Odd unimodular lattices of minimum 4

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Abstract

We prove the non existence of unimodular lattices of minimum 4 and dimension 34 and 35.

1 Introduction

Unimodular lattices have focus interest for a long time. One of its most fascinating properties is that its theta series

$$\theta_\Lambda(\tau) := \sum_{x \in \Lambda} q^{(x \cdot x)} \tag{1}$$

where $\tau \in \mathfrak{h}$ the upper half complex plane and $q := e^{\pi i \tau}$, satisfies an invariance property under the transformation $\tau \to -1/\tau$. If the lattice is moreover even, then its theta series is invariant under the action of the full modular group $SL(2, \mathbb{Z})$, which leads to the upper bound for the minimum of the lattice:

$$\min(\Lambda) \leq 2[n/24] + 2. \tag{2}$$

where n is the dimension of the lattice. The first case where this bound is not known to be tight is $n = 72$.

It is much more difficult to obtain a good bound for the minimum of an odd unimodular lattice, although these lattices are expected to be not so good as the even ones, as is observed in small dimensions. The theta series of such a lattice is only invariant under the congruence subgroup $\Gamma_0(4)$ and the bound derived from this invariance is $\min(\Lambda) \leq 2[n/8] + 2$ which is not sharp. Only recently, E. Rains and N.J.A. Sloane have proved that (2) holds also for the odd lattices, apart from the exceptional case $n = 23$ ([6]). Their proof makes use of the theta series of the *shadow* of the lattice. J. H. Conway and N. J. A. Sloane have given in [3] the exact bound for the minimum of a unimodular lattice of dimension $n \leq 33$. In particular they show that there cannot exist a minimum 4 lattice of dimension 33. We shall extend here this result, proving the

Theorem 1 *There is no unimodular lattice of minimum* 4 *and dimension* 34 *or* 35.

Unimodular lattices of minimum 4 are known in dimensions 36, 38, 39, 40 (see [5]).

Seeking for a contradiction, we shall first compute the theta series of a putative lattice Λ of minimum 4 and of its shadow S. Then we shall compute the number of vectors of the lattice with prescribed scalar product with a fixed minimal vector of S. This amounts to the computation of certain coefficients of some Jacobi theta series associated to the lattice and therefore we shall make use of spherical theta series $\theta_{\Lambda,P}$ where P is a harmonic polynomial.

The paper is organized as follows: Section 2 recalls results on the shadows of unimodular lattices. Section 3 introduces a 40-dimensional even unimodular lattice associated to Λ. Section 4 introduces theta series with spherical coefficients and Section 5 derives equations on the above mentioned numbers. Section 6 ends the proof of Theorem 1.

2 Shadows

Let Λ be a unimodular lattice. The shadow S of Λ is $S = (\Lambda_0)^* \setminus \Lambda$, where Λ_0 denotes the even sublattice of Λ. If Λ is an odd lattice, its theta series has the following expression

$$\theta_\Lambda(\tau) = \sum_{j=0}^{[n/8]} a_j \Delta_8(q)^j \theta_3(q)^{n-8j} \tag{3}$$

and the theta series of the shadow S is

$$\theta_S(\tau) = \sum_{j=0}^{[n/8]} \frac{(-1)^j}{16^j} a_j \theta_4(q^2)^{8j} \theta_2(q)^{n-8j}. \tag{4}$$

where $q := e^{\pi i \tau}$, $\Delta_8(q) = q\prod_{m=1}^{\infty}(1-q^{2m-1})^8(1-q^{4m})^8$, and θ_2, θ_3, θ_4 are the usual Jacobi theta series (see [2, Chap. 4, § 4]).

For the rest of the paper, Λ is a unimodular lattice of minimum 4 and dimension $n \geq 34$. We denote by m the minimum of the shadow S of Λ and by s_m the number of vectors $s \in S$ with $s \cdot s = m$.

If A is any set of vectors, A_r is the set of vectors $a \in A$ with $a \cdot a = r$.

We start with the computation of the theta series. For $n = 34, 35$, the condition that the minimum of the lattice is at least 4 determines the values of a_1, a_2, a_3. If $x \in S$, $2x \in \Lambda$ so the minimum of S must be at least 2. This condition forces $a_4 = 0$. We find the following theta series:

For $n = 34$,

$$\begin{aligned} \theta_\Lambda &= 1 + 60180q^4 + \dots \\ \theta_S &= 204q^{5/2} + 758200q^{9/2} + \dots \end{aligned} \tag{5}$$

For $n = 35$,

$$\begin{aligned} \theta_\Lambda &= 1 + 51030q^4 + \dots \\ \theta_S &= 420q^{11/4} + 1704780q^{19/4} + \dots \end{aligned} \tag{6}$$

We now fix a vector $s \in S_m$. If s' is another minimal vector in S, not equal to $\pm s$, then

$$s \cdot s' \equiv s \cdot s \mod \frac{1}{2}\mathbb{Z} \text{ because } s - s' \in \Lambda,$$

and

$$|s \cdot s'| \leq m - 2 \text{ because } (s \pm s')^2 \geq 4.$$

Hence

$$s \cdot s' \in \{\pm(m - [2m]/2), \dots, \pm(m - 5/2), \pm(m - 2)\}.$$

We get

$$s \cdot s' \in \{0, \pm\frac{1}{2}\} \text{ for } n = 34$$

and

$$s \cdot s' \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\} \text{ for } n = 35.$$

Let $x \in \Lambda_4$. Since $(s \pm x)^2 \geq m$ and $s \cdot x$ is an integer, $s \cdot x \in \{0, \pm 1, \pm 2\}$. We define

$$\begin{aligned} i = 0, 1, 2, \quad & p_i(s) := \operatorname{card}\{x \in \Lambda \mid x^2 = 4, s \cdot x = \pm i\} \\ i = m - [2m]/2, \dots, m - 2, \quad & m_i(s) := \operatorname{card}\{s' \in S \mid s'^2 = m, s \cdot s' = \pm i\} \end{aligned} \tag{7}$$

Our first task is to compute these numbers for $n = 34, 35$. It will turn out that they do not depend on the choice of s. Therefore we need five equations for them; two trivial equations come from the knowledge of θ_Λ:

$$\begin{aligned} \sum p_i(s) &= \mathrm{card}(\Lambda_4) \\ \sum m_i(s) &= \mathrm{card}(S_m) - 2 \end{aligned} \tag{8}$$

Some more equations will come from theta series with spherical coefficients. In order to avoid the use of half integral weight modular forms we do not consider the ones associated directly to Λ but we introduce a 40-dimensional even unimodular lattice constructed from Λ.

3 A certain 40-dimensional even unimodular lattice.

An even unimodular lattice Γ is obtained by gluing the lattice Λ_0 (Λ is assumed to be odd, unimodular, of minimum 4 and dimension $n < 39$) with the root lattice D_{40-n} (if $n = 39$ one should take instead $\sqrt{2}A_1$). Then the discriminant groups $\Lambda_0^*/\Lambda_0 \cong D_{40-n}^*/D_{40-n}$ are isomorphic to $\mathbb{Z}/4\mathbb{Z}$ if n is odd and to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if n is even. To write down this isomorphism explicitely, we again denote by s and x some fixed minimal vectors of respectively S and $\Lambda \backslash \Lambda_0$. Let $\phi : (\Lambda_0)^*/\Lambda_0 \to (D_{40-n})^*/D_{40-n}$ be the isomorphism defined by $\phi(s) = (1/2, \dots, 1/2)$ and $\phi(x) = (1, 0, \dots, 0)$. Then, for all $u \in (\Lambda_0)^*/\Lambda_0$, $\phi(u) \cdot \phi(u) \equiv -u \cdot u \mod 2\mathbb{Z}$ (because $2s$ is a characteristic vector for Λ, we have $4s \cdot s \equiv n \mod 8$). Let

$$\Gamma := \{(u, \phi(u)) \in (\Lambda_0)^* \perp (D_{40-n})^*\}. \tag{9}$$

Clearly, the lattice Γ is an even unimodular lattice of dimension 40. Its root lattice is D_{40-n}. Its vectors of norm 4 are of three types: the ones from D_{40-n}, the ones from Λ_0, and the pairs (s', t) with $s' \in S$ of minimal norm and $t \in \phi(s')$, of minimal norm $10 - n/4$. The number of such vectors t is 2^{39-n}.

4 Theta series with spherical coefficients.

In this section we recall some basic facts about harmonic polynomials and theta series with spherical coefficients associated to even unimodular lattices. We refer to [2, Chapter 18], [4], [7], [8]. The harmonic polynomials are the polynomials in $\mathbb{R}[x_1, \dots, x_n]$ which are homogeneous and satisfy $LP = 0$

where $L = \sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. It is a classical result that the formula

$$P_{k,\alpha}(x) = G_k((x \cdot \alpha), ((x \cdot x)(\alpha \cdot \alpha))^{1/2}) \tag{10}$$

where $G_k(t, 1)$ is the Gegenbauer polynomial of degree k and parameter $n/2 - 1$ defines a harmonic polynomial. For example

$$P_{2,\alpha}(x) = (x \cdot \alpha)^2 - \frac{1}{n}(\alpha \cdot \alpha)(x \cdot x). \tag{11}$$

We shall also need for our computations the polynomial $P_{6,\alpha}(x)$ relative to the dimension 40:

$$\begin{aligned} P_{6,\alpha}(x) =& (x \cdot \alpha)^6 - \frac{5}{16}(x \cdot \alpha)^4(\alpha \cdot \alpha)(x \cdot x) \\ &+ \frac{15}{736}(x \cdot \alpha)^2(\alpha \cdot \alpha)^2(x \cdot x)^2 - \frac{5}{32384}(\alpha \cdot \alpha)^3(x \cdot x)^3. \end{aligned} \tag{12}$$

A classical result due to Hecke asserts that, if P is a harmonic polynomial of degree k and if Γ is an even unimodular lattice, then

$$\theta_{\Gamma,P}(\tau) := \sum_{x \in \Lambda} P(x) q^{(x \cdot x)} \tag{13}$$

defines a modular form for the full modular group $SL(2, \mathbb{Z})$ of weight $n/2+k$. The algebra of modular forms for the full modular group is a polynomial algebra in the elements E_4, E_6 of respective weight 4 and 6:

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{r=1}^{\infty} \sigma_3(r) q^{2r} = 1 + 240q^2 + 240 \cdot 9q^4 + \dots \\ E_6(\tau) &= 1 - 504 \sum_{r=1}^{\infty} \sigma_5(r) q^{2r} = 1 - 504q^2 - 504 \cdot 33q^4 + \dots \end{aligned} \tag{14}$$

The cusp form of lowest weight is the weight 12 form:

$$\Delta_{12} = (E_4^3 - E_6^2)/1728 = q^2 \prod_{r=1}^{\infty} (1 - q^r)^{24}. \tag{15}$$

5 Equations.

We derive in this section some equations satisfied by the numbers $p_i(s)$, $m_i(s)$. We take the notations of Section 2 and consider the lattice Γ constructed in Section 3. We introduce the additional notations:

l_4 is the number of norm 4 vectors in Λ.
d_2, d_4 are the number of norm 2, respectively norm 4 vectors in D_{40-n}.

Let α belong to the vector space spaced by Λ_0, and let $f_k := \theta_{\Gamma,P_{k,\alpha}}$ be defined in the previous section. For all $x \in \Gamma_2$, $x \cdot \alpha = 0$ so the coefficient of q^2 in f_k is $\sum_{x\in\Gamma_2} P_{k,\alpha}(x) = G_k(0, 2^{1/2}(\alpha\cdot\alpha)^{1/2})d_2$.

Taking account of the three types of norm 4 vectors in Γ, the coefficient of q^4 is

$$\begin{aligned}\sum_{x\in\Gamma_4} P_{k,\alpha}(x) =& G_k(0, 2(\alpha\cdot\alpha)^{1/2})d_4 + \sum_{x\in\Lambda_4} G_k((x\cdot\alpha), 2(\alpha\cdot\alpha)^{1/2}) \\ &+ 2^{39-n}\sum_{s'\in S_m} G_k((s'\cdot\alpha), 2(\alpha\cdot\alpha)^{1/2})\end{aligned} \tag{16}$$

If $k = 2$, the weight of f_k is $20+2 = 22$ so f_k is a multiple of $\Delta_{12}E_4E_6 = q^2 - 288q^4 + \ldots$; the multiplicity factor is exactly $G_k(0, 2^{1/2}(\alpha\cdot\alpha)^{1/2})d_2 = -\frac{2(\alpha\cdot\alpha)}{40}d_2$. We now take $\alpha = s$ a minimal vector in S; the equality of the coefficients of q^4 leads, taking account of the expression for G_2, to the equation:

$$\begin{aligned}\sum_{x\in\Lambda_4}((x\cdot s)^2 - 4m/40) + 2^{39-n}\sum_{s'\in S_m}((s'\cdot s)^2 - 4m/40) = \\ \frac{576m}{40}d_2 + \frac{4m}{40}d_4\end{aligned} \tag{17}$$

which leads to the following equation for the $p_i(s)$, $m_i(s)$:

$$\begin{aligned}\sum_i i^2 p_i(s) + 2^{39-n}\sum_i i^2 m_i(s) = \\ \frac{72m}{5}d_2 + \frac{m}{10}d_4 - 2^{40-n}m^2 + \frac{m}{10}(l_4 + 2^{39-n}s_m).\end{aligned} \tag{18}$$

If $k = 4$, we do not get a similar equation because the weight is 24 and the corresponding space of cusp forms is two-dimensional spanned by $\Delta_{12}E_6^2$ and Δ_{12}^2. If $k = 6$ the situation is better because the only cusp form of weight 26 is up to a multiplicative factor $\Delta_{12}E_4^2E_6 = q^2 - 48q^4 + \ldots$. We compute this factor with the coefficient of q^2. We get:

$$\begin{aligned}\sum_{x\in\Lambda_4} G_6((x\cdot\alpha), 2(\alpha\cdot\alpha)^{1/2}) + 2^{39-n}\sum_{s'\in S_m} G_6((s'\cdot\alpha), 2(\alpha\cdot\alpha)^{1/2}) = \\ -\frac{5}{32384}(-2^3\cdot 48d_2 - 4^3d_4)(\alpha\cdot\alpha)^3\end{aligned} \tag{19}$$

This equation leads to an equation in the $p_i(s)$, $m_i(s)$ when $\alpha = s$ belongs to S_m:

$$\sum_i G_6(i, 2m^{1/2})p_i(s) + 2^{39-n}\sum_i G_6(i, 2m^{1/2})m_i(s) = \\ -\frac{5}{32384}(-2^3.48.d_2 - 4^3.d_4)m^3 - 2^{40-n}G_6(m, 2m^{1/2}). \tag{20}$$

The equation (19) is true for all α in the n-dimensional space spanned by Λ, so we can also view it as a polynomial identity in the coordinates of α and apply the Laplace operator corresponding to this space. This will lead to a degree 4 identity.

Let L_n denote the Laplace operator in the n variables of α. We use the following identity, valid for all $y \in \mathbb{R}\Lambda$ ([7]):

$$\begin{aligned} L_n((\alpha\cdot\alpha)^l(\alpha\cdot y)^k) =& 2l(2l+2k+n-2)(\alpha\cdot\alpha)^{l-1}(\alpha\cdot y)^k \\ &+ k(k-1)(y\cdot y)(\alpha\cdot\alpha)^l(\alpha\cdot y)^{k-2}. \end{aligned} \tag{21}$$

and obtain an expression for $L_n(G_6((y\cdot\alpha), 2(\alpha\cdot\alpha)^{1/2}))$:

$$\begin{aligned} L_n(G_6((y\cdot\alpha), 2(\alpha\cdot\alpha)^{1/2})) =& (30(y\cdot y) - 5/2(8+n))(y\cdot\alpha)^4 + \\ & (-15(y\cdot y) + 30/23(6+n))(\alpha\cdot\alpha)(y\cdot\alpha)^2 + \\ & (15/23(y\cdot y) - 15/253(4+n))(\alpha\cdot\alpha)^2. \end{aligned} \tag{22}$$

Then, we again take $\alpha = s$ and find a fifth equation for the $p_i(s)$, $m_i(s)$.

6 Proof of Theorem 1

6.1 Dimension 34.

The system of five equations found from Section 5 on the unknowns $p_0(s)$, $p_1(s)$, $p_2(s)$, $m_0(s)$, $m_{1/2}(s)$ has a unique solution $p_0(s) = 42780$, $p_1(s) = 17300$, $p_2(s) = 100$, $m_0(s) = 102$, $m_{1/2}(s) = 100$.

The quotient $(\Lambda_0)^*/\Lambda_0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The three subgroups of order 2 define three lattices, one is Λ and the two others are dual one of the other; we denote them L and L^*. Clearly two short vectors $s, s' \in S_m$ are both in L or L^* if and only if $s\cdot s' = \pm 1/2$.

Let $s \in S_m$ be a fixed vector. Let

$$X := \{s\} \cup \{s', s' \in S_m \mid s\cdot s' = 1/2\}. \tag{23}$$

From the computation of $m_{1/2}(s)$ we know that the cardinality of X is 51. Let G be the Gram matrix of this set, where s is chosen to be the first of the vectors in X.

Lemma 1 $G^2 = \frac{15}{2}G$.

Proof. We compute G^2: $G^2[s', s''] = \sum_{x \in X}(s' \cdot x)(x \cdot s'')$. The vectors of S_m are either in $\pm X$ or are perpendicular to X, so: $G^2[s', s''] = 1/2 \sum_{x \in S_m}(s' \cdot x)(x \cdot s'')$.

From the values found for $m_0(s)$ and $m_{1/2}(s)$, one can check that the set S_m is a 2-design because $\sum_{s', s'' \in S_m}(s' \cdot s'')^2 = m^2 s_m^2 / n$ (see [7, Theorem 8.1]). Hence, for all α, $\sum_{x \in S_m}(\alpha \cdot x)^2 = \frac{m s_m}{n}(\alpha \cdot \alpha) = 15(\alpha \cdot \alpha)$. Applied to $\alpha + \beta$, this identity leads to $\sum_{x \in S_m}(\alpha \cdot x)(x \cdot \beta) = 15(\alpha \cdot \beta)$ for all α, β, and, when $\alpha = s'$, $\beta = s''$, to the statement $G^2 = \frac{15}{2}G$.

□

We know consider the graph with vertices $X \setminus \{s\}$ and edges the pairs (s', s'') with $(s' \cdot s'') = -1/2$. This graph is regular with valency 22 as can be checked from the computation of the coefficient (s, s') in the identity $G^2 = \frac{15}{2}G$. If A is the incidence matrix of this graph and if A' is the matrix obtained from A by adding a first line of zeros and a first column of zeros, we have

$$G = 2I_{51} + 1/2 J_{51} - A' \tag{24}$$

where I_p denotes the identity matrix of size p, and J_p denotes the matrix with all its coefficients equal to 1 of size p. Replacing in the equation $G^2 = \frac{15}{2}G$ and taking account of the identity $AJ_{50} = J_{50}A = 22J_{50}$, we get

$$A^2 - \frac{7}{2}A - 11J_{50} - 11I_{50} = 0. \tag{25}$$

Of course, this last identity is not possible for a matrix A with entries equal to 0 or 1 so we can conclude of the non existence of the lattice Λ.

6.2 Dimension 35.

The system of five equations found from Section 5 on the unknowns $p_0(s)$, $p_1(s)$, $p_2(s)$, $m_{1/4}(s)$, $m_{3/4}(s)$ has a unique solution $p_0(s) = 35289$, $p_1(s) = 15642$, $p_2(s) = 99$, $m_{1/4}(s) = 319$, $m_{3/4}(s) = 99$. But these numbers should be even so the lattice Λ does not exist.

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