating the Fourier coefficients of 0 and 1. Moreover $H^{(2)} \not\equiv 0$ holds because the Fourier coefficient of $2I$ is 7776.
d) This claim is clear as we deal with elliptic modular forms and elliptic theta series, hence

$$J^{(1)} = 720\Delta,$$

where Δ is the normalized discriminant. □

Remark 3. a) Just as in the case of quaternionic modular forms (cf. [HK]) there is no Hermitian analog of the Schottky form over K.
b) A computation of thousands of Fourier coefficients leads to the conjecture that $F^{(3)} \equiv 0$. In this case the space of cusp forms of weight 12 spanned by Hermitian theta series is one dimensional in the cases of degree $n = 0, 1, 2, 3, 4$. One knows that

$$[\Gamma_4, 12]_0 \neq \{0\}$$

from [I]. Hence it is conjectured that $F^{(4)}$ is a Hermitian Ikeda lift.
c) All the theta series associated with Eisenstein lattices of rank ≤ 12 are symmetric Hermitian modular forms, i.e. $f(Z^{tr}) = f(Z)$, because the lattice $\overline{\Lambda}$ is isometric to Λ.

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