Orthogonal Frobenius reciprocity.

Gabriele Nebe, Lehrstuhl B für Mathematik, RWTH Aachen,
Templergraben 64, 52062 Aachen, Germany, gabi@math.rwth-aachen.de

ABSTRACT: A quantitative version of Frobenius reciprocity for isometries is proved. It is applied to deduce recursion formulas for the rational class of some irreducible orthogonal modules of the symmetric groups.

1 Introduction

Let G be a finite group with subgroup $H \subset G$ and K be a field of characteristic 0. If W is a right KH-module, then the induced module W^G defined as $W^G = W \otimes_{KH} KG$ (cf. [CuR, Section 10]) is a KG-module. On the other hand, by restriction, any KG-module V can also be viewed as a KH-module. Classical Frobenius reciprocity (cf. [CuR, Theorem (10.8)]) establishes a canonical isomorphism between the vector spaces of homomorphisms

$$(F) \quad Hom_{KH}(W, V) \cong Hom_{KG}(W^G, V), \quad \varphi \mapsto \varphi^G.$$

Now assume that W admits a non-degenerate symmetric bilinear form $F_W : W \times W \to K$ that is H-invariant, i.e. $F_W(v, w) = F_W(vh, wh)$ for all $v, w \in W, h \in H$. Call such a pair (W, F_W) an orthogonal KH-module. Then F_W defines a G-invariant form F_W^G on W^G such that (W^G, F_W^G) becomes an orthogonal KG-module.

If (V, F_V) is an orthogonal KG-module, then it is natural to ask what happens with the KH-isometries $Isom_{KH}((W, F_W), (V, F_V)) :=$

$$\{\varphi \in Hom_{KH}(W, V) \mid F_W(w, w') = F_V(\varphi(w), \varphi(w')) \text{ for all } w, w' \in W\}$$

if one applies Frobenius reciprocity. Note that here by definition isometries are injective but not necessarily surjective. Since Frobenius reciprocity does not respect injectivity of the mappings, one has to dualise the right hand side of (F) to again get isometries: F_V induces a KG-isomorphism between V and its dual $V^* := Hom_K(V, K)$. Let F_V^* be the form on V^* such that this isomorphism is an isometry. Assume that V is a uniform KG-module, which means that F_V generates the space of all G-invariant symmetric bilinear forms on V. Then orthogonal Frobenius reciprocity gives a canonical bijection

$$Isom_{KH}((W, F_W), (V, F_V)) \cong Isom_{KG}((V^*, F_V^*), ((W^G)^*, (F_W^G)^*))$$

defined by

$$\varphi \mapsto \frac{dim(V)}{dim(W^G)} (\varphi^G)^*.$$

In Section 2, this orthogonal Frobenius reciprocity and two useful generalisations are proved. It is applied in Section 3 to determine the rational isometry class of some irreducible orthogonal $\mathbb{Q}S_n$-modules.

This paper is one section of my Habilitationsschrift [Neb]. There I give further applications of orthogonal Frobenius reciprocity to the determination of the rational isometry class of orthogonal G-modules.

2 Orthogonal Frobenius reciprocity.

Let K be a field of characteristic 0 and G be a finite group with subgroup $H \subseteq G$. Let $G = \dot\bigcup_{i=1}^{s} Hg_i$ be a decomposition of G into H-cosets.

If W is a KH-module with K-basis $(b_1, \ldots, b_m)$ then $(b_1 \otimes g_1, \ldots, b_m \otimes g_1, b_1 \otimes g_2, \ldots, b_m \otimes g_s)$ is a K-basis for the KG-module W^G. The action of $g \in G$ on W^G is calculated combining the permutation of the cosets induced by g with the action of H on W: If $g_i g = h g_j$ with $h \in H$ then $(w \otimes g_i)g = wh \otimes g_j$ for all $w \in W$.

If $\varphi : W \to V_{|H}$ is a KH-homomorphism, then $\varphi^G : W^G \to V$ defined by $\varphi^G(\sum_{i=1}^s w_i \otimes g_i) = \sum_{i=1}^s \varphi(w_i) g_i$ is a KG-homomorphism. The mapping $\varphi \mapsto \varphi^G$ is independent of the choice of the coset representatives g_i and defines a K-isomorphism $Hom_{KH}(W, V_{|H}) \to Hom_{KG}(W^G, V)$ with inverse $Hom_{KG}(W^G, V) \to Hom_{KH}(W, V_{|H}); \varphi \mapsto \varphi_{|W}$, the restriction to W, where W is identified with $W \otimes 1 \subseteq W^G$ (cf. [CuR, section 10]).

If (W, F_W) is an orthogonal KH-module, then F_W^G defined by

$$F_W^G(w \otimes g_i, w' \otimes g_j) := \delta_{ij} F_W(w, w') \quad (w, w' \in W, 1 \leq i, j \leq s)$$

is a non-degenerate G-invariant symmetric bilinear form on W^G.

Any non-degenerate symmetric G-invariant bilinear form F_V on the KG-module V defines a KG-isomorphism $\tilde{F}_V : v \mapsto F_V(\cdot, v)$ between V and the dual space $V^* = Hom_K(V, K)$. Note that V^* is again a right KG-module, via $(fg)(v) = f(vg^{-1})$ for all $v \in V, f \in V^*, g \in G$. Let F_V^* be the form on V^*, for which $\tilde{F}_V$ is an isometry: For $f \in V^*$ let $v_f := \tilde{F}_V^{-1}(f) \in V$. Then

$$F_V^*(f, h) := F_V(v_f, v_h) \text{ for all } f, h \in V^*.$$

Theorem 2.1 *Let K be a field of characteristic 0, (W, F_W) an orthogonal KH-module, and (V, F_V) a uniform orthogonal KG-module. If $\varphi : (W, F_W) \to (V, F_V)$ is a KH-isometry, then the transposed mapping*

$$(\varphi^G)^* : (V^*, \frac{dim(W^G)}{dim(V)} F_V^*) \to ((W^*)^G, (F_W^*)^G)$$

is a KG-isometry.

Remark 2.2 *The constant $\frac{dim(W^G)}{dim(V)}$ can be easily remembered by taking V and $W = \langle w \rangle$ with $F_W(w, w) = 1$ to be the trivial modules. Then W^G and also $(W^G)^*$ is the permutation module with orthonormal basis $(w \otimes g_1, \ldots, w \otimes g_s)$ and the trivial KG-submodule of W^G is generated by $v := \sum_{j=1}^s w \otimes g_j$ with squared length $(F_W^G)^*(v, v) = s$.*

To prove the theorem it is convenient to choose a basis of W and V and work with matrices. So F_V, F_W also denote the Gram matrices of F_V respectively F_W with respect to the chosen basis, g_k the matrix describing the action of $g_k \in G$, right multiplication with φ the corresponding mapping φ etc.

Lemma 2.3 *Let (W, F_W) be an orthogonal KH-module and (V, F_V) be an orthogonal KG-module. Let $\varphi \in Isom_{KH}((W, F_W), (V, F_V))$. Then the orthogonal projection $P_W \in End_{KH}(V)$ onto $\varphi(W)$ is given by right multiplication with*

$$P_W := F_V \varphi^{tr} F_W^{-1} \varphi.$$

If V is a uniform KG-module, then

$$Tr_{G/H}(P_W) := \sum_{j=1}^{s} g_j^{-1} P_W g_j = \frac{dim(W^G)}{dim(V)} id_V.$$

Proof. A straightforward calculation shows that P_W is the orthogonal projection onto $\varphi(W)$. Therefore $P_W F_V$ is the Gram matrix of a symmetric H-invariant bilinear form on V. By construction $Tr_{G/H}(P_W) \in End_{KG}(V)$ is a KG-endomorphism of V. Since $g_j F_V = F_V (g_j^{tr})^{-1}$, the trace

$$Tr_{G/H}(P_W) F_V = \sum_{j=1}^{s} g_j^{-1} P_W F_V (g_j^{-1})^{tr}$$

is the Gram matrix of a G-invariant symmetric bilinear form on V. Since V is a uniform KG-module, this implies that $Tr_{G/H}(P_W)$ is a scalar matrix. The trace of the matrix P_W is

$$tr(P_W) = tr(F_V \varphi^{tr} F_W^{-1} \varphi) = tr(\varphi F_V \varphi^{tr} F_W^{-1}) = dim(W).$$

Hence the trace of $Tr_{G/H}(P_W)$ is $s \cdot dim(W) = dim(W^G)$ and $Tr_{G/H}(P_W) = \frac{dim(W^G)}{dim(V)} id_V$. □

Proof of Theorem 2.1.
Let $(w_a | 1 \leq a \leq m)$ be a K-basis of W and $v_a := \varphi(w_a)$ $(1 \leq a \leq m)$. Then the set $\{v_a g_i \mid 1 \leq i \leq s, 1 \leq a \leq m\}$ generates the vector space V over K, because V is an irreducible KG-module. Therefore V^* is generated by the functions

$$f_{a,i} = \tilde{F}_V(v_a g_i) \ \ (1 \leq a \leq m, 1 \leq i \leq s).$$

Since F_W^G is non-degenerate there are unique $w_j^{(a,i)} \in W$ $(1 \leq j \leq s)$ such that

$$(\varphi^G)^*(f_{a,i}) = \tilde{F}_W^G(\sum_{j=1}^{s} w_j^{(a,i)} \otimes g_j) \text{ for all } 1 \leq a \leq m, 1 \leq i \leq s.$$

For $1 \leq a, b \leq m$, $1 \leq i, k \leq s$ one has

$$(\varphi^G)^*(f_{a,i})(w_b \otimes g_k) = f_{a,i}(\varphi^G(w_b \otimes g_k)) = f_{a,i}(v_b g_k) = F_V(v_b g_k, v_a g_i).$$

On the other hand

$$(\varphi^G)^*(f_{a,i})(w_b \otimes g_k) = F_W^G(w_b \otimes g_k, \sum_{j=1}^{s} w_j^{(a,i)} \otimes g_j) = F_W(w_b, w_k^{(a,i)}).$$

Choosing K-bases and working with matrices one therefore gets

$$F_W(w_k^{(a,i)})^{tr} = \varphi g_k F_V g_i^{tr} v_a^{tr} \quad \text{for all } 1 \leq i,k \leq s, 1 \leq a \leq m.$$

Hence the scalar product of $(\varphi^G)^*(f_{b,k})$ and $(\varphi^G)^*(f_{a,i})$ with respect to $(F_W^*)^G$ is

$$\sum_{j=1}^{s} w_j^{(b,k)} F_W(w_j^{(a,i)})^{tr} = \sum_{j=1}^{s} v_b g_k (F_V g_j^{tr} \varphi^{tr} F_W^{-1} \varphi g_j F_V) g_i^{tr} v_a^{tr}.$$

By Lemma 2.3

$$\sum_{j=1}^{s} F_V g_j^{tr} \varphi^{tr} F_W^{-1} \varphi g_j F_V = \sum_{j=1}^{s} g_j^{-1} P_W g_j F_V = \frac{dim(W^G)}{dim(V)} F_V$$

and therefore

$$(F_W^G)^*((\varphi^G)^*(f_{b,k}), (\varphi^G)^*(f_{a,i})) = \frac{dim(W^G)}{dim(V)} F_V^*(f_{b,k}, f_{a,i})$$

for all $1 \leq i,k \leq s$, $1 \leq a,b \leq m$, which proves the theorem. □

If W^G contains simple KG-modules with multiplicity > 1 then Theorem 2.1 does not give a complete decomposition of the orthogonal KG-module (W^G, F_W^G).

Theorem 2.4 *Let (W, F_W) be an orthogonal KH-module and let (V, F_V) be an absolutely irreducible orthogonal KG-module. Assume, there is $C = C^{tr} \in GL_n(K)$ such that $(\varphi_1, \ldots, \varphi_n) : (W^n, C \otimes F_W) \to (V, F_V)$ is a KH-isometry. Then*

$$((\varphi_1^G)^*, \ldots, (\varphi_n^G)^*) : ((V^*)^n, \frac{dim(W^G)}{dim(V)} C \otimes F_V^*) \to ((W^*)^G, (F_W^*)^G)$$

is a KG-isometry.

Proof. With the notation from the proof of Theorem 2.1 let $v_a^{(x)} := \varphi_x(w_a)$ and $f_{a,i}^{(x)} = \tilde{F}_V(v_a^{(x)} g_i)$ for $1 \leq a \leq m, 1 \leq x \leq n, 1 \leq i \leq s$. As above one calculates

$$(1) \quad (F_W^*)^G((\varphi_y^G)^*(f_{b,k}^{(y)}), (\varphi_x^G)^*(f_{a,i}^{(x)})) = \sum_{j=1}^{s} (v_b^{(y)} g_k F_V g_j^{tr} \varphi_y^{tr} F_W^{-1})(\varphi_x g_j F_V g_i^{tr} (v_a^{(x)})^{tr}).$$

Now $F_V \varphi_y^{tr} F_W^{-1} \varphi_x \in End_{KH}(V)$ has trace $c_{x,y} dim(W)$, where $c_{x,y}$ is the x,y-entry of C, since $\varphi_x F_V \varphi_y^{tr} = c_{x,y} F_W$. Therefore

$$\sum_{j=1}^{s} F_V g_j^{tr} \varphi_y^{tr} F_W^{-1} \varphi_x g_j \in End_{KG}(V)$$

is an endomorphism of trace $c_{x,y} dim(W^G)$. Since V is an absolutely irreducible KG-module, this endomorphism is scalar, hence the right hand side of (1) is

$$c_{x,y} \frac{dim(W^G)}{dim(V)} v_b^{(y)} g_k F_V g_i^{tr} (v_a^{(x)})^{tr} = c_{x,y} \frac{dim(W^G)}{dim(V)} F_V^*(f_{b,j}^{(y)}, f_{a,i}^{(x)}). \qquad \square$$

In Theorem 2.1 the KG-module V was assumed to be uniform. If one drops this assumption, one has to know more about the KH-isometry $\varphi : W \to V$ to identify the invariant form F_V.

Let (W, F_W) be an orthogonal KH-module and (V, F_V) be an irreducible orthogonal KG-module. Let C be the space of symmetric KG-endomorphisms of V,

$$C := \{\varphi \in End_{KG}(V) \mid \varphi \tilde{F}_V = \tilde{F}_V \varphi^*\}$$

and D the space of symmetric KH-endomorphisms of W. Assume that there is a K-linear mapping $\alpha : C \to D$ satisfying $tr(c)/dim_K(V) = tr(\alpha(c))/dim_K(W)$ for all $c \in C$.

Proposition 2.5 *With the notations above let* $\varphi : (W, \alpha(c)F_W) \to (V, cF_V)$ *be a* KH-*isometry for all* $0 \neq c \in C^+$. *Then*

$$(\varphi^G)^* : (V^*, \frac{dim(W^G)}{dim(V)} F_V^*) \to ((W^*)^G, (F_W^*)^G)$$

is a KG-*isometry.*

Proof. The proof of the proposition is analogous to the one of Theorem 2.1. It only remains to show that the statement of Lemma 2.3 holds with the assumption of the proposition. But the assumptions on φ guarantee that for all $c \in C$

$$tr(cF_V g_j^{tr} \varphi^{tr} F_W^{-1} \varphi g_j) = tr(\varphi c F_V \varphi^{tr} F_W^{-1}) = tr(\alpha(c)) = tr(c) \frac{dim(W)}{dim(V)}.$$

Now C is the eigenspace of the mapping $End_{KG}(V) \to End_{KG}(V), \quad \varphi \mapsto \tilde{F}_V \varphi^* \tilde{F}_V^{-1}$, which is orthogonal with respect to the trace bilinear form. Therefore the restriction of the trace bilinear form of the separable algebra $End_{KG}(V)$ to C is non-degenerate, and therefore

$$\sum_{j=1}^{s} F_V g_j^{tr} \varphi^{tr} F_W^{-1} \varphi g_j = \frac{dim(W^G)}{dim(V)} id_V. \qquad \square$$

3 The Specht modules $S^{(n-k,k)}$.

The representation theory of the symmetric group S_n is very well understood (cf. [Jam], [JaK]). The irreducible representations, so called Specht modules S^λ, of S_n over a field of characteristic 0 are in bijection with the partitions λ of n. They have the remarkable property that S^λ occurs with multiplicity one as a submodule of a permutation module M^λ, such that all the other constituents of M^λ belong to partitions that are smaller than λ for a suitable ordering. So the Specht modules are good candidates to apply orthogonal Frobenius reciprocity (Theorem 2.1).

A particular easy construction for S^λ can be given, if the partition λ of n has only two parts. So let $k, l, n \in \mathbb{N}$ with $1 \leq k \leq l \leq \frac{n}{2}$ and let $S_l \times S_{n-l}$ denote the Young subgroup of the symmetric group S_n, which is the set stabiliser of the subset $\{1,\dots,l\}$ of $\{1,\dots,n\}$.

Let $M^{(n-k,k)}$ be the S_n-permutation module having the k-element subsets of $\{1,\dots,n\}$ as an orthonormal $\mathbb{Q}$-basis. Denote the corresponding S_n-invariant symmetric bilinear form by $I_{\binom{n}{k}}$. Then

$$dim_{\mathbb{Q}}(M^{(n-k,k)}) = \binom{n}{k} \text{ and } M^{(n-k,k)} = 1_{S_k \times S_{n-k}}^{S_n}.$$

For a fixed subset $T \subset \{1,\dots,n\}$ let $\sigma_T : M^{(n-k,k)} \to \mathbb{Q}$ be the $\mathbb{Q}$-linear mapping defined by

$$\sigma_T(S) := \begin{cases} 0 & T \not\subseteq S \\ 1 & T \subseteq S \end{cases}$$

for all k-element subsets $S \subset \{1,\dots,n\}$. Then the Specht module $S^{(n-k,k)} \subseteq M^{(n-k,k)}$ is

$$S^{(n-k,k)} = \bigcap_{T \subseteq \{1,\dots,n\}, |T|<k} Ker(\sigma_T).$$

(cf. [Jam, Corollary (17.18)]).

$S^{(n-k,k)}$ is an absolutely irreducible S_n-submodule of $M^{(n-k,k)}$. Therefore the S_n-invariant symmetric bilinear forms on $S^{(n-k,k)}$ are rational multiples of the restriction F_k of $I_{\binom{n}{k}}$.

Young's rule [Jam, (14.4)] says that the $\mathbb{Q}S_n$-module $M^{(n-l,l)}$ is the direct sum of all $S^{(n-k,k)}$ with $k \leq l$. Hence by classical Frobenius reciprocity the fixed space of $S_l \times S_{n-l}$ on $S^{(n-k,k)}$ is one-dimensional, say spanned by some $v \neq 0$. To apply orthogonal Frobenius reciprocity it suffices to calculate the length of v:

Theorem 3.1 *Let $1 \leq k \leq l \leq \frac{n}{2}$. Then there is $v \in S^{(n-k,k)}$ with $vg = v$ for all $g \in S_l \times S_{n-l}$ satisfying*

$$F_k(v,v) = a(l,k) := \binom{n+1-k}{k}\binom{n-l}{k}\binom{l}{k}^{-1}.$$

To prove this theorem we need 2 lemmata on binomial coefficients.

Lemma 3.2

$$\sum_{j=0}^{k} \binom{l-k+j}{j}\binom{n-l-j}{k-j} = \binom{n+1-k}{k}.$$

Proof. First assume $l = k$. Then the left hand side ist

$$\sum_{j=0}^{k} \binom{n-k-j}{k-j} = \sum_{j=0}^{k} \binom{n-2k+j}{j} = \binom{n-2k+k+1}{k} = \binom{n+1-k}{k}.$$

To show the statement in the general case let $d(l, l+1)$ denote the difference of the left hand sides for l and $l+1$. Then

$$d(l, l+1) = \sum_{j=0}^{k} \binom{l-k+j}{j}\binom{n-l-j}{k-j} - \binom{l-k+j+1}{j}\binom{n-l-j-1}{k-j} =$$

$$\sum_{j=0}^{k} \binom{n-l-j}{k-j}(\binom{l-k+j}{j} - \binom{l-k+j+1}{j}) + \sum_{j=0}^{k} \binom{l-k+j+1}{j}\binom{n-l-j-1}{k-j-1}$$

since $\binom{n-l-j-1}{k-j} = \binom{n-l-j}{k-j} - \binom{n-l-j-1}{k-j-1}$. The difference in brackets is $\binom{l-k+j}{j} - \binom{l-k+j+1}{j} = -\binom{l-k+j}{j-1}$.

If one substitutes the summation index $i = j+1$ in the second sum, one finds

$$d(l, l+1) = -\sum_{j=0}^{k} \binom{n-l-j}{k-j}\binom{l-k+j}{j-1} + \sum_{i=1}^{k+1} \binom{l-k+i}{i-1}\binom{n-l-i}{k-i} = 0.$$

Hence the left hand side is independent of l and the lemma follows. □

Lemma 3.3 *Let $k \leq l \leq \frac{n}{2}$. For $0 \leq i \leq k$ define*

$$a_i := (-1)^i \prod_{j=0}^{i-1} \frac{n-l-k+j+1}{l-j}.$$

Then

$$\sum_{i=x}^{x+b} a_i \binom{n-l-k+x+b}{x+b-i}\binom{l-x}{i-x} = 0 \textit{ for all } 0 \leq x \leq x+b \leq k.$$

Proof. Let $A := \prod_{j=1}^{x+b}(n-l-k+j)$ be the product of the numerator of the first binomial coefficient with the numerator of a_i and let $B := \prod_{j=0}^{x-1}(l-j)^{-1}$ be quotient of the numerator of the second binomial coefficient with the denominator of a_i. Then the sum in the lemma simplifies to

$$BA \sum_{i=x}^{x+b} (-1)^i \frac{1}{(i-x)!(b+x-i)!} = \frac{BA}{b!}(-1)^x \sum_{j=0}^{b} (-1)^j \frac{b!}{j!(b-j)!} = 0. \qquad \square$$

Proof of Theorem 3.1:

The orbits of $S_l \times S_{n-l} = Stab_{S_n}(\{1, \ldots, l\})$ on the k-element subsets T of $\{1, \ldots, n\}$ are parametrised by $|T \cap \{1, \ldots, l\}|$. For $0 \leq j \leq k$ let $v_j \in M^{(n-k,k)}$ be the sum over all k-element subsets of $\{1, \ldots, n\}$ that intersect $\{1, \ldots, l\}$ in j elements. Then $(v_0, \ldots, v_k)$ is a basis of the fixed space of $S_l \times S_{n-l}$ on $M^{(n-k,k)}$. For the standard scalar product one finds

$$I_{\binom{n}{k}}(v_i, v_j) = \delta_{ij} \binom{n-l}{k-j}\binom{l}{j}.$$

Let a_i be as in Lemma 3.3, T be a $(k-b)$-element subset of $\{1,\ldots,n\}$ and $x := |T \cap \{1,\ldots,l\}|$. Then

$$\sigma_T(\sum_{i=0}^{k} a_i v_i) = \sum_{i=x}^{x+b} a_i \binom{n-l-k+x+b}{x+b-i}\binom{l-x}{i-x} = 0$$

by Lemma 3.3. Therefore

$$\sum_{i=0}^{k} a_i v_i \in S^{(n-k,k)}.$$

Now

$$a_i = (-1)^i \prod_{j=0}^{i-1} \frac{n-l-k+j+1}{l-j} = (-1)^i \frac{(n-l-(k-i))!}{(n-l-k)!}\frac{(l-i)!}{l!}.$$

Substituting $j = k-i$, the length of $\sum_{i=0}^{k} a_i v_i$ becomes

$$\sum_{j=0}^{k} a_{k-j}^2 \binom{n-l}{j}\binom{l}{k-j} =$$

$$\sum_{j=0}^{k} \frac{(n-l)!l!((l-k+j)!)^2((n-l-j)!)^2}{j!(n-l-j)!(k-j)!(l-k+j)!(l!)^2((n-l-k)!)^2}$$

$$= \frac{(n-l)!}{l!(n-l-k)!}\sum_{j=0}^{k}\frac{(l-k+j)!(n-l-j)!}{j!(k-j)!(n-l-k)!} =$$

$$\frac{(n-l)!(l-k)!}{l!(n-l-k)!}\sum_{j=0}^{k}\binom{n-l-j}{k-j}\binom{l-k+j}{j}.$$

By Lemma 3.2 this equals $a(l,k)$. □

Orthogonal Frobenius reciprocity (Theorem 2.1) now allows to deduce from Theorem 3.1 the following recursion formula for the rational isometry class of F_k.

Corollary 3.4 *For $0 \le l, k \le \frac{n}{2}$ and $0 \neq a \in \mathbb{Q}$ let $[aF_k]$ respectively $[I_{\binom{n}{l}}]$ denote the class of aF_k respectively $I_{\binom{n}{l}}$ in the Witt group $W(\mathbb{Q})$. Then*

$$[I_{\binom{n}{l}}] = \sum_{k=0}^{l}[\frac{\binom{n}{l}}{\binom{n}{k}-\binom{n}{k-1}}a(l,k)F_k] = \sum_{k=0}^{l}[\binom{n-2k}{l-k}F_k]$$

where $a(l,k)$ is as in Theorem 3.1.

Remark 3.5 *In principle this method can also be used to obtain the rational isometry classes of the other irreducible orthogonal S_n-modules using Theorem 2.4 (see [Neb, Section 2.3]). However the combinatorics to determine explicit bases for the fixed space of the corresponding Young subgroups on these modules gets much more involved so one cannot hope to get formulas for arbitrary n.*

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