An analogue of the Pless symmetry codes

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ABSTRACT. A series of monomial representations of $\mathrm{SL}_2(p)$ is used to construct a new series of self-dual ternary codes of length $2(p+1)$ for all primes $p \equiv 5 \pmod 8$. In particular we find a new extremal self-dual ternary code of length 60.
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1 Introduction.

In 1969 Vera Pless [7] discovered a family of self-dual ternary codes $\mathcal{P}(p)$ of length $2(p+1)$ for primes p with $p \equiv -1 \pmod 6$. Together with the extended quadratic residue codes XQR(q) of length $q+1$ (q prime, $q \equiv \pm 1 \pmod{12}$) they define a series of self-dual ternary codes of high minimum distance (see [4, Chapter 16, §8]). For $p = 5$, the Pless code $\mathcal{P}(5)$ coincides with the Golay code $\mathfrak{g}_{12}$ which is also the extended quadratic residue code XQR(11) of length 12.

Using invariant theory of finite groups, A. Gleason [2] has shown that the minimum distance of a self-dual ternary code of length $4n$ cannot exceed $3\lfloor \frac{n}{3} \rfloor + 3$. Self-dual codes that achieve equality are called *extremal*. Both constructions, the Pless symmetry codes and the extended quadratic residue codes yield extremal ternary self-dual codes for small values of p.

This short note gives an interpretation of the Pless codes using monomial representations of the group $\mathrm{SL}_2(p)$. This construction allows to read off a large subgroup of the automorphism group of the Pless codes (which was already described in [7]). A different but related series of monomial representations of $\mathrm{SL}_2(p)$ is investigated to construct a new series of self-dual ternary codes $\mathcal{V}(p)$ of length $2(p+1)$ for all primes $p \equiv 5 \pmod 8$. The automorphism group of $\mathcal{V}(p)$ contains the group $\mathrm{SL}_2(p)$. For $p = 5$ we again find $\mathcal{V}(5) \cong \mathfrak{g}_{12}$ the Golay code of length 12, but for larger primes these codes are new. In particular the code $\mathcal{V}(29)$ is an extremal ternary code of length 60, so we now know three extremal ternary codes of length 60: XQR(59), $\mathcal{P}(29)$ and $\mathcal{V}(29)$.

2 Codes and monomial groups.

Let K be a field, $n \in \mathbb{N}$. Then the **monomial group** $\mathrm{Mon}_n(K^*) \leq \mathrm{GL}_n(K)$ is the group of monomial $n \times n$-matrices over K, where a matrix is called **monomial**, if it contains exactly one non-zero entry in each row and each column. So $\mathrm{Mon}_n(K^*) \cong K^* \wr S_n \cong (K^*)^n : S_n$ is the semidirect product of the subgroup $(K^*)^n$ of diagonal matrices in $\mathrm{GL}_n(K)$ with the group of permutation matrices. For any subgroup $S \leq K^*$ we define $\mathrm{Mon}_n(S) := S^n \wr S_n$ to be the subgroup of monomial matrices having all non-zero entries in S. There is a natural epimorphism $\pi : \mathrm{Mon}_n(S) \to S_n$ mapping any monomial matrix to the associated permutation.

By MacWilliam's extension theorem ([3], see also [8]) any K-linear weight preserving isomorphism between two subspaces of K^n is the restriction of a monomial transformation in $\mathrm{Mon}_n(K^*)$. This justifies the following commonly used notion of equivalence of codes, which also motivates the investigation of monomial representations of finite groups to find good codes with large automorphism group.

Definition 1. *A K-code C of length n is a subspace of K^n. Two codes C and C' of length n are called* **monomially equivalent***, if there is some $g \in \mathrm{Mon}_n(K^*)$ such that $Cg = C'$. The* **monomial automorphism group** *of C is $\mathrm{Aut}(C) := \{g \in \mathrm{Mon}_n(K^*) \mid Cg = C\}$.*

3 Endomorphism rings of monomial representations.

The theory exposed in this section is well known, a nice explicit formulation is contained in [5, Section I (1)]. Let G be some group. A linear K-representation Δ of degree n is a group homomorphism $\Delta : G \to \mathrm{GL}_n(K)$. The representation is called **monomial**, if its image $\Delta(G)$ is conjugate in $\mathrm{GL}_n(K)$ to some subgroup of $\mathrm{Mon}_n(K^*)$. We call the monomial representation **transitive**, if $\pi(\Delta(G))$ is a transitive subgroup of S_n. In this case the set $\{h \in G \mid 1\pi(\Delta(h)) = 1\} =: H$ is a subgroup of index n in G and Δ is obtained by inducing up a degree 1 representation of H as follows:

Let H be a subgroup of G of index $n := [G : H]$. Choose $g_1, \ldots, g_m \in G$ such that

$$G = \dot{\bigcup}_{\ell=1}^{m} Hg_\ell H$$

and put $H_\ell := H \cap g_\ell^{-1} H g_\ell$. Choose some right transversal $h_{\ell,j}$ of H_ℓ in H, so that $h_{\ell,1} = 1$ and $H = \dot{\bigcup}_{j=1}^{k_\ell} Hh_{\ell,j}$. Then

$$G = \dot{\bigcup}_{\ell=1}^{m} \dot{\bigcup}_{j=1}^{k_\ell} Hg_\ell h_{\ell,j}$$

and the right transversal $\{g_\ell h_{\ell,j} \mid \ell = 1,\ldots,m, k = 1,\ldots,k_\ell\}$ is a set of cardinality n which we will use as an index set of our $n \times n$-matrices.

For a group homomorphism $\lambda : H \to K^*$ the associated **monomial representation** of G is $\Delta := \lambda_H^G : G \to \mathrm{Mon}_n(\lambda(H))$ defined by

$$(\lambda_H^G(g))_{g_\ell h_{\ell j}, g_{\ell'} h_{\ell',j'}} = \begin{cases} 0 & \text{if } g_\ell h_{\ell j} g (g_{\ell'} h_{\ell',j'})^{-1} \notin H \\ \lambda(g_\ell h_{\ell j} g (g_{\ell'} h_{\ell',j'})^{-1}) & \text{if } g_\ell h_{\ell j} g (g_{\ell'} h_{\ell',j'})^{-1} \in H \end{cases}.$$

The representation λ restricts in two obvious ways to a representation of H_ℓ:

$$\lambda_\ell : H_\ell \to K^*, h \mapsto \lambda(h) \text{ and } \lambda_\ell^{g_\ell} : H_\ell \to K^*, h \mapsto \lambda(g_\ell h g_\ell^{-1}).$$

Let $\mathcal{I} := \{\ell \in \{1,\ldots,m\} \mid \lambda_\ell = \lambda_\ell^{g_\ell}\}$ and reorder the double coset representatives so that $\mathcal{I} = \{1,\ldots,d\}$. Then the **endomorphism ring**

$$\mathrm{End}(\Delta) := \{X \in K^{n\times n} \mid X\Delta(g) = \Delta(g)X \text{ for all } g \in G\}$$

has dimension d and the **Schur basis** of $\mathrm{End}(\Delta)$ is $(B_1 = I_n, B_2, \ldots, B_d)$ where $(B_\ell)_{1,g_\ell} = 1$ and $(B_\ell)_{1,g_k h_{k,i}} \neq 0$ if and only if $\ell = k$. As $\Delta(h_{\ell,k})B_\ell = B_\ell\Delta(h_{\ell,k})$ we conclude

$$\lambda(h_{\ell,k})(B_\ell)_{1,g_\ell h_{\ell,j}} = \Delta(h_{\ell,k})_{g_\ell, g_\ell h_{\ell,j}} = \lambda(h_{\ell,k})\lambda(h_{\ell,j}^{-1})$$

so $(B_\ell)_{1,g_\ell h_{\ell,j}} = \lambda(h_{\ell,j})^{-1}$ for all j. More generally we get

Lemma 2. *$(B_\ell)_{g_k h_{k,i}, g_{k'} h_{k',i'}} = 0$ if $g_{k'} h_{k',i'} h_{k,i}^{-1} g_k^{-1} \notin H g_\ell H$. Otherwise write $g_{k'} h_{k',i'} h_{k,i}^{-1} g_k^{-1} = h g_\ell h_{\ell,j}$ for some $h \in H$. Then $(B_\ell)_{g_k h_{k,i}, g_{k'} h_{k',i'}} = \lambda(h)^{-1}\lambda(h_{\ell,j}^{-1})$.*

<u>Proof.</u> To see this we choose $g = (g_k h_{k,i})^{-1} \in G$. Then $\Delta(g)_{g_k h_{k,i},1} = 1$ and hence

$$(\Delta(g)B_\ell)_{g_k h_{k,i}, g_\ell h_{\ell,j}} = \Delta(g)_{g_k h_{k,i},1}(B_\ell)_{1,g_\ell h_{\ell,j}} = \lambda(h_{\ell,j})^{-1}.$$

On the other hand

$$(B_\ell\Delta(g))_{g_k h_{k,i}, g_\ell h_{\ell,j}} = (B_\ell)_{g_k h_{k,i}, g_{k'} h_{k',i'}} \Delta(g)_{g_{k'} h_{k',i'}, g_\ell h_{\ell,j}}$$

for the unique (k', i') such that

$$h := g_{k'} h_{k',i'} (g_k h_{k,i})^{-1} (g_\ell h_{\ell,j})^{-1} \in H$$

and then $\Delta(g)_{g_{k'} h_{k',i'}, g_\ell h_{\ell,j}} = \lambda(h)$. As $\Delta(g)B_\ell = B_\ell\Delta(g)$ we compute

$$\lambda(h_{\ell,j})^{-1} = (B_\ell)_{g_k h_{k,i}, g_{k'} h_{k',i'}} \lambda(h).$$

□

4 Generalized Pless codes.

In this section we reinterpret the construction of the famous Pless symmetry codes $\mathcal{P}(p)$ discovered by Vera Pless [7], [6]. Explicit generator matrices for the Pless codes may be obtained from the endomorphism ring of a monomial representation. Let p be an odd prime and

$$\mathrm{SL}_2(p) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_p^{2\times 2} \mid ad - bc = 1\}$$

the group of 2×2-matrices over the finite field $\mathbb{F}_p$ with determinant 1. Let

$$B := \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(p)\} = \langle h_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \zeta := \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \rangle.$$

Then B is a subgroup of $\mathrm{SL}_2(p)$ or index $p+1$, isomorphic to the semidirect product $C_p : C_{p-1}$, with center $Z(B) = Z(\mathrm{SL}_2(p)) = \langle \zeta^{(p-1)/2} \rangle = \{\pm I_2\}$. Let

$$\lambda : B \to K^*, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto 1, \zeta \mapsto -1$$

Then $\lambda(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) = \left(\frac{a}{p}\right)$ is just the Legendre symbol of the upper left entry. Let

$$\Delta := \lambda_B^{\mathrm{SL}_2(p)} : \mathrm{SL}_2(p) \to \mathrm{Mon}_{p+1}(K^*)$$

be the monomial representation induced by λ. The following facts about this representation are well known, and easily computed from the general description in the previous section.

Remark 3. *(1) (Gauß-Bruhat decomposition)* $\mathrm{SL}_2(p) = B \mathbin{\dot\cup} BwB$ *where* $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(2) $B \cap wBw^{-1} = \langle \zeta \rangle$.

(3) A right transversal of B *in* $\mathrm{SL}_2(p)$ *is given by* $[1, wh_x : x \in \mathbb{F}_p]$ *where* $h_x := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in B$.

(4) The Schur basis of $\mathrm{End}(\Delta)$ *is* (I_{p+1}, P), *where* $P_{1,1} = 0$, $P_{1,wh_x} = 1$ *for all* x. *Then* $P_{wh_x,1} = \left(\frac{-1}{p}\right)$ *and*

$$P_{wh_x,wh_y} = \begin{cases} \left(\frac{x-y}{p}\right) & x \neq y \\ 0 & x = y. \end{cases}$$

(5) $P^2 = \left(\frac{-1}{p}\right) p$ *and* $PP^{tr} = p$.

To construct monomial representations of degree $2(p+1)$ we consider the group

$$\mathcal{G}(p) := \langle \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta(g) \end{pmatrix}, Z := \begin{pmatrix} 0 & I_{p+1} \\ jI_{p+1} & 0 \end{pmatrix} \mid g \in \mathrm{SL}_2(p) \rangle \leq \mathrm{Mon}_{2(p+1)}(K^*)$$

where $j = -\left(\frac{-1}{p}\right) = \begin{cases} 1 & p \equiv 3 \pmod 4 \\ -1 & p \equiv 1 \pmod 4. \end{cases}$

Remark 4. *(1)* $\mathcal{G}(p) \cong \begin{cases} C_4 \times \mathrm{PSL}_2(p) & p \equiv 1 \pmod 4 \\ C_2 \times \mathrm{SL}_2(p) & p \equiv 3 \pmod 4 \end{cases}$

(2) $\mathrm{End}(\mathcal{G}(p)) = \{\begin{pmatrix} A & B \\ jB & A \end{pmatrix} \mid A, B \in \mathrm{End}(\Delta)\}$ *is generated by*

$$I_{2(p+1)}, X := \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, Y := \begin{pmatrix} 0 & I_{p+1} \\ jI_{p+1} & 0 \end{pmatrix}, XY = \begin{pmatrix} 0 & P \\ jP & 0 \end{pmatrix}$$

with $X^2 = -jp$, $Y^2 = j$, $XY = YX$, $(XY)^2 = -p$.

Definition 5. *Let* $K = \mathbb{F}_q$ *be the finite field with* q *elements and assume that there is some* $a \in K^*$ *such that* $a^2 = -p$. *Then we put* $P_q(p) := aI_{2(p+1)} + XY \in \mathrm{End}(\mathcal{G}(p))$ *and define the* **generalized Pless code** $\mathcal{P}_q(p) \leq K^{2(p+1)}$ *to be the code spanned by the rows of* $P_q(p)$.

Theorem 6. *Let* $a \in \mathbb{F}_q^*$ *such that* $a^2 = -p$. *The code* $\mathcal{P}_q(p)$ *has generator matrix* $(aI_{p+1}|P)$ *and is a self-dual code in* $\mathbb{F}_q^{2(p+1)}$.
$d(\mathcal{P}_q(p)) \leq (p+7)/2$ *if* q *is odd and* $d(\mathcal{P}_q(p)) \leq 4$ *if* q *is even.*
The group $\mathcal{G}(p)$ *is a subgroup of* $\mathrm{Aut}(\mathcal{P}_q(p))$.

Proof. By construction the group $\mathcal{G}(p)$ is a subgroup of $\mathrm{Aut}(\mathcal{P}_q(p))$. As $PP^{tr} = pI_{p+1} = -a^2 I_{p+1}$ the code $\mathcal{P}_q(p)$ is self-dual with respect to the standard inner product.

The upper bound on the minimum distance is obtained by adding the first two rows of the generator matrix $(aI_{p+1}|P)$. The sum has weight 4 if q is even. If q is odd then the first row of P is $(0, 1^p)$ and the second row of P is $(-1, 0, v)$ where $v \in \{\pm 1\}^{p-1}$ has exactly $(p-1)/2$ ones and $(p-1)/2$ minus ones. □

Remark 7. *For $K = \mathbb{F}_3$ and $p \equiv -1 \pmod 3$ we may choose $a = 1$ and the generator matrix of $\mathcal{P}_3(p)$ is the one for the Pless symmetry code $\mathcal{P}(p)$ as given in [7].*

With Magma [1] we compute the following invariants of first few Pless codes:

p	5	11	17	23	29	41	47
$2(p+1)$	12	24	36	48	60	84	96
$d(\mathcal{P}_3(p))$	6	9	12	15	18	21	24
$\mathrm{Aut}(\mathcal{P}_3(p))$	$2.M_{12}$	$\mathcal{G}(11).2$	$\mathcal{G}(17).2$	$\mathcal{G}(23).2$	$\mathcal{G}(29).2$	$\geq \mathcal{G}(41)$	$\geq \mathcal{G}(47)$

For $q = 5, 7$, and 11 we computed $d(\mathcal{P}_q(p))$ with Magma:

(p,q)	$(11,5)$	$(19,5)$	$(29,5)$	$(31,5)$	$(3,7)$	$(5,7)$	$(13,7)$	$(17,7)$	$(19,7)$	$(7,11)$	$(13,11)$	$(17,11)$	$(19,11)$
$2(p+1)$	12	40	60	64	8	12	28	36	40	16	28	36	40
$d(\mathcal{P}_q(p))$	9	13	18	18	4	6	10	12	13	7	10	12	13

5 A new series of self-dual codes invariant under $\mathrm{SL}_2(p)$.

Applying the same strategy as in the previous section we now construct a monomial representation of $\mathrm{SL}_2(p)$ of degree $2(p+1)$ where p is a prime so that $p-1 \equiv 4 \pmod 8$. We assume that $\mathrm{char}(K) \neq 2$.

Then the subgroup $B^{(2)} := \{\begin{pmatrix} a^2 & 0 \\ b & a^{-2} \end{pmatrix} \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\} \leq \mathrm{SL}_2(p)$ of index $2(p+1)$ in $\mathrm{SL}_2(p)$ has a unique linear representation $\gamma : B^{(2)} \to K^*$ with $\gamma(B^{(2)}) = \{\pm 1\}$, so $\gamma(\begin{pmatrix} a^2 & 0 \\ b & a^{-2} \end{pmatrix}) = \left(\frac{a}{p}\right)$. Then $\Delta' := \gamma_{B^{(2)}}^{\mathrm{SL}_2(p)}$ is a faithful monomial representation of degree $2(p+1)$.

To obtain explicit matrices we choose $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as above. By assumption $2 \in \mathbb{F}_p^* \setminus (\mathbb{F}_p^*)^2$, put $\epsilon := \mathrm{diag}(2, 2^{-1})$. Then $B = B^{(2)} \dot\cup B^{(2)}\epsilon$ and

$$\mathrm{SL}_2(p) = B \dot\cup BwB = B^{(2)} \dot\cup B^{(2)}wB^{(2)} \dot\cup B^{(2)}\epsilon \dot\cup B^{(2)}\epsilon w B^{(2)}$$

and a right transversal is given by $[1, wh_x, \epsilon, \epsilon wh_x : x \in \mathbb{F}_p]$ where $h_x := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in B^{(2)}$.

Lemma 8. $\mathrm{End}(\Delta')$ *has a Schur basis* $(B_1, B_w, B_\epsilon, B_{\epsilon w} = B_\epsilon B_w)$ *where* $B_\epsilon = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ *and* $B_w = \begin{pmatrix} X & Y \\ -Y^{tr} & X^{tr} \end{pmatrix}$ *with*

$$X = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & & & \\ \vdots & & R_X & \\ -1 & & & \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & R_Y & \\ 0 & & & \end{pmatrix}$$

where rows and columns of R_X *and* R_Y *are indexed by the elements* $\{0, \dots, p-1\}$ *of* $\mathbb{F}_p$ *and*

$$(R_X)_{a,b} = \begin{cases} 0 & b - a \notin (\mathbb{F}_p^*)^2 \\ \left(\frac{c}{p}\right) & b - a = c^2 \in (\mathbb{F}_p^*)^2 \end{cases}, \quad (R_Y)_{a,b} = \begin{cases} 0 & 2(b - a) \notin (\mathbb{F}_p^*)^2 \\ \left(\frac{c}{p}\right) & 2(b - a) = c^2 \in (\mathbb{F}_p^*)^2 \end{cases}$$

Proof. Explicit computations with the general formulas in Lemma 2. For instance $(B_w)_{wh_x, wh_y} \neq 0$ if and only if

$$wh_{x-y}w^{-1} = \begin{pmatrix} 1 & y - x \\ 0 & 1 \end{pmatrix} \in B^{(2)}wB^{(2)}.$$

This is equivalent to $y - x = a^2$ for some $a \in \mathbb{F}_p$ and then

$$wh_{x-y}w^{-1} = \begin{pmatrix} a^2 & 0 \\ 1 & a^{-2} \end{pmatrix} w \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and hence $(B_w)_{wh_x, wh_y} = \left(\frac{a}{p}\right)$. □

Remark 9. *Note that* $(-1) = c^2$ *is a square but not a 4th power, so* $\left(\frac{c}{p}\right) = -1$ *and hence* X *is skew symmetric and* $B_w^{tr} = -B_w$, $B_{\epsilon w}^{tr} = -B_{\epsilon w}$. *We compute that* $B_w^2 = B_{\epsilon w}^2 = -p$ *and* $B_\epsilon^2 = -1$ *so* $\mathrm{End}(\Delta') \cong \left(\frac{-p,-1}{K}\right)$ *is isomorphic to a quaternion algebra over* K. *We also compute that* $(B_w + B_{\epsilon w})^2 = -2p$.

Definition 10. *Let p be a prime $p \equiv_8 5$, $K = \mathbb{F}_q$ so that there is some $a \in K^*$ such that $a^2 = -tp$ for $t = 1$ or $t = 2$. We then put*

$$V_t(p) := \begin{cases} aI_{2(p+1)} + B_w & t = 1 \\ aI_{2(p+1)} + B_w + B_{\epsilon w} & t = 2 \end{cases}$$

and let $\mathcal{V}_q(p)$ be the linear code spanned by the rows of $V_t(p)$.

Theorem 11. *$\mathcal{V}_q(p)$ is a self-dual code in $\mathbb{F}_q^{2(p+1)}$. Its monomial automorphism group contains the group $\mathrm{SL}_2(p)$.*

Proof. By construction the code $\mathcal{V}_q(p) \leq \mathbb{F}_q^{2(p+1)}$ is invariant under $\mathrm{SL}_2(p) \cong \Delta'(\mathrm{SL}_2(p))$. To see that $\mathcal{V}_q(p)$ is self-orthogonal we check that

$$V_1(p)V_1(p)^{tr} = (a + B_w)(a + B_w^{tr}) = a^2 + a(B_w + B_w^{tr}) + B_w B_w^{tr} = a^2 - B_w^2 = 0$$
$$V_2(p)V_2(p)^{tr} = (a + B_w + B_{\epsilon w})(a + B_w^{tr} + B_{\epsilon w}^{tr}) = a^2 - (B_w + B_{\epsilon w})^2 = 0.$$

To obtain the rank of the matrix $V_t(p)$ we note that $\mathrm{End}(\Delta') \cong \left(\frac{-p,-1}{\mathbb{F}_q}\right) \cong \mathbb{F}_q^{2\times 2}$. So the representation Δ' is the sum of two equivalent representations over $\mathbb{F}_q$. These have the same degree, $p+1$, half of the degree of Δ' and therefore $p+1$ divides the rank of any matrix in $\mathrm{End}(\Delta')$. □

Remark 12. *The matrices of rank $p+1$ in $\mathrm{End}(\Delta')$ yield $q+1$ different self-dual codes invariant under $\Delta'(\mathrm{SL}_2(p))$. In general these fall into different equivalence classes. For instance for $q = 7$, where 2 is a square mod 7, the codes spanned by the rows of $V_1(p)$ and $V_2(p)$ are inequivalent for $p = 5$ and $p = 13$ but have the same minimum distance.*

The first few codes $\mathcal{V}_3(p)$ have the following parameters (computed with MAGMA [1]):

p	5	13	29	37	53
$2(p+1)$	12	28	60	76	108
$d(\mathcal{V}_3(p))$	6	9	18	18	24
$\mathrm{Aut}(\mathcal{V}_3(p))$	$2.M_{12}$	$\mathrm{SL}_2(13)$	$\mathrm{SL}_2(29)$	$\geq \mathrm{SL}_2(37)$	$\geq \mathrm{SL}_2(53)$

For $q = 5, 7$, and 11 and small lengths we computed $d(\mathcal{V}_q(p))$ with MAGMA:

(p,q)	$(13,5)$	$(29,5)$	$(5,7)$	$(13,7)$	$(5,11)$	$(13,11)$
$2(p+1)$	28	60	12	28	12	28
$d(\mathcal{V}_q(p))$	10	16	6	9	7	11

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