The root lattices of the complex reflection groups

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Abstract

Four questions on the classification of root lattices, root systems and bad primes of the finite irreducible complex reflection groups raised in a letter by Broué are fully answered.

1 Introduction.

Let K be a number field and V a finite dimensional K-vector space. A pseudo-reflection is an element of finite order of $GL(V)$ which has exactly 1 eigenvalue $\lambda \neq 1$. A finite complex reflection group G on V is a finite subgroup of $GL(V)$ generated by pseudo-reflections. G is called irreducible, if V is an irreducible G-module. The finite complex reflection groups have been classified by Shephard and Todd [ShT 54]. According to their classification there are 3 infinite series of irreducible finite complex reflection groups and 34 exceptional groups. These exceptional groups come in 17 families according to the isomorphism type of the factor group modulo the center.

To recognize the complex reflection groups efficiently, one wants to know invariants of the finite matrix groups. One possibility is to look at the invariant lattices. Let G be an irreducible finite complex reflection group. Eigenvectors of pseudo-reflections to eigenvalues $\neq 1$ in G are called roots of G. There are distinguished lattices, the so called root lattices, which are the G-invariant lattices in V spanned by roots of G. All root lattices can be build up from primitive root lattices, which are in some sense the 'smallest' G-invariant root lattices.

If K is the character field of the character of G afforded by V and R is its ring of integers, a root lattice of G is called primitive, if it is spanned as RG-lattice by one root. Section 4 describes the root lattices of the finite complex

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reflection groups with the help of primitive root lattices, of which the inclusion patterns are given in Theorem 18. It turns out that the primitive root lattices are free over R. Passing to completions $R_\wp$ of R at maximal ideals $\wp$ of R one notices that there is at most one $\wp$ such that the lattices $R_\wp L$, where L runs through the primitive root lattice of G, are not all isomorphic. Then these lattices $R_\wp L$ fall in at most 3 isomorphism classes.

In analogy to the real reflection groups one may define a notion of root system for an irreducible complex reflection group. The invariant root systems are described in section 5.

Section 6 deals with bad primes. These are primes dividing the index of a sublattice of a root lattice L of G spanned by all the roots in L that are roots of a reflection subgroup of G. From the classification of the root lattices of all irreducible finite complex reflection groups one gets that these bad primes divide the order of G.

The article is written to answer a letter of M. Broué ([Bro 97]). I would like to thank him for the interesting questions. I am also grateful to the referee for pointing out useful references.

2 First Definitions

Let K be an abelian number field with ring of integers $\mathbb{Z}_K$ and V a finite dimensional K-vector space. Let G be a finite subgroup of $GL(V)$ generated by pseudo-reflections such that the representation of G afforded by V is (absolutely) irreducible. Since K is an abelian number field, there is a unique complex conjugation $^-$ on K, induced by the Galois automorphism of any cyclotomic field containing K that inverts all roots of unity. Note that, since G is a finite group, there is a totally positive definite G-invariant Hermitian (with respect to $^-$) form $(\cdot,\cdot)$ on V (cf. [Fei 74]). Since V is an absolutely irreducible G-module, this form is unique up to multiplication with totally positive elements in the maximal real subfield $Fix(^-)$ of K. Consider V as Hermitian vector space over K.

Let K_0 be the field generated over $\mathbb{Q}$ by the traces of the elements in G and R be the ring of integers in K_0. It is well known (cf. [Bou 81, Proposition V.2.1]), that there is a K_0-vector space V_0 such that the representation of G on V can be realized over V_0, i.e. $V = K \otimes V_0$ and G is conjugate in $GL(V)$ to a subgroup of $GL(V_0) \leq GL(V)$. Here and in the following V_0 is identified with $1 \otimes V_0 \subseteq V$ and the signs $\otimes$ are omitted to describe extensions of scalars.

If $\sigma \in G$ is a pseudo-reflection then the unique eigenvalue $\lambda \neq 1$ of σ lies in K_0. Therefore, V_0 contains an eigenvector v_0 of σ with $v_0\sigma = \lambda v_0$. A vector $0 \neq v \in V$ such that $v\sigma = \lambda v$ is called a *root* of σ and also a *root* of G. The one-dimensional subspace Kv of V spanned by a root of G (or of σ) is called a *root line* of G (or of σ).

Definition 1 *A* $\mathbb{Z}_K$-root lattice *of* G *or* $\mathbb{Z}_K G$-root lattice *is a* $\mathbb{Z}_K G$*-lattice in* V *that is generated by roots of* G.
An RG*-root lattice* L *is called* primitive *(for* G*) if* $L \leq V_0$ *is spanned as* RG*-lattice by one root.*

The following trivial observation reduces the classification of the root lattices of G to the one of the primitive root lattices of G.

Remark 2 *Let* $L \leq V$ *be a* $\mathbb{Z}_K$*-root lattice of* G*. Then there are fractional ideals* $\mathcal{A}_1, \ldots, \mathcal{A}_s$ *of* $\mathbb{Z}_K$ *and primitive* RG*-root lattices* $L_1, \ldots, L_s$ *in* V_0 *such that* $L = \mathcal{A}_1 L_1 + \ldots + \mathcal{A}_s L_s$.

Proof: Let $v_1, \ldots, v_s \in V$ be representatives of the orbits of G on the roots that span L. Since V_0 contains eigenvectors of the pseudo-reflections of G there are $w_i \in V_0$ and $a_i \in K$ such that $v_i = a_i w_i$ $(1 \leq i \leq s)$. For $1 \leq i \leq s$ let $L_i \leq V$ be the RG-lattices generated by w_i and $\mathcal{A}_i := a_i \mathbb{Z}_K$. Then $L = \mathcal{A}_1 L_1 + \ldots + \mathcal{A}_s L_s$. □

Two $\mathbb{Z}_K G$-lattices L, L' in V lie in the same *genus*, if there is a fractional $\mathbb{Z}_K$-ideal $\mathcal{A}$ with $L' = \mathcal{A}L$. If $\mathcal{A}$ is a principal ideal, then the two lattices L and L' are *isomorphic* $\mathbb{Z}_K G$-lattices.

If L is a root lattice in V, then clearly all the lattices in the genus of L are root lattices.

Now let $L_1, \ldots, L_s$ be a system of representatives of the isomorphism classes of primitive RG-root lattices in V_0. Let $\Lambda := \cap_{i=1}^{s} End_R(L_i)$ be the biggest R-order in $End_{K_0}(V_0)$ that preserves all the L_i $(1 \leq i \leq s)$. Then the $\mathbb{Z}_K$-order $\mathbb{Z}_K \Lambda$ in $End_K(V)$ preserves all the root lattices of G in V. This idea will be used to describe the root lattices in V cf. Theorem 18.

To classify the primitive root lattices of G in V_0 representatives for the orbits of root lines of G are needed. This is not a question about the conjugacy classes of pseudo-reflections in G but about the conjugacy classes of maximal cyclic (complex) reflection subgroups of G as shown in the next lemma which well known (cf. [Coh 76, (1.8)]).

Lemma 3 *Let σ_1, σ_2 be two pseudo-reflections in G. Then σ_1 and σ_2 are conjugate in G to elements of the same cyclic reflection subgroup of G, if and only if the root lines of σ_1 and σ_2 are in the same orbit under G.*

Proof: Since the elements of a cyclic reflection subgroup of G have the same roots, the "only if" part is clear. On the other hand let v_i be roots of σ_i $(i = 1, 2)$ and $g \in G$ such that $v_1 = v_2 g$. Then g maps the orthogonal complement $v_1^\perp$, which is the fixed space of σ_1, of $\langle v_1 \rangle_K$ onto the one of v_2, the fixed space of σ_2. Hence the subgroup of G generated by the two pseudo-reflections σ_2 and $g\sigma_1 g^{-1}$ is isomorphic to a finite subgroup of K^* and therefore cyclic. □

The numeration of Shephard and Todd ([ShT 54]) is used to denote the irreducible finite complex reflection groups. Shephard and Todd distinguish three infinite series $G_1(n)$, $G_2(m,p,n) \neq G(2,2,2)$, $G_3(m)$ $(n, m \in \mathbb{N}_{>1}, p \mid m)$, of irreducible finite complex reflection groups and 34 exceptional groups $G_4, \ldots, G_{37}$.

3 The conjugacy classes of maximal cyclic reflection subgroups

In this section the conjugacy classes of maximal cyclic reflection subgroups of the irreducible finite complex reflection groups are described. This information can be deduced from [ShT 54] or [Coh 76]. The correctness can be checked with the following lemma.

Lemma 4 *([Coh 76, Corollary (1.9)]) Let G be a finite irreducible complex reflection group and $n_1, \ldots, n_h$ be the orders of representatives of the conjugacy classes of maximal cyclic reflection subgroups of G. Then the commutator factor group $G/[G,G] \cong \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_h\mathbb{Z}$.*

The three infinite series.

Lemma 5 *Let $n \in \mathbb{N}$.*

(i) The group $G_1(n) = W(A_n) \cong S_{n+1}$ has a unique conjugacy class of reflections.

(ii) *Let $G = G_2(m,p,n)$, with $m = pq > 1$, $n > 1$. Then G is isomorphic to a subgroup of index p of $C_m \wr S_n$.*
If $n \geq 3$ then G has 2 respectively 1 conjugacy classes of maximal cyclic reflection subgroups according to $m \neq p$ or $m = p$.
Now let $n = 2$. If p is odd then G has 2 respectively 1 conjugacy classes of maximal cyclic reflection subgroups according to $m \neq p$ or $m = p$. Otherwise G has 3 respectively 2 conjugacy classes of maximal cyclic reflection subgroups according to $m \neq p$ or $m = p$.

(iii) *The group $G_3(m) \cong C_m$ is cyclic and its unique maximal cyclic reflection subgroup.*

Proof: Only (ii) needs a proof. So let $\sigma \in G_2(m,1,n) \leq GL(V)$ be a pseudo-reflection. Then K contains the m-th roots of unity. Choose a basis of V such that the elements in $C_m^n \trianglelefteq G_2(m,1,n)$ act diagonally and some complement isomorphic to S_n of C_m^n acts as permutation matrices. The fact that the rank of the matrix $\sigma - id$ is 1 implies that either $\sigma \in C_m^n$ is conjugate in $G_2(m,m,n) \leq G$ to a matrix $d(\zeta') := diag(\zeta', 1, \ldots, 1)$ or the permutation induced by σ is a transposition and $\sigma \sim \pi(\zeta) := diag(\begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix}, 1, \ldots, 1)$ for some m-th root of unity ζ, ζ' with $\zeta' \neq 1$.
Now let $G = G_2(m,p,n)$ be a certain subgroup of index p of $G_2(m,1,n)$, and ζ_m be a primitive m-th root of unity in K. If $m \neq p$ then $\langle d(\zeta_m^p) \rangle$ contains all the $d(\zeta') \in G$. If $m = p$, then there is no pseudo-reflection conjugate to $d(\zeta')$ in G. If $n \geq 3$ then $diag(\zeta, 1, \zeta^{-1}, 1, \ldots, 1) \in G_2(m,m,n)$ conjugates $\pi(1)$ to $\pi(\zeta)$.

Now let $n = 2$. If $p = 2a+1$ is odd $diag(\zeta^{a+1}, \zeta^a) \in G_2(m,p,2)$ conjugates $\pi(1)$ to $\pi(\zeta)$.

If p is even, let $b := p/2 - 1$. Then $diag(\zeta^{b+2}, \zeta^b) \in G_2(m,p,2)$ conjugates $\pi(1)$ to $\pi(\zeta^2)$. Since $\pi(1)$ and $\pi(\zeta_m)$ are clearly not conjugate in $G_2(m,p,2)$, the subgroups $\langle \pi(1) \rangle$ and $\langle \pi(\zeta_m) \rangle$ represent the conjugacy classes of maximal cyclic reflection subgroups of $G_2(m,p,n)$ not contained in C_m^n for even p.
□

The 34 exceptional groups G_4, …, G_{37}.

An inspection of the character tables of the 34 exceptional complex reflection groups yields the following Lemmata.

Lemma 6 *Let G be one of the groups G_4, G_5, G_6, or G_7. Let a represent the conjugacy class of elements of order 4 in G_4 and x, x^2 represent the two conjugacy classes of elements of order 3 in G_4. Let $-\omega$, i, respectively ωi denote suitable generators of the center of G_5, G_6, respectively G_7. Then the maximal cyclic reflection subgroups in G are conjugate to $\langle x\rangle$; $\langle x\rangle$ or $\langle \omega^2 x\rangle$; $\langle x\rangle$ or $\langle ia\rangle$; respectively $\langle x\rangle$, $\langle \omega^2 x\rangle$, or $\langle ia\rangle$ according to $G = G_4$; G_5; G_6; respectively G_7.*

Lemma 7 *G_8 and G_9 have one resp. two conjugacy classes of maximal cyclic reflection subgroups. The groups $G_{10} \cong G_8 \times C_3$ and $G_{11} \cong G_9 \times C_3$ contain two resp. three such conjugacy classes.*

Lemma 8 *Let G be one of the groups G_{12}, G_{13}, G_{14}, or G_{15}. Let σ, a, respectively x represent the conjugacy class of elements of order 2, 4, respectively 3 in G_{12}. Let i, $-\omega$, respectively ωi denote suitable generators of the center of G_{13}, G_{14}, respectively G_{15}. Then the maximal cyclic reflection subgroups in G are conjugate to groups generated by σ; σ or ia; σ or ωx; respectively σ, ωx, or ia, according to $G = G_{12}$; G_{13}; G_{14}; respectively G_{15}.*

Lemma 9 *Let $16 \leq j \leq 22$. Let a, x, respectively y and y^2 represent a conjugacy class of elements of order 4, 3, respectively 5 in G_{16}. Let i, $-\omega$, respectively $-\tau$ denote suitable generators of the center of G_{22}, G_{20}, respectively G_{16} such that the center of G_j is generated by $i\tau$, $-\tau\omega$, $i\tau\omega$, $i\omega$ if $j = 17, 18, 19, 21$. Then the maximal cyclic reflection subgroups in G are conjugate to groups generated by τy; τy or ia; τy or ωx; τy, ia, or ωx; ωx; ia or ωx; respectively ia, according to $j = 16$; 17; 18; 19; 20; 21; respectively 22.*

Lemma 10 *The groups G_{23}, G_{24}, G_{25}, G_{27} and $G_{29}, \ldots, G_{37}$ contain only one conjugacy class of (maximal) cyclic reflection subgroups. The groups G_{26} and G_{28} contain two such classes.*

4 The root lattices.

The three infinite series.

Proposition 11 *Let $G = G_2(m, m, 2)$ be the dihedral group of order $2m$.*

(i) $K_0 = \mathbb{Q}[\theta_m]$*, where* $\theta_m = \zeta_m + \zeta_m^{-1}$*, is the maximal real subfield of the* m*-th cyclotomic number field.*

(ii) If m *is odd then* G *has only one genus of* $\mathbb{Z}_K$*-root lattices.*

(iii) If m *and* $m/2$ *are no prime powers, then all* $\mathbb{Z}_K G$*-lattices in* V *lie in one genus.*

(iv) If $m = 2^a$ *with* $a \geq 2$ *then* G *has two isomorphism classes of primitive root lattices, representatives* L_1*,* L_2 *of which can be chosen such that* $L_1 \supset L_2 \supset \wp L_1$ *where* $\wp$ *is the maximal ideal dividing* 2 *in* R *(inclusion pattern ② in Theorem 18).*

(v) If $m = 2l^a$ *with* $a \geq 1$ *for some odd prime* l *then* G *has two isomorphism classes of primitive root lattices, representatives* L_1*,* L_2 *of which can be chosen such that* $L_1 \supset L_2 \supset \wp L_1$ *where* $\wp$ *is the maximal ideal dividing* l *in* R *(inclusion pattern ② in Theorem 18).*

Proof: (i) is clear and (ii) follows from Lemma 5.
(iii) Let l be a prime and $m = m' l^\alpha$ with $m' > 2$ and $l \not| m'$. Then $G/O_l(G) \cong D_{2m'}$. Since $l \not| m'$ the l-modular constituent of the representation of G on V is of degree 2. Now (iii) follows from Lemma 15 below.
(iv) The two reflections $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \theta_m & -1 \end{pmatrix}$ acting from the right with respect to some K_0-basis (b_1, b_2) of V_0 represent the conjugacy classes of pseudo-reflections in G. Roots of these reflections are $b_1 - b_2$ resp. $\theta_m b_1 - 2b_2$. Their G-orbits generate the R-lattices $L_1 := \langle b_1 - b_2, (2-\theta_m) b_2 \rangle_R$ respectively $L_2 := \langle \theta_m b_1 - 2b_2, 2b_1 - \theta_m b_2 \rangle_R$.
(v) Now the two reflections $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ represent the conjugacy classes of pseudo-reflections in G. Roots of these reflections are $b_1 - b_2$ resp. $b_1 + b_2$. Their G-orbits generate the R-lattices $L_1 := \langle b_1 - b_2, (2-\theta_m) b_2 \rangle_R$ respectively $L_2 := \langle b_1 + b_2, (2+\theta_m) b_2 \rangle_R$. Now $2 - \theta_m = (1-\zeta_m)(1-\zeta_m^{-1})$ is a unit in R since $1 - \zeta_m$ is a unit in $\mathbb{Z}[\zeta_m]$ (cf. [Was 82, Proposition 2.8]) and $2 + \theta_m = (1+\zeta_m)(1+\zeta_m^{-1})$ generates the maximal ideal over l in R, because $1 + \zeta_m = 1 - \zeta_{l^a}$ is a prime element over l in $\mathbb{Z}[\zeta_m]$. Hence $L_1 = \langle b_1, b_2 \rangle_R$.
□

Proposition 12 *Let* $G = G_2(m, p, n) \not\cong G_2(m, m, 2)$.

(i) $K_0 = \mathbb{Q}[\zeta_m]$.

(ii) If m is not a prime power, then the $\mathbb{Z}_K G$-lattices in V lie in one genus and there is only one isomorphism class of primitive root lattices.

(iii) If $m = p$ then G has only one isomorphism class of primitive root lattices in V_0.

(iv) If $m = 2^a$ with $a \geq 2$ and $n = 2$ then G has three (resp. two) isomorphism classes of primitive root lattices according to $p > 1$ or $p = 1$, representatives L_1, L_2, L_3 (resp. L_1, L_2) of which can be chosen as in inclusion pattern ③ (resp. ②) in Theorem 18 where $\wp$ is the maximal ideal dividing 2 in R.

(v) If $n = 2$ and $m = l^a$ for some odd prime l then G has two isomorphism classes of primitive root lattices, representatives L_1, L_2 of which can be chosen as in inclusion pattern ② in Theorem 18 where $\wp$ is the maximal ideal dividing l in R.

(vi) If $n \geq 3$ and $l^a = m \neq p$ is a prime power then G has two isomorphism classes of primitive root lattices, representatives L_1, L_2 of which can be chosen as in inclusion pattern ② in Theorem 18 where $\wp$ is the maximal ideal dividing l in R.

Proof: (i) is clear, (ii) follows as in Proposition 11 (iii) and (iii) follows from Lemma 5.
(iv) Assume first that $1 < p < m$. Using the notation of the proof of Lemma 5, the maximal cyclic reflection subgroups of G are generated by $\pi(1)$, $\pi(\zeta_m)$, and $d(\zeta_m^p)$. These three pseudo-reflections also generate G. The corresponding primitive root lattices may be chosen as $L_1 := \langle b_1 - b_2, (1 - \zeta_m^2) b_2 \rangle_R$, $L_2 := \langle b_1 - \zeta_m b_2, (1 - \zeta_m^2) b_2 \rangle_R$, and $L_3 := \langle (1 - \zeta_m) b_1, (1 - \zeta_m) b_2 \rangle_R$. If $L := \langle b_1 - b_2, (1 - \zeta_m) b_2 \rangle_R$ denotes the lattice generated by each two of these three primitive root lattices, then L_1, L_2 and L_3 are the full preimages of the 1-dimensional subspaces of $L/(1 - \zeta_m)L \cong \mathbb{F}_2^2$.
If $p = 1$ then $\pi(1)$ and $\pi(\zeta_m)$ are conjugate in G and the root lattices are $M_1 := \langle b_1 - b_2, (1 - \zeta_m) b_2 \rangle_R$ and $M_2 := \langle b_1, b_2 \rangle_R$, satisfying $M_2 \supset M_1 \supset (1 - \zeta_m) M_2$.
(v) The lattices M_1 and M_2 of (iv) (with the new m) represent the genera of the primitive root lattices of G.

(vi) Now $\pi(1)$ and $d(\zeta_m^p)$ generate representatives for the maximal cyclic reflection subgroups of G. Primitive root lattices can be chosen as $M_1 := \langle b_1 - b_2, b_2 - b_3, \ldots, (1-\zeta_m)b_n\rangle_R$ and $M_2 := \langle b_1, b_2, \ldots, b_n\rangle_R$, satisfying $M_2 \supset M_1 \supset (1-\zeta_m)M_2$ and $M_2/M_1 \cong R/(1-\zeta_m)R \cong \mathbb{F}_l$. □

Corollary 13 *If G is a finite irreducible complex reflection group, then the primitive root lattices of G are free.*

Proof: If G is one of the groups in the three infinite series, then explicit bases of the primitive RG-root lattices have been constructed above. If G is one of the 34 exceptional irreducible finite complex reflection groups, then it can be checked with the computer algebra system Pari ([Coh 93]) or with the tables in [Was 82] for the cyclotomic number fields K_0 that the class number of R is one. Therefore all R-lattices are free. □

The 34 exceptional groups G_4, ..., G_{37}.

By the definition of a primitive root lattice, it is clear that all the exceptional groups that have only one orbit of root lines fix only one isomorphism class of primitive root lattices in V_0. Therefore it follows from section 3 that the groups G_4, G_8, G_{12}, G_{16}, G_{20}, G_{22}, and G_j with $j \geq 23$, $j \neq 26, 28$ have up to isomorphism only one primitive root lattice.

The invariant RG-lattices in V_0 for the 34 exceptional groups can be easily calculated with the help of a computer.

Proposition 14 *G_4 has only one isomorphism class of primitive root lattices in V_0. G_5 has two such classes, representatives L_1, L_2 of which can be chosen such that $L_1 \supseteq L_2 \supseteq 2L_1$ (inclusion pattern ② in Theorem 18). G_6 has two isomorphism classes of primitive root lattices in V_0, representatives L_1, L_2 of which can be chosen such that $L_1 \supseteq L_2 \supseteq (1+i)L_1$ (inclusion pattern ② in Theorem 18). G_7 has three such classes, representatives L_1, L_2, L_3 of which can be chosen such that $L_1 \supseteq L_2 \supseteq (1+i)L_1$ and $L_2 \supseteq L_3 \supseteq (1+i)L_2$ (inclusion pattern ④ in Theorem 18).*

The root lattices of the groups G_8 up to G_{22} can be easily obtained using the following lemma which is only true for lattices over the maximal order $\mathbb{Z}_K$ in K.

Lemma 15 *Let H be a finite subgroup of $GL(V)$, such that for some $\mathbb{Z}_K H$-lattice L in V the $\mathbb{Z}_K H$-module $L/\wp L$ is absolutely simple for all prime ideals $\wp$ of $\mathbb{Z}_K$. Then there is only one genus of $\mathbb{Z}_K H$-lattices in V.*

Proof: For a (finite) prime $\wp$ of K let $\mathbb{Z}_\wp$ denote the completion of $\mathbb{Z}_K$ at $\wp$. Let $K_\wp := frac(\mathbb{Z}_\wp)$ be the completion of K at $\wp$. It is clearly enough to show that $\Lambda := \mathbb{Z}_\wp H$ is a maximal order in the completion $A := End(K_\wp \otimes_K V)$. Since $L/\wp L$ is absolutely simple, the semisimple algebra $\Lambda/J(\Lambda)$ is absolutely simple and isomorphic to a matrix ring over $\mathbb{Z}_K/\wp =: k$ with $dim_k(\Lambda/J(\Lambda)) = dim_K(V)^2 =: n^2$. By [Zas 54] (or [Rei 75]) one may lift a system of orthogonal primitive idempotents of $\Lambda/J(\Lambda)$ to orthogonal primitive idempotents $e_1, \ldots, e_n$ of Λ with $\sum_{i=1}^n e_i = 1$. Now $\Lambda = \oplus_{i,j} e_i \Lambda e_j$ where the $e_i \Lambda e_j$ are $\mathbb{Z}_\wp$-modules in $e_i A e_j \cong K_\wp$. Hence they are of the form $\wp^{n_{ij}} \mathbb{Z}_\wp$ for some $n_{ij} \in \mathbb{Z}$. Since Λ is an order one has $n_{ii} = 0$ and $n_{ij} + n_{ji} \geq 0$ $(1 \leq i, j \leq n)$. Our assumptions on $\Lambda/J(\Lambda)$ imply that $n_{ij} + n_{ji} = 0$ for all $(1 \leq i, j \leq n)$ (cf. [Ple 83, Remark II.4]). Hence Λ is a maximal order in A. □

Now one only has to consider the p-modular constituents of the natural characters of the irreducible reflection groups to see the following

Corollary 16 *Let G be one of the groups $G_8, \ldots, G_{15}$, $G_{16}, \ldots, G_{22}$, G_{24}, G_{27}, G_{29}, G_{31}, G_{30}, G_{32}, G_{34}, or G_{37}. Then there is only one genus of $\mathbb{Z}_K G$-lattices in V.*

Proposition 17 *For G_{26} one maximal cyclic reflection subgroup is of order 3 and already contained in the subgroup G_{25}. Let L_1 be a primitive root lattice spanned by roots for this group. Then one finds a primitive root lattice L_2, spanned by roots reflections of order 2 in G_{26} containing L_1 of index 3 (inclusion pattern ② in Theorem 18).*
The two conjugacy classes of reflections in $G_{28} = W(F_4)$ are interchanged by the outer automorphism of G_{28}. The corresponding primitive root lattices in V_0 are represented by F_4 and its dual (which is similar to F_4) (inclusion pattern ② in Theorem 18).

The results of this section are summarized in the following theorem.

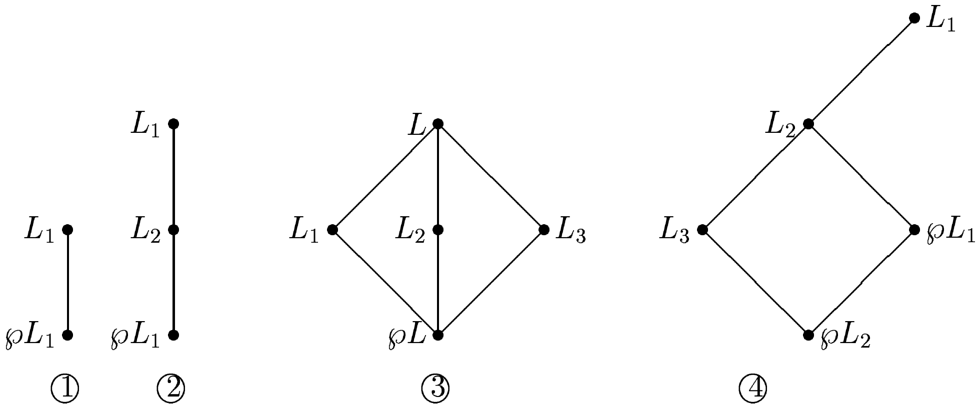
Theorem 18 *From the preceding discussion it follows that there are three possible inclusion patterns for the primitive root lattices in V_0. For all exceptional groups $\neq G_5, G_6, G_7, G_{26}$ and G_{28} and also for the groups $G_1(n)$ and $G_3(m)$ there is up to isomorphism a unique primitive root lattice in V_0.*

(i) There is only one isomorphism class of primitive root lattices in V_0. Then there is only one genus of $\mathbb{Z}_K G$-root lattices in V (inclusion pattern ①).

(ii) (G_5, G_6, G_{26}, G_{28}, $G_2(2l^a, 2l^a, 2)$ (l prime), $G_2(2^a, 1, 2)$, $G_2(l^a, l^b, 2)$ (l odd prime, $b < a$), $G_2(l^a, l^b, n)$ ($n \geq 3$, $b < a$, l prime)) There are 2 isomorphism classes of primitive root lattices in V_0 representatives of which can be chosen as L_1, L_2 such that $L_1 \supset L_2 \supset \wp L_1$ for some (principal) prime ideal $\wp$ in R (inclusion pattern ②). Choosing compatible bases one obtains a partition $B = B_1 \,\dot\cup\, B_2$ of an R-basis B of L_1 such that $L_2 = \langle \wp B_1, B_2 \rangle_R$. Let $\wp \mathbb{Z}_K = \wp_1^{i_1} \cdots \wp_s^{i_s}$ be the decomposition of $\wp \mathbb{Z}_K$ into powers of distinct prime ideals in $\mathbb{Z}_K$. Then the $\mathbb{Z}_K G$-lattices $L_{j_1,\ldots,j_s} := \wp_1^{j_1} \cdots \wp_s^{j_s} \langle B_1 \rangle_{\mathbb{Z}_K} \oplus \langle B_2 \rangle_{\mathbb{Z}_K} = \wp_1^{j_1} \cdots \wp_s^{j_s} L_1 + \mathbb{Z}_K L_2$ ($0 \leq j_k \leq i_k$, for all $1 \leq k \leq s$) form a system of representatives of the genera of $\mathbb{Z}_K G$-root lattices in V.

(iii) ($G_2(2^q, 2^s, 2)$, $0 < s < q$) There are 3 isomorphism classes of primitive root lattices in V_0 representatives of which can be chosen as full preimages L_1, L_2, L_3 of the 1-dimensional subspaces of $L/\wp L \cong (R/\wp)^2 \cong \mathbb{F}_2^2$ for some RG-lattice L and the unique (principal) prime ideal $\wp \trianglelefteq R$ over 2 (inclusion pattern ③). Then the root lattices $\wp_1 L_1 + \wp_2 L_2 + \wp_3 L_3$ for ideals $\wp_i \trianglelefteq \mathbb{Z}_K$ dividing $\wp \mathbb{Z}_K$ represent all genera of $\mathbb{Z}_K G$-root lattices in V.

(iv) For $G = G_7$ there are 3 isomorphism classes of primitive root lattices in V_0 (cf. Proposition 14). If one takes $\wp = (1+i)R$ the primitive RG_7-root lattices are as in inclusion pattern ④. Choosing compatible bases one obtains a partition $B = B_1 \,\dot\cup\, B_2$ of an R-basis B of L_1 such that $L_2 = \langle \wp B_1, B_2 \rangle_R$ and $L_3 = \langle \wp^2 B_1, B_2 \rangle_R$. Let $\wp^2 \mathbb{Z}_K = \wp_1^{i_1} \cdots \wp_s^{i_s}$ be the decomposition of $2\mathbb{Z}_K = \wp^2 \mathbb{Z}_K$ into powers of distinct prime ideals in $\mathbb{Z}_K$. Then the $\mathbb{Z}_K G$-lattices $L_{j_1,\ldots,j_s}$ defined as in (ii) above form a system of representatives of the genera of $\mathbb{Z}_K G$-root lattices in V.



5 Root systems

Definition 19 *(cf. [Bro 97],[Coh 76, Definition 4.9]) Let $G \leq GL(V)$ be an irreducible finite complex reflection group. The pair $(\mathcal{R}, e)$ is called a* reduced K-root system *for G if the following conditions hold:*

R0 *$\mathcal{R}$ is a subset of V, $e : \mathcal{R} \to \mathbb{N}_{>1}$.*

R1 *The unit group $\mathbb{Z}_K^*$ acts by multiplication on $\mathcal{R}$ with finitely many orbits and e is constant on these orbits. $\mathcal{R}$ generates V and $K\alpha \cap \mathcal{R} = \mathbb{Z}_K^* \alpha$* for all *$\alpha \in \mathcal{R}$.*

R2 *For all $\alpha \in \mathcal{R}$ there is $\alpha^\vee \in V^* := Hom_K(V, K)$ with*
$\alpha^\vee(\alpha) = 1 - exp(2\pi i / e(\alpha))$.
$\alpha^\vee(\mathcal{R}) \subseteq \mathbb{Z}_K$.
The pseudo-reflection $\rho_\alpha \in GL(V)$ defined by $x\rho_\alpha := x - \alpha^\vee(x)\alpha$ for all *$x \in V$ maps $\mathcal{R}$ into itself.*

R3 *The pseudo-reflections ρ_α with $\alpha \in \mathcal{R}$ generate G.*

Two reduced K-root systems $(\mathcal{R}, e)$ and $(\mathcal{R}', e')$ of G are called equivalent, *if there is $0 \neq a \in K$ such that $\mathcal{R}' = a\mathcal{R}$ and for all $\alpha \in \mathcal{R}$ $e(\alpha) = e'(a\alpha)$.*

As the referee pointed out, it can be seen from [Coh 76, (1.8),(1.9)] that all finite irreducible complex reflection groups G have the following property:

Let $\{\sigma_1, \ldots, \sigma_s\}$ be a generating set of reflections for G. Then there are $J_1, \ldots, J_k \subseteq \{1, \ldots, s\}$ and $g_1, \ldots, g_s \in G$ such that $\langle \sigma_i^{g_i} \mid i \in J_j \rangle$ $(1 \leq j \leq k)$ represent all conjugacy classes of maximal cyclic reflection subgroups of G.

This implies that no proper subset of a reduced K-root system $\mathcal{R}$ is a K-root system of G and that the function $e : \mathcal{R} \to \mathbb{N}_{>1}$ is already determined by the pair $(\mathcal{R}, G)$.

Remark 20 *The reduced K-root systems consist of unions of orbits of roots of G in V. If G has only one conjugacy class of maximal cyclic reflection subgroups then there is only one equivalence class of reduced K-root systems for G.*

Corollary 21 *The groups $G_1(n)$, $G_2(m, m, n)$ (with either m odd or $n \geq 3$) $G_3(m)$, G_4, G_8, G_{12}, G_{16}, G_{20}, G_{22}, G_{23}, G_{24}, G_{25}, G_{27}, $G_{29}, \ldots, G_{37}$ have only one equivalence class of reduced K-root systems.*

Denote by $(\cdot, \cdot)$ the (up to scalar multiples unique) G-invariant totally positive definite Hermitian scalar product on V and let $^-$ denote the complex conjugation of K. Then for $x, y \in V$, $\alpha \in \mathcal{R}$ one has $(x, y) = (x\rho_\alpha, y\rho_\alpha) = (x, y) - \alpha^\vee(x)(\alpha, y) - \overline{\alpha^\vee(y)}(x, \alpha) + \alpha^\vee(x)\overline{\alpha^\vee(y)}(\alpha, \alpha)$. Choosing $y = \alpha$ one finds $(x, \alpha) = ((\alpha, \alpha)(1 - \alpha^\vee(\alpha))/\alpha^\vee(\alpha))\alpha^\vee(x)$. Since $\alpha^\vee \neq 0, 1$ this implies $\alpha^\vee(x) = 0 \Leftrightarrow (\alpha, x) = 0$. Identify V^* with V using the G-invariant Hermitian form. Then $\alpha^\vee = (\alpha^*, \cdot)$ for some $\alpha^* \in V$. One gets $\alpha^* \in (\alpha^\perp)^\perp = \langle \alpha \rangle_K$ and $(\alpha^*, \alpha) = 1 - exp(2\pi i/e(\alpha))$ so

$$\alpha^* = (1 - exp(2\pi i/e(\alpha)))(\alpha, \alpha)^{-1}\alpha.$$

Let L be the $\mathbb{Z}_K$-lattice spanned by $\mathcal{R}$. From **R2** one finds that for all $\alpha \in \mathcal{R}$ the dual root α^* lies in the Hermitian dual lattice

$$L^* := \{x \in V \mid (x, L) \subseteq \mathbb{Z}_K\}.$$

Hence $\mathcal{R}^* := \{\alpha^* \mid \alpha \in \mathcal{R}\} \subseteq L^*$.

Definition 22 *Let L be a $\mathbb{Z}_K$-lattice in V. A vector $v \in L$ is called* primitive *in L if $\frac{1}{a}v \notin L$* for all $0 \neq a \in \mathbb{Z}_K \setminus \mathbb{Z}_K^*$.

To classify all reduced K-root systems for G the following lemma is helpful:

Lemma 23 *Assume that there is an RG-lattice L in V_0 such that for every root α of G, that is a primitive vector in L, the orbit αG spans L as R-lattice and $\alpha^* G$ spans the dual lattice L^* as R-lattice. Then G has only one reduced K-root system up to equivalence.*

Proof: Let $\mathcal{R}$ be a reduced K-root system for G. Since the ρ_α with $\alpha \in \mathcal{R}$ generate G, all root lines of G are represented in $\mathcal{R}$. Replacing $\mathcal{R}$ by an equivalent root system we may assume that $\mathcal{R}$ contains some primitive vector $\alpha \in L$ of $\mathbb{Z}_K L$. Since $\alpha^* G$ spans $(\mathbb{Z}_K L)^* = \mathbb{Z}_K L^*$ as a $\mathbb{Z}_K$-lattice, $\mathcal{R}$ is contained in $\mathbb{Z}_K L$. Now let $\beta \in \mathcal{R}$. Since the root lines are already contained in V_0, there is some $b \in \mathbb{Z}_K$ such that $b\beta \in L$ is a primitive vector in L. Then $\beta^* = \frac{1}{b}(b\beta)^*$ and the orbit of β^* spans $\frac{1}{b}\mathbb{Z}_K L^*$. Since β^* is contained in $\mathbb{Z}_K L^*$ this implies that b is a unit in $\mathbb{Z}_K$. Hence $\mathcal{R}$ precisely consists of the orbits of $\mathbb{Z}_K^*$ on the primitive root vectors in L. □

The root systems of the groups $G_2(m,p,n)$.

Let G be an imprimitive group $G = G_2(m,p,n) \not\cong G_2(m,m,2)$. Let L be the standard monomial RG-lattice in V_0. Then $L = L^*$

Using Lemma 23 one finds the following Corollary.

Corollary 24 *Let $G = G_2(m,p,n)$ such that $m/p \neq 1$ is not a prime power. Then G has an up to equivalence unique reduced K-root system.*

Proof: Representatives for the conjugacy classes of maximal cyclic reflection subgroups of G are generated by $\pi(1)$, $\pi(\zeta_m)$ (if p is even and $n = 2$) and $d(\zeta_m^p)$ (notation as in the proof of Lemma 5). Let α, α' and β be the corresponding roots that are primitive vectors in L. Then $(\alpha,\alpha) = (\alpha',\alpha') = 2$ and $(\beta,\beta) = 1$. Since m/p is no prime power, the element $(1-\zeta_m^p)$ is a unit in R (cf. [Was 82, Proposition 2.8]). Therefore $\alpha = \alpha^*$ and $\alpha' = \alpha'^*$ span $L = L^*$ as an RG-lattice. Now $e(\beta) = m/p$ implies $\beta^* = (1-\zeta_m^p)\beta$ and also β and β^* span L as an RG-lattice. Therefore the Corollary follows from Lemma 23. □

Proposition 25 *Let $n \geq 3$ and $G = G_2(m,p,n)$ with $m \neq p$. Let $a_1, \ldots, a_s$ be the $\mathbb{Z}_K^*$-orbits on the divisors of $(1-\zeta_m^p)$ in $\mathbb{Z}_K$. Let α respectively β be a primitive root of $\pi(1)$ respectively $d(\zeta_m^p)$ in L. Then the equivalence*

classes of reduced K-root systems are represented by $\mathcal{R}(a_i) := a_i\beta G \cup \mathbb{Z}_K^*\alpha G$ $(1 \le i \le s)$.
$G_2(m,m,n)$ has only one equivalence class of reduced K-root systems.

Proof: Let $\mathcal{R}$ be a reduced K-root system. Up to equivalence one may assume that $\beta \in \mathcal{R}$. Then $\mathcal{R} \subseteq \langle \beta^* \rangle_{\mathbb{Z}_K}^* = (1-\zeta_m^p)^{-1}L$ and $\mathcal{R}^* \subseteq L$. Let $a \in K$ such that $\frac{1}{a}\alpha \in \mathcal{R}$. Then $(\frac{1}{a}\alpha)^* = a\alpha^* \in \mathcal{R}^*$ and therefore $a \in \mathbb{Z}_K$. Moreover $\frac{1}{a}\alpha \in (1-\zeta_m^p)^{-1}L$ implies that a divides $(1-\zeta_m^p)$. □

Proposition 26 *Let $G = G_2(m,p,2)$. Let $a_1,\ldots,a_s$ be the $\mathbb{Z}_K^*$-orbits on the divisors of $(1-\zeta_m^p)$ in $\mathbb{Z}_K$. Let α, α', respectively β be a primitive root of $\pi(1)$, $\pi(\zeta_m)$, respectively $d(\zeta_m^p)$ in L.*

a) If $m = p$ is odd, then G has only one equivalence class of reduced K-root systems.

b) If $m \neq p$ then the equivalence classes of reduced K-root systems are represented by $\mathcal{R}(a_i) := a_i\beta G \cup \mathbb{Z}_K^*\alpha G \cup \mathbb{Z}_K^*\alpha' G$ $(1 \le i \le s)$.

c) If $m = p$ is even, then let $a_1,\ldots,a_s$ be the $\mathbb{Z}_K^*$-orbits on the divisors of $(2+\theta_m) = (1+\zeta_m)(1+\zeta_m^{-1})$ (which is a unit unless $\frac{m}{2}$ is a prime power). Then the equivalence classes of reduced K-root systems are represented by $\mathcal{R}(a_i) := a_i\alpha G \cup \mathbb{Z}_K^*\gamma G$ $(1 \le i \le s)$ for some roots α and γ of G.

Proof: a) Follows from Remark 20 and Lemma 5.
b) Let $\mathcal{R}$ be a reduced K-root system for G. Up to equivalence one may assume that $\beta \in \mathcal{R}$. Then $\mathcal{R} \subseteq \langle \beta^* \rangle_{\mathbb{Z}_K}^* = (1-\zeta_m^p)^{-1}L$ and $\mathcal{R}^* \subseteq L$. Let $a \in K$ such that $\frac{1}{a}\alpha \in \mathcal{R}$. Then $(\frac{1}{a}\alpha)^* = a\alpha^* \in \mathcal{R}^*$ and therefore $a \in \mathbb{Z}_K$. Moreover $\frac{1}{a}\alpha \in (1-\zeta_m^p)^{-1}L$ implies that a divides $(1-\zeta_m^p)$. This already implies b) if p is odd, since $\alpha' \in \mathbb{Z}_K^*\alpha G$ in this case. If p is even let L_α be the $\mathbb{Z}_K G$-lattice generated by α. Now let $a' \in K$ such that $\frac{1}{a'}\alpha' \in \mathcal{R}$. Then $\mathcal{R} \subseteq \frac{1}{a}L_\alpha^*$ implies that a divides a' and $\mathcal{R}^* \subseteq aL_\alpha^*$ implies that a' divides a.
c) With respect to the basis (b_1,b_2) in the proof of Lemma 11, the Gram matrix of $(\cdot,\cdot)$ is $\begin{pmatrix} 2 & \theta_m \\ \theta_m & 2 \end{pmatrix}$ (unique up to totally positive multiples).

Assume first that $\frac{m}{2}$ is odd. Then $\alpha := b_1 - b_2$ and $\gamma := b_1 + b_2$ are roots of non conjugate reflections in G. Note that αG spans $L := \langle b_1, b_2 \rangle_R$ and γG a sublattice L' of index $(2+\theta_m)$ of L. Now $\alpha^* = \frac{1}{2-\theta_m}\alpha$ is a unit times α

and hence $\alpha^* G$ also spans L. Moreover $\gamma^* = \frac{1}{2+\theta_m}\gamma$ spans the dual lattice $L^* = (2+\theta_m)^{-1}L'$ of L. Let $\mathcal{R}$ be a reduced K-root system for G. Assume that $\alpha \in \mathcal{R}$. Then $\mathcal{R}, \mathcal{R}^* \subseteq L^*$. There is some $a \in K$ such that $\frac{1}{a}\gamma \in \mathcal{R}$. Then $\frac{a}{2+\theta_m}\gamma \in L^*$ implies that $a \in \mathbb{Z}_K$ and $\frac{1}{a}\gamma \in L$ shows that a divides $2+\theta_m$. This gives c) if $\frac{m}{2}$ is odd.

Now assume that $\frac{m}{2}$ is even. Then $\alpha := \frac{1}{2-\theta_m}(b_1 - b_2)$ and $\gamma := \frac{1}{2-\theta_m}(\theta_m b_1 - 2b_2)$ are roots of non conjugate reflections in G. If m is not a power of 2 then $2-\theta_m = (1-\zeta_m)(1-\zeta_m^{-1})$ and $2+\theta_m = (1+\zeta_m)(1+\zeta_m^{-1})$ are units in R. Hence $L = L^*$ is unimodular, αG and γG span L and $\alpha^* \in \mathbb{Z}_K^*\alpha$ as well as $\gamma^* \in \mathbb{Z}_K^*\gamma$. Therefore c) follows from Lemma 23 in this case.

If $m = 2^a$ ($a \geq 2$) then $(2+\theta_m)$ and $(2-\theta_m)$ both generate the maximal ideal in R over 2. αG generates a unimodular lattice $M = M^*$ and γG a sublattice of index $2-\theta_m$ of M that is isometric to the lattice L above. Let $\mathcal{R}$ be a reduced K-root system for G. Assume that $\alpha \in \mathcal{R}$. There is $a \in K$ such that $\frac{1}{a}\gamma \in \mathcal{R}$. Since $(\frac{1}{a}\gamma)^* = a\gamma^* = a\gamma \in M$ one has $a \in \mathbb{Z}_K$. Moreover $\alpha^* = (2-\theta_m)\alpha$ generates $(2-\theta_m)M$ so $\mathcal{R} \subseteq (2-\theta_m)^{-1}M$. Therefore a divides $(2-\theta_m)$. □

The root systems of the 34 exceptional groups.

The exceptional groups that have only one orbit of root lines have only one equivalence class of reduced root systems and are listed in Corollary 21. So we only deal with the other exceptional finite irreducible complex reflection groups. The most difficult situation occurs for the family $G_4, \ldots, G_7$.

Proposition 27 *Let $a_1, \ldots, a_s$ be the orbits of $\mathbb{Z}_K^*$ on the divisors of 2. Then there are roots α_3 and β_3 of G_5 such that $a_i\alpha_3 G_5 \cup \mathbb{Z}_K^*\beta_3 G_5$ represent the equivalence classes of reduced K-root systems of G_5.*
Let $a_1, \ldots, a_s$ be the orbits of $\mathbb{Z}_K^*$ on the divisors of $(1+i)$. Then there are roots α_2, α_3, and β_3 of G_7 (resp. α_2, α_3 of G_6) such that $\mathbb{Z}_K^*\alpha_2 G_7 \cup a_i\alpha_3 G_7 \cup a_i\beta_3 G_7$ (resp. $\mathbb{Z}_K^*\alpha_2 G_6 \cup a_i\alpha_3 G_6$) $(1 \leq i \leq s)$ represent the equivalence classes of reduced K-root systems of G_7 (resp. G_6).

<u>Proof:</u> Let $G = G_5$ and L be a RG-lattice in V_0 such that $(1-\zeta_3)L^*/L \cong \mathbb{F}_4$. One calculates that one may choose β_3 such that $\beta_3 G$ spans L and α_3 such that $\alpha_3 G$ spans $(1-\zeta_3)L^*$. Moreover $(\beta_3, \beta_3) = 3$ and $(\alpha_3, \alpha_3) = \frac{3}{2}$. Hence $\beta_3^* = (1-\zeta_3^{-1})^{-1}\beta_3$ and $\alpha_3^* = 2(1-\zeta_3^{-1})^{-1}\alpha_3$. Let $\mathcal{R}$ be a reduced K-root

system of G. Up to equivalence one may assume that $\beta_3 \in \mathcal{R}$. Then $\mathcal{R}$ is contained in $\langle \beta_3^* G\rangle^* = \mathbb{Z}_K(1-\zeta_3)L^*$. Hence there is an $a \in \mathbb{Z}_K$ such that $a\alpha_3 \in \mathcal{R}$. But $(a\alpha_3)^* = \frac{2}{a}(1-\zeta_3^{-1})^{-1}\alpha_3$ must lie in L^*. Hence a divides 2. Therefore $\mathcal{R}$ is one of the root systems in the proposition.
Let $G = G_7$ and L be a RG-lattice in V_0 such that $L^* = (1-\zeta_3)^{-1}L$. Calculations show that one may choose a root α_2 of a reflection of order 2 in G such that $\alpha_2 G$ spans L. Furthermore there are roots α_3 and β_3 of pseudo-reflections of order 3 in G that generate non conjugate maximal cyclic reflection subgroups of G, such that $\alpha_3 G$ and $\beta_3 G$ generate lattices L_α and L_β of L with $L_\alpha/L \cong L_\beta/L \cong \mathbb{F}_4$, $L_\alpha \cap L_\beta = L$, and $L_\alpha + L_\beta = (1+i)^{-1}L$. One calculates $(\alpha_2,\alpha_2) = 3+\sqrt{3}$, $(\alpha_3,\alpha_3) = (\beta_3,\beta_3) = \frac{3}{2}$. Therefore $\alpha_2^* = u(1+i)(1-\zeta_3)^{-1}\alpha_2$ for some $u \in R^*$ and $\alpha_3^* = 2(1-\zeta_3^{-1})^{-1}\alpha_3$, $\beta_3^* = 2(1-\zeta_3^{-1})^{-1}\beta_3$. Let $\mathcal{R}$ be a reduced K-root system of G. Up to equivalence one may assume that $\alpha_2 \in \mathcal{R}$. Then $\mathcal{R}$ is contained in $\langle \alpha_2^* G\rangle^* = \mathbb{Z}_K(1+i)^{-1}L$. One concludes that there are $a, b \in \mathbb{Z}_K$ such that $a\alpha_3$ and $b\beta$ lie in $\mathcal{R}$. But $(a\alpha_3)^* = \frac{2}{a}(1-\zeta_3^{-1})^{-1}\alpha_3$ must lie in L^*. Hence a divides $1+i$. Analogously b divides $1+i$. Now $x\alpha_3$ lies in the dual lattice L_β^* if and only if 2 divides x. This implies that a divides b. By symmetry b divides a and therefore $\mathcal{R}$ is one of the root systems in the proposition.
The case $G = G_6$ easily follows from the discussion of the case $G = G_7$.
□

Proposition 28 *The groups $G_9, \ldots, G_{11}$ have up to equivalence only one reduced K-root system.*

<u>Proof:</u> Let $G = G_9$ and L be the unimodular RG-lattice in V_0. Let α be a root of G that is primitive in L. Since L is up to multiples the only RG-lattice in V_0, αG spans L and one calculates that $\alpha^* = \alpha$ spans L^* as RG-lattice. Hence the proposition follows from Lemma 23.
Similarly G_{10} (where one may choose $L \leq V_0$ such that $L^* = (1+i)^{-1}L$) and G_{11} satisfy the assumptions of Lemma 23 and have therefore a unique reduced K-root system up to equivalence. □

Proposition 29 *The group G_{14} has up to equivalence only one reduced K-root system.*
Let $a_1, \ldots, a_s$ be $\mathbb{Z}_K^*$-orbits on the divisors of $(2-\sqrt{2}) \in \mathbb{Z}_K$. Then there are roots α, α_2 of G_{13} resp. α, α_2, and α_3 of G_{15} such that $\mathbb{Z}_K^*\alpha G_{13} \cup a_i\alpha_2 G_{13}$

$(1 \leq i \leq s)$ resp. $\mathbb{Z}_K^*\alpha G_{15} \cup a_i\alpha_2 G_{15} \cup \mathbb{Z}_K^*\alpha_3 G_{15}$ $(1 \leq i \leq s)$ represent the equivalence classes of reduced K-root systems of G_{13} resp. G_{15} in V.

Proof: Let $G = G_{14}$ and L be the unimodular RG_{14}-lattice in V_0. Let α be a root of G that is primitive in L. Since L is up to multiples the only RG-lattice in V_0, αG spans L and one calculates that $\alpha^* = u\alpha$ for some unit $u \in R$ spans L^* as RG-lattice. Hence the proposition follows from Lemma 23.
Now let $G = G_{13}$ and L be the unimodular RG-lattice in V_0. Then G contains two conjugacy classes of reflections. Let α resp. α_2 be primitive vectors in L such that these conjugacy classes are represented by ρ_α and ρ_{α_2}. Since L is up to multiples the only RG-lattice in V_0, αG and also $\alpha_2 G$ spans L. One calculates that $\alpha^* = \alpha$ and $\alpha_2^* = (2-\sqrt{2})\alpha_2$. Let $\mathcal{R}$ be a reduced K-root system for G. Replacing $\mathcal{R}$ by an equivalent root system we may assume that $\mathcal{R}$ contains the primitive vector $\alpha \in L$ of $\mathbb{Z}_K L$. Since $\alpha^* G$ spans $(\mathbb{Z}_K L)^* = \mathbb{Z}_K L^*$ as a $\mathbb{Z}_K$-lattice, $\mathcal{R}$ is contained in $\mathbb{Z}_K L$. Now let $a \in \mathbb{Z}_K$ such that $\beta := a\alpha_2 \in \mathcal{R}$. Then $\beta^* = \frac{1}{a}(\alpha_2)^* = \frac{2-\sqrt{2}}{a}\alpha_2$. Since β^* is contained in $\mathbb{Z}_K L^* = \mathbb{Z}_K L$ this implies that a divides $2-\sqrt{2}$. Now the proposition follows for G_{13} and G_{15} can be dealt with similarly if one notes that $\alpha_3^* = u\alpha_3$ for some unit $u \in R^*$. □

Proposition 30 *If $G = G_{17}$, G_{18}, G_{19}, or G_{21}, then G has up to equivalence only one reduced K-root system.*

Proof: Let L be an RG-lattice in V_0 and α be a primitive vector in L that is a root of G. Since all RG-lattices in V_0 are isomorphic, αG spans L as R-lattice. One calculates that in all cases $\alpha^* G$ spans the dual lattice L^*. Hence the proposition follows from Lemma 23. □

Proposition 31 *The group $G := G_{28} = W(F_4)$ has two orbits of root lines. If $L := F_4 \leq V_0$ denotes the root lattice F_4 with $L^*/L \cong \mathbb{F}_2^2$, then the orbits of primitive roots in L are represented by α and β such that $(\alpha,\alpha) = 2$ and $(\beta,\beta) = 4$. Note that αG generates L and βG generates the sublattices $2L^*$ of L. There are up to equivalence two reduced K_0-root systems $\mathcal{R}$ in V_0: $R^*\alpha G \cup R^*\beta G$ and $(2R^*\alpha)G \cup R^*\beta G$. In general let $a_1, \ldots, a_s$ be the orbits of $\mathbb{Z}_K^*$ on the divisors of $2 \in \mathbb{Z}_K$. Then the reduced K-root systems are equivalent to one of $(a_i\alpha)G \cup \mathbb{Z}_K^*\beta G$ $(1 \leq i \leq s)$.*

Similarly one finds the following

Proposition 32 *Let $G = G_{26}$ and L the RG-lattice that is a $\mathbb{Z}$-lattice isometric to ${}^{(3)}E_6^{\#}$ a rescaling of the dual lattice of E_6. L is the integral RG-lattice with smallest determinant. Then the orbits on the primitive roots of G in L are represented by α and β such that $(\alpha, \alpha) = 2$ and $(\beta, \beta) = 3$. Note that αG generates L and βG generates the sublattice $(1 - \zeta_3)L^*$ of L. Since ρ_α^G generates a subgroup of index 3 of G and ρ_β^G only generates the subgroup G_{25} of G, there are up to equivalence two reduced K_0-root systems $\mathcal{R}$ in V_0: $R^*\alpha G \cup R^*\beta G$ and $((1 - \zeta_3)R^*\alpha)G \cup R^*\beta G$. In general let $a_1, \ldots, a_s$ be the orbits of $\mathbb{Z}_K^*$ on the divisors of $(1 - \zeta_3) \in \mathbb{Z}_K$. Then the reduced K-root systems are equivalent to $(a_i\alpha)G \cup \mathbb{Z}_K^*\beta G$ $(1 \leq i \leq s)$.*

6 Bad primes.

The following definition is a slight modification of the definition given in [Bro 97].

Definition 33 *Let G be an irreducible complex reflection group. A prime $\wp$ of $\mathbb{Z}_K$ is called* bad *for G, if there is a root lattice L of G and a reflection subgroup U of G such that $L(U) := \sum_v v \cap L$, where v runs over the root lines of U has finite index $[L : L(U)] := |L/L(U)|$ divisible by $\wp$. The subgroup U of G that gives rise to the bad prime $\wp$ is called a* bad subgroup of G for $\wp$.

It is well known that the natural representation of an irreducible complex reflection group is absolutely irreducible. For the reducible groups U such that the root lines of U generate the vector space V, one gets a similar result (cf. Proposition 5, [Bou 81, V.3.7]).

Lemma 34 *Let U be a bad subgroup of G for some prime $\wp$. As a KU-module V decomposes in the orthogonal sum of irreducible KU-modules affording pairwise distinct absolutely irreducible representations.*

Since $[L : L(U)] < \infty$ one immediately has that U is the direct product of irreducible complex reflection groups and $L(U)$ decomposes as an orthogonal sum of root lattices of the irreducible factors of U.

From the classification of the root lattices of the irreducible reflection groups one finds the following.

Remark 35 *Bad primes divide the group order.*

Proof: Let L be a $\mathbb{Z}_K G$-root lattice in V. Then there is a primitive root lattice L_0 in V_0, an ideal $\mathcal{A}_0 \trianglelefteq R$ dividing the group order and an ideal $\mathcal{A}$ of $\mathbb{Z}_K$ such that $\mathbb{Z}_K \mathcal{A}_0 L_0 \subseteq \mathcal{A}L \subseteq \mathbb{Z}_K L_0$. Let U be a reflection subgroup of G such that $L(U)$ is of finite index in L. Clearly $\mathcal{A}L(U) = \sum_v v \cap \mathcal{A}L$. Since the root lines are already generated by vectors in V_0, one gets $\mathbb{Z}_K \mathcal{A}_0(\sum_v v \cap L_0) \subseteq \mathcal{A}L(U) \subseteq \mathbb{Z}_K(\sum_v v \cap L_0)$. Therefore it suffices to show that $[L_0 : \sum_v v \cap L_0]$ only involves primes dividing the group order. For all irreducible complex reflection groups there are representatives of the isomorphism classes of primitive root lattices in V_0 such that the determinants and the lengths of the primitive root vectors only involve primes dividing the group order.

Let $\wp$ be a prime of R such that $\wp \mid [L_0 : L_0(U)]$. Assume that $\wp$ does not divide $|G|$. Then the localization $R_\wp \otimes_R L_0(U)$ is of the shape $\mathcal{A}_1 M_1 \perp \ldots \perp \mathcal{A}_s M_s$ where the M_j are localizations of primitive U-root lattices in the corresponding irreducible U-module and the $\mathcal{A}_j$ are ideals of $R_\wp$. Since $\wp$ does not divide the length of a primitive root in L_0 and $L_0(U)$ is generated by primitive roots in L_0, one gets that $\wp$ does not divide any of the ideals $\mathcal{A}_j$. Now the determinant of the M_j is not divisible by $\wp$ which contradicts the fact that $\wp \mid [L_0 : L_0(U)]$. □

In the same way one proves:

Lemma 36 *Let $p \in \mathbb{Z}$ be a prime number and $\wp \trianglelefteq R$ some divisor of pR. Assume that $\wp$ does not divide the determinant of L and the lengths of the primitive roots of G in the root lattice L. If $\wp$ is a bad prime for G, then either the bad subgroups for $\wp$ are absolutely irreducible or p^2 divides the order of G.*

Proof: Let U be a reducible reflection subgroup of G yielding the bad prime $\wp$. Then $L(U)$ is an orthogonal sum $L_1 \perp \ldots \perp L_s$ of root lattices of the irreducible components U_i of $U = U_1 \times \ldots \times U_s$. Since $\wp$ divides the index of $L(U)$ in L, it divides $det(L_i)$ for some $1 \leq i \leq s$. Since the root lengths or the generating roots of L_i are not divisible by $\wp$, it follows that $p \mid |U_i|$. Since $\wp$ does not divide $det(L)$ it divides the determinant of the orthogonal complement $L_i^\perp$ of L_i in L. Therefore there is a second index $1 \leq j \neq i \leq s$ such that $\wp \mid det(L_j)$. As above one concludes that $p \mid |U_j|$ and therefore $p^2 \mid |U| \mid |G|$. □

Table 37 The primitive root lattices of the irreducible complex reflection groups in the three infinite series.

d	group	det	lengths	K_0
n	$G_1(n)$	$n+1$	2	$\mathbb{Q}$
2	$G_2(2l^a, 2l^a, 2)$ $a \geq 1$, l odd	$4-\theta_{2l^a}^2$, $(2+\theta_{2l^a})^3(2-\theta_{2l^a})$	$2(2-\theta_{2l^a})$, $2(2+\theta_{2l^a})$	$\mathbb{Q}[\theta_{2l^a}]$
2	$G_2(2^a, 2^a, 2)$ $a \geq 2$	$(2+\theta_{2^a})(2-\theta_{2^a})^{-1}$, $4-\theta_{2^a}^2$	$2(2-\theta_{2^a})^{-1}$, $2(2+\theta_{2^a})(2-\theta_{2^a})^{-1}$	$\mathbb{Q}[\theta_{2^a}]$
2	$G_2(m, m, 2)$ m odd	$4-\theta_m^2$	$2(2-\theta_m)$	$\mathbb{Q}[\theta_m]$
2	$G_2(m, m, 2)$ $m = 2x$, $x \neq l^a$	$4-\theta_m^2$, $4-\theta_m^2$	$2(2-\theta_m)$, $2(4-\theta_m^2)$	$\mathbb{Q}[\theta_m]$
2	$G_2(2^a, 2^b, 2)$ $0 < b < a$	$2-\theta_{2^{a-1}}$, $2-\theta_{2^{a-1}}$, 1	2, 2, 1	$\mathbb{Q}[\zeta_{2^a}]$
2	$G_2(2^a, 1, 2)$ $1 < a$	$2-\theta_{2^a}$, 1	2, 1	$\mathbb{Q}[\zeta_{2^a}]$
2	$G_2(l^a, l^b, 2)$ $b < a$, l odd	$2-\theta_{l^a}$, 1	2, 1	$\mathbb{Q}[\zeta_{l^a}]$
2	$G_2(m, p, 2)$ $m \neq p$, $m \neq l^a$	1, 1	1, 2	$\mathbb{Q}[\zeta_m]$
$n \geq 3$	$G_2(l^a, l^b, n)$ $a > b$	$2-\theta_{l^a}$, 1	2, 1	$\mathbb{Q}[\zeta_{l^a}]$
$n \geq 3$	$G_2(l^a, l^a, n)$	$2-\theta_{l^a}$	2	$\mathbb{Q}[\zeta_{l^a}]$
$n \geq 3$	$G_2(m, p, n)$ $m \neq p$, $m \neq l^a$	1, 1	2, 1	$\mathbb{Q}[\zeta_m]$
$n \geq 3$	$G_2(m, m, n)$ $m \neq l^a$	1	2	$\mathbb{Q}[\zeta_m]$
1	$G_3(n)$	1	1	$\mathbb{Q}[\zeta_n]$

The first column gives the dimension $dim(V_0)$, the second the name of the group G and conditions on the parameters. Here l denotes a prime number and $m \neq l^a$ means that m is not a prime power. The third column contains the determinants of representatives of the isomorphism classes of the primitive root lattices L of G in V_0, followed by a column that indicates the lengths of the respective roots that span L. The last column gives the character field K_0 of the reflection representation of G.

Theorem 38 *The groups* $G_1(n) = W(A_n)$ $(n \geq 2)$ *and the groups* $G_3(m) \cong C_m$ *have no bad primes.*

The bad primes for the groups $G_2(m,p,n)$ *not isomorphic to* $G_2(l,l,2)$ *or* $G_2(l,l,3)$ *for some prime number* l *are exactly the prime divisors of* m. $G_2(l,l,2)$ *and* $G_2(l,l,3)$ *have no bad primes.*

Proof: $G_1(n)$ has no bad primes, as one sees as follows: The orthogonal rank $OR(X)$ of a root lattice of a real reflection group is the maximal number of pairwise orthogonal roots in X. It holds that $OR(A_n) = \frac{n}{2}$ if n is even and $OR(A_n) = \frac{n+1}{2}$ if n is odd. $OR(D_n) = n$ if n is even and $n-1$ if n is odd, $OR(F_4) = 4$, $OR(E_6) = 4$, $OR(E_7) = 7$, $OR(E_8) = 8$. Assume that $A_n \geq X_1 \perp \ldots \perp X_s =: X$ contains a root lattice X. Then $\sum_{i=1}^{s} OR(X_i) \leq OR(A_n)$. If follows that all X_i are of type A_{n_i} for some n_i with $\sum_{i=1}^{s} n_i = n$. But then the group generated by reflections along the roots of X is $\prod_{i=1}^{s} S_{n_i+1}$. This is only a subgroup of S_{n+1} if $s = 1$. Therefore $A_n = X$ and $G_1(n)$ has no bad primes.
Since the degree of the natural character of $G_3(m)$ is 1, it is clear that $G_3(m)$ has no bad primes.
Let H be one of the 34 exceptional finite irreducible complex reflection groups. Then H has no abelian normal subgroup of index dividing $dim(V_0)!$. Therefore the irreducible components of the bad subgroups of the groups $G_2(m,p,n)$ are among the groups $G_3(m')$ and $G_2(m',p',n')$ with m' dividing m. Comparing the determinants of the primitive root lattices (rescaled in such a way that they are spanned by roots of length 1) of these groups one finds that the bad primes for $G_2(m,p,n)$ divide $2m$.

We first treat the case where $G = G_2(m,p,n)$ and m is not prime. Let $l \in \mathbb{N}$ be a prime divisor of m. Then the group $U := G_2(l,l,n)$ is a subgroup of $G_2(m,p,n)$. From Table 37 it follows that there is a primitive root lattice L of $G_2(m,p,n)$ such that l divides $[L : L(U)]$.

If $m = l$ is a prime, then $G = G_2(l,1,n)$ or $G = G_2(l,l,n)$. The group $G_2(l,1,n)$ contains the bad subgroup $U := G_3(l)^n \cong C_l^n$. If L is the primitive root lattice of determinant $2 - \theta_l$ then l divides the index of $L(U)$ in L. If $n \geq 4$ then the group $G_2(l,l,n)$ contains the pseudo reflection subgroup $G_2(l,l,n-2) \times G_2(l,l,2)$ which is bad for the prime l.

Since the determinants of the primitive root lattices generated by roots of length 2 respectively 1 are odd, one sees that for odd m the prime 2 is not bad for the groups $G_2(m,p,n)$, by comparing determinants and root lengths in Table 37.

It remains to show that for primes l the groups $G_2(l,l,2)$ and $G_2(l,l,3)$ have no bad primes. Let U be a bad subgroup of $G_2(l,l,2)$. Then U contains at least two reflections. But any two reflections in $G_2(l,l,2)$ generate the whole group. Similarly if one chooses 3 reflections in $G_2(l,l,3)$ such that the root vectors are linearly independent, they generate $G_2(l,l,3)$. $\square$

Table 39 The primitive root lattices of the 34 exceptional irreducible complex reflection groups.

d	group	det	lengths	K_0	refl. orders
2	G_4	6	3	$\mathbb{Q}[\zeta_3]$	3
2	G_5	6, $\frac{3}{2}$	3, $\frac{3}{2}$	$\mathbb{Q}[\zeta_3]$	3,3
2	G_6	3, 6	$3+\sqrt{3}$, 3	$\mathbb{Q}[\zeta_{12}]$	2,3
2	G_7	3, 6, 6	$3+\sqrt{3}$, 3, 3	$\mathbb{Q}[\zeta_{12}]$	2,3,3
2	G_8	2	2	$\mathbb{Q}[\zeta_4]$	2
2	G_9	1	2, $2+\sqrt{2}$	$\mathbb{Q}[\zeta_8]$	2,2
2	G_{10}	2	2, $3+\sqrt{3}$	$\mathbb{Q}[\zeta_{12}]$	2,3
2	G_{11}	1	2, $2+\sqrt{2}$, $3+\sqrt{6}$	$\mathbb{Q}[\zeta_{24}]$	2,2,3
2	G_{12}	1	2	$\mathbb{Q}[\sqrt{-2}]$	2
2	G_{13}	1	2, $2+\sqrt{2}$	$\mathbb{Q}[\zeta_8]$	2,2
2	G_{14}	1	2, $3+\sqrt{6}$	$\mathbb{Q}[\zeta_3,\sqrt{-2}]$	2,3
2	G_{15}	1	2, $2+\sqrt{2}$, $3+\sqrt{6}$	$\mathbb{Q}[\zeta_{24}]$	2,2,3
2	G_{16}	$\frac{5+\sqrt{5}}{2}$	$5+2\sqrt{5}$	$\mathbb{Q}[\zeta_5]$	5
2	G_{17}	$\frac{5+\sqrt{5}}{2}$	$5+2\sqrt{5}$, $2u(1-\zeta_5)$	$\mathbb{Q}[\zeta_{20}]$	5,2
2	G_{18}	$\frac{5+\sqrt{5}}{2}$	$5+2\sqrt{5}$, $u(1-\zeta_3)(1-\zeta_5)$	$\mathbb{Q}[\zeta_{15}]$	5, 3
2	G_{19}	$\frac{5+\sqrt{5}}{2}$	$5+2\sqrt{5}$, $2u(1-\zeta_5)$,	$\mathbb{Q}[\zeta_{60}]$	5,2,
			$u(1-\zeta_3)(1-\zeta_5)$		3
2	G_{20}	3	3	$\mathbb{Q}[\zeta_3,\sqrt{5}]$	3
2	G_{21}	1	2, $u(1-\zeta_3)$	$\mathbb{Q}[\zeta_{12},\sqrt{5}]$	2,3
2	G_{22}	1	2	$\mathbb{Q}[\zeta_4,\sqrt{5}]$	2

d	group	det	lengths	K_0	refl. orders
3	G_{23}	2	2	$\mathbb{Q}[\sqrt{5}]$	2
3	G_{24}	1	2	$\mathbb{Q}[\sqrt{-7}]$	2
3	G_{25}	3^2	3	$\mathbb{Q}[\zeta_3]$	3
3	G_{26}	3^2, 3	3, 2	$\mathbb{Q}[\zeta_3]$	3,2
3	G_{27}	1	2	$\mathbb{Q}[\zeta_3, \sqrt{5}]$	2
4	G_{28}	2^2, $2^2{\cdot}4^2$	2, 4	$\mathbb{Q}$	2,2
4	G_{29}	1	2	$\mathbb{Q}[\zeta_4]$	2
4	G_{30}	1	2	$\mathbb{Q}[\sqrt{5}]$	2
4	G_{31}	1	2	$\mathbb{Q}[\zeta_4]$	2
4	G_{32}	3^2	3	$\mathbb{Q}[\zeta_3]$	3
5	G_{33}	2	2	$\mathbb{Q}[\zeta_3]$	2
6	G_{34}	1	2	$\mathbb{Q}[\zeta_3]$	2
6	G_{35}	3	2	$\mathbb{Q}$	2
7	G_{36}	2	2	$\mathbb{Q}$	2
8	G_{37}	1	2	$\mathbb{Q}$	2

The first column gives the dimension $dim(V_0)$, the second the number of the group G in [ShT 54]. The third column contains the determinants of representatives of the isomorphism classes of the primitive root lattices L of G in V_0, followed by a column that indicates the lengths of the respective roots that span L. Here u stands for suitable units in R. The last column gives the orders of the corresponding reflections.

Theorem 40 *The bad primes for the 34 exceptional groups are exactly the primes of $\mathbb{Z}_K$ that divide the integral primes in the last column of Table 41.*

Table 41 Subgroups yielding the bad primes of the 34 exceptional finite irreducible complex reflection groups.
The first column gives the the name of the group G in [ShT 54] followed by the order of G. The bad primes for G are precisely the primes of $\mathbb{Z}_K$ that divide the rational primes in the boldface brackets after the bad reflection subgroup of G in the last column. Note that this column does not contain all bad subgroups of G but only one for each bad (rational) prime.

G	$\lvert G\rvert$	bad primes
G_4	$2^3{\cdot}3$	–
G_5	$2^3{\cdot}3^2$	G_4 **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_6	$2^4{\cdot}3$	$G_3(2)\times G_3(2)$ **(2)**
G_7	$2^4{\cdot}3^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_8	$2^5{\cdot}3$	$G_3(2)\times G_3(2)$ **(2)**
G_9	$2^6{\cdot}3$	$G_3(2)\times G_3(2)$ **(2)**, $G_1(2)$ **(3)**
G_{10}	$2^5{\cdot}3^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_{11}	$2^6{\cdot}3^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_{12}	$2^4{\cdot}3$	$G_3(2)\times G_3(2)$ **(2)**, $G_1(2)$ **(3)**
G_{13}	$2^5{\cdot}3$	$G_3(2)\times G_3(2)$ **(2)**, $G_1(2)$ **(3)**
G_{14}	$2^4{\cdot}3^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_{15}	$2^5{\cdot}3^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_{16}	$2^3{\cdot}3{\cdot}5^2$	$G_3(5)\times G_3(5)$ **(5)**
G_{17}	$2^4{\cdot}3{\cdot}5^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_1(2)$ **(3)**, $G_3(5)\times G_3(5)$ **(5)**
G_{18}	$2^3{\cdot}3^2{\cdot}5^2$	G_5 **(2)**, $G_3(3)\times G_3(3)$ **(3)**, $G_3(5)\times G_3(5)$ **(5)**
G_{19}	$2^4{\cdot}3^2{\cdot}5^2$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**, $G_3(5)\times G_3(5)$ **(5)**
G_{20}	$2^3{\cdot}3^2{\cdot}5$	G_5 **(2)**, $G_3(3)\times G_3(3)$ **(3)**
G_{21}	$2^4{\cdot}3^2{\cdot}5$	$G_3(2)\times G_3(2)$ **(2)**, $G_3(3)\times G_3(3)$ **(3)**, $G_2(5,5,2)$ **(5)**
G_{22}	$2^4{\cdot}3{\cdot}5$	$G_3(2)\times G_3(2)$ **(2)**, $G_1(2)$ **(3)**, $G_2(5,5,2)$ **(5)**
G_{23}	$2^3{\cdot}3{\cdot}5$	$G_3(2)\times G_3(2)\times G_3(2)$ **(2)**
G_{24}	$2^4{\cdot}3{\cdot}7$	$G_1(3)$ **(2)**
G_{25}	$2^3{\cdot}3^4$	$G_3(3)\times G_3(3)\times G_3(3)$ **(3)**
G_{26}	$2^4{\cdot}3^4$	G_{25} **(2)**, $G_3(3)\times G_3(3)\times G_3(3)$ **(3)**
G_{27}	$2^4{\cdot}3^3{\cdot}5$	G_{23} **(2)**, $G_2(3,3,3)$ **(3)**
G_{28}	$2^7{\cdot}3^2$	$G_2(2,1,4)$ **(2)**, $G_1(2)\times G_1(2)$ **(3)**
G_{29}	$2^9{\cdot}3{\cdot}5$	$G_3(2)^4$ **(2)**, $G_1(4)$ **(5)**
G_{30}	$2^6{\cdot}3^2{\cdot}5^2$	$G_{23}\times G_3(2)$ **(2)**, $G_1(2)\times G_1(2)$ **(3)**, $G_2(5,5,2)\times G_2(5,5,2)$ **(5)**
G_{31}	$2^{10}{\cdot}3^2{\cdot}5$	G_{28} **(2)**, $G_1(2)\times G_1(2)$ **(3)**, $G_1(4)$ **(5)**
G_{32}	$2^7{\cdot}3^5{\cdot}5$	$G_4\times G_4$ **(2)**, $G_{25}\times G_3(3)$ **(3)**
G_{33}	$2^7{\cdot}3^4{\cdot}5$	$G_{28}\times G_3(2)$ **(2)**, $G_1(5)$ **(3)**
G_{34}	$2^9{\cdot}3^7{\cdot}5{\cdot}7$	$G_{33}\times G_3(2)$ **(2)**, G_{35} **(3)**, $G_1(6)$ **(7)**
G_{35}	$2^7{\cdot}3^4{\cdot}5$	$G_1(5)\times G_3(2)$ **(2)**, $G_1(2)^3$ **(3)**
G_{36}	$2^{10}{\cdot}3^4{\cdot}5{\cdot}7$	$G_3(2)^7$ **(2)**, $G_1(5)\times G_1(2)$ **(3)**
G_{37}	$2^{14}{\cdot}3^5{\cdot}5^2{\cdot}7$	$G_3(2)^8$ **(2)**, $G_{35}\times G_1(2)$ **(3)**, $G_1(4)\times G_1(4)$ **(5)**

Proof: That the primes in Table 41 are in fact bad primes for the corresponding groups is proved by finding bad subgroups. So we only prove that the list exhausts the bad primes. By Remark 35 it suffices to show that the other prime divisors of the group order are not bad. This is done with the classification of all finite irreducible complex reflection groups.

We first deal with the 2-dimensional exceptional groups $G_4, \ldots, G_{22}$. There one has to rule out the possibilities of the bad primes $2, 3$ for G_4, 3 for G_6 and G_8, $2, 3$ for G_{16}, and 5 for G_{20}:

G_4 is the unique two-dimensional reflection group containing only reflections of order 3 of which the group order is not divisible by 3^2. Hence G_4 has no bad primes.

Similarly the unique candidate for a bad subgroup of G_6 for the prime 3 is $G_3(2) \times G_3(3)$ which is not contained in G_6.

Since G_8 contains only reflections of order 2, the only candidate for a bad subgroup yielding the bad prime 3 for the group G_8 is $G_1(2)$. Since the determinant of the primitive root lattice L of G_8 is 2, this implies that the corresponding root lattice of $G_1(2)$ spanned by the intersections of the root lines of $G_1(2)$ with L is the rescaled lattice ${}^{(2)}A_2$ of determinant $2^2 \cdot 3$. But the primitive roots in L have length 2 which is a contradiction.

Since G_{16} only contains reflections of order 5, one easily sees that 2 and 3 are no bad primes fo G_{16}.

Since $D_{10} = G_2(5, 5, 2)$ is not contained in G_{20} and G_{20} has no reflections of order 5, it follows that 5 is not bad for G_{20}. Hence the theorem is proved for the 2-dimensional exceptional reflection groups.

With Lemma 36 one shows that 3 and 7 resp. 3 and 5 are not bad for G_{24} resp. G_{23}.

Let U be a bad reflection subgroup of G_{25} for the prime 2. Then U only contains reflections of order 3, and $|U|$ is only divisible by 2 and 3. From the classification of the complex reflection groups one finds that U contains a subgroup $G_4 \times G_3(3)$. Therefore U normalizes a Sylow 2-subgroup P of G_{25}. But $N_G(P) = Z(G) \times SL_2(3) \cong C_3 \times G_4$ is not a reflection subgroup.

With Lemma 36 one shows that 5 is not bad for G_{27} and that 3 is not bad for G_{29}.

Let U be a bad subgroup of G_{32} for $p = 5$. Then by Lemma 36 U is absolutely irreducible. One finds that $U = W(A_4) = G_1(4)$ is the only candidate. But G_{32} has no reflections of order 2, so $U \not\leq G_{32}$.

The prime 5 is not bad for G_{33} since there are no irreducible reflection groups

of degree 5 of order dividing $|G_{33}|$ involving the prime 5 in the determinant of a primitive root lattice. Since $\nu_5(|G_{33}|) = 1$, 5 is not bad by Lemma 36. Analogously one sees that 5 is not bad for G_{34}.
The last three groups are the real reflection groups $W(E_6)$, $W(E_7)$, and $W(E_8)$. The bad primes are exactly the ones dividing the indices of root sublattices of full rank in the three root lattices. To show that there are no other bad primes one uses Lemma 36. □

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