On extremal lattices in jump dimensions

Gabriele Nebe

Let (L, Q) be an even unimodular lattice, so L is a free $\mathbb{Z}$-module of rank n, and $Q : L \to \mathbb{Z}$ a positive definite regular integral quadratic form. Then L can be embedded into Euclidean n-space $(\mathbb{R}^n, (,))$ with bilinear form defined by $(x, y) := Q(x + y) - Q(x) - Q(y)$ and L defines a lattice sphere packing, whose density measures its error correcting properties. One of the main goals in lattice theory is to find dense lattices. This is a very difficult problem, the densest lattices are known only in dimension $n \leq 8$ and in dimension 24 [3], for $n = 8$ and $n = 24$ the densest lattices are even unimodular lattices. The density of a unimodular lattice is proportional to its **minimum**, $\min(L) := \min\{Q(\ell) \mid 0 \neq \ell \in L\}$. For even unimodular lattices the theory of modular forms allows to bound this minimum $\min(L) \leq 1 + \lfloor \frac{n}{24} \rfloor$ and **extremal lattices** are those even unimodular lattices L that achieve equality. The link is the **theta series** of L,

$$\theta_L = \sum_{\ell \in L} q^{Q(\ell)} = 1 + \sum_{k=\min(L)}^{\infty} a_k q^k$$

where $a_k = |\{\ell \in L \mid Q(\ell) = k\}|$. After substituting the formal variable q by the holomorphic function $\exp(2\pi i z)$ with $z \in \mathbb{C}$, $\Im(z) > 0$, $\theta_L(z)$ becomes a modular form of weight $\frac{n}{2}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. So one may apply explicit transformation rules to conclude that the dimension n is always a multiple of 8 (see [4, Theorem 2.1]), which also follows from the theory of quadratic forms. The space of modular forms of weight $4k$ has dimension $m_k := \lfloor \frac{k}{3} \rfloor + 1$ and contains a unique form

$$f^{(k)} := 1 + 0q + 0q^2 + \ldots + 0q^{m_k - 1} + a(f^{(k)})q^{m_k} + b(f^{(k)})q^{m_k+1} + \ldots$$

the **extremal modular form** of weight $4k$. Already Siegel [12, end of proof of Satz 2] has shown that $a(f^{(k)}) > 0$ for all k, therefore $\min(L) \leq 1 + \lfloor \frac{n}{24} \rfloor$ for all even unimodular lattices of rank n. Lattices achieving equality are called **extremal**. Recently Jenkins and Rouse [5] have shown that the next coefficient $b(f^{(k)})$ of the extremal modular form becomes negative for all $k \geq 20408$, so there are no extremal lattices of dimension $n \geq 163, 264$.

Extremal even unimodular lattices L$\leq \mathbb{R}^n$

n	8	16	**24**	32	40	**48**	**72**	80	$\geq 163, 264$
min(L)	1	1	**2**	2	2	**3**	**4**	4	
number of extremal lattices	1	2	**1**	$\geq 10^7$	$\geq 10^{51}$	$\geq$ **3**	$\geq$**1**	≥ 4	0

Of particular interest are extremal even unimodular lattices L in the **jump dimensions** $24m$. Then $\theta_{L,p} = 0$ for all harmonic polynomials of degree $1 \leq \deg(p) \leq 11$ hence all non-empty layers $\{\ell \in L \mid Q(\ell) = a\}$ form spherical 11-designs. In particular the minimal vectors of L form a spherical 4-design, so all these lattices are strongly perfect [14] and their density realises a local maximum of the density function on the space of all $24m$-dimensional lattices. For $m = 1$ there is a unique extremal even unimodular lattice, the **Leech lattice**, which is the densest 24-dimensional lattice [3]. The 196560 minimal vectors of the Leech lattice form the unique tight spherical 11-design and realise the maximal kissing number in dimension 24. Also $m = 2, 3$ these lattices are the densest known lattices and realise the maximal known kissing number. There are only 5 extremal lattices known in jump dimensions. Using the classification of finite simple groups, one may show that the automorphism groups of these lattices are [11]

$\mathrm{Aut}(\Lambda_{24}) \cong 2.Co_1$	order	8315553613086720000
	$=$	$2^{22}3^9 5^4 7^2 \cdot 11 \cdot 13 \cdot 23$
$\mathrm{Aut}(P_{48p}) \cong (\mathrm{SL}_2(23) \times S_3) : 2$	order	$72864 = 2^5 3^2 11 \cdot 23$
$\mathrm{Aut}(P_{48q}) \cong \mathrm{SL}_2(47)$	order	$103776 = 2^5 3 \cdot 23 \cdot 47$
$\mathrm{Aut}(P_{48n}) \cong (\mathrm{SL}_2(13) \curlyvee \mathrm{SL}_2(5)).2^2$	order	$524160 = 2^7 3^2 5 \cdot 7 \cdot 13$
$\mathrm{Aut}(\Gamma_{72}) \cong (\mathrm{SL}_2(25) \times \mathrm{PSL}_2(7)) : 2$	order	$5241600 = 2^8 3^2 5^2 7 \cdot 13$

A **canonical construction** of a lattice is a construction that is respected by (a big subgroup of) its automorphism group. Two of the 48-dimensional extremal lattices have a canonical construction with codes:

Let $(e_1, \ldots, e_n)$ be a p-**frame**, so $(e_i, e_j) = p\delta_{ij}$. Given $C \leq \mathbb{F}_p^n$ the **codelattice** is $\Lambda(C) := \{\frac{1}{p}\sum c_i e_i \mid (\overline{c}_1, \ldots, \overline{c}_n) \in C\}$.

Theorem [6], [7]

Let $C = C^\perp \leq \mathbb{F}_3^{48}$ with $d(C) = 15$. Then one of the two even neighbors of the codelattice $\Lambda(C)$ is an extremal even unimodular lattice. The other even neighbor has minimum 4, its minimal vectors form a 4-frame and hence this is a codelattice for some extremal code modulo 4. This is one explanation of the surprising bijection between Hadamard matrices mod 4 and mod 3 given in [7].

Having this application to extremal lattices in mind, I classified all extremal ternary codes of length 48 that have an automorphism prime order ≥ 5 in [9]. It turned out that the two known codes are the only such codes: the extended quadratic residue code Q_{48} with $\mathrm{Aut}(Q_{48}) \cong \mathrm{SL}_2(47)$ and the Pless code P_{48} with $\mathrm{Aut}(P_{48}) \cong (\mathrm{SL}_2(23) \times C_2) : 2$. These codes yield the two lattices P_{48q} and P_{48p}.

In [8] I found the third lattice P_{48n} which has a canonical construction as a tensor product of lattices over quaternions which is very similar to the construction of Γ_{72} as a Hermitian tensor product over $\mathbb{Z}[\alpha]$ where $\alpha = \frac{1+\sqrt{-7}}{2}$. For sake of brevity I will only comment on Γ_{72} and show how one may apply the theory from [1] to obtain the minimum of Γ_{72}. A $\mathbb{Z}[\alpha]$-lattice P is a free $\mathbb{Z}[\alpha]$ module of rank n together with a positive definite Hermitian form $h : P \times P \to \mathbb{Q}[\alpha]$. The **minimum** of P is $\min(P) := \min\{h(\ell, \ell) \mid 0 \neq \ell \in P\}$, the **determinant** of P is the determinant of any Gram matrix of P and the **Hermitian dual lattice** is $P^* := \{v \in V \mid h(v, \ell) \in \mathbb{Z}[\alpha] \text{ for all } \ell \in P\}$ We call P **Hermitian unimodular**, if $P = P^*$ One example of such a lattice is the **Barnes lattice** P_b with Hermitian

Gram matrix $\begin{pmatrix} 2 & \alpha & -1 \\ \beta & 2 & \alpha \\ -1 & \beta & 2 \end{pmatrix}$ where $\beta = \overline{\alpha} = 1 - \alpha$. Then P_b is Hermitian unimodular, $\det(P_b) = 1$, $\min(P_b) = 2$ and $\mathrm{Aut}(P_b) = \pm \mathrm{PSL}_2(7)$.

Any Hermitian $\mathbb{Z}[\alpha]$-lattice (P, h) is also a $\mathbb{Z}$-lattice (L, Q) of dimension $2n$, where $L = P$ and $Q(x) := h(x, x) \in \mathbb{R} \cap \mathbb{Q}[\alpha] = \mathbb{Q}$. Then the polar form of Q is $(x, y) = \mathrm{Trace}_{\mathbb{Q}[\alpha]/\mathbb{Q}}(h(x, y))$ and (L, Q) is called the **trace lattice** of (P, h). We have $\min(L) = \min(P)$, $L^{\#} = \frac{1}{\sqrt{-7}} P^*$ and $\det(L) = 7^n \det(P)^2$.

Transfering ideas of Kitaoka, Renaud Coulangeon [1] obtained bounds on the minimum of the tensor product of Hermitian lattices: Let K be an imaginary quadratic field and (L, h_L) and (M, h_M) be Hermitian $\mathbb{Z}_K$-lattices, $n = \dim_{\mathbb{Z}_K}(L) \leq m := \dim_{\mathbb{Z}_K}(M)$. Each $v \in L \otimes M$ is the sum of at most n pure tensors $v = \sum_{i=1}^r \ell_i \otimes m_i$ where r is minimal. Put $A := (h_L(\ell_i, \ell_j))$ and $B := (h_M(m_i, m_j))$, then $h(v, v) = \mathrm{Trace}\, A\overline{B} \geq r \det(A)^{1/r} \det(B)^{1/r}$. so

$$\min(L \otimes M) \geq \min\{r d_r(L)^{1/r} d_r(M)^{1/r} \mid r = 1, \ldots, n\}$$

where $d_r(L) = \min\{\det(T) \mid T \leq L, Rg(T) = r\}$.

Theorem [2]
Let P be an Hermitian $\mathbb{Z}[\alpha]$-lattice with $\min(P) = 2$. Then $\min(P \otimes P_b) \geq 3$ and $\min(P \otimes P_b) > 3$ if and only if P has no sublattice isometric to P_b.
Proof: Clearly $d_1(P_b) = \min(P_b) = 2$, $d_3(P_b) = \det(P_b) = 1$ and $d_2(P_b) = d_1(P_b^*) = 2$. By assumption $d_1(P) = \min(P) = 2$ and so $d_2(P) \geq 2^2 \frac{3}{7}$ and $d_3(P) \geq 1$, as these are the minimal determinants of the densest $\mathbb{Z}[\alpha]$-lattices of minimum 2 and dimension 2 respectively 3. So

$$r d_r(P_b)^{1/r} d_r(P)^{1/r} \begin{cases} = 4 & r = 1 \\ \geq 3.7 & r = 2 \\ \geq 3 & r = 3 \end{cases}$$

So the bound on $\min(P \otimes P_b)$ is strictly bigger than 3, if P does not represent the lattice P_b.

The nine $\mathbb{Z}[\alpha]$ structures of the Leech lattice

i	group	$\#P_b \leq P_i$
1	$\mathrm{SL}_2(25)$	0
2	$2.A_6 \times D_8$	$2 \cdot 20,160$
3	$\mathrm{SL}_2(13).2$	$2 \cdot 52,416$
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
6	$2^9 3^3$	$2 \cdot 177,408$
7	$\pm \mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2 \cdot 306,432$
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2 \cdot 504,000$
9	$2.J_2.2$	$2 \cdot 1,209,600$

In particular we may apply this to the nine 12-dimensional $\mathbb{Z}[\alpha]$-lattices P_i given in the table such that $\mathrm{Trace}(P_i) \cong \Lambda_{24}$. The representation number of P_b

in P_i can be obtained by computations within the set of minimal vectors of the Leech latticed and is given in the last column of this table. It gives the number of vectors of norm 3 in $P_i \otimes P_b$. Therefore the trace lattice $\mathrm{Trace}(P_1 \otimes P_b) =: \Gamma_{72}$ is an extremal even unimodular lattice. Two computational proofs of the extremality of Γ_{72} have been given in [10] a third proof by M. Watkins is based on the following idea.

Theorem. [13]
Let L be an even unimodular lattice of dimension 72 with $\min(L) \geq 3$. Then L is extremal, if and only if it contains at least $6,218,175,600$ vectors v with $Q(v) = 4$.

Proof: L is an even unimodular lattice of minimum ≥ 3, so its theta series is

$$\theta_L = 1 + a_3 q^3 + a_4 q^4 + \ldots = f^{(9)} + a_3 \Delta^3.$$

$$\begin{array}{rcccccc} f^{(9)} & = & 1 & + & 6,218,175,600q^4 & + \ldots \\ \Delta^3 & = & & q^3 & -72q^4 & + \ldots \end{array}$$

So $a_4 = 6,218,175,600 - 72a_3 \geq 6,218,175,600$ if and only if $a_3 = 0$.

References

[1] R. Coulangeon, *Tensor products of Hermitian lattices.* Acta Arith. 92 (2000) 115-130.

[2] R. Coulangeon, G. Nebe, *Dense lattices as Hermitian tensor products.* Proceedings of the BIRS workshop “Diophantine Methods, Lattices, and Arithmetic Theory of Quadratic Forms”, Contemporary Mathematics (to appear)

[3] H. Cohn, A. Kumar, *Optimality and uniqueness of the Leech lattice among lattices.* Annals of Mathematics 170 (2009) 1003-1050.

[4] W. Ebeling, *Lattices and Codes*, Vieweg 1994

[5] P. Jenkins, J. Rouse, *Bounds for coefficients of cusp forms and extremal lattices.* Bulletin LMS 43 (2011) 927-938.

[6] H. Koch, *The* 48*-dimensional analogues of the Leech lattice.* Trudy Mat. Inst. Steklov. 208 (1995) 193-201.

[7] Akihiro Munemasa, Hiroki Tamura, *The codes and the lattices of Hadamard matrices.* European J. Combin. 33 (2012) 519-533.

[8] G. Nebe, *Some cyclo-quaternionic lattices.* J. Algebra 199 (1998) 474-498.

[9] G. Nebe, *On extremal self-dual ternary codes of length 48.* International Journal of Combinatorics, vol. 2012

[10] G. Nebe, *An even unimodular 72-dimensional lattice of minimum 8.* J. Reine und Angew. Math. (to appear)

[11] G. Nebe, *On automorphisms of extremal lattices.* (in preparation)

[12] C.L. Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen.* Nachr. Akad. Wiss. Göttingen **10** (1969) 87-102.

[13] D. Stehlé, M. Watkins, *On the Extremality of an 80-Dimensional Lattice.* ANTS 2010: 340-356.

[14] B. Venkov, *Even unimodular extremal lattices.* Trudy Mat. Inst. Steklov. 165 (1984) 43-48.