

AUTOMORPHISMS OF MODULAR LATTICES.

GABRIELE NEBE

ABSTRACT. The methods to classify extremal unimodular lattices with given automorphisms are extended to the situation of modular lattices. A slightly more general notion than the type from the PhD thesis [11] is the det-type. The det-type of an automorphism on L determines the one of all partial dual lattices of L . This easy observation allows to exclude quite a few det-types of automorphisms left open in [11]. Passing to suitable p -maximal lattices, extremal ℓ -modular lattices of composite level $\ell = 14$ and $\ell = 15$ of dimension 12 and the ones of level $\ell = 6$ and dimension 16 are classified.

MSC: 11H56, 52C17, 11H50

1. INTRODUCTION

The study of densest sphere packings in Euclidean space is a classical mathematical problem. Whereas trivial in dimension 1 and easy in dimension 2 this is already a very hard problem in dimension 3, known as the Kepler problem. The (uncountable many) densest packings in Euclidean 3-space have been classified only 15 years ago by Thomas Hales [10]. The sphere-packing problem becomes much easier if one restricts to regular (or lattice) packings, where the centers of the spheres form a finitely generated subgroup of Euclidean space, a so called *lattice*. The densest lattices are known up to dimension 8 and in dimension 24 (see [4]). The papers [35] and [5] prove that the lattice packings in dimension 8 and 24 realize in fact the densest sphere packings in their respective dimensions. The underlying lattices, the famous E_8 lattice and the Leech lattice Λ_{24} , are extremal even unimodular lattices. Even unimodular lattices are positive definite regular integral quadratic forms. They exist only when the dimension is a multiple of 8. A full classification is known in dimension 8, 16 and 24, see Section 2.3 below. The theory of modular forms allows to show that the minimum of an even unimodular lattice of dimension n is bounded from above by $2 + 2\lfloor \frac{n}{24} \rfloor$, lattices achieving equality are called extremal. As the minimum of a unimodular lattice determines its density, the extremal lattices are the densest even unimodular lattices in their dimension. In the jump dimensions, the multiples of 24, one knows only 6 extremal lattices, the Leech lattice Λ_{24} , the unique 24-dimensional unimodular lattice of minimum 4, four lattices, P_{48q} , P_{48p} [6], P_{48n} [21], P_{48m} [18], in dimension 48 and one lattice, Γ_{72} [23], of dimension 72. In [17] the

author started a long term project to classify extremal lattices with given symmetry which also led to the discovery of P_{48m} . The thesis [11] applied the techniques from [17] to the more general situation of extremal ℓ -modular lattices (see Definition 2.2).

The present paper aims to give a few easy more general techniques for the use of automorphisms to classify such extremal lattices. We try to be very brief and not to overload the paper with definitions. The interested reader is referred to the textbooks [15] (for more geometric properties of lattices), [7] (for the relations between lattices and modular forms), [14] and [27] (for the arithmetic theory of quadratic forms) and also the famous collection [6]. The most important notions are given in Section 2, which also lists the current state of knowledge on extremal lattices in Section 2.3. One highlight of this paper is Section 4 which does not rely on any automorphism. The easy idea to pass to a maximal lattice at a suitable prime divisor of ℓ allows to classify certain extremal ℓ -modular lattices of composite level ℓ in a few minutes computer calculations, thus providing new complete classifications to Section 2.3.

The main notion to deal with automorphisms is the one of the *type* of an automorphism σ of prime order p of the lattice L introduced in [17] (see Section 5.1). It is independent of the quadratic form on L and determines the $\mathbb{Z}_p[\sigma]$ -module structure of $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$.

A finer information than the type is the *det-type*, introduced in [11], where it is called type. The det-type of a lattice determines the det-type of all its partial dual lattices (Theorem 5.9). For extremal strongly ℓ -modular lattices all these partial dual lattices are again extremal. This observation allows to exclude quite a few possibilities for det-types of automorphisms that are left open in [11], see Section 8 for examples. To illustrate the new methods we apply them to the classification of 11-modular lattices of dimension 14 in Section 7 and to the important open problem of the existence of an extremal 3-modular lattice of dimension 36 in Section 8. Similar computations (partly already in [11]) could be done in many of the other open cases marked by a “?” in the table of known extremal modular lattices of dimension ≤ 48 .

2. LATTICES

Throughout this paper let (V, Q) be a positive definite rational quadratic space of dimension n . So $Q : V \rightarrow \mathbb{Q}$ is a rational quadratic form such that $Q(v) \geq 0$ for all $v \in V$ with equality if and only if $v = 0$. The associated bilinear form will be denoted by (\cdot, \cdot) , so

$$(x, y) = Q(x + y) - Q(x) - Q(y) \text{ for all } x, y \in V.$$

A *lattice* L in V is a finitely generated subgroup of V of full rank. As \mathbb{Z} is a principal ideal domain, there hence exists a basis $B := (b_1, \dots, b_n)$

of V such

$$L = \left\{ \sum_{i=1}^n a_i b_i \mid a_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n \right\}.$$

The basis B is then called a lattice basis of L and its Gram matrix $((b_i, b_j))_{1 \leq i, j \leq n}$ a Gram matrix of L . The *determinant* of L is the determinant of any of its Gram matrices and the *minimum* of L is

$$\min(L) := \min\{(\lambda, \lambda) \mid 0 \neq \lambda \in L\}.$$

The lattice L is called *even*, if $Q(\lambda) = \frac{1}{2}(\lambda, \lambda) \in \mathbb{Z}$ for all $\lambda \in L$. The *dual lattice* of L is

$$L^\# := \{v \in V \mid (v, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in L\}.$$

The dual basis of any lattice basis of L is a lattice basis of $L^\#$. The *level* ℓ of an even lattice L is the minimal natural number ℓ such that the rescaled dual lattice

$${}^{(\ell)}L^\# := (L^\#, \ell(\cdot, \cdot))$$

is an even lattice. In particular even lattices of level 1 are precisely the even unimodular lattices, they satisfy $L = L^\#$.

The following result is well known (see for instance [15, Proposition 1.3.4]):

Proposition 2.1. *Let L be a lattice in V , $U \leq V$ and $\pi_U \in \text{End}(V)$ the orthogonal projection onto U (with kernel U^\perp). Put $L_U := L \cap U$. Then L_U is a lattice in U with $(L_U)^\# = \pi_U(L^\#)$.*

2.1. Genera of lattices and the mass formula. Two lattices L, M in (V, Q) are called *isometric*, if there is an orthogonal mapping

$$\sigma \in O(V, Q) := \{\sigma \in \text{GL}(V) \mid Q(\sigma(x)) = Q(x) \text{ for all } x \in V\}$$

with $\sigma(L) = M$. The stabilizer of L in $O(V, Q)$ is called the *automorphism group* $\text{Aut}(L)$. This is always a finite unimodular group.

The lattices L and M in V are said to be *in the same genus*, if the p -adic completions $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ and $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$ are isometric for all primes p . A genus of lattices contains only finitely many isometry classes; if L_1, \dots, L_h represent these classes then

$$\sum_{i=1}^h |\text{Aut}(L_i)|^{-1}$$

is called the *mass of the genus*. This rational number can be calculated directly from the local invariants of the genus, without knowing the L_i . As $|\text{Aut}(L_i)| \geq 2$ for all i , the class number, h , is always at least twice the mass of the genus. For more details we refer to [27, Section 102] or [14, Chapter VII-X]. The latter reference, [14, Section 28], also describes an algorithm, the Kneser-neighbor-algorithm, that is used to enumerate all isometry classes of lattices in a genus. This algorithm is available in MAGMA [3].

An even lattice L is said to be ℓ -*elementary* if the exponent of $L^\# / L$ is equal to the square free number ℓ . When ℓ is odd, this is equivalent to having level ℓ . It is well known (see [6, Chapter 15, Theorem 13]) that for an odd prime p there is at most one genus of even p -elementary lattices of a given dimension and determinant.

2.2. Extremal strongly modular lattices. Let ℓ be a square-free integer and let L be an even lattice of level ℓ . For a divisor d of ℓ , the partial dual lattice, $L^{\#,d}$ is defined as $L^{\#,d} := L^\# \cap \frac{1}{d}L$.

The lattice L is called *strongly ℓ -modular*, if L is isometric to all its rescaled partial dual lattices, i.e. for all $d \mid \ell$ we have $L \cong {}^{(d)}L^{\#,d}$.

Note that strongly ℓ -modular lattices are ℓ -elementary lattices. Not all lattices in the genus of an ℓ -modular lattice are again ℓ -modular. However, being ℓ -elementary is a local property, so all the lattices in the genus of a strongly ℓ -modular lattice are again ℓ -elementary.

The notion of strongly ℓ -modular lattices generalizes the one of p -modular lattices (p a prime) from [28] and was introduced in [29]. In this paper Quebbemann shows the following generalization of Siegel's [34] result for unimodular lattices.

Theorem 2.2. [29] *Let ℓ be a square-free integer such that the divisor sum $\sigma_1(\ell) := \sum_{d \mid \ell} d$ divides 24 and let L be a strongly ℓ -modular lattice of dimension n . Then*

$$\min(L) \leq 2 + 2 \lfloor \frac{n\sigma_1(\ell)}{24\sigma_0(\ell)} \rfloor$$

where $\sigma_0(\ell) := \sum_{d \mid \ell} 1$ is the number of divisors of ℓ . Strongly ℓ -modular lattices achieving equality are called *extremal*.

As $\sigma_1(\ell)$ divides 24, the number $J(\ell) := \frac{24\sigma_0(\ell)}{\sigma_1(\ell)}$ is an integer. The extremal ℓ -modular lattices where the dimension n is a multiple of $J(\ell)$ (these dimensions are also called the *jump dimensions*) are of particular interest, as their minimum is strictly bigger than the one of smaller dimensional ℓ -modular lattices.

For a survey of the relation between lattices, modular forms and spherical designs the reader might want to consider my two articles [22] and [24].

2.3. The known extremal lattices. As mentioned in the introduction, even unimodular lattices only exist in dimensions a multiple of 8. Up to dimension 24 all even unimodular lattices are classified, for higher dimensions, the mass formula ([32], [33], see also [6, Chapter 16]) gives a lower bound for the number of such lattices. Oliver King [12] refined this mass formula to count lattices of minimum > 2 , which improves these bounds and also provides lower bounds on the number of extremal lattices in dimension 32. For dimension 40-80 we applied the Minkowski-Siegel mass formula to obtain the rough lower bounds in the table below. In the jump dimensions, the multiples of $J(1) = 24$,

we list the number of known extremal lattices. In the other dimensions the symbol \exists indicates that there are explicit extremal lattices known.

n	8	16	24	32	40	48	56	64	72	80
$\min(L)$	2	2	4	4	4	6	6	6	8	8
$\#$	1	2	$24 \geq 10^9$	$\geq 10^{51}$	$\geq 10^{121}$	$\geq 10^{219}$	$\geq 10^{346}$	$\geq 10^{506}$	$\geq 10^{700}$	
$\# \text{ ext}$	1	2	$1 \geq 10^7$	\exists	≥ 4	\exists	\exists	≥ 1	\exists	

The following table displays the known classification of strongly ℓ -modular lattices for the relevant values of $\ell \geq 2$. The entries in the table either display the exact number of all extremal lattices or a lower bound. A “-” sign indicates that the extremal modular form has a negative coefficient, so no such extremal lattice exists. If an entry is empty, then there are no strongly ℓ -modular lattices of the given dimension. We also use different colours to indicate the extremal minimum, $\text{min}=2,10$, $\text{min}=4,12$, $\text{min}=6,14$, $\text{min}=8,16$ where, of course, the minimum increases downwards in a column.

Table of known extremal even strongly modular lattices of levels $\ell = 2, 3, 5, 6, 7, 11, 14, 15, 23$ and dimension ≤ 48 .

$J(\ell)$	16	12	8	8	6	4	4	4	2
ℓ	2	3	5	6	7	11	14	15	23
2		1			1	1			1
4	1	1	1	1	1	1	1	1	1
6		1			1	1			-
8	1	2	1	1	1	1	1	2	-
10		3			4	2			-
12	3	1	4	10	0	0*	1	3	-
14		1			1	?a			-
16	1	6	1*	8	≥ 18	-	?	≥ 1	-
18		37			0*	?			-
20	3*	≥ 100	$\geq 97a$	≥ 13	$\geq 1a$	-	?	?	-
22		$\geq 10^3$?a	?			-
24	≥ 8	$\geq 1a$	$\geq 1a$	≥ 2	0*	-	?	?	-
26		≥ 6			?	-			-
28	≥ 24	≥ 9	≥ 1	?	?	-	?	?	-
30		≥ 2			-	-			-
32	$\geq 7a$	≥ 33	?	?	?	-	-	-	-
34		≥ 100			?	-			-
36	≥ 3	?a	?	?	-	-	-	-	-
38		?			?	-			-
40	≥ 6	≥ 1	?	?	?	-	-	-	-
42		?			-	-			-
44	≥ 1	?	?	?	-	-	-	-	-
46		?			?	-			-
48	≥ 6	?	?	?	-	-	-	-	-

This table is available in the Catalogue of Lattices [25].

The complete classification results in this table are mostly obtained by a complete enumeration of the strongly ℓ -modular genus (see [30], [31]). For certain medium size dimensions, the class number of this genus is far too high to enumerate all lattices. Nevertheless a clever application of modular forms (see [2]) allows to construct all extremal lattices in the cases marked by *. An “a” indicates that [11] gives restrictions on the order of the automorphism group of such lattices. For more precise results see below and the thesis [11].

The classification of the extremal strongly 6-modular lattices of dimension 16 and the one of the extremal strongly 14- and 15-modular lattices of dimension 12 is described in Section 4 below. In particular the classification of extremal strongly ℓ -modular lattices is complete up to dimension m for (ℓ, m) as in the following table:

ℓ	1	2	3	5	6	7	11	14	15	23
m	24	20	18	16	16	14	12	12	12	∞

In the jump dimensions there is a complete classification of all extremal ℓ -modular lattices for $\ell = 7, 11$, and 23.

3. MAXIMAL LATTICES

A lattice L in (V, Q) is called *maximal* if L is even (i.e. $Q(L) \subseteq \mathbb{Z}$) and L is maximal with this property, i.e. all proper overlattices M of L satisfy $Q(M) \not\subseteq \mathbb{Z}$. It is well known (see [27, Section 82H]) that the set of maximal lattices in (V, Q) forms a single genus of lattices. In general this maximal genus has the smallest possible mass among all genera of lattices in (V, Q) , and one strategy to find all lattices in a given genus is to construct them as sublattices of the ones in the maximal genus. A related strategy is applied in Section 4 below.

Corresponding maximality questions in presence of a finite group G are treated in [16]. In particular [16, Lemma (1.2)] shows that a maximal even G -lattice (M, Q) in (V, Q) has a semisimple anisotropic discriminant group, so $M^\# / M$ is a direct sum of simple $\mathbb{Z}G$ -modules and $\overline{Q} : M^\# / M \rightarrow \mathbb{Q} / \mathbb{Z}$ defined by $\overline{Q}(v + M) := Q(v) + \mathbb{Z}$ does not vanish on any of the non-zero submodules of $M^\# / M$. If G is a p -group then the simple $\mathbb{F}_p G$ -modules are trivial, so in this case $(\mathbb{F}_p \otimes_{\mathbb{Z}} M^\# / M, \overline{Q})$ is an anisotropic quadratic \mathbb{F}_p -space allowing to conclude the following theorem.

Theorem 3.1. *Let L be an even lattice and $\sigma \in \text{Aut}(L)$ of prime order p . Then there is a σ -invariant overlattice M containing L of p -power index such that $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$ is a maximal lattice in $\mathbb{Q}_p \otimes_{\mathbb{Q}} V$.*

4. STRONGLY MODULAR LATTICES OF COMPOSITE LEVEL

To classify extremal strongly ℓ -modular lattices L of composite level $\ell = 6, 14, 15$ in medium size dimensions one may use the following strategy. If the genus containing the strongly ℓ -modular lattices is too

big to be enumerated but the classification of p -modular lattices M is known for some p dividing ℓ , then one may construct L as a sublattice of ℓ/p -power index in M as illustrated in this section.

Theorem 4.1. *There are exactly three extremal even strongly 15-modular lattices of dimension 12.*

Proof. Let L be an even 15-elementary lattice of determinant 15^6 in dimension 12 such that $\min(L) = 8$ and $\min(L^{\#,3}) = 8/3$. Then there is a 5-elementary lattice M of determinant 5^6 such that $L \leq M = M^{\#,3} \leq L^{\#,3}$. In particular $\min(M)$ is an even number $\geq 8/3$ so $\min(M) \geq 4$. There are 4 such lattices M , all are extremal 5-modular lattices. Successively computing 15-elementary sublattices X of index 3^i , $i = 1, \dots, 3$ of these 4 lattices M , satisfying $\min(X^{\#,3}) \geq 8/3$ we finally obtain three such lattices L . \square

Remark 4.2. More precisely we checked that these three extremal even strongly 15-modular lattices of dimension 12 are the only even 15-elementary lattices of minimum 8 and dimension 12 of determinant 15^6 such that the 3-dual has minimum $\geq 8/3$.

Theorem 4.3. *There is a unique extremal strongly 14-modular lattice in dimension 12.*

Proof. Let L be such an extremal lattice. Then $\min(L) = 8$ and hence $\min(L^{\#,2}) = 4$ and there is a lattice M in the genus of even 7-modular lattices with $L \subseteq M = M^{\#,2} \subseteq L^{\#,2}$, in particular $\min(M) \geq 4$. The genus of M has class number 395 ([30], [3]) and contains no lattice with minimum 6 (which would be extremal) and 49 lattices M of minimum 4 such that also the rescaled dual lattice ${}^{(7)}M^\#$ has minimum 4. Successively computing sublattices X of 2-power index in one of these 49 lattices M such that $\min(X^{\#,2}) \geq 4$ one finally reaches a unique strongly 14-modular extremal L . \square

Theorem 4.4. *There are exactly 8 extremal even strongly 6-modular lattices in dimension 16.*

Proof. Let L be such an extremal lattice. Then $\min(L) = 6$ and hence $\min(L^{\#,2}) = 3$ and there is a lattice M in the genus of even 3-modular lattices with $L \subseteq M = M^{\#,2} \subseteq L^{\#,2}$ with $\min(M) \geq 4$. These lattices M are hence extremal and there are 6 such lattices. Successively computing sublattices X of 2-power index in one of these 6 lattices M such that $\min(X^{\#,2}) \geq 3$ one finally ends with 8 isometry classes of strongly 6-modular extremal lattices L . \square

5. AUTOMORPHISMS OF PRIME ORDER

5.1. The type of an automorphism. Let L be a lattice and $\sigma \in \text{Aut}(L)$ an automorphism of L of prime order p . As $\sigma^p = 1$ the elements $e_1 := \frac{1}{p}(1 + \sigma + \dots + \sigma^{p-1})$ and $e_\zeta := 1 - e_1$ are orthogonal idempotents

in the endomorphism ring of the $\mathbb{Q}\langle\sigma\rangle$ -module $V := \mathbb{Q}L$ giving the σ -invariant decomposition

$V = Ve_1 \oplus Ve_\zeta = V_1 \oplus V_\zeta$ of dimensions $n_1 := \dim(V_1)$, $n_\zeta := \dim(V_\zeta)$ such that the restriction of σ to V_1 is the identity and the restriction of σ to V_ζ has minimal polynomial $\Phi_p := X^{p-1} + X^{p-2} + \dots + X + 1$. In particular V_ζ is a vector space over $\mathbb{Q}[\zeta_p]$, so $n_\zeta = r_\zeta(p-1)$ is divisible by $p-1$.

Put $L_1 := L \cap V_1 := \{\lambda \in L \mid \lambda\sigma = \lambda\}$ the fixed lattice of σ in L and

$$L_\zeta := L \cap V_\zeta = \{\lambda \in L \mid (\lambda, \lambda_1) = 0 \text{ for all } \lambda_1 \in L_1\},$$

its orthogonal lattice.

As $Le_1 = \pi_{V_1}(L)$ and $Le_\zeta = \pi_{V_\zeta}(L)$ Proposition 2.1 yields the following corollary.

Corollary 5.1. $L^\#e_1 = (L_1)^\#$ and $L^\#e_\zeta = (L_\zeta)^\#$.

The basis of the definition of the type is given in the following lemma (see for instance [17], [11]).

Lemma 5.2. *With the notation above we have*

$$pLe_1 \perp Le_\zeta(1 - \sigma) \subseteq L_1 \perp L_\zeta \subseteq L \subseteq Le_1 \perp Le_\zeta$$

and L is a full subdirect product of Le_1 and Le_ζ in particular $Le_1/L_1 \cong Le_\zeta/L_\zeta$ as $\mathbb{F}_p\langle\sigma\rangle$ -modules. Moreover the integer

$$s := \dim_{\mathbb{F}_p}(Le_1/L_1) = \dim_{\mathbb{F}_p}(Le_\zeta/L_\zeta)$$

satisfies $s \leq \min(n_1, r_\zeta)$

Proof. Let $R = \mathbb{Z}_p[\sigma] \cong \mathbb{Z}_p C_p \cong \mathbb{Z}_p[X]/(X^p - 1)$ be the group ring of the cyclic group C_p of prime order p over the ring of p -adic integers \mathbb{Z}_p . Then the indecomposable R -lattices are the free R -module R , the trivial R -lattice \mathbb{Z}_p and the lattice $\mathbb{Z}_p[\zeta_p]$ in the irreducible faithful \mathbb{Q}_p -representation of C_p .

By the theorem of Krull-Schmidt, the R -lattice $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ is isomorphic to a unique direct sum of indecomposable lattices:

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} L \cong R^{s'} \oplus \mathbb{Z}_p[\zeta_p]^r \oplus \mathbb{Z}_p^t.$$

The lattice $\mathbb{Z}_p \otimes_{\mathbb{Z}} L_1$ is then a sublattice of dimension $s' + t$ of $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ and

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} Le_1 / \mathbb{Z}_p \otimes_{\mathbb{Z}} L_1 \cong \mathbb{Z}_p / p\mathbb{Z}_p^{s'} \cong Le_1/L_1 \cong \mathbb{F}_p^s,$$

so $s = s'$ and $n_1 = s + t \geq s$. Similarly $r_\zeta = r + s \geq s$. \square

Definition 5.3. *The tuple $p - (r_\zeta, n_1) - s$ is called the type associated to (L, σ) .*

From the proof of Lemma 5.2 we obtain the following corollary (see also [11, Section 4.1 and 4.2], in particular [11, Proposition 4.1.8] for part (d)).

Corollary 5.4. *Let $p - (r_\zeta, n_1) - s$ be the type of (L, σ) .*

- (a) $pLe_1 \subseteq L_1$ and $Le_\zeta(1 - \sigma) \subseteq L_\zeta$.
- (b) If $s = n_1$ then $pLe_1 = L_1$.
- (c) If $s = r_\zeta$ then $Le_\zeta(1 - \sigma) = L_\zeta$.
- (d) If p does not divide $\det(L)$ then $s \equiv r_\zeta \pmod{2}$.

Remark 5.5. The type determines the isomorphism class of L as a $\mathbb{Z}_p[\sigma]$ -module. In particular the type of L is the same as the one of any of its σ -invariant sublattices of index prime to p ; and if L is an even lattice such that p does not divide its determinant, then the types of (L, σ) and $(L^\#, \sigma)$ are the same.

The last sentence of the previous remark also holds if p divides the determinant of L :

Lemma 5.6. *The type of (L, σ) equals the type of $(L^\#, \sigma)$ and also to the type of all its partial dual lattices.*

Proof. The σ -invariant quadratic form identifies the $L^\#$ with $\text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ and also $\mathbb{Z}_p \otimes L^\#$ with the dual module $\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes L, \mathbb{Z}_p)$. As the indecomposable direct summands of $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ are self-dual, hence isomorphic to their dual as a $\mathbb{Z}_p[\sigma]$ -module, one sees that $\mathbb{Z}_p \otimes_{\mathbb{Z}} L \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} L^\#$ as a $\mathbb{Z}_p[\sigma]$ -module. For the partial dual lattice $L^{\#,d}$ it is enough to note that

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} L^{\#,d} = \begin{cases} \mathbb{Z}_p \otimes_{\mathbb{Z}} L^\# & \text{if } p \mid d \\ \mathbb{Z}_p \otimes_{\mathbb{Z}} L & \text{otherwise.} \end{cases}$$

□

5.2. The det-type of an automorphism. Whereas the type of an automorphism determines the $\mathbb{Z}_p[\sigma]$ -module structure of $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$, the det-type depends on the $\mathbb{F}_q[\sigma]$ -module structure of the Sylow- q -subgroup $L^{\#,q}/L$ of the discriminant group $L^\#/L$. Here we assume that L is an even lattice of square-free level ℓ . For all prime divisors q of ℓ with $p \neq q$ the Sylow- q -subgroup of the discriminant groups of L and its sublattice of p -power index $L_1 \perp L_\zeta$ coincide. So

$$L^{\#,q}/L \cong (L^{\#,q})_1/L_1 \perp (L^{\#,q})_\zeta/L_\zeta \cong (L_1)^{\#,q}/L_1 \perp (L_\zeta)^{\#,q}/L_\zeta$$

and we put $d_1(q) := \dim_{\mathbb{F}_q}((L_1)^{\#,q}/L_1)$ and $d_\zeta(q) := \dim_{\mathbb{F}_q}((L_\zeta)^{\#,q}/L_\zeta)$.

The irreducible factors of $\Phi_p \in \mathbb{F}_q[X]$ are of degree $o_p(q)$, the order of q in \mathbb{F}_p^* . As $(L_\zeta)^{\#,q}/L_\zeta$ is a self-dual $\mathbb{F}_q[\sigma]$ -module we find that

Remark 5.7. The least common multiple of 2 and $o_p(q)$ divides $d_\zeta(q)$.

Definition 5.8. *Let σ be an automorphism of order p of an even lattice L of square-free level ℓ . Let $\ell = q_1 \cdots q_r$ be the prime factorization of ℓ and put $d_1(q_i) := \dim_{\mathbb{F}_{q_i}}((L^{\#,q_i})_1/L_1)$ and $d_\zeta(q_i) := \dim_{\mathbb{F}_{q_i}}((L^{\#,q_i})_\zeta/L_\zeta)$ for $1 \leq i \leq r$. Then the det-type of (L, σ) is*

$$[p - (r_\zeta, n_1) - s, q_1 - (d_\zeta(q_1), d_1(q_1)), \dots, q_r - (d_\zeta(q_r), d_1(q_r))].$$

Theorem 5.9. *Let σ be an automorphism of order p of an even lattice L of square-free level ℓ . If*

$$[p - (r_\zeta, n_1) - s, q_1 - (d_\zeta(q_1), d_1(q_1)), \dots, q_r - (d_\zeta(q_r), d_1(q_r))]$$

is the det-type of (L, σ) then the det-type of $({}^{(\ell)}L^\#, \sigma)$ is

$$[p - (r_\zeta, n_1) - s, q_1 - (n_\zeta - d_\zeta(q_1), n_1 - d_1(q_1)), \dots, q_r - (n_\zeta - d_\zeta(q_r), n_1 - d_1(q_r))].$$

Proof. By Lemma 5.6 the type of (L, σ) and the one of $(L^\#, \sigma)$ are the same. To compute the det-type it is enough to deal with one prime at a time (formally we could just take the tensor product with \mathbb{Z}_q and deal with q -adic lattices). So assume that $\ell = q$ is a prime and put $M := {}^{(q)}L^\#$. Then $M^\# = M^{\#,q} = {}^{(q)}(\frac{1}{q}L)$ and forgetting the quadratic form $(M^\#)_1 = \frac{1}{q}L_1$ and $(M^\#)_\zeta = \frac{1}{q}L_\zeta$. We have

$$\frac{1}{q}L_1 \supseteq (L^\#)_1 \supseteq L_1 \quad \text{and} \quad \frac{1}{q}L_\zeta \supseteq (L^\#)_\zeta \supseteq L_\zeta$$

so the theorem follows from the fact that

$$q^{n_1} = |\frac{1}{q}L_1/L_1| = |\frac{1}{q}L_1/(L^\#)_1| |(L^\#)_1/L_1|$$

and similarly for L_ζ . □

6. AUTOMORPHISMS OF ORDER 2

Of course -1 is an automorphism of any lattice, the *trivial* automorphism of order 2, so it is impossible to exclude automorphisms of order 2. However, non-trivial automorphisms of order 2 usually yield quite restrictive conditions, e.g. for the extremal even unimodular lattices of dimension 48 there is only one possible type of such automorphisms, $2 - (24, 24) - 24$, whereas for automorphisms of order 5 there are three different types occurring in the known lattices. One reason is the following lemma, which is a direct generalization of [17, Lemma 4.9].

Lemma 6.1. *Let M be an even lattice such that $M^\#/M$ has exponent $2d$ with d odd. Then M contains a sublattice N of 2-power index such that ${}^{(1/2)}N =: U$ is an integral lattice and the exponent of $U^\#/U$ is d . Moreover if N is a proper sublattice of M then U can be chosen to be an odd lattice.*

Proof. Since this is a statement about 2-adic lattices, we pass to $M_2 := \mathbb{Z}_2 \otimes_{\mathbb{Z}} M$. This lattice has a Jordan decomposition $M_2 = M_0 \perp M_1$ (see for instance [27, Section 91C], [6, Chapter 15]), where (M_0, Q) is a regular quadratic \mathbb{Z}_2 -lattice of dimension, say, $2m$, and ${}^{(1/2)}M_1$ is a regular bilinear \mathbb{Z}_2 -lattice. If $m = 0$, then $M = N \cong {}^{(2)}U$ for some integral lattice U of odd determinant and we are done. So assume $m \geq 1$. Then M_0 contains vectors v, w such that $(v, w) \in \mathbb{Z}_2^*$ and we

may choose v such that $(v, v) \in 2\mathbb{Z}_2^*$. So $\langle v, w \rangle$ is a regular sublattice of M_0 and hence $M_0 = \langle v, w \rangle \perp N_0$ for some lattice N_0 . Then

$$M_0 \perp M_1 \geq_2 N_0 \perp (\langle v, 2w \rangle \perp M_1) \cong N_0 \perp N_1$$

with $\dim(N_1) = \dim(M_1) + 2$ and $\dim(N_0) = 2(m - 1)$. Note that $(^{1/2})N_1$ is an odd lattice as $1/2(v, v) \in \mathbb{Z}_2^*$. Since (N_0, Q) is again regular, we may proceed by induction, until $N_0 = \{0\}$. Then the sublattice N of the lattice M is constructed as the unique lattice N with $\mathbb{Z}_p \otimes_{\mathbb{Z}} N = \mathbb{Z}_p \otimes_{\mathbb{Z}} M$ for all primes $p > 2$ and $\mathbb{Z}_2 \otimes_{\mathbb{Z}} N = N_1$. \square

If σ is an automorphism of type $2 - (n_{-1}, n_1) - s$ of a lattice L of odd determinant, then $M = L_1(\sigma) \perp L_\zeta(\sigma)$ satisfies the assumption of Lemma 6.1, in particular there is a sublattice $(^2)(U_1 \perp U_\zeta) \leq M$ with $\dim(U_1) = n_1$, $\dim(U_\zeta) = n_{-1}$ such that $\min(U_1) \geq \frac{1}{2} \min(L)$, $\min(U_\zeta) \geq \frac{1}{2} \min(L)$ and $U_1^\# / U_1 \oplus U_\zeta^\# / U_\zeta \cong L^\# / L$ as abelian groups.

7. EXTREMAL 11-MODULAR LATTICES OF DIMENSION 14

The paper [26] uses Siegel modular forms of degree 2 to conclude the non-existence of an even 11-modular lattice of dimension 12 and minimum 8, so the first dimension where an 11-modular lattice of minimum 8 might exist is dimension 14. Applying the strategy of Section 3 and Section 6 we obtain:

Theorem 7.1. *Let L be an extremal 11-modular lattice of dimension 14. Then $\text{Aut}(L) = \{\pm 1\}$ is trivial.*

Proof. Satz 4.2.1 in [11] concludes that the only primes that divide the order of $\text{Aut}(L)$ are 2 and 11.

First we assume that there is $\sigma \in \text{Aut}(L)$ of order 11. Then by Theorem 3.1 there is some maximal lattice M with $\sigma \in \text{Aut}(M)$. As $\det(L) = 11^7$ is an odd power of 11 the determinant of M is 11. There is one genus of even lattices of dimension 14 and determinant 11, its class number is 8 and there are 4 lattices admitting an automorphism of order 11, 2 of which have dual minimum $\geq 8/11$, these have dual minimum $12/11$ and the Sylow 11-subgroup of their automorphism group G is of order 11. For both lattices we take σ to be a generator of the Sylow 11-subgroup of G and compute the maximal σ -invariant sublattices N of M . It turns out to be easier to dualize the picture: Let $D := (^{11})M^\#$ denote the rescaled dual lattice of M and denote the action of σ on $D^\# / D \cong \mathbb{F}_{11}^{13}$ by $X \in \mathbb{F}_{11}^{13 \times 13}$. Then $(^{11})N^\# / D$ is a minimal σ -invariant isotropic subspace of $D^\# / D$. As $(^{11})N^\# / D$ is a simple $\mathbb{F}_{11}\langle\sigma\rangle$ -module, the action of σ on $(^{11})N^\# / D$ is trivial, so $(^{11})N^\# = \langle D, u \rangle$ where $\langle u + D \rangle \leq D^\# / D$ is a one-dimensional subspace in the kernel K of $X - 1 \in \mathbb{F}_{11}^{13 \times 13}$. The normaliser $N_G(\langle\sigma\rangle)$ acts on these one-dimensional subspaces in K , admitting only one orbit represented by an isotropic subspace (in both cases). This orbit corresponds to a

lattice ${}^{(11)}N^\# = \langle D, u \rangle$ of minimum 2. As this is smaller than 8, this gives a contradiction.

It remains to treat automorphisms of 2-power order. We first assume that there is $\sigma \in \text{Aut}(L)$ with $\sigma^2 = -1$. Then σ acts on $L^\# / L \cong \mathbb{F}_{11}^7$ with irreducible minimal polynomial $X^2 + 1 \in \mathbb{F}_{11}[X]$ contradicting the fact that 7 is odd (see also Remark 5.7). To finish the proof we need to exclude non-trivial automorphisms of order 2. Let σ be such an automorphism of det-type $[2 - (n_1, 14 - n_{-1}) - s, 11 - (d_1, 7 - d_1)]$ (with $s \leq \min(n_1, 14 - n_1)$, $s \equiv_2 n_1$). Then the det-type of $({}^{(11)}L^\#, \sigma)$ is $[2 - (n_1, 14 - n_{-1}) - s, 11 - (n_1 - d_1, 7 - n_1 + d_1)]$. Applying the strategy of Section 6 we hence need to classify quadruples of 11-elementary lattices $(U_{-1}, U'_{-1}, U_1, U'_1)$ of minimum ≥ 4 and dimensions $(n_1, n_1, 14 - n_1, 14 - n_1)$ and determinants $(11^{d_1}, 11^{n_1 - d_1}, 11^{7 - d_1}, 11^{7 - n_1 + d_1})$. By symmetry we hence may assume that $n_1 \leq 7$ and $d_1 \leq n_1/2$.

By [6, Chapter 15, Theorem 13] there are always two genera of positive definite odd 11-elementary lattices of given dimension n and determinant 11^d (except for $n \leq 2$ or $n = d$, where there is just one such genus). There is an additional genus of positive definite even lattices if $n = 2k$ is even and $k \equiv d \pmod{2}$.

Enumerating all these genera we find one possible lattice U_{-1} of dimension 4, determinant 11^2 and minimum 4 and three such lattices U_{-1} of dimension 6, determinant 11^3 and minimum 4. For all lattices U_{-1} no proper overlattice of index 2 has minimum 4, so $L_{-1} = {}^{(2)}U_{-1}$ and $s = n_{-1} = \dim(U_{-1})$ in all four cases and L_1 contains ${}^{(2)}U_1$ as a proper sublattice. In particular we may choose U_1 as an odd lattice of dimension 10 respectively 8 and determinant 11^5 respectively 11^4 . The 11-adic lattice $\mathbb{Z}_{11} \otimes_{\mathbb{Z}} ({}^{(2)}U_{-1} \perp {}^{(2)}U_1) = \mathbb{Z}_{11} \otimes_{\mathbb{Z}} L$ then determines the genus of U_1 completely. This allows to enumerate the respective genera for U_1 and to construct all candidates for the lattices L_1 as overlattices of ${}^{(2)}U_1$. The lattice L is a full subdirect product of $L_{-1} \perp L_1$ and is constructed using the gluing strategy described for instance in [13, Remark 2.5] and [18]. The computations are done in the master-thesis [36]; no extremal lattice is found. \square

8. EXTREMAL 3-MODULAR LATTICES OF DIMENSION 36

The existence of an extremal 3-modular lattice of dimension 36 is an interesting open problem. On one hand 36 is the only jump dimension, where the existence of an extremal lattice of minimum 8 is still open. On the other hand such a lattice would yield the densest known sphere packing in dimension 36. The third reason comes from a beautiful observation by B.Gross (see [1, Introduction] for a more detailed explanation): There is a connection between extremal modular lattices

of level 7, 3, and 1 based on the following chain of division algebras

$$E = \mathbb{Q}[\sqrt{-7}] \subseteq Q = \left(\frac{-1, -3}{\mathbb{Q}} \right) \subseteq \mathbf{O}$$

where Q is the definite quaternion algebra over \mathbb{Q} with discriminant 9 and \mathbf{O} is the non-associative Cayley octonion algebra over \mathbb{Q} . All three algebras have a unique conjugacy class of maximal orders and we obtain unique embeddings

$$\mathbb{Z}_E \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{O}$$

of these maximal orders. For a Hermitian unimodular \mathbb{Z}_E -lattice B of dimension m , the trace lattices

$$\mathrm{Tr}(B), \mathrm{Tr}(B \otimes_{\mathbb{Z}_E} \mathcal{M}), \mathrm{Tr}(B \otimes_{\mathbb{Z}_E} \mathcal{O})$$

are \mathbb{Z} -lattices which are 7-modular of dimension $2m$, 3-modular of dimension $4m$ respectively unimodular of dimension $8m$. The sequence of dimensions is compatible to the jump dimensions for the respective p -modular lattices (multiples of 6, 12, resp. 24). For $m = 3$ one obtains the 7-modular Barnes lattice in dimension 6, the 3-modular Coxeter-Todd lattice in dimension 12 and the unimodular Leech lattice of dimension 24. These three lattices are extremal of minimum 4 and they are the densest (known) lattices in their respective dimension. The same construction is applied to a rank 10 lattice in [1] to obtain extremal lattices of minimum 8 in dimension 20, 40 and 80. For $m = 6$ it is known that no extremal 7-modular lattice of dimension 12 exists [30] and no extremal 3-modular \mathbb{Z} -lattice of dimension 24 is the trace lattice of a Hermitian unimodular $\mathbb{Z}[\zeta_3]$ -lattice (see [9]). Also for $m = 9$ the non-existence of an extremal 7-modular lattice in dimension 18 is known [2], but the discovery of a 72-dimensional extremal unimodular lattice in [23] could nevertheless yield the hope to find a shorter sequence of extremal 3-modular and unimodular lattices in dimension 36 and 72.

The dissertation of Michael Jürgens [11] exhibits the possible automorphisms of an extremal 3-modular lattice in dimension 36, whose existence is still open. In particular [11, Section 4.2.3] shows that such an extremal lattice has no automorphisms of order 11, 13, or any prime $p \geq 23$ and specifies a unique possible det-type for automorphisms of order 17 and 19. The paper [13] constructs binary Hermitian lattices over $\mathbb{Z}[\zeta_p]$ for $p = 17$ and $p = 19$ to conclude that such automorphisms do not exist. So we know that the only primes that might divide the order of the automorphism group of an extremal 3-modular lattice of dimension 36 are ≤ 7 .

Proposition 8.1. *Let L be an extremal 3-modular lattice of dimension 36 and let $\sigma \in \mathrm{Aut}(L)$ be an automorphism of order 7. Then $\mu_\sigma = \Phi_7$, i.e. L is a lattice of dimension 6 over $\mathbb{Z}[\zeta_7]$.*

Proof. By [11] the only possible det-types of (L, σ) are

$$[7-(4,12)-4,3-(12,6)], [7-(5,6)-3,3-(12,6)], \\ [7-(5,6)-5,3-(12,6)], \text{ and } [7-(6,0)-0,3-(18,0)].$$

As both lattices, L and ${}^{(3)}L^\#$ are extremal, this allows us to apply Theorem 5.9 to conclude that the second and the third case is not possible. It remains to deal with the first case. Here the lattice L_ζ is in the genus $\text{II}_{24}(3^{+12}7^{+4})$ (see [11]) and σ acts with minimal polynomial Φ_7 on L_ζ , so we may consider L_ζ as a lattice of rank $r_\zeta = 4$ over $\mathbb{Z}[\zeta_7]$. Let M_ζ be a maximal even $\mathbb{Z}[\zeta_7]$ -overlattice of L_ζ . Then M_ζ is an even unimodular lattice of dimension 24 having an automorphism σ with minimal polynomial Φ_7 . There are two such lattices, the Leech lattice Λ_{24} and the lattice with root system A_6^4 . For both lattices there is up to conjugacy a unique automorphism σ with the correct minimal polynomial. So L_ζ is a sublattice of index $p_7 p_3$ of one of these two maximal $\mathbb{Z}[\zeta_7]$ -lattices. We first construct all representatives of the isometry classes in the set of sublattices of index p_7 , there are five such isometry classes of $\mathbb{Z}[\zeta_7]$ -lattices of determinant 7^4 (the mass of the genus of Hermitian lattices is $\frac{395}{345744}$). For all these five lattices we construct all 14, 393, 320 sublattices L_ζ of index p_3 of level 21, without testing isometry. None of them has minimum 8. \square

Corollary 8.2. *The genus $\text{II}_{24}(3^{+12}7^{+4})$ contains no lattice L_ζ of minimum 8 admitting an automorphism σ with minimal polynomial Φ_7 .*

The possible det-types of automorphisms of order 5 of an extremal 3-modular lattice of dimension 36 are listed in [11] as

$$[5-(5,16)-5, 3-(8,10)], [5-(6,12)-6, 3-(8,10)], [5-(6,12)-6, 3-(12,6)], \\ [5-(7,8)-5, 3-(12,6)], [5-(7,8)-7, 3-(12,6)], \text{ and } [5-(8,4)-4, 3-(16,2)].$$

By Theorem 5.9 only the third and the last possibility can occur. A computation excluding the possibilities for the third case hence shows the following theorem.

Theorem 8.3. *Let L be an extremal 3-modular lattice of dimension 36 and $\sigma \in \text{Aut}(L)$ be an automorphism of order 5. Then the det-type of (L, σ) is $[5 - (8, 4) - 4, 3 - (16, 2)]$.*

Proof. As explained above we only need to exclude the possibility that the det-type of (L, σ) is $[5 - (6, 12) - 6, 3 - (12, 6)]$. So assume that we have such an automorphism σ of order 5.

Then the lattice L_ζ is in the genus $\text{II}_{24}(5^{-6}3^{-12})$ (see [11]). To classify the relevant $\mathbb{Z}[\zeta_5]$ -lattices in this genus we first classify the $\mathbb{Z}[\zeta_5]$ -lattices in the genus $\text{II}_{24}(5^{-6}3^{-4})$. The mass of this Hermitian genus is $577524389/405000000$ and its class number is 222. Only 132 of these 222 lattices L have the additional property that $\min(L^{\#,3}) \geq 8/3$. For these 132 lattices we compute the maximal ζ_5 -invariant sublattices of index 3^4 that have minimum 8. There are in total 3 isometry classes of

such lattices M , all satisfying $\min(M^{\#,3}) \geq 8/3$. These three lattices are candidates for L_ζ .

For these lattices we computed all the 3-modular overlattices of $L_\zeta \perp L_1$ where L_1 is one of the three extremal strongly 15-modular lattices from Theorem 4.1. None of the 3-modular overlattices has minimum 8. \square

The paper [19] develops methods to deal with automorphisms of order p of extremal p -modular lattices. These are used in [8] to show that the only known extremal 3-modular lattice of dimension 24 is the only one admitting an automorphism of order 3. Combining this with the results in the master-thesis [36] and [11] we hence have

Theorem 8.4. *Let L be an extremal 3-modular lattice of dimension 24. Then either $L \cong L_{24,2}$ discovered in [20] (see also [21]) with $\text{Aut}(L_{24,2}) \cong \text{SL}_2(13)\text{YSL}_2(3)$ or the automorphism group of L is a 2-group containing no non-trivial automorphism of order 2.*

The methods of [19] are also used to study extremal 3-modular lattices of dimension 36 with an automorphism of order 3. Unpublished computations of the author allow to show that if there is such a lattice with an automorphism of order 3, then this automorphism has type $[3 - (18, 0) - 0]$, $[3 - (12, 12) - 12]$, or $[3 - (12, 12) - 10]$.

The last theorem summarizes our present knowledge about automorphisms of a possible extremal 3-modular lattice in dimension 36:

Theorem 8.5. *Let σ be an automorphisms of prime order of an extremal 3-modular lattice L of dimension 36. Then the order of σ is 2, 3, 5, or 7. If σ has order 7 then it acts fixed point freely, i.e. with minimal polynomial Φ_7 . If the order of σ is 5, then the det-type of (L, σ) is $[5 - (8, 4) - 4, 3 - (16, 2)]$. If σ has order 3 then it has type $[3 - (18, 0) - 0]$, $[3 - (12, 12) - 12]$, or $[3 - (12, 12) - 10]$.*

REFERENCES

- [1] Christine Bachoc and Gabriele Nebe. Extremal lattices of minimum 8 related to the Mathieu group M_{22} . *J. Reine Angew. Math.*, 494:155–171, 1998. Dedicated to Martin Kneser on the occasion of his 70th birthday.
- [2] Christine Bachoc and Boris Venkov. Modular forms, lattices and spherical designs. In *Réseaux euclidiens, designs sphériques et formes modulaires*, volume 37 of *Monogr. Enseign. Math.*, pages 87–111. Enseignement Math., Geneva, 2001.
- [3] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.
- [4] Henry Cohn and Abhinav Kumar. Optimality and uniqueness of the Leech lattice among lattices. *Ann. of Math. (2)*, 170(3):1003–1050, 2009.
- [5] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska. The sphere packing problem in dimension 24. *Ann. of Math. (2)*, 185(3):1017–1033, 2017.

- [6] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1988. With contributions by E. Bannai, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.
- [7] Wolfgang Ebeling. *Lattices and codes*. Advanced Lectures in Mathematics. Springer Spektrum, Wiesbaden, third edition, 2013. A course partially based on lectures by Friedrich Hirzebruch.
- [8] Simon Eisenbarth. Gitter und Codes über Kettenringen. *Thesis, RWTH Aachen University*, 2020.
- [9] Walter Feit. Some lattices over $\mathbf{Q}(\sqrt{-3})$. *J. Algebra*, 52(1):248–263, 1978.
- [10] Thomas C. Hales. A proof of the Kepler conjecture. *Ann. of Math. (2)*, 162(3):1065–1185, 2005.
- [11] Michael Jürgens. Nicht-Existenz und Konstruktion extremaler Gitter. *Dissertation, TU Dortmund*, <http://dx.doi.org/10.17877/DE290R-7882>, 2015.
- [12] Oliver D. King. A mass formula for unimodular lattices with no roots. *Math. Comp.*, 72(242):839–863, 2003.
- [13] Markus Kirschmer and Gabriele Nebe. Binary Hermitian lattices over number fields. *Experimental Mathematics*, (to appear).
- [14] M. Kneser. *Quadratische Formen*. Springer-Verlag, Berlin, 2002. Revised and edited in collaboration with Rudolf Scharlau.
- [15] Jacques Martinet. *Perfect lattices in Euclidean spaces*, volume 327 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.
- [16] Jorge F. Morales. Maximal Hermitian forms over $\mathbb{Z}G$. *Comment. Math. Helv.*, 63(2):209–225, 1988.
- [17] G. Nebe. On automorphisms of extremal even unimodular lattices. *Int. J. Number Theory*, 9(8):1933–1959, 2013.
- [18] G. Nebe. A fourth extremal even unimodular lattice of dimension 48. *Discrete Math.*, 331:133–136, 2014.
- [19] G. Nebe. On elementary $\mathbb{Z}_p C_p$ -lattices. *preprint*, 2020.
- [20] Gabriele Nebe. Endliche rationale Matrixgruppen vom Grad 24. *Thesis, RWTH Aachen University*, 1995.
- [21] Gabriele Nebe. Some cyclo-quaternionic lattices. *J. Algebra*, 199(2):472–498, 1998.
- [22] Gabriele Nebe. Gitter und Modulformen. *Jahresber. Deutsch. Math.-Verein.*, 104(3):125–144, 2002.
- [23] Gabriele Nebe. An even unimodular 72-dimensional lattice of minimum 8. *J. Reine Angew. Math.*, 673:237–247, 2012.
- [24] Gabriele Nebe. Boris Venkov’s theory of lattices and spherical designs. In *Diophantine methods, lattices, and arithmetic theory of quadratic forms*, volume 587 of *Contemp. Math.*, pages 1–19. Amer. Math. Soc., Providence, RI, 2013.
- [25] Gabriele Nebe and Neil Sloane. Catalogue of lattices. <http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/>.
- [26] Gabriele Nebe and Boris B. Venkov. Nonexistence of extremal lattices in certain genera of modular lattices. *J. Number Theory*, 60(2):310–317, 1996.
- [27] O. T. O’Meara. *Introduction to Quadratic Forms*. Springer, 1973.
- [28] H.-G. Quebbemann. Modular lattices in euclidean spaces. *J. Number Theory*, 54:190–202, 1995.
- [29] H.-G. Quebbemann. Atkin-Lehner eigenforms and strongly modular lattices. *Enseign. Math. (2)*, 43(1-2):55–65, 1997.

- [30] R. Scharlau and B. Hemkemeier. Classification of integral lattices with large class number. *Math. Comp.*, 67(222):737–749, 1998.
- [31] R. Scharlau and R. Schulze-Pillot. Extremal lattices. In *Algorithmic algebra and number theory (Heidelberg, 1997)*, pages 139–170. Springer, Berlin, 1999.
- [32] C. L. Siegel. Über die Analytische Theorie der quadratischen Formen. *Annals of Mathematics*, 36(3):527–606, 1935.
- [33] C. L. Siegel. Über die Analytische Theorie der quadratischen Formen III. *Annals of Mathematics*, 38(1):212–291, 1937.
- [34] Carl Ludwig Siegel. Berechnung von Zetafunktionen an ganzzahligen Stellen. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1969:87–102, 1969.
- [35] Maryna S. Viazovska. The sphere packing problem in dimension 8. *Ann. of Math. (2)*, 185(3):991–1015, 2017.
- [36] Fabian Voullie. Involutionen extremaler modularer Gitter. *Masterthesis, RWTH Aachen University*, 2019.

Email address: `nebe@math.rwth-aachen.de`

LEHRSTUHL FÜR ALGEBRA UND ZAHLENTHEORIE, RWTH AACHEN UNIVERSITY, 52056 AACHEN, GERMANY