

PSEUDO-NORMALIZED HECKE EIGENFORM AND ITS APPLICATION TO EXTREMAL 2-MODULAR LATTICES

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ABSTRACT. It is shown that extremal 2-modular lattices of ranks 32 and 48 are generated by their vectors of minimal norm. In the proof, we use certain properties of the difference of normalized Hecke eigenforms. We refer to them as the pseudo-normalized Hecke eigenform, the concept of which is introduced in this paper.

1. INTRODUCTION

A lattice in \mathbb{R}^n is a subset $L \subset \mathbb{R}^n$ containing a basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n such that $L = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$, i.e., L consists of all integral linear combinations of the vectors e_1, \dots, e_n . The dual lattice L is defined as

$$L^\sharp := \{y \in \mathbb{R}^n \mid (y, x) \in \mathbb{Z}, \forall x \in L\},$$

where (x, y) is the standard inner product. Herein, we assume that the lattice L is integral, i.e., $(x, y) \in \mathbb{Z}$ for all $x, y \in L$. An integral lattice L is even if $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. An integral lattice L is unimodular if $L^\sharp = L$.

The notion of ℓ -modular lattices is introduced in [11]. An n -dimensional integral lattice L is modular if a similarity σ of \mathbb{R}^n exists such that

$$\sigma(L^\sharp) = L,$$

where L^\sharp is the dual lattice of L . If σ multiplies norms by ℓ , it is regarded as ℓ -modular. For example, the root lattices E_8, D_4, A_2 are 1-, 2-, 3-modular, respectively. The 1-modular lattices are better known as unimodular lattices.

Let L be an even 2-modular lattice of rank n . Then n is divisible by 4 and [11, 12] give the following bound on the minimum norm of a nonzero vector in L :

$$(1.1) \quad \min(L) \leq 2 \left\lfloor \frac{n}{16} \right\rfloor + 2.$$

A 2-modular lattice L that achieves equality in (1.1) is called extremal.

Herein, we investigate the following problem:

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Problem 1.1. Let L be a lattice and

$$L_{\ell_1, \dots, \ell_s} := \{x \in L \mid (x, x) = \ell_1 \text{ or } \dots \text{ or } (x, x) = \ell_s\}.$$

Is L generated by $L_{\ell_1, \dots, \ell_s}$?

Let us introduce the known results of Problem 1.1.

- (1) If L is an extremal even unimodular lattice of rank 32 or 48, then L is generated by its vectors of minimal norm [8, 9].
- (2) If L is an extremal even unimodular lattice of rank 56, 72 or 96, then L is generated by its vectors of minimal norm [7].
- (3) If L is an extremal even unimodular lattice of rank 40, then L is generated by its vectors of norms 4 and 6 [10].
- (4) If L is an extremal even unimodular lattice of rank 80 (resp. 120), then L is generated by its vectors of norms 8 and 10 (resp. norms 12 and 14) [6].

The main result of this paper is the following theorem:

- Theorem 1.1.**
- (1) *Let L be an extremal even 2-modular lattice of rank 32, then L is generated by its vectors of minimal norm.*
 - (2) *Let L be an extremal even 2-modular lattice of rank 48, then L is generated by its vectors of minimal norm.*
 - (3) *Let L be an extremal even 2-modular lattice of rank 24, then L is generated by its vectors of norms 4 and 6.*
 - (4) *Let L be an extremal even 2-modular lattice of rank 36, then L is generated by its vectors of norms 6 and 8.*

Remark 1.2. Let L be an extremal even 2-modular lattice of rank n with $4 \leq n \leq 20$. Then for $n \neq 12$, we can show that L is generated by its vectors of minimal norm and for $n = 12$, we can show that L is generated by its vectors of norms 2 and 4 with the same arguments of Theorem 1.1. However, we omit to describe it here because these can be obtained by direct computations as all even extremal 2-modular lattices in dimensions up to 20 are known [13, 1].

For the proof of Theorem 1.1, we will introduce the notion of a pseudo-normalized Hecke eigenform.

This paper is organized as follows. In Section 2, we provide the definitions and basic properties of 2-modular lattices, as well as the spherical t -designs used in this study. In Section 3, the proof of Theorem 1.1 is provided, along with concluding remarks.

All computer calculations in this paper were done with the help of MAGMA [2] and MATHEMATICA [15].

2. PRELIMINARIES

2.1. Spherical t -designs. The concept of a spherical t -design originated from Delsarte, Goethals, and Seidel [3]. For a positive integer t , a finite

nonempty set X in the unit sphere

$$S^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}$$

is known as a spherical t -design in S^{n-1} if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x),$$

for all polynomials $f(x) = f(x_1, \dots, x_n)$ of degree not exceeding t . A finite subset X in $S^{n-1}(r)$ for a sphere of radius r centered at the origin is also called a spherical t -design if the appropriately rescaled set $(1/r)X$ is a spherical t -design on the unit sphere S^{n-1} . Hence, we say that L_ℓ is a spherical t -design if $(1/\sqrt{\ell})L_\ell$ is a spherical t -design.

Let L be an extremal even 2-modular lattice of dimension n , and let us set

$$L_\ell := \{x \in L \mid (x, x) = \ell\}.$$

If the set is non-empty then L_ℓ forms a spherical t -design ([1, Corollary 3.1]), where

$$t = \begin{cases} 7 & \text{if } n \equiv 0 \pmod{16}, \\ 5 & \text{if } n \equiv 4 \pmod{16}, \\ 3 & \text{if } n \equiv 8 \pmod{16}. \end{cases}$$

Let $\text{Harm}_j(\mathbb{R}^n)$ denote the set of homogeneous harmonic polynomials of degree j on \mathbb{R}^n . It is well known that X is a spherical t -design if and only if the condition

$$\sum_{x \in X} P(x) = 0$$

holds for all $P \in \text{Harm}_j(\mathbb{R}^n)$ with $1 \leq j \leq t$. If the set X is antipodal, i.e., $-X = X$, and j is odd, then the aforementioned condition is fulfilled automatically. Hence, we can reformulate the condition of spherical t -design on an antipodal set as follows:

Proposition 2.1. *A nonempty finite antipodal subset $X \subset S_m^{n-1}$ is a spherical $2s+1$ -design if the condition*

$$\sum_{x \in X} P(x) = 0$$

holds for all $P \in \text{Harm}_{2j}(\mathbb{R}^n)$ with $2 \leq 2j \leq 2s$.

2.2. Spherical theta series. Let $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half-plane.

Definition 2.2. Let L be the lattice of \mathbb{R}^n . Then, for a polynomial P , the function

$$\theta_{L,P}(z) := \sum_{x \in L} P(x) e^{i\pi z(x,x)}$$

is known as the theta series of L weighted by P .

Remark 2.3. The weighted theta series can be written as

$$(2.1) \quad \theta_{L,P}(z) = \theta_{L,P}(q) = \sum_{m \geq 0} a_m^{(P)} q^m,$$

where $a_m^{(P)} := \sum_{x \in L_m} P(x)$ and $q = e^{\pi iz}$.

For example, we consider an even 2-modular lattice L . Subsequently, the weighted theta series $\theta_{L,P}$ of L weighted by a harmonic polynomial P is of a modular form with respect to Γ . In general, we have the following:

Proposition 2.4 ([11, 1]). *Let L be an even 2-modular lattice L of rank n and $L' := \sqrt{\ell}L$. Then, for $P \in \text{Harm}_{4k}(\mathbb{R}^n)$, we have*

$$\begin{cases} \theta_{L,P}(z) + \theta_{L',P}(z) & \in \mathbb{C}[\theta_{D_4}(q), \Delta_{16}(q)], \\ \theta_{L,P}(z) - \theta_{L',P}(z) & \in \Phi_{24}(q)\mathbb{C}[\theta_{D_4}(q), \Delta_{16}(q)], \end{cases}$$

for $P \in \text{Harm}_{4k+2}(\mathbb{R}^n)$,

$$\begin{cases} \theta_{L,P}(z) + \theta_{L',P}(z) & \in \Phi_{24}(q)\mathbb{C}[\theta_{D_4}(q), \Delta_{16}(q)], \\ \theta_{L,P}(z) - \theta_{L',P}(z) & \in \mathbb{C}[\theta_{D_4}(q), \Delta_{16}(q)], \end{cases}$$

where

$$\Delta_{16}(q) = (\eta(q)\eta(2q))^8 = q^2 + \dots,$$

$\eta(q)$ is the Dedekind η -function, and $\Phi_{24}(q) = q^2 + \dots$ is a modular form of weight 12. For more details, see [11, 1].

Using Proposition 2.4, for $P \in \text{Harm}_8(\mathbb{R}^{16m})$ and $P \in \text{Harm}_{10}(\mathbb{R}^{16m})$, $\theta_{L,P}(z)$ can be written explicitly as follows:

Lemma 2.5. (1) *Let*

$$d(r) = \begin{cases} 8 & (\text{if } r = 0), \\ 6 & (\text{if } r = 4), \\ 4 & (\text{if } r = 8). \end{cases}$$

Let L be an extremal even 2-modular lattice of rank $16m + r$ ($r = 0, 4, 8$) and $P \in \text{Harm}_{d(r)}(\mathbb{R}^{16m+r})$. Subsequently, we have

$$\begin{cases} \theta_{L',P} = \theta_{L,P}(z) = c_1 \Delta_{16}(q)^{m+1} = c_1 (q^{2m+2} - 8(m+1)q^{2m+4} + \dots) & \text{if } r = 0, 8, \\ -\theta_{L',P} = \theta_{L,P}(z) = c_1 \Delta_{16}(q)^{m+1} = c_1 (q^{2m+2} - 8(m+1)q^{2m+4} + \dots) & \text{if } r = 4. \end{cases}$$

(2) *Let L be an extremal even 2-modular lattice of rank $16m + r$ ($r = 0, 4, 8$) and $P \in \text{Harm}_{d(r)+2}(\mathbb{R}^{16m+r})$. Subsequently, we have*

$$\begin{cases} -\theta_{L',P} = \theta_{L,P} = c_2 \theta_{D_4}(z) \Delta_{16}(z)^{m+1} = c_2 (q^{2m+2} - 8(m-2)q^{2m+4} + \dots) & \text{if } r = 0, 8, \\ \theta_{L',P} = \theta_{L,P} = c_2 \theta_{D_4}(z) \Delta_{16}(z)^{m+1} = c_2 (q^{2m+2} - 8(m-2)q^{2m+4} + \dots) & \text{if } r = 4. \end{cases}$$

Moreover, for $m = 2$ we have

$$\begin{aligned}\theta_{L,P} &= c_2 \theta_{D_4}(z) \Delta_{16}(z)^3 = c_2 (q^6 - 324q^{10} + 4096q^{12} + \dots) \\ &= c_2 \sum_{m=1}^{\infty} a(m) q^m \text{ (say)}\end{aligned}$$

and

- (i) $a(2^i) = 0$ for all $i \in \mathbb{N}$,
- (ii) $a(2^i 3) = 2^{12i}$ for all $i \in \mathbb{N}$.

Proof. (1) We prove the statement for $n = 16m$ case only, the other cases can be proved in the same way. Let L be an extremal even 2-modular lattice of rank $16m$ and $P \in \text{Harm}_8(\mathbb{R}^{16m})$. Therefore, $\theta_{L,P}(z)$ and $\theta_{L',P}(z)$ are modular forms of weight $8m + 8$. We remark that by the extremality, the leading term is $cq^{2m+2} + \dots$ for some constant $c \in \mathbb{R}$. By Proposition 2.4, for some constant $c \in \mathbb{R}$, we have

$$\begin{aligned}\theta_{L,P}(z) + \theta_{L',P}(z) &= c \Delta_{16}(q)^{m+1}, \\ \theta_{L,P}(z) - \theta_{L',P}(z) &= 0.\end{aligned}$$

Subsequently,

$$\theta_{L',P} = \theta_{L,P}(z) = c_1 \Delta_{16}(q)^{m+1}.$$

(2) We prove the statement for $n = 16m$ case only, the other cases can be proved in the same way. Let L be an extremal even 2-modular lattice of rank $16m$ and $P \in \text{Harm}_{10}(\mathbb{R}^{16m})$. Therefore, $\theta_{L,P}(z)$ and $\theta_{L',P}(z)$ are modular forms of weight $8m + 10$. We remark that by the extremality, the leading term is $cq^{2m+2} + \dots$ for some constant $c \in \mathbb{R}$. By Proposition 2.4, for some constant $c \in \mathbb{R}$, we have

$$\begin{aligned}\theta_{L,P}(z) + \theta_{L',P}(z) &= 0, \\ \theta_{L,P}(z) - \theta_{L',P}(z) &= c \theta_{D_4} \Delta_{16}(q)^{m+1}.\end{aligned}$$

Subsequently,

$$\theta_{L,P}(z) = c_2 \theta_{D_4} \Delta_{16}(q)^{m+1}.$$

We remark that $\theta_{D_4}, \Delta_{16}(q)$ are modular forms for $\Gamma_0(2)$ [5]. The dimension of the space of cusp forms of weight 26 for $\Gamma_0(2)$ is five; using MAGMA [2], we obtain the basis as follows:

$$\begin{aligned}f_1 &= q^2 + 2657760q^{12} - 21963256q^{14} + 1015627776q^{16} - 8615579463q^{18} + \dots, \\ f_2 &= q^4 - 252252q^{12} - 1032192q^{14} - 42991616q^{16} - 54853632q^{18} - \dots, \\ f_3 &= q^6 + 19648q^{12} + 256770q^{14} + 2654208q^{16} + 16097088q^{18} + \dots, \\ f_4 &= q^8 - 1176q^{12} - 21504q^{14} - 196656q^{16} - 1142784q^{18} - \dots, \\ f_5 &= q^{10} + 48q^{12} + 852q^{14} + 8192q^{16} + 48510q^{18} + \dots.\end{aligned}$$

Subsequently, the following are normalized Hecke eigenforms:

$$\begin{aligned}
h_1 &= f_1 + 4096f_2 + 12 \left(15827 + 400\sqrt{106705} \right) f_3 \\
&\quad + 16777216f_4 + 150 \left(2473177 - 10368\sqrt{106705} \right) f_5 \\
&= q^2 + 4096q^4 + 12 \left(15827 - 400\sqrt{106705} \right) q^6 + 16777216q^8 \\
&\quad + 150 \left(2473177 + 10368\sqrt{106705} \right) q^{10} + \dots, \\
h_2 &= f_1 + 4096f_2 + 12 \left(15827 - 400\sqrt{106705} \right) f_3 \\
&\quad + 16777216f_4 + 150 \left(2473177 + 10368\sqrt{106705} \right) f_5 \\
&= q^2 + 4096q^4 + 12 \left(15827 + 400\sqrt{106705} \right) q^6 + 16777216q^8 \\
&\quad + 150 \left(2473177 - 10368\sqrt{106705} \right) q^{10} + \dots.
\end{aligned}$$

By comparing the Fourier coefficients, we have

$$\theta_{D_4}(z)\Delta_{16}(z)^3 = \frac{h_2 - h_1}{9600\sqrt{106705}}.$$

For $i = 1, 2$, we denote by $c_{h_i}(n)$ the coefficient of h_i as follows:

$$h_i = \sum_{n=1}^{\infty} c_{h_i}(n)q^n.$$

Because h_1 and h_2 are normalized Hecke eigenforms, the Fourier coefficients of h_i satisfy the following equations:

$$(2.2) \quad c_{h_i}(mn) = c_{h_i}(m)c_{h_i}(n) \quad (m, n \text{ coprime}),$$

$$(2.3) \quad c_{h_i}(p^{\alpha+1}) = c_{h_i}(p)c_{h_i}(p^\alpha) - p^{k-1}c_{h_i}(p^{\alpha-1}) \quad (p \text{ is a prime with } p \neq 2),$$

$$(2.4) \quad c_{h_i}(2^{\alpha+1}) = c_{h_i}(2)c_{h_i}(2^\alpha).$$

Because $c_{h_1}(2) = c_{h_2}(2) = 4096$ and by applying (2.3), we obtain (i).

Using (2.4), we obtain

$$c_{h_i}(2^i) = c_{h_i}(2)c_{h_i}(2^{i-1}) = c_{h_i}(2)^i = 4096^i.$$

Using (2.2), for $i = 1, 2$, $c_{h_i}(2^i 3) = c_{h_i}(2^i)c_{h_i}(3)$. Subsequently,

$$\begin{aligned}
c_{2i}(2^i 3) - c_{h_1}(2^i 3) &= c_{h_2}(2^i)c_{h_2}(3) - c_{h_1}(2^i)c_{h_1}(3) \\
&= c_{h_2}(2^i)(9600\sqrt{106705})
\end{aligned}$$

Namely,

$$a(2^i 3) = 4096^i = 2^{12i}.$$

The proof of (ii) is completed. \square

Next, we define the concept of a pseudo-normalized Hecke eigenform.

Definition 2.6. Let f be a pseudo-normalized Hecke eigenform of weight k for some group Γ if

$$f = g_1 - g_2,$$

where g_1 and g_2 are normalized Hecke eigenforms of weight k for Γ .

Example 2.7. By Lemma 2.5, $\theta_{D_4} \Delta_{16}^3$ is a pseudo-normalized Hecke eigenform. The coefficients of a pseudo-normalized Hecke eigenform have desirable properties similar to the Hecke eigenforms.

3. PROOF OF THEOREM 1.1

Let $\mathcal{L}(L_m)$ be the lattice generated by L_m , and $\mathcal{L}(L_{m_1, m_2})$ be that generated by L_{m_1} and L_{m_2} . Let $m_0 := \min(L)$. For the proof, suppose an equivalence class $[x'] \in L/\mathcal{L}(L_{m_0})$ exists such that x' is a minimal-norm representative with norm $(x', x') = s > m_0$. For $j \in \mathbb{Z}$, we write

$$\begin{aligned} M_j &:= M_j(L; x') := \#\{x \in L_{\min(L)} \mid |(x, x')| = j\}, \\ M'_j &:= M'_j(L; x') := \#\{x \in L_{\min(L)^\sharp}^\sharp \mid |(x, x')| = j\}, \\ N_j &:= N_j(L; x') := \#\{x \in L_{\min(L)+2} \mid |(x, x')| = j\}. \end{aligned}$$

Then we have the following results:

Lemma 3.1 ([14, 6]). (1) For all $x \in L_{m_0}$, we have the inequality

$$|(x, x')| \leq \frac{m_0}{2}.$$

(2) For all $x \in L_{m_0+2}$, we have the inequality

$$|(x, x')| \leq \frac{m_0}{2} + 1.$$

Lemma 3.2 ([14, 6]). Let

$$\theta_L(q) = \sum_{i=0}^{\infty} a_i(L) q^i.$$

(1)

$$\begin{aligned} a_{m_0}(L) &= \sum_{j=0}^{m_0/2} M_j(L; x'), \\ a_{m_0+2}(L) &= \sum_{j=0}^{m_0/2+1} N_j(L; x'). \end{aligned}$$

(2) Let L be a lattice such that for $m \in \{m_0, m_0 + 2\}$, L_m is a spherical $(2t + 1)$ -design. Then we have that

$$\sum_{j=1}^{m_0/2} j^{2k} M_j(L; x') = a_{m_0}(L) \frac{1 \cdot 3 \cdots (2k-1)}{n \cdot (n+2) \cdots (n+2k-2)} m_0^k (x', x')^k,$$

$$\sum_{j=1}^{m_0/2+1} j^{2k} N_j(L; x') = a_{m_0+2}(L) \frac{1 \cdot 3 \cdots (2k-1)}{n \cdot (n+2) \cdots (n+2k-2)} (m_0+2)^k (x', x')^k,$$

for $k \in \{1, \dots, t\}$.

(3) If L_{m_0} is a spherical 2-design, then $(x', x') \leq \frac{n \cdot m_0}{4}$.

Note that part (3) gives a general upper bound on the minimal norm in a class of $L/\mathcal{L}(L_{m_0})$ for 2-design lattices. In any concrete case (see for example below) we find much better upper bounds.

To conclude part (3) we read part (2) for $k = 1$ and note that the left hand side is $\leq (m_0/2)^2 a_{m_0}(L)$. So $a_{m_0}(L) m_0(x', x')/n \leq a_{m_0}(L) m_0^2/4$, whence $(x', x') \leq n m_0/4$.

Next, we present a proof of Theorem 1.1.

Proof of Theorem 1.1 (1). Let L be an extremal even 2-modular lattice of rank 32. Let $m_0 := \min(L) = 6$ and $t(32) := 3$.

We show that any class $[x] \in L/\mathcal{L}(L_{m_0})$, $[x]$ is represented by a vector $x' \in [x]$ with norm $(x', x') \leq m_0$.

Suppose that an equivalence class $[x'] \in L/\mathcal{L}(L_{m_0})$ exists such that $[x']$ is a minimal-norm representative with norm $(x', x') = s > m_0$.

Then by Lemma 3.1 we have $|(x', v)| \leq 3 = m_0/2$ for all $v \in L_6$. For the values $M_i := M_i(L, x') := |\{v \in L_6 \mid |(x', v)| = i\}|$ from Lemma 3.2 we have that $M_i \neq 0$ only if $i = 0, 1, 2, 3$ and

$$\begin{aligned} M_0 + M_1 + M_2 + M_3 &= a_6(L) = |L_6| = 261120, \\ \sum_{v \in L_6} (x', v)^2 &= \sum_{i=1}^3 i^2 M_i = 6s a_6(L)/32, \\ \sum_{v \in L_6} (x', v)^4 &= \sum_{i=1}^3 i^4 M_i = 6^2 s^2 3a_6(L)/(32 \cdot 34), \\ \sum_{v \in L_6} (x', v)^6 &= \sum_{i=1}^3 i^6 M_i = 6^3 s^3 3 \cdot 5a_6(L)/(32 \cdot 34 \cdot 36). \end{aligned}$$

Regarding s as a parameter we have 4 equations with 4 unknowns having a unique solution:

$$\begin{aligned} M_0 &= -600s^3 + 10080s^2 - 66640s + 261120, \\ M_1 &= 900s^3 - 14040s^2 + 73440s, \\ M_2 &= -360s^3 + 4320s^2 - 7344s, \\ M_3 &= 60s^3 - 360s^2 + 544s. \end{aligned}$$

The polynomial M_2/s is only positive when $s \in [2.05, 9.94]$. As $s = (x', x')$ is an even positive integer $> m_0 = 6$ we conclude that $s = 8$ is the only possible solution. Putting $s = 8$ we obtain

$$M_0 = 65920, \quad M_1 = 149760, \quad M_2 = 33408, \quad M_3 = 12032.$$

Now we need the harmonic polynomial $P_{8,x'}$ of degree 8:

$$P_{8,x'}(x) = (x, x')^8 - \frac{7}{11}(x, x')^6(x', x')(x, x) + \frac{5}{44}(x, x')^4(x', x')^2(x, x)^2 \\ - \frac{1}{176}(x, x')^2(x', x')^3(x, x)^3 + \frac{1}{26752}(x', x')^4(x, x)^4.$$

By Lemma 2.5 (1),

$$\sum_{v \in L_6} P_{8,x'}(v) = \sum_{w \in L'_6} P_{8,x'}(w).$$

Recall that $L' = \sqrt{2}L^\sharp$. For $w \in L_3^\sharp$ and $x' \in L_8$ we have

$$|(w, x')| \in \{0, 1, 2, 3\}$$

because if $(w, x') = 4$ then $x' - 2w \in L$ is a vector of norm 4 in L contradicting the extremality of L . Also L_3^\sharp is a spherical 6-design, so putting $M'_j := |\{w \in L_3^\sharp \mid |(x', w)| = j\}|$ we find with the equations in Lemma 3.2 that

$$M'_0 = 117440, \quad M'_1 = 126720, \quad M'_2 = 16704, \quad M'_3 = 256.$$

From these numbers we compute

$$\sum_{w \in L'_6} P_{8,x'}(w) = -8847360/19$$

which implies that

$$\sum_{v \in L_6} (x', v)^8 = 97320960 = 2^{16}3^35 \cdot 11.$$

This equation is not satisfied by the M_i above. \square

Remark 3.3. From our calculations we obtain that for $x' \in L_8$ the unique solution for $M_i := |\{v \in L_6 \mid |(x', v)| = i\}|$ is

$$M_0 = 82720, \quad M_1 = 122880, \quad M_2 = 46848, \quad M_3 = 8192, \quad M_4 = 480.$$

Proof of Theorem 1.1 (2). Let L be an extremal even 2-modular lattice of rank 48. Let $m_0 := \min(L) = 8$ and $t(48) := 3$.

We show that any class $[x] \in L/\mathcal{L}(L_{m_0})$, $[x]$ is represented by a vector $x' \in [x]$ with norm $(x', x') \leq m_0$.

Suppose that an equivalence class $[x'] \in L/\mathcal{L}(L_{m_0})$ exists such that $[x']$ is a minimal-norm representative with norm $(x', x') = s > m_0$.

Claim 1: $s \leq 18$

First, we show that $s \leq 18$. By Lemma 3.1 we have $|(x', v)| \leq 4 = m_0/2$ for all $v \in L_8$. For the values $M_i := M_i(L, x') := |\{v \in L_8 \mid |(x', v)| = i\}|$ from Lemma 3.2 (2), we have that $M_i \neq 0$ only if $i = 0, 1, 2, 3, 4$. Since L_8 is a

6-design we obtain

$$\begin{aligned} M_0 + M_1 + M_2 + M_3 + M_4 &= a_8(L) = |L_8| = 9828000, \\ \sum_{v \in L_8} (x', v)^2 &= \sum_{i=1}^4 i^2 M_i = 8s a_8(L)/48, \\ \sum_{v \in L_8} (x', v)^4 &= \sum_{i=1}^4 i^4 M_i = 8^2 s^2 3 a_8(L)/(48 \cdot 50), \\ \sum_{v \in L_8} (x', v)^6 &= \sum_{i=1}^4 i^6 M_i = 8^3 s^3 3 \cdot 5 a_8(L)/(48 \cdot 50 \cdot 52). \end{aligned}$$

Then we have:

$$\begin{aligned} M_0 &= -16800s^3 + 305760s^2 - 2229500s + 35M_4 + 9828000, \\ M_1 &= 25200s^3 - 425880s^2 + 2457000s - 56M_4, \\ M_2 &= -10080s^3 + 131040s^2 - 245700s + 28M_4, \\ M_3 &= 1680s^3 - 10920s^2 + 18200s - 8M_4. \end{aligned}$$

Since $M_2 \geq 0$ and $M_3 \geq 0$, we have

$$360s^3 - 4680s^2 + 8775s \leq M_4 \leq 210s^3 - 1365s^2 + 2275s.$$

As $s \geq 0$ this implies that $-s(s - 221/10) \geq 130/3$ showing that $s \leq 18$.

The harmonic polynomials:

Now we need the harmonic polynomials $P_{8,x'}$ of degree 8 and $P_{10,x'}$ of degree 10:

$$\begin{aligned} P_{8,x'}(x) &= (x, x')^8 - \frac{7}{15}(x, x')^6(x', x')(x, x) + \frac{7}{116}(x, x')^4(x', x')^2(x, x)^2 \\ &\quad - \frac{1}{464}(x, x')^2(x', x')^3(x, x)^3 + \frac{1}{100224}(x', x')^4(x, x)^4, \\ P_{10,x'}(x) &= (x, x')^{10} - \frac{45}{64}(x, x')^8(x', x')(x, x) + \frac{315}{1984}(x, x')^6(x', x')^2(x, x)^2 \\ &\quad - \frac{105}{7936}(x, x')^4(x', x')^3(x, x)^3 + \frac{315}{902576}(x, x')^2(x', x')^4(x, x)^4 \\ &\quad - \frac{9}{7364608}(x', x')^5(x, x)^5. \end{aligned}$$

For the rescaled dual lattice $L' = \sqrt{2}L^\sharp$ we obtain by Lemma 2.5 (1) and (2),

$$(3.1) \quad \begin{cases} \sum_{v \in L_8} P_{8,x'}(v) = \sum_{w \in L'_8} P_{8,x'}(w), \\ \sum_{v \in L_8} P_{10,x'}(v) = - \sum_{w \in L'_8} P_{10,x'}(w). \end{cases}$$

Claim 2: $\mathcal{L}(L_8) = \mathcal{L}(L_{8,10,12})$.

So assume that $s \leq 12$. Then $s = 10$ or $s = 12$ and for $v \in L_4^\sharp$ we have $|(v, x')| \leq 5$. Otherwise there is $v \in L_4^\sharp$ with $(v, x') \geq 6$ yielding a vector $2v - x' \in L$ of norm

$$(2v - x', 2v - x') = 16 - 24 + s = s - 8 \leq 4.$$

Also L_4^\sharp is a spherical 6-design, so putting $M'_j := |\{w \in L_4^\sharp \mid |(x', w)| = j\}|$ a system of 10 equations in the 12 variables s, M_j ($0 \leq j \leq 4$), M'_j ($0 \leq j \leq 5$)

is obtained, implying that

$$f(s) + (32 - 4s)M'_4 + (384 - 40s)M'_5 = 0$$

where

$$f(s) = 6s^5 - 210s^4 + 2681s^3 - 14742s^2 + 29120s.$$

For $s = 10$ and $s = 12$ the coefficients $32 - 4s$ and $384 - 40s$ are negative and so is $f(10)$ and $f(12)$ yielding the desired contradiction.

Claim 3: $\mathcal{L}(L_8)$ contains all elements $2v$ with $v \in L_4^\sharp$.

Let $v \in L_4^\sharp$. Then $2v \in L$ has norm 16. We show that there is a vector $x \in L_8$ such that either $(x, v) = 4$ or $(x, v) = 3$. In the first case $2v - x \in L_8$ and in the second case $2v - x \in L_{12}$, so by Claim 2 we obtain $2v \in \mathcal{L}(L_8)$. So in the notation above we need to show that $M'_3(L, x) \neq 0$ or $M'_4(L, x) \neq 0$. Assume that $M'_4(L, x) = 0$. Then the fact that L_4^\sharp is a 6-design yields 4 equations on the 4 unknowns $M'_j(L, x)$ for $j = 0, 1, 2, 3$ having the unique solution $M'_0 = 4726960, M'_1 = 4626720, M'_2 = 468720, M'_3 = 5600$. In particular $M'_3 > 0$.

Claim 4: $\mathcal{L}(L_8)$ contains L_{14}, L_{16} and L_{18} .

So let $x' \in L$ be a minimal representative of $x' + \mathcal{L}(L_8)$ of norm $s = (x', x')$. By the above we know that $s = 14, 16$, or 18 . Also $M_j \neq 0$ only for $j = 0, 1, 2, 3, 4$ by Lemma 3.1. By Claim 3 we have $(x' - 2v, x' - 2v) \geq s$ for all $v \in L_4^\sharp$, so $M'_j \neq 0$ only for $j = 0, 1, 2, 3, 4$. We have 4 equations for the M_j because L_8 is a 6-design, 4 equations for the M'_j because L_8' is a 6-design, as well as the two equations from Lemma 2.5 which admit a unique solution in all three cases:

s	M_0	M_1	M_2	M_3	M_4
14	12872510/3	3361792/3	12184144/3	50176/3	1015378/3
16	7466480	-4652032	7406336	-1074176	681392
18	13864158	-78591744/5	68553072/5	-16296192/5	6154074/5

This is clearly a contradiction as all M_j are non-negative integers. \square

Proof of Theorem 1.1 (3). Let L be an extremal even 2-modular lattice of rank 24. Let $m_0 := \min(L) = 4$ and $t(24) := 1$.

We show that for any class $[x] \in L/\mathcal{L}(L_{m_0, m_0+2})$, $[x]$ is represented by a vector $x' \in [x]$ with norm $(x', x') \leq m_0 + 2$.

Suppose that an equivalence class $[x'] \in L/\mathcal{L}(L_{m_0, m_0+2})$ exists such that $[x']$ is a minimal-norm representative with norm $(x', x') = s > m_0 + 2 = 6$.

By Lemma 3.2, a system of $2(t(24) + 1) = 4$ equations in the variables $s, M_j(L; x')$ ($0 \leq j \leq 2$) and $N_j(L; x')$ ($0 \leq j \leq 3$) is provided.

Now we need the harmonic polynomials $P_{4,x'}$ of degree 4 and $P_{6,x'}$ of degree 6:

$$\begin{aligned} P_{4,x'}(x) &= (x, x')^4 - \frac{3}{14}(x, x')^2(x', x')(x, x) + \frac{3}{728}(x', x')^2(x, x)^2, \\ P_{6,x'}(x) &= (x, x')^6 - \frac{15}{32}(x, x')^4(x', x')(x, x) + \frac{3}{64}(x, x')^2(x', x')^2(x, x)^2 \\ &\quad - \frac{1}{1792}(x', x')^3(x, x)^3. \end{aligned}$$

By Lemma 2.5 (1),

$$\sum_{x \in L_6} P_{4,x'}(x) = -16 \sum_{x \in L_4} P_{4,x'}(x).$$

By Lemma 2.5 (2),

$$\sum_{x \in L_6} P_{6,x'}(x) = 8 \sum_{x \in L_4} P_{6,x'}(x).$$

By Lemmas 3.2 and 2.5, a system of 6 equations in the 8 variables s, M_j ($0 \leq j \leq 2$), N_j ($0 \leq j \leq 3$) is obtained and we have

$$\begin{aligned} M_1 &= -336(-4 + s)s + (2N_3)/(-2 + s), \\ N_0 &= 8064(32 + (-9 + s)s) + (2(22 - 5s)N_3)/(-2 + s). \end{aligned}$$

Since $M_1 \geq 0$ and $N_0 \geq 0$, we have

$$1344s - 1008s^2 + 168s^3 \leq N_3 \leq \frac{-258048 + 201600s - 44352s^2 + 4032s^3}{-22 + 5s}.$$

As $s > 0$ this implies that $s \leq 8$, in particular for $s = 8$ we have $M_0 = 2016, M_1 = 0, M_2 = 1008, N_0 = 0, N_1 = 225792, N_2 = 0, N_3 = 32256$.

So (x', v) is even for all $v \in L_4$ and (x', w) is odd for all $w \in L_6$. This implies that no vector of norm 6 in L is the sum of two vectors of norm 4. Therefore the inner products of all vectors $v_1, v_2 \in L_4$ lie in $\{0, \pm 2, \pm 4\}$. So $\sqrt{2}^{-1}L_4$ is a root system in 24-dimensional space consisting of 3024 roots, which is impossible, by the classification of irreducible root systems. \square

Proof of Theorem 1.1 (4). Let L be an extremal even 2-modular lattice of rank 36. Let $m_0 := \min(L) = 6$ and $t(36) := 2$.

We show that for any class $[x] \in L/\mathcal{L}(L_{m_0, m_0+2})$, $[x]$ is represented by a vector $x' \in [x]$ with norm $(x', x') \leq m_0 + 2$.

Suppose that an equivalence class $[x'] \in L/\mathcal{L}(L_{m_0, m_0+2})$ exists such that $[x']$ is a minimal-norm representative with norm $(x', x') = s > m_0 + 2$.

First, we show that $s \leq 10$. By Lemma 3.2, a system of $2(t(36) + 1) = 6$ equations in the variables $s, M_j(L; x')$ ($0 \leq j \leq 3$) and $N_j(L; x')$ ($0 \leq j \leq 4$) is provided.

Now we need the harmonic polynomials $P_{6,x'}$ of degree 6 and $P_{8,x'}$ of degree 8:

$$\begin{aligned} P_{6,x'}(x) &= (x, x')^6 - \frac{15}{44}(x, x')^4(x', x')(x, x) + \frac{15}{616}(x, x')^2(x', x')^2(x, x)^2 \\ &\quad - \frac{1}{4928}(x', x')^3(x, x)^3, \\ P_{8,x'}(x) &= (x, x')^8 - \frac{7}{12}(x, x')^6(x', x')(x, x) + \frac{35}{368}(x, x')^4(x', x')^2(x, x)^2 \\ &\quad - \frac{35}{8096}(x, x')^2(x', x')^3(x, x)^3 + \frac{5}{194304}(x', x')^4(x, x)^4. \end{aligned}$$

By Lemma 2.5 (1),

$$\sum_{x \in L_8} P_{6,x'}(x) = -24 \sum_{x \in L_6} P_{6,x'}(x).$$

By Lemma 2.5 (2),

$$\sum_{x \in L_8} P_{8,x'}(x) = 0.$$

By Lemmas 3.2 and 2.5, a system of 8 equations in the 10 variables s, M_j ($0 \leq j \leq 3$), N_j ($0 \leq j \leq 4$) is obtained and we have

$$\begin{aligned} M_2 &= -9s(1425 - 750s + 86s^2) + (3N_4)/(-3 + s), \\ N_3 &= 1020s(19 + 6(-4 + s)s) + (4(9 - 2s)N_4)/(-3 + s). \end{aligned}$$

Since $M_2 \geq 0$ and $N_3 \geq 0$, we have

$$\begin{aligned} -12825s + 11025s^2 - 3024s^3 + 258s^4 &\leq \\ N_3 \leq (-14535s + 23205s^2 - 10710s^3 + 1530s^4)/(-9 + 2s) \end{aligned}$$

As $s > 0$ this implies that $s \leq 10$.

Finally, we show that $s \leq 8$. Let $x' \in L$ be a minimal representative of $x' + \mathcal{L}(L_{6,8})$ of norm $s = (x', x')$. Assume that $s = 10$. Also $M_j \neq 0$ only for $j = 0, 1, 2, 3$ by Lemma 3.1. For $v \in L_3^\sharp$ we have $2v \in L_{12} \subset \mathcal{L}(L_{6,8})$ and $(x' - 2v, x' - 2v) \geq s$ for all $v \in L_3^\sharp$, so $M'_j \neq 0$ only for $j = 0, 1, 2, 3$. We have 3 equations for the M_j because L_8 is a 4-design, 3 equations for the M'_j because L'_6 is a 4-design, as well as the two equations from Lemma 2.5 which admit a unique solution:

$$M'_3 = 575.$$

This is a contradiction since $M'_3 \in 2\mathbb{Z}$. \square

Remark 3.4. Michael Jürgens used similar methods in his thesis to prove that an extremal 3-modular lattice of dimension 36 is generated by its minimal vectors ([4, Satz 2.6.2]). We checked the results of Theorem 1.1 with a slight modification of Jürgens' program, by which we also found unique solutions for the configuration numbers in the case of dimension 24 and 36.

If L is an extremal 2-modular lattice of dimension 24 and $x' \in L \setminus \mathcal{L}(L_4)$ a vector of norm 6, then

$$\begin{aligned} M_0 &= 1116, & M_1 &= 1536, & M_2 &= 372, & N_0 &= 83052, \\ N_1 &= 119040, & N_2 &= 47646, & N_3 &= 7936, & N_4 &= 372 \\ M'_0 &= 1602, & M'_1 &= 1392, & \text{and} & & M'_2 &= 30. \end{aligned}$$

If L is an extremal 2-modular lattice of dimension 36 and $x' \in L \setminus \mathcal{L}(L_6)$ a vector of norm 8, then

$$\begin{aligned} M_0 &= 56320, & M_1 &= 77760, & M_2 &= 25920, & M_3 &= 4160, & N_0 &= 6416070 \\ N_1 &= 9953920, & N_2 &= 4439040, & N_3 &= 1051968, & N_4 &= 111760, & N_5 &= 4160 \\ M'_0 &= 77840, & M'_1 &= 78840, & M'_2 &= 7344, & \text{and} & & M'_3 &= 136. \end{aligned}$$

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REFERENCES

- [1] C. Bachoc and B. Venkov, Modular forms, lattices and spherical designs, Réseaux euclidiens, designs sphériques et formes modulaires, 87–111, Monogr. Enseign. Math., 37, Enseignement Math., Geneva, 2001.
- [2] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system I: The user language. *J. Symb. Comp.*, 24, 3/4:235–265, 1997.
- [3] P. Delsarte, J.-M. Goethals, and J.J. Seidel, Spherical codes and designs, *Geom. Dedicata* **6** (1977), 363–388.
- [4] M. Jürgens, *Nicht-Existenz und Konstruktion extremaler Gitter*. Dissertatio, TU Dortmund 2015.
- [5] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, Berlin/New York, 1984.
- [6] S.D. Kominers and Z. Abel, Configurations of rank- $40r$ extremal even unimodular lattices ($r = 1, 2, 3$), *J. Théor. Nombres Bordeaux* **20** (2008), no. 2, 365–371.
- [7] S.D. Kominers, Configurations of extremal even unimodular lattices, *Int. J. Number Theory* **5** (2009), no. 3, 457–464.
- [8] M. Ozeki, On even unimodular positive definite quadratic lattices of rank 32, *Math. Z.* **191** (1986), 283–291.
- [9] M. Ozeki, On the configurations of even unimodular lattices of rank 48, *Arch. Math.* **46** (1986), 247–248.
- [10] M. Ozeki, On the structure of even unimodular extremal lattices of rank 40, *Rocky Mtn. J. Math.* **19** (1989), 847–862.
- [11] H.-G. Quebbemann, Modular lattices in Euclidean spaces, *J. Number Theory* **54** (1995), no. 2, 190–202.
- [12] E.M. Rains and N.J.A. Sloane, The shadow theory of modular and unimodular lattices, *J. Number Theory* **73** (1998), no. 2, 359–389.
- [13] R. Scharlau and B.B. Venkov, The genus of the Barnes–Wall lattice, *Comment. Math. Helv.* **69** (1994), no. 2, 322–333.
- [14] B.B. Venkov, Réseaux et designs sphériques, (French) [Lattices and spherical designs] *Réseaux euclidiens, designs sphériques et formes modulaires*, 10–86, Monogr. Enseign. Math., 37, Enseignement Math., Geneva, 2001.
- [15] Wolfram Research, Inc., Mathematica, Version 11.2, Champaign, IL (2017).

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