Strongly modular lattices with long shadow.

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RÉSUMÉ: Cet article donne une classification des réseaux fortement modulaires donc la longueur de l'ombre prends les deux plus grandes valeurs possibles.

ABSTRACT:* This article classifies the strongly modular lattices with longest and second longest possible shadow.

1 Introduction

To an integral lattice L in the euclidean space $(\mathbb{R}^n, (,))$, one associates the set of characteristic vectors $v \in \mathbb{R}^n$ with $(v,x) \equiv (x,x) \bmod 2\mathbb{Z}$ for all $x \in L$. They form a coset modulo $2L^*$, where

$$L^* = \{v \in \mathbb{R}^n \mid (v,x) \in \mathbb{Z}\ \forall x \in L\}$$

is the dual lattice of L. Recall that L is called integral, if $L \subset L^*$ and unimodular, if $L = L^*$. For a unimodular lattice, the square length of a characteristic vector is congruent to n modulo 8 and there is always a characteristic vector of square length $\leq n$. In [1] Elkies characterized the standard lattice $\mathbb{Z}^n$ as the unique unimodular lattice of dimension n, for which all characteristic vectors have square length $\geq n$. [2] gives the short list of unimodular lattices L with $\min(L) \geq 2$ such that all characteristic vectors of L have length $\geq n-8$. The largest dimension n is 23 and in dimension 23 this lattice is the shorter Leech lattice O_{23} of minimum 3. In this paper, these theorems are generalized to certain strongly modular lattices. Following [7] and [8], an integral lattice L is called N-modular, if L is isometric to its rescaled dual lattice $\sqrt{N}L^*$. A N-modular lattice L is called strongly N-modular, if L is isometric to all rescaled partial dual lattices $\sqrt{m}L^{*,m}$, for all exact divisors m of N, where

$$L^{*,m} := L^* \cap \frac{1}{m}L.$$

The simplest strongly N-modular lattice is

$$C_N := \perp_{d|N} \sqrt{d}\mathbb{Z}$$

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of dimension $\sigma_0(N) := \sum_{d|N} 1$ the number of divisors of N. The lattice C_N plays the role of $\mathbb{Z} = C_1$ for square free $N > 1$.

With the help of modular forms Quebbemann [8] shows that for

$$N \in \mathcal{L} := \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\}$$

(which is the set of all positive integers N such that the sum of divisors

$$\sigma_1(N) := \sum_{d|N} d$$

divides 24), the minimum of an even strongly N-modular lattice L of dimension n satisfies

$$\min(L) \leq 2 + 2\left\lfloor \frac{n\ \sigma_1(N)}{24\ \sigma_0(N)} \right\rfloor.$$

Strongly modular lattices meeting this bound are called **extremal**. Whereas Quebbemann restricts to even lattices, [9] shows that the same bound also holds for odd strongly modular lattices, where there is one exceptional dimension $n = \sigma_0(N)(\frac{24}{\sigma_1(N)} - 1)$, where the bound on the minimum is 3 (and not 2). In this dimension, there is a unique lattice $S^{(N)}$ of minimum 3. For $N = 1$, this is again the shorter Leech lattice O_{23}. The main tool to get the bound for odd lattices is the **shadow**

$$S(L) := \left\{\frac{v}{2} \mid v \text{ is a characteristic vector of } L\right\}.$$

If L is even, then $S(L) = L^*$ and if L is odd, $S(L) = L_0^* - L^*$, where

$$L_0 := \{v \in L \mid (v, v) \in 2\mathbb{Z}\}$$

is the even sublattice of L.

The main result of this paper is Theorem 3. It is shown that for a strongly N-modular lattice L that is rationally equivalent to C_N^k, the minimum

$$\min_0(S(L)) := \min\{(v, v) \mid v \in S(L)\}$$

equals

$$M^{(N)}(m, k) := \begin{cases} \frac{1}{N}(k\frac{\sigma_1(N)}{4} - 2m) & \text{if } N \text{ is odd} \\ \frac{1}{N}(k\frac{\sigma_1(N/2)}{2} - m) & \text{if } N \text{ is even} \end{cases}$$

for some $m \in \mathbb{Z}_{\geq 0}$. If $\min_0(S(L)) = M^{(N)}(0, k)$, then $L \cong C_N^k$. For the next smaller possible minimum $\min_0(S(L)) = M^{(N)}(1, k)$ one gets that $L \cong C_N^l \perp L'$, where $\min(L') > 1$ and $\dim(L') \leq \sigma_0(N)(s(N) - 1)$ for odd N resp. $\dim(L') \leq \sigma_0(N)s(N)$ for even N. The lattices L' of maximal possible dimensions have minimum 3 and are uniquely determined: $L' = S^{(N)}$, if N is odd and $L' = O^{(N)}$ (the "odd analogue" of the unique extremal strongly N-modular lattice of dimension $\sigma_0(N)s(N)$) if N is even (see [9, Table 1]).

The main tool to prove this theorem are the formulas for the theta series of a strongly N-modular lattice L and of its shadow $S(L)$ developed in [9]. Therefore we briefly repeat these formulas in the next section.

2 Theta series

For a subset $S \subset \mathbb{R}^n$, which is a finite union of cosets of an integral lattice we put its theta series

$$\Theta_S(z) := \sum_{v \in S} q^{(v,v)}, \quad q = \exp(\pi i z).$$

The theta series of strongly N-modular lattices are modular forms for a certain discrete subgroup Γ_N of $SL_2(\mathbb{R})$ (see [9]). Fix $N \in \mathcal{L}$ and put

$$g_1^{(N)}(z) := \Theta_{C_N}(z) = 1 + 2q + 2\mathrm{ev}(N)q^2 + \dots$$

where

$$\mathrm{ev}(N) := \begin{cases} 1 & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd} \end{cases}.$$

Let η be the Dedekind eta-function

$$\eta(z) := q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 - q^{2m}), \quad q = \exp(\pi i z).$$

and put

$$\eta^{(N)}(z) := \prod_{d|N} \eta(dz).$$

If N is odd define

$$g_2^{(N)}(z) := \left(\frac{\eta^{(N)}(z/2)\eta^{(N)}(2z)}{\eta^{(N)}(z)^2} \right)^s$$

and if N is even then

$$g_2^{(N)}(z) := \left(\frac{\eta^{(N/2)}(z/2)\eta^{(N/2)}(4z)}{\eta^{(N/2)}(z)\eta^{(N/2)}(2z)} \right)^s.$$

Then $g_2^{(N)}$ generates the field of modular functions of Γ_N. It is a power series in q starting with

$$g_2^{(N)}(z) = q - sq^2 + \dots.$$

Theorem 1 *([9, Theorem 9, Corollary 3]) Let $N \in \mathcal{L}$ and L be a strongly N-modular lattice that is rational equivalent to C_N^k. Define $l_N := \frac{1}{8}\sigma_1(N)$, if N is odd and $l_N := \frac{1}{6}\sigma_1(N)$, if N is even. Then*

$$\Theta_L(z) = g_1^{(N)}(z)^k \sum_{i=0}^{\lfloor kl_N \rfloor} c_i g_2^{(N)}(z)^i$$

for $c_i \in \mathbb{R}$. The theta series of the rescaled shadow $S := \sqrt{N} S(L)$ of L is

$$\Theta_S(z) = s_1^{(N)}(z)^k \sum_{i=0}^{\lfloor kl_N \rfloor} c_i s_2^{(N)}(z)^i$$

where $s_1^{(N)}$ and $s_2^{(N)}$ are the corresponding "shadows" of $g_1^{(N)}$ and $g_2^{(N)}$.

For odd N
$$s_1^{(N)}(z) = 2^{\sigma_0(N)}\frac{\eta^{(N)}(2z)^2}{\eta^{(N)}(z)}$$
and
$$s_2^{(N)}(z) = -2^{-s(N)\sigma_0(N)/2}\Big(\frac{\eta^{(N)}(z)}{\eta^{(N)}(2z)}\Big)^{s(N)}$$
For $N = 2$ one has
$$s_1^{(2)}(z) = \frac{2\eta(z)^5\eta(4z)^2}{\eta(z/2)^2\eta(2z)^3}$$
and
$$s_2^{(2)}(z) = -\frac{1}{16}\Big(\frac{\eta(z/2)\eta(2z)^2}{\eta(z)^2\eta(4z)}\Big)^8$$
which yields $s_1^{(N)}$ and $s_2^{(N)}$ for $N = 6, 14$ as
$$s_1^{(N)} = s_1^{(2)}(z)s_1^{(2)}\Big(\frac{N}{2}z\Big)$$
and
$$s_2^{(N)} = -(s_2^{(2)}(z)s_2^{(2)}\Big(\frac{N}{2}z\Big))^{s(N)/s(2)}.$$
If N is odd, then $s_1^{(N)}$ starts with $q^{\sigma_1(N)/4}$ and $s_2^{(N)}$ starts with q^{-2}. If N is even, then $s_1^{(N)}$ starts with $q^{\sigma_1(\frac{N}{2})/2}$ and $s_2^{(N)}$ starts with q^{-1}.

3 Strongly modular lattices with long shadow.

Proposition 2 *Let $N \in \mathbb{N}$ be square free and let L be a strongly N-modular lattice. If L contains a vector of length 1, then L has an orthogonal summand C_N.*

Proof. Since L is an integral lattice that contains a vector of length 1, the unimodular lattice $\mathbb{Z}$ is an orthogonal summand of L. Hence $L = \mathbb{Z} \perp L'$. If d is a divisor of N, then
$$L \cong \sqrt{d}L^{*,d} = \sqrt{d}\mathbb{Z} \perp \sqrt{d}(L')^{*,d}$$
by assumption. Hence L contains an orthogonal summand $\sqrt{d}\mathbb{Z}$ for all divisors d of N and therefore C_N is an orthogonal summand of L. □

Theorem 3 *(see [2] for $N = 1$) Let $N \in \mathcal{L}$ and L be a strongly N-modular lattice that is rational equivalent to C_N^k. Let $M^{(N)}(m,k)$ be as defined in the introduction.*

(i) $\min_0(S(L)) = M^{(N)}(m,k)$ *for some* $m \in \mathbb{Z}_{\geq 0}$.

(ii) If $\min_0(S(L)) = M^{(N)}(0,k)$ *then* $L \cong C_N^k$.

(iii) *If* $\min_0(S(L)) = M^{(N)}(m,k)$ *then* $L \cong C_N^a \perp L'$, *where* L' *is a strongly* N*-modular lattice rational equivalent to* C_N^{k-a} *with* $\min(L') \geq 2$ *and* $\min_0(S(L')) = M^{(N)}(m, k-a)$.

(iv) *If* $\min_0(S(L)) = M^{(N)}(m,1)$ *and* $\min(L) \geq 2$, *then the number of vectors of length* 2 *in* L *is*

$$2k(s(N) + \mathrm{ev}(N) - (k+1)).$$

In particular $k \leq k_{max}(N)$ *with*

$$k_{max}(N) = s(N) - 1 + \mathrm{ev}(N)$$

and if $k = k_{max}(N)$, *then* $\min(L) \geq 3$.

Proof. (i) Follows immediately from Theorem 1.
(ii) In this case the theta series of L is g_1^k. In particular L contains $2k$ vectors of norm 1. Applying Proposition 2 one finds that $L \cong C_N$.
(iii) Follows from Proposition 2 and Theorem 1.
(iv) Since $\min(L) > 1$, $\Theta_L = g_1^k - 2kg_1^k g_2$. Explicit calculations give the number of norm-2-vectors in L. □

The following table gives the maximal dimension $n_{max}(N) = \sigma_0(N) k_{max}(N)$ of a lattice in Theorem 3 (iv).

N	1	2	3	5	6	7	11	14	15	23
$\sigma_1(N)$	1	3	4	6	12	8	12	24	24	24
$k_{max}(N)$	23	8	5	3	2	2	1	1	0	0
$n_{max}(N)$	23	16	10	6	8	4	2	4	0	0

The lattices L with $\min_0(S(L)) = M(1,k)$ are listed in an appendix. These are only finitely many since k is bounded by k_{max}. In general it is an open problem whether for all m, there are only finitely many strongly N-modular lattices L rational equivalent to C_N^k for some k and of minimum $\min(L) > 1$ such that $\min_0(S(L)) = M(m,k)$. For $N = 1$, Gaulter [3] proved that $k \leq 2907$ for $m = 2$ and $k \leq 8388630$ for $m = 3$.

Theorem 4 *(cf. [2] for $N = 1$) Let $N \in \mathcal{L}$ be odd and $k \in \mathbb{N}$ such that*

$$\frac{8}{\sigma_1(N)} \leq k \leq k_{max}(N) = \frac{24}{\sigma_1(N)} - 1.$$

Then there is a unique strongly N*-modular lattice* $L := L_k(N)$ *that is rational equivalent to* C_N^k *such that* $\min(L) > 1$ *and* $\min_0(S(L)) = M^{(N)}(1,k)$, *except for* $N = 1$, *where there is no such lattice in dimension 9, 10, 11, 13 and there are two lattices in dimension 18 and 20 (see [2]). If* $k = k_{max}(N)$, *then* L *is the shorter lattice* $L = S^{(N)}$ *described in [9, Table 1] and* $\min(L) = 3$.

Proof. For $N = 15$ and $N = 23$ there is nothing to show since $k_{max}(N) = 0$. The case $N = 1$ is already shown in [2]. It remains to consider $N \in \{3, 5, 7, 11\}$. Since N is a prime, there are only 2 genera of strongly modular lattices, one consisting of even lattices and one of odd lattices. With a short MAGMA program using Kneser's neighboring method, one obtains a list of all lattices in the relevant genus. In all cases there is a unique lattice with the right number of vectors of length 2. Gram matrices of these lattices are given in the appendix. □

Remark 5 *For $N = 1$ and dimension $n = 9, 10, 11$ the theta series of the hypothetical shadow has non integral resp. odd coefficients, so there is no lattice $L_n(1)$.*

Theorem 6 *Let $N \in \mathcal{L}$ be even and $k \in \mathbb{N}$ such that*

$$\frac{2}{\sigma_1(N/2)} \leq k \leq k_{max}(N) = \frac{24}{\sigma_1(N)}.$$

If $(k, N) \neq (3, 2)$ then there are strongly N-modular lattices $L := L_k(N)$ that are rational equivalent to C_N^k such that $\min(L) > 1$ and $\min_0(S(L)) = M^{(N)}(1, k)$. If $k = k_{max}(N)$, then $L_k(N)$ is unique. It is the odd lattice $L = O^{(N)}$ described in [9, Table 1] and $\min(L) = 3$.

Remark 7 *For $N = 2$ and $k = 3$ the corresponding shadow modular form has non integral coefficients, so there is no lattice $L_3(2)$.*

Remark 8 *All odd lattices $L_k(N)$ in Theorem 6 lie in the genus of C_N^k.*

4 Appendix: The lattices $L_k(N)$.

The lattices $L_k(1)$:

The lattices $L_k(1)$ are already listed in [2]. They are uniquely determined by their root-sublattices R_k and given in the following table:

k	8	12	14	15	16	17	18	19	20	21	22	23
R_k	E_8	D_{12}	E_7^2	A_{15}	D_8^2	$A_{11}E_6$	D_6^3, A_9^2	$A_7^2 D_5$	D_4^5, A_5^4	A_3^7	A_1^{22}	0

The lattices $L_k(N)$ for $N > 1$ odd:

$L_2(3)$: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cong A_2 \perp A_2$, Automorphism group: $D_{12} \wr C_2$.

$L_3(3)$: $\begin{pmatrix} 211111 \\ 121111 \\ 112111 \\ 111300 \\ 111030 \\ 111003 \end{pmatrix}$ Automorphism group: order 1152.

$L_4(3)$: $\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 & 3 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 & 3 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 1 & 3 \end{pmatrix}$ Automorphism group: order 6144.

$L_5(3)$: $\begin{pmatrix} 3 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 1 & 0 \\ -1 & 3 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 3 & 1 & 1 & 1 & -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 3 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 3 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 & 1 & 3 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 1 & 0 & -1 & 3 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 0 & 0 & 3 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 3 \end{pmatrix}$ Automorphism group: $\pm U_4(2).2$ of order 103680

$L_2(5)$: $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, Automorphism group: $(\pm C_2) \wr C_2$ of order 32.

$L_3(5)$: $\begin{pmatrix} 3 & -1 & 1 & -1 & 1 & 0 \\ -1 & 3 & -1 & 0 & 1 & 1 \\ 1 & -1 & 3 & 1 & 0 & 1 \\ -1 & 0 & 1 & 3 & -1 & 1 \\ 1 & 1 & 0 & -1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}$. Automorphism group: $\pm S_5$ of order 240.

$L_1(7)$: $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$. Automorphism group: $\pm C_2$.

$L_2(7)$: $\begin{pmatrix} 3 & -1 & 1 & 0 \\ -1 & 3 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}$. Automorphism group: order 16.

$L_1(11)$: $\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$. Automorphism group: $\pm C_2$.

The lattices $L_k(N)$ for N even:

For $N = 2$ there is only one genus of odd lattices to be considered. Also for $N = 14$ there is only one odd genus for each k, since 2 is a square modulo 7. For $N = 6$, there are 2 such genera, since $L := (\sqrt{2}\mathbb{Z})^2 \perp (\sqrt{3}\mathbb{Z})^2$ is not in the genus of C_6. The genus of L contains no strongly modular lattices. The genus of $L \perp C_6$ contains 3 lattices with minimum 3, none of which is strongly modular.

$L_2(2)$: $L_2(2) = D_4$ with automorphism group $W(F_4)$ of order 1152.

$$L_4(2): \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 & 1 & 3 & 2 & -2 \\ -1 & 1 & 1 & 0 & 0 & 2 & 3 & -1 \\ 1 & -1 & -1 & 1 & -1 & -2 & -1 & 3 \end{pmatrix}.$$

The root sublattice is $D_4 \perp A_1^4$ and the automorphism group of $L_4(2)$ is $W(F_4) \times (C_2^4 : D_8)$ of order 147456.

$$L_5(2): \begin{pmatrix} 2&1&1&1&1&1&1&1&1&1 \\ 1&2&1&1&1&1&1&1&1&1 \\ 1&1&2&1&1&1&1&1&1&1 \\ 1&1&1&2&1&0&0&0&0&0 \\ 1&1&1&1&2&1&1&1&1&1 \\ 1&1&1&0&1&3&1&1&1&1 \\ 1&1&1&0&1&1&3&1&1&1 \\ 1&1&1&0&1&1&1&3&1&1 \\ 1&1&1&0&1&1&1&1&3&1 \\ 1&1&1&0&1&1&1&1&1&3 \end{pmatrix}.$$

The root sublattice is A_5 and the automorphism group of $L_5(2)$ is $\pm S_6 \times S_6$ of order 1036800.

$L_6(2)$: There are two such lattices:

$$L_{6a}(2): \begin{pmatrix} 2&0&1&1&0&0&1&1&1&1&-1&-1 \\ 0&2&0&0&-1&1&1&1&-1&1&0&-1 \\ 1&0&2&1&0&0&1&1&1&1&-1&-1 \\ 1&0&1&2&0&0&1&1&1&1&-1&-1 \\ 0&-1&0&0&2&-1&0&-1&1&-1&1&1 \\ 0&1&0&0&-1&2&0&1&-1&1&0&-1 \\ 1&1&1&1&0&0&3&1&0&1&0&-1 \\ 1&1&1&1&-1&1&1&3&0&1&-1&-1 \\ 1&-1&1&1&1&-1&0&0&3&0&0&0 \\ 1&1&1&1&-1&1&1&1&0&3&-1&-1 \\ -1&0&-1&-1&1&0&0&-1&0&-1&3&1 \\ -1&-1&-1&-1&1&-1&-1&-1&0&-1&1&3 \end{pmatrix}, \text{ and } L_{6b}(2): \begin{pmatrix} 2&0&0&0&0&0&1&1&-1&-1&1&1 \\ 0&2&0&0&0&0&0&0&0&0&1&1 \\ 0&0&2&0&0&0&1&-1&0&1&-1&-1 \\ 0&0&0&2&0&0&0&0&1&0&0&0 \\ 0&0&0&0&2&0&-1&1&0&-1&1&1 \\ 0&0&0&0&0&2&0&0&-1&0&1&1 \\ 1&0&1&0&-1&0&3&0&0&1&-1&-1 \\ 1&0&-1&0&1&0&0&3&-1&-2&2&2 \\ -1&0&0&1&0&-1&0&-1&3&1&-2&-2 \\ -1&0&1&0&-1&0&1&-2&1&3&-2&-2 \\ 1&1&-1&0&1&1&-1&2&-2&-2&4&3 \\ 1&1&-1&0&1&1&-1&2&-2&-2&3&4 \end{pmatrix}$$

with automorphism group of order $2^{15}3^4$ resp. $2^{21}3$.

$$L_7(2): \begin{pmatrix} 2&0&0&0&0&1&1&1&1&1&0&0&-1&0 \\ 0&2&0&0&1&1&1&1&0&0&0&0&0&1 \\ 0&0&2&0&1&1&0&0&-1&1&1&-1&0&0 \\ 0&0&0&2&-1&0&-1&-1&0&0&-1&-1&1&0 \\ 0&1&1&-1&3&1&1&1&0&1&1&0&0&1 \\ 1&1&1&0&1&3&1&1&0&1&1&0&-1&0 \\ 1&1&0&-1&1&1&3&2&0&0&0&0&-1&1 \\ 1&1&0&-1&1&1&2&3&1&1&1&1&-1&0 \\ 1&0&-1&0&0&0&0&1&3&0&0&1&0&0 \\ 1&0&1&0&1&1&0&1&0&3&1&0&0&0 \\ 0&0&1&-1&1&1&0&1&0&1&3&0&-1&-1 \\ 0&0&-1&-1&0&0&0&1&1&0&0&3&-1&-1 \\ -1&0&0&1&0&-1&-1&-1&0&0&-1&-1&3&1 \\ 0&1&0&0&1&0&1&0&0&0&-1&-1&1&3 \end{pmatrix}$$

Automorphism group of order 2752512.

$L_8(2)$: $L_8(2)$ is the odd version of the Barnes–Wall lattice BW_{16} (see [6]). It is unique by [9, Theorem 8].

$L_1(6)$: $\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$. Automorphism group C_2^4.

$L_2(6)$: $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 3 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 3 \end{pmatrix}$ Automorphism group $SL_2(3).2^2$ of order 96.

$L_1(14)$: Gram matrix $\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \perp \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$. Automorphism group D_8.

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