Subspaces Fixed by a Nilpotent Matrix

Marvin Hahn, Gabriele Nebe, Mima Stanojkovski, and Bernd Sturmfels

Abstract

The linear spaces that are fixed by a given nilpotent $n \times n$ matrix form a subvariety of the Grassmannian. We classify these varieties for small n. Mutiah, Weekes and Yacobi conjectured that their radical ideals are generated by certain linear forms known as shuffle equations. We prove this conjecture for $n \leq 7$, and we disprove it for $n = 8$. The question remains open for nilpotent matrices arising from the affine Grassmannian.

1 Introduction

For an arbitrary field K, the Grassmannian $\mathrm{Gr}(\ell, n)$ parametrizes ℓ-dimensional subspaces L of the vector space K^n. Given any matrix $T \in K^{n \times n}$, we write LT for the image of L under the map given by T. This right action is compatible with representing L as the row space of an $\ell \times n$ matrix $\mathbf{L}$. The Plücker embedding of $\mathrm{Gr}(\ell, n)$ into $\mathbb{P}^{\binom{n}{\ell}-1}$ arises by representing L with the vector of maximal minors $p_{i_1 i_2 \cdots i_\ell}$ of $\mathbf{L}$. Its homogeneous prime ideal has a natural Gröbner basis of quadrics [5, Theorem 3.1.7]. These are known as the *Plücker quadrics*.

In this paper we assume that T is nilpotent, i.e. $T^n = 0$, and we study the subvariety

$$\mathrm{Gr}(\ell, n)^T \;=\; \bigl\{ L \in \mathrm{Gr}(\ell, n) \,:\, LT \subseteq L \bigr\}.$$

We are interested in its homogeneous radical ideal in the Plücker coordinates $p_{i_1 i_2 \cdots i_\ell}$.

Example 1 $(n = 4, \ell = 2)$**.** Fix a nonzero scalar ϵ and consider the nilpotent 4×4 matrix

$$T \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fixed point locus $\mathrm{Gr}(2,4)^T$ is a singular surface in the 4-dimensional Grassmannian $\mathrm{Gr}(2,4) = V(p_{12}p_{34} - p_{13}p_{23} + p_{14}p_{23})$. It is the quadratic cone in $\mathbb{P}^3$ defined by the prime ideal

$$\langle\, p_{13},\, p_{14} + \epsilon\, p_{23},\, p_{12}p_{34} - \epsilon p_{23}^2 \,\rangle \quad = \quad \langle\, p_{13},\, p_{14} + \epsilon\, p_{23} \,\rangle \,+\, \text{ideal of } \mathrm{Gr}(2,4). \qquad (1)$$

On an affine chart of $\mathrm{Gr}(2,4)$, each plane L that is fixed by T is the row span of a matrix

$$\mathbf{L} \;=\; \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & 0 & \epsilon x \end{pmatrix}, \quad \text{or} \quad \mathbf{L} \;=\; \begin{pmatrix} \epsilon z & w & 1 & 0 \\ 0 & z & 0 & 1 \end{pmatrix} \;\text{ after setting }\; x = \frac{1}{\epsilon z},\, y = -\frac{w}{\epsilon z^2}.$$

Next consider the special case $\epsilon = 0$. The ideal (1) is still radical, but it now decomposes:

$$\langle\, p_{13}\,,\, p_{14}\,,\, p_{12}p_{34}\,\rangle \quad = \quad \langle\, p_{13}, p_{14}, p_{34}\,\rangle \,\cap\, \langle\, p_{12}, p_{13}, p_{14}\,\rangle. \tag{2}$$

The quadratic cone degenerates into two planes $\mathbb{P}^2$ in $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$. They are given by

$$\mathbf{L} \;=\; \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{L} \;=\; \begin{pmatrix} 0 & w & 1 & 0 \\ 0 & z & 0 & 1 \end{pmatrix}.$$

We conclude that $\mathrm{Gr}(2,4)^T$ can be singular or reducible. For all values of $\epsilon \in K$, its radical ideal is generated by two linear forms plus the Plücker quadric $p_{12}p_{34} - p_{13}p_{23} + p_{14}p_{23}$. $\diamond$

We assume from now on that the nilpotent matrix T is in Jordan canonical form. The necessary change of basis in K^n works over an arbitrary field K because all the eigenvalues of T are zero. The matrix T in Example 1 is in Jordan canonical form when $\epsilon = 0$ or $\epsilon = 1$.

Kreiman, Lakshmibai, Magyar, and Weyman [2] identified a natural set of linear forms in Plücker coordinates that vanish on $\mathrm{Gr}(\ell,n)^T$. These are called *shuffle equations* and they generalize the two linear forms seen in (1). It was conjectured in [2] that the shuffle equations cut out certain models of the affine Grassmannian. Muthiah, Weekes and Yacobi [4] gave a reformulation of the shuffle equations, and they proved the main conjecture of [2]. We refer to [4, Section 6] for that proof and for a conceptual discussion of the shuffle equations. It was subsequently conjectured in [4, Section 7] that the shuffle equations plus the Plücker quadrics generate the radical ideal of $\mathrm{Gr}(\ell,n)^T$. The present paper settles that conjecture.

Our presentation is organized as follows. In Section 2 we review the shuffle equations and we show how to generate them in `Macaulay2` [1]. The duality result in Theorem 5 allows us to swap ℓ and $n-\ell$ in these computations. In Section 3 we present the classification of all varieties $\mathrm{Gr}(\ell,n)^T$ for $n \leq 8$. We compute their dimensions, degrees, irreducible components, and defining equations. We disprove the conjecture of Muthiah, Weekes and Yacobi [4, Conjecture 7.6] for $n = 8$, and we show that it holds for $n \leq 7$. Section 4 is devoted to finite-dimensional models of the affine Grassmannian. Here T is the nilpotent matrix given by a partition of rectangular shape. We prove that $\mathrm{Gr}(\ell,n)^T$ is irreducible for such T, and we give a matrix parametrization. We believe that Conjecture 7.1 in [4] holds. This is equivalent to [4, Conjecture 7.6] for rectangular shapes. We offer supporting evidence.

2 Shuffle Equations

Fix a nilpotent $n\times n$ matrix $T = T_\lambda$ in Jordan canonical form. Here $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$ is any partition of the integer n. Each entry of the matrix T_λ is either 0 or 1. The entries 1 are located in positions $(j, j+1)$ where $j \in \{1,2,\ldots,n-1\}\backslash\{\lambda_1, \lambda_1+\lambda_2, \ldots, \lambda_1+\cdots+\lambda_{s-1}\}$. In other words, T_λ is the nilpotent matrix in Jordan canonical form where the sizes of the Jordan blocks are given by the parts λ_i of the partition λ. The rank of T_λ equals $n-s$. We regard $\ker(T_\lambda)$ as a linear subspace of dimension $s-1$ in the projective space $\mathbb{P}^{n-1}$.

The shuffle relations are defined as follows. Consider the $n \times n$ matrix $\mathrm{Id}_n + zT$ where z is a parameter. For $z \in K$, this is an automorphism of the vector space K^n. A subspace

L of K^n satisfies $LT \subseteq L$ if and only if $L\,(\mathrm{Id}_n + zT) = L$ for all z. Writing $P \in K^{\binom{n}{\ell}}$ for the row vector of Plücker coordinates of L, the last equation is equivalent to the identity

$$P \cdot \wedge_\ell(\mathrm{Id}_n + zT) \;=\; P. \tag{3}$$

Here $\wedge_\ell(\mathrm{Id}_n + zT)$ is the ℓth exterior power of the $n \times n$ matrix $\mathrm{Id_n} + zT$. This is an $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix whose entries are polynomials in $\mathbb{Z}[z]$ of degree $\leq \ell$. Equivalently, we can write

$$\wedge_\ell(\mathrm{Id}_n + zT) \;=\; \wedge_\ell \mathrm{Id}_n \;+\; \sum_{i=1}^{\ell} [\, \wedge_\ell(\mathrm{Id}_n + zT)\,]_i \, z^i, \tag{4}$$

where the coefficient of z^i is an integer matrix of format $\binom{n}{\ell} \times \binom{n}{\ell}$. From (3) we then obtain

$$P \cdot [\, \wedge_\ell(\mathrm{Id}_n + zT)\,]_i \;=\; 0 \quad \text{for } \; i = 1, 2, \ldots, \ell. \tag{5}$$

This is a finite collection of linear forms in the $\binom{n}{\ell}$ Plücker coordinates $p_{i_1 i_2 \cdots i_\ell}$. These are the *shuffle equations* of T. The following was proved by Muthiah et al. in [4, Proposition 6.6].

Proposition 2. *The variety* $\mathrm{Gr}(\ell, n)^T$ *is the intersection of the Grassmannian* $\mathrm{Gr}(\ell, n)$ *with a linear subspace in* $\mathbb{P}^{\binom{n}{\ell}-1}$. *That linear subspace is defined by the shuffle equations.*

Example 3 ($n = 4, \ell = 2$)**.** We compute the shuffle equations for the matrix T in Example 1. Write $P = (p_{12}, p_{13}, p_{23}, p_{14}, p_{24}, p_{34})$. With this ordering of the Plücker coordinates, we have

$$\wedge_2(\mathrm{Id}_4 + zT) \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & z & \epsilon z & \epsilon z^2 & 0 \\ 0 & 0 & 1 & 0 & \epsilon z & 0 \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In (5), we find $P \cdot [\, \wedge_2(\mathrm{Id}_4 + zT)\,]_1 \;=\; \big(0, 0, p_{13}, \epsilon p_{13}, (\epsilon p_{23} + p_{14}), 0\big)$ and $P \cdot [\, \wedge_2(\mathrm{Id}_4 + zT)\,]_2 = \big(\,0,\; 0,\; 0,\; 0,\; \epsilon p_{13},\; 0\,\big)$. The coordinates are the shuffle equations. We saw these in (1). $\diamond$

In the next example we demonstrate how the shuffle equations can be computed and analyzed within the computer algebra system `Macaulay2` [1]. All computations of the varieties $\mathrm{Gr}(\ell, n)^T$ in this paper were carried out by this code, with `n`, `l` and `U` $= \mathrm{Id}_n + zT$ adjusted.

Example 4. We examine the smallest instance where $\mathrm{Gr}(\ell, n)^T$ has three irreducible components, namely $n{=}6, \ell{=}3$ and $\lambda = (3, 1, 1, 1)$, as seen in Table 2 below. The following `Macaulay2` code outputs the ideal `J` generated by the shuffle equations and Plücker quadrics:

```
n=6; l=3;
R = ring Grassmannian(l-1,n-1,CoefficientRing => QQ);
P = matrix{gens R}; S = R[z];
U = matrix {{1,z,0,0,0,0},
            {0,1,z,0,0,0},
```

```
            {0,0,1,0,0,0},
            {0,0,0,1,0,0},
            {0,0,0,0,1,0},
            {0,0,0,0,0,1}};

M = (toList coefficients(P*exteriorPower(l,U)))_1;
rowws = toList(0..((# entries M)-2));
I = minors(1,submatrix(M,rowws,))
J = I+Grassmannian(l-1,n-1,R); toString mingens J
betti mingens J, (dim J)-2, degree J
J == radical(J), isPrime J
```

The output of this code shows that the ideal J is radical but not prime. It is minimally generated by 12 linear forms and 8 quadrics. Its variety $\mathrm{Gr}(3,6)^T$ has dimension 4 and degree 2 in the ambient space $\mathbb{P}^{19}$ of $\mathrm{Gr}(3,6)$. We next compute the prime decomposition:

```
DJ =  decompose J; #DJ, betti mingens radical J
apply(DJ, T -> {T,codim T, degree T, betti mingens T})
```

The fixed point locus $\mathrm{Gr}(3,6)^T$ has three irreducible components. The largest component is defined by a quadric in a subspace $\mathbb{P}^5$. In addition, there are two coordinate subspaces $\mathbb{P}^3$. $\diamond$

We next come to a duality result which will aid our computations in Section 3.

Theorem 5. *The varieties $\mathrm{Gr}(\ell,n)^T$ and $\mathrm{Gr}(n-\ell,n)^T$ coincide after a linear change of coordinates in the ambient space $\mathbb{P}^{\binom{n}{\ell}-1}$. This holds for all ℓ, n and all nilpotent $n\times n$ matrices T. Under this coordinate change, which depends on T, the shuffle equations coincide.*

Proof. Let $\mathrm{B}_m = (b_{ij})$ denote the $m\times m$ matrix with 1's on the antidiagonal and 0's elsewhere, i.e. $b_{ij} = 1$ if $i+j = m+1$ and $b_{ij} = 0$ otherwise. Given a partition λ of n and its matrix $T = T_\lambda$, we define $B = B_\lambda$ to be the block-diagonal $n\times n$ matrix $B_\lambda = \mathrm{diag}(\mathrm{B}_{\lambda_1},\ldots,\mathrm{B}_{\lambda_s})$. Note that $B^2 = \mathrm{Id}_n$ and $TB = BT^{\mathrm{t}}$. Here T^{t} denotes the transpose of the matrix T.

We consider the non-degenerate inner product on K^n that is defined by the invertible symmetric matrix B. The orthogonal complement of a given ℓ-dimensional subspace L with respect to this inner product is the $(n-\ell)$-dimensional subspace

$$L^\perp \;=\; \ker(\mathbf{L}B) \;=\; \big\{\, v\in K^n \,:\, uBv^t = 0 \text{ for all } u\in L \big\}.$$

Suppose L is T-fixed. We claim that $L^\perp$ is T-fixed. Indeed, suppose $v\in L^\perp$, i.e. $uBv^t = 0$ for all $u\in L$. This implies $uB(vT)^t = uBT^tv^t = (uT)Bv^t = 0$ for all $u\in L$, and so $vT\in L^\perp$. This shows that taking the orthogonal component defines the desired linear isomorphism

$$\mathrm{Gr}(\ell,n)^T \longrightarrow \mathrm{Gr}(n-\ell,n)^T, \quad L\longmapsto L^\perp. \tag{6}$$

For $T = 0_n$, this is the familiar isomorphism between the Grassmannians $\mathrm{Gr}(\ell,n)$ and $\mathrm{Gr}(n-\ell,n)$. A subtle point is that duality is taken relative to the inner product given by B.

We shall explicitly describe the linear change of coordinates on $\mathbb{P}^{\binom{n}{\ell}-1}$ that induces the isomorphism (6). We start with the *Hodge star isomorphism* $P \mapsto P^*$ that takes the vector $P = (p_{i_1\cdots i_\ell})_{1\leq i_1<\cdots<i_\ell\leq n}$ to the vector $P^* = (p^*_{j_1\cdots j_{n-\ell}})_{1\leq j_1<\cdots<j_{n-\ell}\leq n}$. If I is an ordered ℓ-subset of $[n]=\{1,2,\ldots,n\}$ and $J=[n]\backslash I$ is the complementary ordered $(n-\ell)$-subset then

$$p^*_J \;:=\; \mathrm{sign}(I,J)\cdot p_I.$$

Here $\mathrm{sign}(I,J)$ is the sign of the permutation of $[n]$ given by the ordered sequence (I,J).

To be completely explicit, here is an example. For n=4 the formula for the Hodge star is

$$\begin{aligned} P^* \;&=\; (\,p^*_1,p^*_2,p^*_3,p^*_4 \mid p^*_{12},p^*_{13},p^*_{23},p^*_{14},p^*_{24},p^*_{34} \mid p^*_{123},p^*_{124},p^*_{134},p^*_{234}\,) \\ &=\; (p_{234},-p_{134},p_{124},-p_{123} \mid p_{34},-p_{24},p_{14},p_{23},-p_{13},p_{12} \mid p_4,-p_3,p_2,-p_1). \end{aligned} \tag{7}$$

The restriction of the Hodge star to the Grassmannian $\mathrm{Gr}(\ell,n)$ in $\mathbb{P}^{\binom{n}{\ell}-1}$ takes a linear space to its orthogonal complement with respect to the standard inner product. To incorporate the quadratic form B, we consider the automorphism of $\mathbb{P}^{\binom{n}{\ell}-1}$ that takes P to $\big(P\cdot(\wedge_\ell B)\big)^*$. The restriction of this automorphism to the Grassmannian $\mathrm{Gr}(\ell,n)$ is the isomorphism (6).

It remains to be shown that the map $P\mapsto \big(P\cdot(\wedge_\ell B)\big)^*$ preserves the shuffle equations. The images of these equations for $\mathrm{Gr}(\ell,n)^T$ under our automorphism of $\mathbb{P}^{\binom{n}{\ell}-1}$ are given by

$$\begin{aligned} \big(P\cdot(\wedge_\ell B)\big)^*\cdot\wedge_\ell(\mathrm{Id}_n+zT)^* \;&=\; P^*\cdot\big((\wedge_\ell B)\cdot\wedge_\ell(\mathrm{Id}_n+zT)\big)^* \\ &=\; P^*\cdot(\wedge_{n-\ell}B)\cdot\wedge_{n-\ell}(\mathrm{Id}_n+zT^t) \;=\; P^*\cdot\wedge_{n-\ell}(B+zBT^t) \\ &=\; P^*\cdot\wedge_{n-\ell}(B+zTB) \;=\; P^*\cdot\wedge_{n-\ell}(\mathrm{Id}_n+zT)\cdot\wedge_{n-\ell}B. \end{aligned}$$

The rightmost factor is an invertible matrix and can hence be removed. We conclude that our automorphism takes the space of shuffle equations for $\mathrm{Gr}(\ell,n)^T$ to the space of the shuffle equations for $\mathrm{Gr}(n-\ell,n)^T$. This was the claim, and the proof of Theorem 5 is complete. $\square$

Example 6 $(n=4)$**.** Fix $\epsilon=0$ in Example 1. Then $T=T_\lambda$ for $\lambda=(2,1,1)$, and we have

$$B = B_\lambda \;=\; \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}.$$

Our map $P\mapsto\big(P\cdot(\wedge_\ell B)\big)^*$, written for $\ell=1,2,3$, is the following signed permutation of (7):

$$P \;\mapsto\; (\,-p_{134},p_{234},p_{124},-p_{123} \mid -p_{34},p_{14},-p_{24},-p_{13},p_{23},p_{12} \mid -p_4,p_3,-p_1,p_2\,). \tag{8}$$

For $\ell=1$ the unique shuffle equation is p_1. This is mapped to $-p_{134}$, which is the unique shuffle equation for $\ell=3$. Likewise, p_{134} is mapped to $-p_1$. This makes sense because $\mathrm{Gr}(1,4)^T = V(p_1) = \mathrm{span}(e_2,e_3,e_4) = \ker(T)$, whereas $\mathrm{Gr}(3,4)^T = V(p_{134})$ consists of all hyperplanes in K^4 that contain e_2. Both are projective planes $\mathbb{P}^2$. Our involution swaps them.

For $\ell = n-\ell = 2$, there are two shuffle equations, namely p_{13} and p_{14}, as seen in Example 3. These two Plücker coordinates are swapped (up to sign) in (8), so our involution fixes $\mathrm{Gr}(2,4)^T$. Moreover, this involution interchanges the two irreducible components in (2). We see this in the coordinate change (8) which sends $p_{12}\mapsto -p_{34}$ and $p_{34}\mapsto p_{12}$. $\diamond$

3 Classification and Counterexample

The main result in this article is the determination of all fixed point loci $\mathrm{Gr}(\ell, n)^T$ for $n \leq 8$. From this computational result, we extract the following theorem about the shuffle equations.

Theorem 7. *Fix $1 \leq \ell < n \leq 7$ and let T be any nilpotent $n \times n$ matrix. Then the shuffle equations generate the radical ideal of the fixed point locus $\mathrm{Gr}(\ell, n)^T$. The same does not hold for $n = 8$: there is a unique partition, namely $\lambda = (4, 2, 2)$, and a unique dimension, namely $\ell = 4$, such that the radical ideal of $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is not generated by the shuffle equations.*

Proof. The proof is carried out by exhaustive computation of all varieties $\mathrm{Gr}(\ell, n)^{T_\lambda}$ where λ is any partition of $n \leq 8$. Here we use the `Macaulay2` code from Example 4 and Theorem 5.

The results are summarized in Tables 1, 2 and 3. For each instance (λ, ℓ), we report a triple $[\sigma, \delta, \gamma]$ or $[\sigma, \delta, \gamma]^\kappa$. Here σ is the number of linearly independent shuffle equations. The entries δ and γ are the dimension and degree of $\mathrm{Gr}(\ell, n)^{T_\lambda}$ in its Plücker embedding into $\mathbb{P}^{\binom{n}{\ell}-1}$. The upper index κ is the number of irreducible components of $\mathrm{Gr}(\ell, n)^T$, and this index is dropped if $\kappa = 1$. The columns for $\ell > n/2$ are omitted because of Theorem 5. In any given row of one of our tables, the entry for $n - \ell$ would be identical to that for ℓ.

λ	$\ell=1$	$\ell=2$
(1,1,1,1)	[0,3,1]	[0,4,2]
(2,1,1)	[1,2,1]	$[2,2,2]^2$
(2,2)	[2,1,1]	[2,2,2]
(3,1)	[2,1,1]	[4,1,1]
(4)	[3,0,1]	[5,0,1]

λ	$\ell=1$	$\ell=2$
(1,1,1,1,1)	[0,4,1]	[0,6,5]
(2,1,1,1)	[1,3,1]	$[3,4,2]^2$
(2,2,1)	[2,2,1]	[4,3,3]
(3,1,1)	[2,2,1]	$[6,2,2]^2$
(3,2)	[3,1,1]	[6,2,2]
(4,1)	[3,1,1]	[8,1,1]
(5)	[4,0,1]	[9,0,1]

Table 1: Fixed point loci $\mathrm{Gr}(\ell, n)^T$ for $n = 4$ and $n = 5$.

In each case, we computed the irreducible components of the shuffle ideal. We recorded the prime ideal for each component, and we determined degree, dimension, singularities etc. The intersection of these primes is the radical ideal of $\mathrm{Gr}(\ell, n)^T$. In all cases but one, we found that the radical ideal is generated by the shuffle equations plus the Plücker quadrics. The unique exceptional case is $\lambda = (4, 2, 2)$ and $\ell = 4$, with the highlighted entry $\mathbf{[54,4,24]^3}$. This means that there are 54 linearly independent shuffle relations plus 4 additional Plücker quadrics. However, this ideal is not radical. To generate the radical, we need one more linear form. Further below, we shall examine the geometry of this counterexample in detail.

An easy `Macaulay2` proof for the failure of `J` to be radical is running the following line:

```
apply(first entries promote(P,S),p -> {p % J, p^2 % J})
```

This reveals that the variable p_{1468} is not in `J` but its square is in `J`. Note that this coordinate corresponds to `p0357` in the zero-based indexing of `Macaulay2`. This concludes the proof. □

λ	$\ell=1$	$\ell=2$	$\ell=3$
(1,1,1,1,1,1)	[0,5,1]	[0,8,14]	[0,9,42]
(2,1,1,1,1)	[1,4,1]	$[4,6,5]^2$	$[6,6,10]^2$
(2,2,1,1)	[2,3,1]	$[6,4,6]^2$	[8,5,10]
(3,1,1,1)	[2,3,1]	$[8,4,2]^2$	$[12,4,2]^3$
(2,2,2)	[3,2,1]	[6,4,6]	[11,4,6]
(3,2,1)	[3,2,1]	[9,3,3]	$[12,3,6]^2$
(4,1,1)	[3,2,1]	$[11,2,2]^2$	$[16,2,2]^2$
(3,3)	[4,1,1]	[11,2,2]	[12,3,6]
(4,2)	[4,1,1]	[11,2,2]	[16,2,2]
(5,1)	[4,1,1]	[13,1,1]	[18,1,1]
(6)	[5,0,1]	[14,0,1]	[19,0,1]

λ	$\ell=1$	$\ell=2$	$\ell=3$
(1,1,1,1,1,1,1)	[0,6,1]	[35,10,42]	[140,12,462]
(2,1,1,1,1,1)	[1,5,1]	$[5,8,14]^2$	$[10,9,42]^2$
(2,2,1,1,1)	[2,4,1]	$[8,6,5]^2$	$[14,7,35]^2$
(3,1,1,1,1)	[2,4,1]	$[10,6,5]^2$	$[20,6,10]^3$
(2,2,2,1)	[3,3,1]	[9,5,10]	[17,6,30]
(3,2,1,1)	[3,3,1]	$[12,4,6]^2$	$[21,5,10]^2$
(4,1,1,1)	[3,3,1]	$[14,4,2]^2$	$[27,4,2]^3$
(3,2,2)	[4,2,1]	[12,4,6]	$[23,4,12]^2$
(3,3,1)	[4,2,1]	[15,3,3]	[23,4,12]
(4,2,1)	[4,2,1]	[15,3,3]	$[27,3,6]^2$
(5,1,1)	[4,2,1]	$[17,2,2]^2$	$[31,2,2]^2$
(4,3)	[5,1,1]	[17,2,2]	[27,3,6]
(5,2)	[5,1,1]	[17,2,2]	[31,2,2]
(6,1)	[5,1,1]	[19,1,1]	[33,1,1]
(7)	[6,0,1]	[20,0,1]	[34,0,1]

Table 2: Fixed point loci $\mathrm{Gr}(\ell,n)^T$ for $n=6$ and $n=7$.

Example 8 ($n=6$)**.** Consider the lower right entry on the left in Table 2. Here $\ell=3$ and $\lambda=(6)$, so T is the nilpotent matrix that maps $e_1\mapsto e_2\mapsto e_3\mapsto e_4\mapsto e_5\mapsto e_6\mapsto 0$. The variety $\mathrm{Gr}(3,6)^T$ consists of a single point e_{456}. It is instructive to revisit the construction of the shuffle equations for this case. The 20 coordinates of the row vector $P\cdot\wedge_3(\mathrm{Id}_6+zT)$ are

$$
\begin{gathered}
p_{123},\ p_{123}z+p_{124},\ p_{123}z^2+p_{124}z+p_{134},\ p_{123}z^3+p_{124}z^2+p_{134}z+p_{234},\ p_{124}z+p_{125},\\
p_{124}z^2+(p_{134}+p_{125})z+p_{135},\ p_{124}z^3+(p_{134}+p_{125})z^2+(p_{234}+p_{135})z+p_{235},\ p_{134}z^2+p_{135}z\\
+p_{145},\ p_{134}z^3+(p_{234}+p_{135})z^2+(p_{235}+p_{145})z+p_{245},\ p_{234}z^3+p_{235}z^2+p_{245}z+p_{345},\\
p_{125}z+p_{126},\ p_{125}z^2+(p_{135}+p_{126})z+p_{136},\ p_{125}z^3+(p_{135}+p_{126})z^2+(p_{235}+p_{136})z+p_{236},\\
p_{135}z^2+(p_{145}+p_{136})z+p_{146},\ p_{135}z^3+(p_{235}+p_{145}+p_{136})z^2+(p_{245}+p_{236}+p_{146})z+p_{246},\\
p_{235}z^3+p_{245}+p_{236}z^2+(p_{345}+p_{246})z+p_{346},\ p_{145}z^2+p_{146}z+p_{156},\ p_{145}z^3+(p_{245}+p_{146})z^2+\\
(p_{246}+p_{156})z+p_{256},\ p_{245}z^3+(p_{345}+p_{246})z^2+(p_{346}+p_{256})z+p_{356},\ p_{345}z^3+p_{346}z^2+p_{356}z+p_{456}.
\end{gathered}
$$

The shuffle equations are the coefficients of z^3, z^2 and z. They span the ideal of all Plücker coordinates except p_{456}. This is the homogeneous maximal ideal of $\mathrm{Gr}(3,6)^T=\{e_{456}\}$. $\diamond$

Example 9 (n=8)**.** The smallest instance of a variety $\mathrm{Gr}(\ell,n)^{T_\lambda}$ with four irreducible components occurs for $n=8$, $\lambda=(4,1,1,1,1)$, and $\ell=4$. There are 54 linearly independent shuffle equations, and 46 Plücker quadrics remain modulo these linear forms. The variety $\mathrm{Gr}(4,8)^{T_\lambda}$ has dimension 6 and degree 10 in $\mathbb{P}^{69}$. It is the union of four irreducible components, two of dimension 6 and degree 5, and two linear spaces of dimension 4. $\diamond$

We now present a detailed study of our counterexample to [4, Conjecture 7.6]. We have $n=8$, $\ell=4$, and the matrix $T=T_\lambda$ given by the partition $\lambda=(4,2,2)$, i.e. operating as

$$
e_1\mapsto e_2\mapsto e_3\mapsto e_4\mapsto 0,\quad e_5\mapsto e_6\mapsto 0,\quad e_7\mapsto e_8\mapsto 0.
$$

λ	$\ell=2$	$\ell=3$	$\ell=4$
(1,1,1,1,1,1,1,1)	[0,12,132]	[420,15,6006]	[721,16,24024]
(2,1,1,1,1,1,1)	$[6,10,42]^2$	$[15,12,462]^2$	$[20,12,924]^2$
(2,2,1,1,1,1)	$[10,8,14]^2$	$[22,9,168]^2$	$[28,10,420]^3$
(3,1,1,1,1,1)	$[12,8,14]^2$	$[30,9,42]^3$	$[40,9,42]^3$
(2,2,2,1,1)	$[12,6,20]^2$	[26,8,140]	$[34,8,280]^2$
(3,2,1,1,1)	$[15,6,5]^2$	$[33,7,35]^3$	$[42,7,70]^2$
(4,1,1,1,1)	$[17,6,5]^2$	$[41,6,10]^3$	$[54,6,10]^4$
(2,2,2,2)	[12,6,20]	[32,7,70]	[34,8,280]
(3,2,2,1)	[16,5,10]	$[35,6,30]^2$	$[46,6,60]^2$
(3,3,1,1)	$[19,4,6]^2$	$[38,5,30]^2$	[46,6,60]
(4,2,1,1)	$[19,4,6]^2$	$[42,5,10]^2$	$[54,5,10]^3$
(5,1,1,1)	$[21,4,2]^2$	$[48,4,2]^3$	$[62,4,2]^3$
(3,3,2)	[19,4,6]	[38,5,30]	[52,5,30]
(4,2,2)	[19,4,6]	$[44,4,12]^2$	$\mathbf{[54,4,24]^3}$
(4,3,1)	[22,3,3]	[44,4,12]	$[54,4,24]^2$
(5,2,1)	[22,3,3]	$[48,3,6]^2$	$[62,3,6]^2$
(6,1,1)	$[24,2,2]^2$	$[52,2,2]^2$	$[66,2,2]^2$
(4,4)	[24,2,2]	[48,3,6]	[54,4,24]
(5,3)	[24,2,2]	[48,3,6]	[62,3,6]
(6,2)	[24,2,2]	[52,2,2]	[66,2,2]
(7,1)	[26,1,1]	[54,1,1]	[68,1,1]
(8)	[27,0,1]	[55,0,1]	[69,0,1]

Table 3: Fixed point loci $\mathrm{Gr}(\ell,n)^T$ for $n=8$.

We consider the scheme structure on $\mathrm{Gr}(4,8)^T$ given by the shuffle ideal $\mathtt{J}$. There are three minimal primes, each of dimension 4 and degree 6. One component is non-reduced of multiplicity 2, so the degree of our scheme is $24 = 6+6+2\cdot 6$. It has no embedded primes.

We begin with the two reduced components. Each of these is a Segre fourfold $\mathbb{P}^2\times\mathbb{P}^2$ lying in a $\mathbb{P}^8$ inside a coordinate subspace $\mathbb{P}^{11}$. The two ambient coordinate subspaces are

$$\begin{aligned}&\mathrm{span}\{e_{1234}, e_{1346}, e_{1348}, e_{2345}, e_{2346}, e_{2347}, e_{2348}, e_{3456}, e_{3458}, e_{3467}, e_{3468}, e_{3478}\},\\ &\mathrm{span}\{e_{3456}, e_{3458}, e_{3467}, e_{3468}, e_{3478}, e_{3568}, e_{3678}, e_{4567}, e_{4568}, e_{4578}, e_{4678}, e_{5678}\}.\end{aligned}$$

In suitable affine coordinates, the two reduced components are parametrized by

$$\begin{pmatrix}1&0&0&0&a&b&c&d\\0&1&0&0&0&a&0&c\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0&0&a&b&1&0&0&0\\0&0&0&a&0&1&0&0\\0&0&c&d&0&0&1&0\\0&0&0&c&0&0&0&1\end{pmatrix}.$$

The T-module structures on these subspaces L are given by the partitions (4) and $(2,2)$.

We now study the non-reduced component. It lies in a $\mathbb{P}^8$ inside the coordinate subspace

$$\mathrm{span}\{e_{3468}, e_{2346}, e_{2348}, e_{2468}, e_{3456}, e_{3458}, e_{3467}, e_{3478}, e_{4568}, e_{4678}\} \;\simeq\; \mathbb{P}^9.$$

Geometrically, it is a cone over a hyperplane slice of $\mathbb{P}^2 \times \mathbb{P}^2$. It has the matrix representation

$$\begin{pmatrix} 0 & a & 1 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & e & 1 & f & 0 \\ 0 & g & 0 & 0 & h & 0 & i & 1 \end{pmatrix} \text{ where the } 3 \times 3 \text{ block } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ has trace 0 and rank} \leq 1.$$

The zero matrix gives a singular point on this component. There are moreover three distinct T-module structures on the subspaces L in this component, namely $(3, 1)$, $(2, 2)$ and $(2, 1, 1)$.

Remark 10. The first nontrivial entry in each table is $\lambda = (2, 1, 1, \ldots, 1)$. Each irreducible component of $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is a Grassmannian. This is obvious for $\ell = 1$. We sketch a proof for $2 \leq \ell \leq \lceil n/2 \rceil$. The matrix T maps $e_1 \mapsto e_2$ and $e_i \mapsto 0$ for $i \geq 2$. Thus, $\ker(T)$ is a hyperplane in K^n and each $L \in \mathrm{Gr}(\ell, n)^T$ possesses a minimal subspace $\tilde{L}$ satisfying $L = \tilde{L} + \tilde{L}T$. The space $\tilde{L}$ might not be unique, but its dimension is, being either ℓ or $\ell - 1$. In the first case, $\tilde{L} = L$ and L is a subspace of $\ker(T)$. In the second case, $\tilde{L}$ is an $(\ell - 1)$-dimensional subspace of K^n with $\tilde{L} \not\subseteq \ker T$ and $\tilde{L}T = \mathrm{span}(e_2)$. This implies that $\mathrm{Gr}(\ell, n)^T$ has two irreducible components, namely the Grassmannians $\mathrm{Gr}(\ell, n-1)$ and $\mathrm{Gr}(\ell - 1, n)$.

4 The Affine Grassmannian

The key player in the articles [2] and [4] is the affine Grassmannian, which is an infinite-dimensional variety. Our varieties $\mathrm{Gr}(\ell, n)^T$ serve as finite-dimensional models, when restricting to $T = T_\lambda$ where λ is a *rectangular partition*. By this we mean partitions $\lambda = (r, r, \ldots, r)$ with d parts, so that $dr = n$ and $d, r \geq 2$. This section revolves around the next two points.

Theorem 11. *If λ is a rectangular partition then the variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is irreducible.*

Conjecture 12. Conjecture 7.6 in [4] holds for rectangular partitions λ. In other words, for rectangular partitions, the shuffle equations plus Plücker quadrics generate a prime ideal.

Remark 13. Tables 1, 2 and 3 show that Theorem 11 and Conjecture 12 are true for $n \leq 8$. In that range, the only rectangular partitions λ are $(2, 2), (2, 2, 2), (3, 3), (2, 2, 2, 2)$ and $(4, 4)$. We see that the shuffle ideals that cut out their varieties $\mathrm{Gr}(\ell, n)^{T_\lambda}$ are prime for all ℓ.

We shall derive Theorem 11 from known facts about Schubert varieties in affine Grassmannians. We aim to explain this approach in a manner that is as self-contained as possible. The section concludes with some further evidence in support of Conjecture 12.

Let $\mathbb{K} = K((t))$ be the field of Laurent series with coefficients in K. Its valuation ring $\mathcal{O}_\mathbb{K} = K[[t]]$ consists of formal power series with nonnegative integer exponents. The residue field is K. The $\mathbb{K}$-vector space $\mathbb{K}^d$ is a module over $\mathcal{O}_\mathbb{K}$. A *lattice* L is an $\mathcal{O}_\mathbb{K}$-submodule of $\mathbb{K}^d$ of maximal rank d. Two lattices L and L' are *equivalent* if $L' = t^a L$ for some $a \in \mathbb{Z}$.

To parametrize all lattices, we consider the groups $\mathrm{GL}_d(\mathbb{K})$ and $\mathrm{GL}_d(\mathcal{O}_\mathbb{K})$ of invertible $d \times d$ matrices, with entries in $\mathbb{K}$ and $\mathcal{O}_\mathbb{K}$ respectively. The *affine Grassmannian* is the coset space

$$\mathrm{GL}_d(\mathbb{K}) \,/\, \mathrm{GL}_d(\mathcal{O}_\mathbb{K}). \tag{9}$$

Its points are the lattices L. Indeed, every L is the column span over $\mathcal{O}_\mathbb{K}$ of a matrix in $\mathrm{GL}_d(\mathbb{K})$. Two matrices define the same L if they differ via right multiplication by a matrix in $\mathrm{GL}_d(\mathcal{O}_\mathbb{K})$. To obtain finite-dimensional varieties we can study with a computer, we set

$$\mathcal{B}_r \;=\; \{\, L \text{ lattice } : \; t^r\mathcal{O}_\mathbb{K}^d \subseteq L \subseteq \mathcal{O}_\mathbb{K}^d \,\}. \tag{10}$$

We note that (9) modulo equivalence of lattices equals the *Bruhat-Tits building* for $\mathrm{GL}_d(\mathbb{K})$. The set $\mathcal{B}_r$ represents the ball of radius r around the standard lattice $\mathcal{O}_\mathbb{K}^d$ in that building.

Both $t^r\mathcal{O}_\mathbb{K}^d$ and $\mathcal{O}_\mathbb{K}^d$ are infinite-dimensional vector spaces over K. Their quotient is a finite-dimensional vector space over K. This space has dimension $n = dr$ and we identify

$$K^n \;=\; \mathcal{O}_\mathbb{K}^d \,/\, t^r\mathcal{O}_\mathbb{K}^d. \tag{11}$$

Writing $e_1, e_2, \ldots, e_d$ for the standard basis of K^d, we shall use the following basis for K^n:

$$e_1, te_1, \ldots, t^{r-1}e_1,\; e_2, te_2, \ldots, t^{r-1}e_2,\; \ldots\ldots,\; e_d, te_d, \ldots, t^{r-1}e_d. \tag{12}$$

In this basis, multiplication with t is given by the nilpotent $n \times n$ matrix T_λ for $\lambda = (r, \ldots, r)$.

Every lattice $L \in \mathcal{B}_r$ is determined by its image in (11). We also write L for that image. Hence L is a subspace of K^n that satisfies $LT_\lambda \subseteq L$. Conversely, every subspace L of K^n satisfying $LT_\lambda \subseteq L$ comes from a unique lattice in $\mathcal{B}_r$. This establishes the following result.

Proposition 14. *The radius r ball in* (10) *is the following finite union of projective varieties:*

$$\mathcal{B}_r \;=\; \bigcup_{\ell=0}^{dr} \mathrm{Gr}(\ell, n)^{T_\lambda}, \qquad \textit{where } \lambda \;=\; (r, r, \ldots, r). \tag{13}$$

Example 15 ($d = r = 2$)**.** Here $n = rd = 4$, $T = T_{(2,2)}$, and the disjoint union in (13) equals

$$\mathcal{B}_2 \;=\; \mathrm{Gr}(0,4)^T \,\cup\, \mathrm{Gr}(1,4)^T \,\cup\, \mathrm{Gr}(2,4)^T \,\cup\, \mathrm{Gr}(3,4)^T \,\cup\, \mathrm{Gr}(4,4)^T.$$

The first and last Grassmannian are the points that represent the lattices $\mathcal{O}_\mathbb{K}^2$ and $t^2\mathcal{O}_\mathbb{K}^2$. The second and fourth Grassmannian are projective lines $\mathbb{P}^1$. The middle Grassmannian is a quadratic cone in $\mathbb{P}^3$. We saw this in (1) for $\epsilon = 1$. Note the row $\lambda = (2, 2)$ in Table 1. $\diamond$

Example 16 ($n = 8$)**.** The two options are $d = 4, r = 2$ and $d = 2, r = 4$. These are the rows $\lambda = (2, 2, 2, 2)$ and $\lambda = (4, 4)$ of Table 3. In either case, $\mathcal{B}_r$ is the disjoint union of nine irreducible varieties, indexed by $\ell = 0, 1, 2, 3, 4, 5, 6, 7, 8$, and soon to be called Schubert varieties. Their dimensions are $0, 3, 6, 7, 8, 7, 6, 3, 0$ and $0, 1, 2, 3, 4, 3, 2, 1, 0$ respectively. $\diamond$

We now turn towards the proof of Theorem 11. We shall give a polynomial parametrization for each variety $\mathrm{Gr}(\ell,n)^{T_\lambda}$ in (13). The elements of the group $\mathrm{GL}_d(\mathcal{O}_{\mathbb{K}})$ are $d\times d$-matrices $\mathbf{A} = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \cdots$, where each A_i is a $d \times d$ matrix with entries in K, and $\det(A_0) \neq 0$. This group acts naturally on (10) and on (11). The $d \times d$ matrix $\mathbf{A}$ with entries in $\mathcal{O}_{\mathbb{K}} \subset \mathbb{K}$ admits the following representation by an $n \times n$ matrix over the residue field K:

$$A \;=\; \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{r-2} & A_{r-1} \\ 0 & A_0 & A_1 & \cdots & A_{r-3} & A_{r-2} \\ 0 & 0 & A_0 & \cdots & A_{r-4} & A_{r-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}. \tag{14}$$

To get this nice block form, the basis of K^n shown in (12) has to be reordered as follows:

$$e_1, e_2, \ldots, e_d,\; te_1, te_2, \ldots, te_d,\; \ldots\ldots,\; t^{r-1}e_1, t^{r-1}e_2, \ldots, t^{r-1}e_d. \tag{15}$$

The matrices A act on each of the components in (13). We are interested in their orbits.

Let μ be a partition of the integer ℓ with at most d parts and largest part at most r. To be precise, we write $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ where $r \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_d \geq 0$ and $\sum_{i=1}^d \mu_i = \ell$. With this partition we associate the lattice $L_\mu = t^{r-\mu_1}\mathcal{O}_{\mathbb{K}}\, e_1 \oplus t^{r-\mu_2}\mathcal{O}_{\mathbb{K}}\, e_2 \oplus \cdots \oplus t^{r-\mu_d}\mathcal{O}_{\mathbb{K}}\, e_d$. The corresponding subspace of K^n is spanned by standard basis vectors:

$$L_\mu \;=\; K\bigl\{t^{r-i}e_j \,:\, 1 \leq i \leq \mu_j \text{ and } 1 \leq j \leq d\bigr\}.$$

By construction, we have $L_\mu \in \mathrm{Gr}(\ell,n)^T$. Since L_μ is a coordinate subspace, its Plücker coordinates are given by one of the basis points in $\mathbb{P}^{\binom{n}{\ell}-1}$, here denoted e_μ for simplicity.

The orbit of L_μ under the above group action is a constructible subset of $\mathrm{Gr}(\ell,n)^T \subset \mathbb{P}^{\binom{n}{\ell}-1}$. It consists of all points $e_\mu \cdot \wedge_\ell A$ that represent the subspaces $L_\mu A$, where A runs over all matrices of the form (14). Let W_μ denote the Zariski closure of this orbit. In symbols,

$$W_\mu \;=\; \overline{\mathrm{GL}_d(\mathcal{O}_{\mathbb{K}}) \cdot L_\mu} \quad\subseteq\; \mathrm{Gr}(\ell,n)^T.$$

The variety W_μ is called a *Schubert variety*. We immediately obtain the following lemma.

Remark 17. For each partition μ of ℓ, the Schubert variety W_μ is irreducible. It is given by an explicit polynomial parametrization, namely $A \mapsto e_\mu \cdot \wedge_\ell A$, which encodes $A \mapsto L_\mu A$.

Example 18 (d=3, r=2, ℓ=3)**.** Let $\mu = (2,1,0)$ with the basis (15) of K^6. The subspace L_μ corresponds to the point $e_\mu = e_{145}$ in $\mathrm{Gr}(3,6)^T \subset \mathbb{P}^{19}$. Its image under A is the row space of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot A \;=\; \begin{pmatrix} a_{011} & a_{012} & a_{013} & a_{111} & a_{112} & a_{113} \\ 0 & 0 & 0 & a_{011} & a_{012} & a_{013} \\ 0 & 0 & 0 & a_{021} & a_{022} & a_{023} \end{pmatrix}. \tag{16}$$

The action of the group $\mathrm{GL}_3(\mathcal{O}_{\mathbb{K}})$ on K^6 is given by the matrix in (14), here written as

$$A \;=\; \begin{pmatrix} A_0 & A_1 \\ 0 & A_0 \end{pmatrix} \;=\; \begin{pmatrix} a_{011} & a_{012} & a_{013} & a_{111} & a_{112} & a_{113} \\ a_{021} & a_{022} & a_{023} & a_{121} & a_{122} & a_{123} \\ a_{031} & a_{032} & a_{033} & a_{131} & a_{132} & a_{133} \\ 0 & 0 & 0 & a_{011} & a_{012} & a_{013} \\ 0 & 0 & 0 & a_{021} & a_{022} & a_{023} \\ 0 & 0 & 0 & a_{031} & a_{032} & a_{033} \end{pmatrix}.$$

The Schubert variety W_μ is parametrized by all matrices (16). As a subvariety of the Grassmannian $\mathrm{Gr}(3,6)$, it is defined by the following 11 linear forms in the 20 Plücker coordinates:

$$p_{123},\; p_{124}, p_{134}, p_{234},\; p_{125}, p_{135}, p_{235},\; p_{126}, p_{136}, p_{236},\; p_{156} - p_{246} + p_{345}. \tag{17}$$

This subvariety has dimension 4 and degree 6, and we find that $\mathrm{Gr}(3,6)^T = W_\mu$. It is the entry $[11,4,6]$ for $\lambda = (2,2,2)$ of Table 2. The expressions (17) are the shuffle equations. $\diamond$

The duality of Theorem 5 acts on the Schubert varieties as follows. The complement to $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ is the partition $\mu^c = (r-\mu_d, r-\mu_{d-1}, \ldots, r-\mu_1)$ of the integer $n-\ell$. Then the inclusion $W_{\mu^c} \subseteq \mathrm{Gr}(n-\ell, n)^T$ is isomorphic to the inclusion $W_\mu \subseteq \mathrm{Gr}(\ell, n)^T$.

We summarize the above discussion as follows: for any partition μ of ℓ with $\leq d$ parts of size $\leq r$, we have constructed an irreducible subvariety W_μ of $\mathrm{Gr}(\ell,n)^T$. Here $n = dr$ and $T = T_\lambda$ for $\lambda = (r, r, \ldots, r)$. The union of these varieties equals $\mathrm{Gr}(\ell,n)^T$ because every lattice in the ball $\mathcal{B}_r$ lies in the $\mathrm{GL}(\mathcal{O}_{\mathbb{K}})$-orbit of some lattice $L_\mu = t^{r-\mu_1}\mathcal{O}_{\mathbb{K}}\, e_1 \oplus \cdots \oplus t^{r-\mu_d}\mathcal{O}_{\mathbb{K}}\, e_d$.

To proceed further, we record the following fact about inclusions of Schubert varieties.

Lemma 19. *Let μ, ν be two partitions of ℓ with at most d parts and largest part at most r. Then the inclusion $W_\mu \subseteq W_\nu$ holds if and only if $\mu \leq \nu$ in the dominance order on partitions.*

Proof. This is well known in algebraic combinatorics; see e.g. [3, Remark 5.3.4]. $\square$

Deriving the irreducibility of $\mathrm{Gr}(\ell,n)^T$ is now reduced to a combinatorial argument.

Proof of Theorem 11. Two partitions satisfy $\mu \leq \nu$ in dominance order if and only if $\mu_1 + \cdots + \mu_i \leq \nu_1 + \cdots + \nu_i$ for all i. Consider the set $\mathcal{P}$ of all partitions of ℓ with at most d parts whose largest part has size at most r. The restriction of dominance order to this set has a unique largest element $\mu_{\max}$. Namely, this largest partition equals $\mu_{\max} = (r, r, \ldots, r, b)$. The partition $\mu_{\max}$ has a blocks of size r, where $\ell = ar + b$ and $0 \leq b < r$. Lemma 19 implies

$$W_{\mu_{\max}} \;=\; \bigcup_{\mu \in \mathcal{P}} W_\mu \;=\; \mathrm{Gr}(\ell, n)^T. \tag{18}$$

In light of Remark 17, this proves the irreducibility of $\mathrm{Gr}(\ell,n)^T$, i.e. Theorem 11 holds. $\square$

Corollary 20. *Fix $T = T_\lambda$ where $\lambda = (r, r, \ldots, r)$ and set $a = \lfloor \ell/r \rfloor$ and $b = \ell - ar$. The dimension of the irreducible variety $\mathrm{Gr}(\ell,n)^T$ is equal to $(d-a)\ell - (a+1)b$.*

Proof. We compute the dimension of W_μ for any $\mu \in \mathcal{P}$. For the action of $\mathrm{GL}_d(\mathcal{O}_\mathbb{K})$ by the group of matrices A, we determine the stabilizer of the distinguished point L_μ. Every matrix in this stabilizer has $A_1 = A_2 = \cdots = A_{r-1} = 0$. The matrix A_0 breaks into blocks according to various levels given by powers of t. This can be expressed conveniently by the partition $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_r^*)$ that is conjugate to $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$. Here μ_i^* is the number of indices j such that $\mu_j \geq i$. Note that μ^* is a partition of ℓ with at most r parts and largest part at most d. The desired stabilizer is the product of the matrix groups $\mathrm{GL}_{\mu_i^*}(K)$, for $i = 1, 2, \ldots, r$. In particular, the dimension of the stabilizer is $\sum_{i=1}^r (\mu_i^*)^2$. This implies

$$\dim(W_\mu) \;=\; d\ell - \sum_{i=1}^{r} (\mu_i^*)^2 \;=\; \sum_{1 \leq i \leq j \leq d} (\mu_i - \mu_j).$$

We learned the last identity from [6, eqn (26)]. We shall apply the middle formula to the maximal partition $\mu = \mu_{\max} = (r, r, \ldots, r, b)$. Its conjugate partition is $\mu^* = (a+1, \ldots, a+1, a, \ldots, a)$, with b blocks of size $a+1$ and $r-b$ blocks of size a. The middle formula yields

$$\dim(W_\mu) \;=\; d\ell - b(a+1)^2 - (r-b)a^2 \;=\; (d-a)\ell - (a+1)b. \tag{19}$$

The assertion now follows from (18). ☐

We now present further evidence in favor of Conjecture 12, beginning with the computational results shown in Table 4. For each table entry we verified that the shuffle equations span the space of linear forms that vanish on $\mathrm{Gr}(\ell, n)^T$. For all entries marked with a star, the `Macaulay2` command `isPrime J` terminated and proved that the shuffle ideal is prime.

λ	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
(3,3,3)	[27, 4, 6]*	[57, 6, 90]*	[99, 6, 90]*	[99, 6, 90]*
(5,5)	[41, 2, 2]*	[112, 3, 6]*	[194, 4, 24]*	[220, 5, 120]*
(2,2,2,2,2)	[20, 8, 70]*	[70, 10, 1050]*	[110, 12, 23100]	[152, 12, 23100]
(6,6)	[62, 2, 2]*	[212, 3, 6]*	[479, 4, 24]	[760, 5, ??]
(4,4,4)	[57, 4, 6]*	[193, 6, 90]*	[414, 8, 2520]	[711, 8, ??]
(3,3,3,3)	[50, 6, 20]*	[156, 9, 1680]	[399, 10, 8400]	[648, 11, ??]
(2,2,2,2,2,2)	[30, 10, 252]*	[130, 13, 18018]	[270, 16, ??]	[492, 17, ??]

Table 4: Fixed point loci for rectangular partitions of $n = 9, 10, 12$.

The extremal cases $\lambda = (1, 1, \ldots, 1)$ and $\lambda = (n)$ had been excluded from the definition of rectangular partition, but it is worthwhile to consider these now. Conjecture 12 holds for both of these cases. Indeed for $\lambda = (1, 1 \ldots, 1)$, we have $T = 0_n$, so there are no shuffle equations. The corresponding variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ agrees with the Grassmannian $\mathrm{Gr}(\ell, n)$. This is defined by the Plücker quadrics which are well-known to generate a prime ideal.

We conclude by addressing the case $\lambda = (n)$. This was studied for $n = 6$ in Example 8. We now generalize what we saw there, namely Conjecture 12 holds for the one-part partition.

Proposition 21. *For $\lambda = (n)$ and any ℓ, the shuffle equations are all Plücker coordinates p_I except for $I = (n-\ell+1, \dots, n)$. These generate the prime ideal of the point* $\mathrm{Gr}(\ell, n)^T = \{e_I\}$.

Proof. Consider any ordered ℓ-set I in $[n]$. If $n \notin I$ then p_I equals the coefficient of z^ℓ in the coordinate of the row vector $P \cdot \wedge_\ell(I + zT)$ that is indexed by $I + (1, 1, \dots, 1, 1)$. If $I = (J, n)$ but $n - 1 \notin J$ then, modulo the above Plücker coordinates, p_I equals the coefficient of $z^{\ell-1}$ of the coordinate in $P \cdot \wedge_\ell(I + zT)$ that is indexed by $I + (1, 1, \dots, 1, 0)$. If $I = (J, n-1, n)$ but $n - 2 \notin J$ then, modulo the above Plücker coordinates, p_I equals the coefficient of $z^{\ell-2}$ of the coordinate that is indexed by $I + (1, ..., 1, 0, 0)$ etc... Iterating this process yields all Plücker coordinates other than the last one, $I = (n - \ell + 1, n - \ell + 2, ..., n - 1, n)$. For $n = 6$ and $\ell = 3$, our argument can be checked by looking at the 20 expressions in Example 8. □

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Authors' addresses:

Marvin Hahn, Sorbonne Université, Paris `hahn@imj-prg.fr`
Gabriele Nebe, RWTH Aachen `gabriele.nebe@rwth-aachen.de`
Mima Stanojkovski, RWTH Aachen and MPI-MiS Leipzig `mima.stanojkovski@mis.mpg.de`
Bernd Sturmfels, MPI-MiS Leipzig and UC Berkeley `bernd@mis.mpg.de`