THE ORTHOGONAL CHARACTER TABLE OF $\mathbf{SL}_2(q)$

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Abstract. The rational invariants of the $\mathrm{SL}_2(q)$-invariant quadratic forms on the real irreducible representations are determined. There is still one open question (see Remark 6.5) if q is an even square.

1. Introduction

Throughout the paper let G be a finite group. The isomorphism classes of $\mathbb{C}G$-modules are parametrized by their characters. Our aim is to extend this connection in order to also determine the G-invariant quadratic forms from the character table of G. The ordinary character table displays the characters χ_V of the absolutely irreducible $\mathbb{C}G$-modules V. For each χ_V let K be the maximal real subfield of the character field of V and W the irreducible KG-module such that V occurs in $W \otimes_K \mathbb{C}$. Then the space

$$\mathcal{F}_G(W) := \left\{ F : W \times W \to K \mid {\scriptstyle F(v,w)=F(w,v) \text{ and } \atop \scriptstyle F(gw,gv)=F(w,v) \text{ for all } g\in G, v,w \in W} \right\}$$

of G-invariant symmetric bilinear forms on W is at least one-dimensional and every non-zero $F \in \mathcal{F}_G(W)$ is non-degenerate. The character χ_V also determines the K-isometry classes of the elements of $\mathcal{F}_G(W)$. The orthogonal character table additionally contains the invariants (see Section 2) that determine the K-isometry classes of (W, F) for in the case where these are independent of the choice of the non-zero $F \in \mathcal{F}_G(W)$.

For

$$G = \mathrm{SL}_2(q) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_q^{2\times 2} \mid ad - bc = 1 \}$$

the ordinary character table was already known to Schur, [16]. This paper determines the orthogonal character tables of $\mathrm{SL}_2(q)$ for all prime powers q. For $q = 2^n$ with n even and the characters of degree $q + 1$ we could not specify which even primes ramify in the Clifford algebra (see Section 6).

This work grew out of the first author's PhD thesis [2] written under the supervision of the second author. In this thesis, the first author also determines the ordinary orthogonal character tables for all (non-abelian) finite quasisimple groups of order up to $200,000$.

Date: March 13, 2017.

2010 *Mathematics Subject Classification.* 20C15; 20C33, 11E12, 11E81.

The first author was supported by the DFG Research Training Group "Experimental and Constructive Algebra" at RWTH Aachen University.

2. Invariants of quadratic spaces

Let K be a field of characteristic 0, V an n-dimensional vector space over K and $F : V \times V \to K$ a non-degenerate symmetric bilinear form. The two most important invariants attached to such a space (V, F) are the discriminant and the Clifford invariant.

The *discriminant* of (V, F) is

$$\mathrm{d}_{\pm}(V, F) := (-1)^{n(n-1)/2} \det(V, F)$$

where the determinant $\det(V, F) \in K^{\times}/(K^{\times})^2$ is defined as the square class of the determinant of a Gram matrix of F with respect to any basis.

The Clifford algebra $\mathcal{C}(V, F)$ is the quotient of the tensor algebra by the two-sided ideal $\langle v \otimes v - F(v, v) \cdot 1 \mid v \in V \rangle$. A K-basis of $\mathcal{C}(V, F)$ is given by the ordered tensors $(\overline{b_{i_1} \otimes \ldots \otimes b_{i_k}} \mid 1 \leq i_1 < \ldots < i_k \leq n)$ of any basis $(b_1, \ldots, b_n)$ of V, in particular $\dim(\mathcal{C}(V, F)) = 2^n$. Put

$$c(V, F) := \begin{cases} \mathcal{C}(V, F) & \text{if } n \text{ is even,} \\ \mathcal{C}_0(V, F) := \langle \overline{b_{i_1} \otimes \ldots \otimes b_{i_k}} \mid k \text{ even } \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Then $c(V, F) \cong \mathcal{D}^{r \times r}$ is a central simple K-algebra with involution and therefore it has order 1 or 2 in the Brauer group. The *Clifford invariant* of (V, F) is defined as the Brauer class of $c(V, F)$:

$$\mathfrak{c}(V, F) := [c(V, F)] = [\mathcal{D}] \in \mathrm{Br}(K).$$

A more detailed exposition of this material may be found e.g. in [15].

Our interest in these two isometry invariants of quadratic spaces is mainly due to the following classical result by Helmut Hasse.

Theorem 2.1 ([7]). *Over a number field K the isometry class of a quadratic space is uniquely determined by its dimension, its determinant, its Clifford invariant and its signature at all real places of K.*

We are mainly interested in the case where K is a number field. Then $\mathcal{D}$ is either K or a quaternion division algebra over K. We use two notations for these $\mathcal{D}$, either as a symbol algebra or by giving all the local invariants of $\mathcal{D}$:

Definition 2.2. *For $a, b \in K$ let $(a, b) := [\left(\frac{a,b}{K}\right)] \in \mathrm{Br}(K)$ where*

$$\left(\frac{a, b}{K}\right) := \langle 1, i, j, k \mid i^2 = a, j^2 = b, ij = -ji = k \rangle.$$

By the Theorem of Hasse, Brauer, Noether, Albert (see [14, Theorem (32.11)]*) any quaternion algebra $\mathcal{D}$ over K is determined by the set of places $\wp_1, \ldots, \wp_s$ (the* ramified *places) of K, for which the completion of $\mathcal{D}$ stays a division algebra. Therefore we also describe $\mathcal{D} = \mathcal{Q}_{\wp_1, \ldots, \wp_s}$ by its ramified places, where we assume that the center K is clear from the context.*

Remark 2.3. *Let (V, F) be a bilinear space and $a \in K^\times$. Then the scaled space (V, aF) has the following algebraic invariants (see* [10, Chapter 5, (3.16)] *for the Clifford invariant):*

$$\mathrm{d}_\pm(V, aF) = \begin{cases} \mathrm{d}_\pm(V, F) & \text{if } \dim(V) \text{ is even,} \\ a\mathrm{d}_\pm(V, F) & \text{if } \dim(V) \text{ is odd.} \end{cases}$$

and

$$\mathfrak{c}(V, aF) = \begin{cases} \mathfrak{c}(V, F)(a, \mathrm{d}_\pm(V, F)) & \text{if } \dim(V) \text{ is even,} \\ \mathfrak{c}(V, F) & \text{if } \dim(V) \text{ is odd.} \end{cases}$$

If

$$(V, F) = (V_1, F_1) \perp (V_2, F_2)$$

is the orthogonal direct sum of two subspaces the determinant is just the product $\det(V, F) = \det(V_1, F_1) \cdot \det(V_2, F_2)$.
The behavior of the Clifford invariant is more complicated, cf. [10, Chapter five (3.13)]*:* $\mathfrak{c}(V, F) =$

$$\begin{cases} \mathfrak{c}(V_1, F_1)\mathfrak{c}(V_2, F_2)(\mathrm{d}_\pm(V_1, F_1), \mathrm{d}_\pm(V_2, F_2)), & \dim(V_1) \equiv \dim(V_2) \pmod 2, \\ \mathfrak{c}(V_1, F_1)\mathfrak{c}(V_2, F_2)(-\mathrm{d}_\pm(V_1, F_1), \mathrm{d}_\pm(V_2, F_2)), & \dim(V) \equiv \dim(V_1) \equiv 1 \pmod 2. \end{cases}$$

Example 2.4. *Let $\mathbb{I}_n$ be the n-dimensional $\mathbb{Q}$-vector space that has an orthonormal basis $(e_1, \ldots, e_n)$. Then $\mathrm{d}_\pm(\mathbb{I}_n) = (-1)^{n(n-1)/2}(\mathbb{Q}^\times)^2$ and*

$$\mathfrak{c}(\mathbb{I}_n) = \begin{cases} (1, 1) & n \equiv 0, 1, 2, 7 \pmod 8, \\ (-1, -1) & n \equiv 3, 4, 5, 6 \pmod 8. \end{cases}$$

The space $\mathbb{A}_{n-1} := \langle \sum_{i=1}^n e_i \rangle^\perp \leq \mathbb{I}_n$ is the orthogonal complement of a space of discriminant n in $\mathbb{I}_n$. This allows to compute the discriminant and Clifford invariant of $\mathbb{A}_{n-1}$ using the formulas from the previous example: $\mathrm{d}_\pm(\mathbb{A}_{n-1}) = (-1)^{(n-1)(n-2)/2} n(\mathbb{Q}^\times)^2$ *and $\mathfrak{c}(\mathbb{A}_{n-1})$ depends on the value of n modulo 8:*

$n \pmod 8$	$0, 1$	$2, 3$	$4, 5$	$6, 7$
$\mathfrak{c}(\mathbb{A}_{n-1})$	1	$(-1, n)$	$(-1, -1)$	$(-1, -n)$

3. Methods

3.1. **Orthogonal character tables.** Let χ be a complex irreducible character of the finite group G and let $K = \mathbb{Q}(\chi)^+$ be the maximal real subfield of the character field $\mathbb{Q}(\chi)$. Let V be the irreducible $\mathbb{C}G$-module affording the character χ and let W be the irreducible KG-module such that V is a constituent of $W_\mathbb{C} := \mathbb{C} \otimes_K W$. Put

$$\mathcal{F}_G(W) := \left\{ F : W \times W \to K \mid {}_{F(gw,gv)=F(w,v) \text{ for all } g \in G, v, w \in W}^{F(v,w)=F(w,v) \text{ and}} \right\}$$

the space of G-invariant symmetric bilinear forms on W. As W is irreducible, all non-zero elements of $\mathcal{F}_G(W)$ are non-degenerate and an easy averaging argument shows that $\mathcal{F}_G(W)$ always contains a totally positive definite form F_0. We call W *uniform* if $\mathcal{F}_G(W) = \{aF_0 \mid a \in K\}$ is one-dimensional over K.

Remark 3.1. *There are three different situations to be considered:*

(a) $K = \mathbb{Q}(\chi)$ *and* $V = W_{\mathbb{C}}$*: Then* W *is an absolutely irreducible* KG*-module and hence uniform.*

(b) $K = \mathbb{Q}(\chi)$ *and* $W_{\mathbb{C}} \cong V \oplus V$*: Then the Schur index of* χ *over* K *is* 2*,* $\chi(1)$ *is even, and* [18] *tells us that* $\mathrm{d}_{\pm}(F) \in (K^{\times})^2$ *for all non-zero* $F \in \mathcal{F}_G(W)$*. If the real Schur index of* χ *is one, then* $\dim(\mathcal{F}_G(W)) = 3$*. If the real Schur index of* χ *is* 2*, then* W *is uniform and* [18, Theorem B] *also gives the Clifford invariant of* (W, F)*:*

$$\mathfrak{c}(W,F) = \begin{cases} 1 & \text{if } \dim_K(W) \equiv 0 \pmod 8 \\ [\mathrm{End}_{KG}(W)] & \text{if } \dim_K(W) \equiv 4 \pmod 8. \end{cases}$$

(c) $[\mathbb{Q}(\chi) : K] = 2$*. Then* $\chi_W = m(\chi + \overline{\chi})$ *for some* $m \in \mathbb{N}$ *and* W *is uniform if and only if* $m = 1$*. Choose* $\delta \in K$ *such that* $\mathbb{Q}(\chi) = K(\sqrt{\delta})$*, then* $\mathrm{d}_{\pm}(F) = \delta^{m\chi(1)}(K^{\times})^2$ *for all* $0 \neq F \in \mathcal{F}_G(W)$ *(see* [15, Chapter 10, Remark 1.4]*,* [2, Theorem 4.3.9]*).*

Definition 3.2. *Let* χ*,* $K := \mathbb{Q}(\chi)^+$*,* W *be as above and put* $n := \dim_K(W)$*. Assume that* W *is uniform and choose* $0 \neq F \in \mathcal{F}_G(W)$*.*
If n *is even then we define*

$$\mathrm{d}_{\pm}(\chi) := \mathrm{d}_{\pm}(W, F).$$

If n *is odd, or* n *is even and* $\mathrm{d}_{\pm}(\chi) = 1$*, then we put*

$$\mathfrak{c}(\chi) := \mathfrak{c}(W, F).$$

The orthogonal character table *of* G *is the complex character table of* G *with this additional information added.*

As $\mathcal{F}_G(W) = \{aF \mid a \in K\}$ Remark 2.3 and Remark 3.1 show that the values $\mathrm{d}_{\pm}(\chi)$ and $\mathfrak{c}(\chi)$ are well defined, i.e. independent of the choice of the non-zero $F \in \mathcal{F}_G(W)$.

3.2. **Clifford orders.** Let us now assume that K is a local or global field of characteristic 0, i.e. K is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{Q}$, and let R denote the ring of integers in K. Let V be a finite dimensional vector space over K and $F : V \times V \to K$ a symmetric bilinear form with associated quadratic form

$$Q_F : V \to K, v \mapsto Q_F(v) = \frac{1}{2}F(v,v).$$

Definition 3.3. *A* lattice L *in* V *is a finitely generated* R*-submodule of* V *that contains a* K*-basis of* V*. The lattice* L *is called* integral*, if* $F(L,L) \subseteq R$ *and* even*, if* $Q_F(L) \subseteq R$*. The* dual lattice *of* L *is* $L^{\#} := \{v \in V \mid F(v,L) \subseteq R\}$ *and* L *is called* unimodular*, if* $L = L^{\#}$*.*

Even unimodular lattices are called regular quadratic R-modules in [9]. If $2 \notin R^{\times}$, then there are no regular R-modules L of odd dimension. Kneser calls an even lattice L of odd dimension such that $L^{\#}/L \cong R/2R$ a semi-regular quadratic R-module.

Theorem 3.4 ([9, Satz 15.8]). *Assume that R is a complete discrete valuation ring (with finite residue class field) and let L be a regular or semi-regular quadratic R-module in (V, Q_F). If $\dim(V) \geq 3$ then $L \cong \mathbb{H}(R) \perp M$ for some regular or semi-regular quadratic R-module M. Here $\mathbb{H}(R)$ is the hyperbolic plane, the regular free R-lattice with basis (e, f) such that $Q_F(e) = Q_F(f) = 0$ and $F(e, f) = 1$.*

As both invariants, the Clifford invariant and the discriminant, of the hyperbolic plane $\mathbb{H}(K) = K\mathbb{H}(R)$ are trivial, we obtain the following corollary.

Corollary 3.5. *Additionally to the assumptions of the theorem let $\dim(V)$ be odd. Then $\mathfrak{c}(V, F) = 1$.*

Proof. We proceed by induction on the dimension of V. If $\dim(V) = 1$ then $c(V, F) = K$ and so $\mathfrak{c}(V, F) = 1$. So assume that $\dim(V) \geq 3$ and let L be a semi-regular quadratic R-lattice in (V, Q_F). Then $L \cong \mathbb{H}(R) \perp M$ and hence $V \cong \mathbb{H}(K) \perp KM$ for some semi-regular lattice M in KM. By induction we have $\mathfrak{c}(KM, F_{|KM}) = 1$. So

$$\mathfrak{c}(V, F) = \mathfrak{c}(KM, F_{|KM})\mathfrak{c}(\mathbb{H}(K))(-\mathrm{d}_\pm(KM, F_{|KM}), \mathrm{d}_\pm(\mathbb{H}(K))) = 1. \quad \square$$

Remark 3.6. *Let L be an even lattice in V. Then the* Clifford order *$\mathcal{C}(L, F)$ of L is defined to be the R-subalgebra of $\mathcal{C}(V, F)$ generated by L. As $Q_F(L) \subseteq R$, the Clifford order is an R-lattice in $\mathcal{C}(V, F)$, in particular finitely generated over R. If L has an orthogonal basis $(b_1, \ldots, b_n)$, then the ordered tensors $(\overline{b_{i_1} \otimes \ldots \otimes b_{i_k}} \mid 1 \leq i_1 < \ldots < i_k \leq n)$ form an R-basis of $\mathcal{C}(L, F)$. In this case it is easy to compute the determinant of $\mathcal{C}(L, F)$ and of $\mathcal{C}_0(L, F)$ with respect to the reduced trace bilinear form (see* [2, Theorem 7.2.2]*): Up to some power of 2 they are both powers of $Q_F(b_1) \cdots Q_F(b_n)$.*

Corollary 3.7. *Assume that K is a number field, $2 \neq p \in \mathbb{Z}$ is some odd prime and $\wp$ is a prime ideal of K containing p. Denote the completion of K at $\wp$ by $K_\wp$ and its valuation ring by $R_\wp$. Assume that there is a lattice L in V such that $L_\wp = R_\wp \otimes L$ is a unimodular $R_\wp$-lattice. Then*

$$[c(V, F) \otimes K_\wp] = 1 \in \mathrm{Br}(K_\wp).$$

Proof. Since $2 \in R_\wp^\times$ the lattice $L_\wp$ has an orthogonal basis and Remark 3.6 shows that the determinant of the Clifford order $\mathcal{C}(L_\wp, F)$ and also of $\mathcal{C}_0(L_\wp, F)$ is a unit in $R_\wp$. In particular the determinant of a maximal order in $c(V, F) \otimes K_\wp$ is a unit in $R_\wp$, which shows that this central simple $K_\wp$-algebra is a matrix ring over $K_\wp$ (see for instance [14, Theorem (20.3)]). $\square$

A bit more generally we may also compute the Clifford invariant of a bilinear space that contains a lattice of prime determinant:

Corollary 3.8. *Keep the assumptions of Corollary 3.7 and let $(W_\wp, E_\wp)$ be a 1-dimensional bilinear $K_\wp$ vector space such that the $\wp$-adic valuation of the discriminant of $E_\wp$ is odd. Then*

$$\mathfrak{c}((V \otimes K_\wp, F) \perp (W_\wp, E_\wp)) = 1 \in \mathrm{Br}(K_\wp) \textit{ if and only if } \mathrm{d}_\pm(V \otimes K_\wp, F) \in (K_\wp^\times)^2.$$

Proof. Clearly the Clifford invariant of the 1-dimensional space is trivial and also $\mathfrak{c}(V \otimes K_\wp, F)$ is trivial by Corollary 3.7. So the formulas in Remark 2.3 give us the Clifford invariant of the orthogonal sum as

$$\mathfrak{c}((V \otimes K_\wp, F) \perp (W_\wp, E_\wp)) = (\mathrm{d}_\pm(V \otimes K_\wp, F), u\pi)$$

where u is a unit and π is a prime element in the valuation ring $R_\wp$. As $d := \mathrm{d}_\pm(V \otimes K_\wp, F)$ has even valuation, this quaternion symbol is trivial if and only if d is a square. □

3.3. **A Clifford theory of orthogonal representations.** Let $N \trianglelefteq G$ be a normal subgroup. Clifford theory explains the interplay between irreducible representations of N and G (see for instance [4, Section 11.1]). We want to describe the behavior of invariant forms under this correspondence.

Let K be a totally real number field and V an irreducible KG-module with a non-degenerate invariant form F. We will then call (V, F) an *orthogonal representation* of G. Let U be an irreducible KN-module occurring as a direct summand of $V|_N$ with multiplicity e. Let I be the inertia group of U, of index $t := [G : I]$ in G, and let $G = \bigsqcup_{i=1}^t g_i I$ be a decomposition of G into left I-cosets. We then have the following decomposition of $V|_N$ into pairwise non-isomorphic irreducible KN-modules ${}^{g_i}U$ $(i = 1, \ldots, t)$:

$$V|_N \cong \bigoplus_{i=1}^{t} ({}^{g_i}U)^e\,, \tag{1}$$

In this situation we obtain the following lemma

Lemma 3.9. *The decomposition* (1) *is orthogonal*

$$(V|_N, F) = ({}^{g_1}U^e, F_1) \perp ({}^{g_2}U^e, F_2) \perp \ldots \perp ({}^{g_t}U^e, F_t)$$

and the forms F_i are non-degenerate and pairwise K-isometric.

Proof. Clearly, the restriction of F to the direct summand ${}^{g_i}U^e$ is N-invariant. For $i \neq j$ we have

$${}^{g_i}U \cong {}^{g_i}U^* \ncong {}^{g_j}U$$

so the summands $({}^{g_i}U)^e$ are orthogonal to each other and the F_i are non-degenerate. The elements $g_j^{-1}g_i \in G \leq O(V, F)$ induce isometries between F_i and F_j. □

Example 3.10. *Consider an odd prime p, a natural number n and abbreviate $q := p^n$. Let $C_{(q-1)/2} \cong H \leq \mathrm{GL}_n(\mathbb{F}_p)$ be a subgroup acting with regular orbits on $\mathbb{F}_p^n \setminus \{0\}$ in its natural action. Then the group $G := C_p^n \rtimes H$, which is isomorphic to the normalizer of a Sylow p-subgroup in $\mathrm{PSL}_2(q)$ has $(q-1)/2$ linear characters and two non-linear characters ψ_1, ψ_2 of degree $(q-1)/2$ with Schur index 1 and character field*

$$\mathbb{Q}(\psi_1) = \mathbb{Q}(\psi_2) = \begin{cases} \mathbb{Q}(\sqrt{q}) & \text{if } q \equiv 1 \pmod 4, \\ \mathbb{Q}(\sqrt{-q}) & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

Let H_1 be the unique subgroup of H of order $\frac{p-1}{2}$ and put $N := C_p^n \rtimes H_1$. Then $N \trianglelefteq G$ and we will apply Lemma 3.9 to this normal subgroup in order to compute the discriminant $\mathrm{d}_\pm(\psi_i)$ in the case $q \equiv 1 \pmod 4$.
Let $\psi \in \{\psi_1, \psi_2\}$, $K = \mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{q})$ and (V, F) an orthogonal KG-module whose character is ψ.
There is a character $\mathbb{1} \neq \chi \in \mathrm{Irr}(C_p^n)$ such that $\psi = \mathrm{ind}_{C_p^n}^G(\chi) = \mathrm{ind}_N^G(\mathrm{ind}_{C_p^n}^N(\chi))$. Ordinary Clifford theory shows that $\kappa := \mathrm{ind}_{C_p^n}^N(\chi)$ is irreducible and an easy application of Frobenius reciprocity reveals $(\psi|_N, \kappa)_N = 1$.
Thus we obtain an orthogonal decomposition

$$(V|_N, F) \cong (V_1, F_1) \perp \ldots \perp (V_t, F_t)$$

where $F_1 \cong \ldots \cong F_t$ by Lemma 3.9. We have $t = \frac{1}{2}\frac{q-1}{p-1}$ if $K = \mathbb{Q}$ and $t = \frac{q-1}{p-1}$ if $K = \mathbb{Q}(\sqrt{p})$.
Notice that κ is a faithful character of a group isomorphic to $C_p \rtimes C_{\frac{p-1}{2}}$. As the trace forms of cyclotomic fields are well understood (cf. [11, Section 3.3.2]*), we can find the determinants of the (V_i, F_i) as*

$$\det(V_i, F_i) = \det(V_1, F_1) = \begin{cases} p(\mathbb{Q}^\times)^2 & \text{if } n \text{ is even} \\ u\sqrt{p}(\mathbb{Q}(\sqrt{p})^\times)^2 & \text{if } n \text{ is odd} \end{cases}$$

for some unit u of the ring of integers of $\mathbb{Q}(\sqrt{p})$. In conclusion, we obtain

$$\det(\psi) = \begin{cases} 1(\mathbb{Q}^\times)^2 & \text{if } n \equiv 0 \pmod 4 \text{ or } p \equiv 3 \pmod 4, \\ p(\mathbb{Q}^\times)^2 & \text{if } n \equiv 2 \pmod 4 \text{ and } p \equiv 1 \pmod 4, \\ u\sqrt{p}(\mathbb{Q}(\sqrt{p})^\times)^2 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

In the case $q \equiv 3 \pmod 4$ the character ψ has non-real values and we find $\mathrm{d}_\pm(\psi) = -p(\mathbb{Q}^\times)^2$.

4. The orthogonal character table of $\mathrm{SL}_2(q)$ for odd q

Let p be an odd prime, n a natural number, put $q := p^n$ and let $G := \mathrm{SL}_2(q)$ be the group of all 2×2 matrices of determinant 1 over the field with q elements. A reference for the ordinary (and modular) representation theory of this group is, for example, [1]. We use the ordinary character table and the notation for the absolutely irreducible characters from [5]:

Theorem 4.1 ([5, Theorem 38.1])**.** *Let $\langle \nu \rangle = \mathbb{F}_q^\times$. Consider*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ d = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix},\ a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

and let $b \in \mathrm{SL}_2(q)$ be an element of order $q+1$.
For $x \in \mathrm{SL}_2(q)$, let (x) denote the conjugacy class containing x. $\mathrm{SL}_2(q)$ has the following $q+4$ conjugacy classes of elements, listed together with the size of the classes.

x	1	z	c	d	zc	zd	a^ℓ	b^m
$\lvert(x)\rvert$	1	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$

where $1 \leq \ell \leq \frac{q-3}{2}$, $1 \leq m \leq \frac{q-1}{2}$.
Put

$$\varepsilon := (-1)^{(q-1)/2},\ \zeta_r := \exp(2\pi i/r) \text{ and } \vartheta_r^{(s)} := \zeta_r^s + \zeta_r^{-s} \text{ for } r, s \in \mathbb{N}.$$

Then the character table of $\mathrm{SL}_2(q)$ *reads as*

	1	z	c	d	a^ℓ	b^m
$\mathbb{1}$	1	1	1	1	1	1
ψ	q	q	0	0	1	-1
χ_i	$q+1$	$(-1)^i(q+1)$	1	1	$\vartheta_{q-1}^{(i\ell)}$	0
θ_j	$q-1$	$(-1)^j(q-1)$	-1	-1	0	$-\vartheta_{q+1}^{(jm)}$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}\left(1+\sqrt{\varepsilon q}\right)$	$\frac{1}{2}\left(1-\sqrt{\varepsilon q}\right)$	$(-1)^\ell$	0
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}\left(1-\sqrt{\varepsilon q}\right)$	$\frac{1}{2}\left(1+\sqrt{\varepsilon q}\right)$	$(-1)^\ell$	0
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}\left(-1+\sqrt{\varepsilon q}\right)$	$\frac{1}{2}\left(-1-\sqrt{\varepsilon q}\right)$	0	$(-1)^{m+1}$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}\left(-1-\sqrt{\varepsilon q}\right)$	$\frac{1}{2}\left(-1+\sqrt{\varepsilon q}\right)$	0	$(-1)^{m+1}$

where $1 \leq i \leq \frac{q-3}{2}$, $1 \leq j \leq \frac{q-1}{2}$, $1 \leq \ell \leq \frac{q-3}{2}$, $1 \leq m \leq \frac{q-1}{2}$.
The columns for the classes (zc) *and* (zd) *are omitted because for any irreducible character* χ *the relation* $\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c)$ *holds.*

Theorem 4.2. *The following table gives the orthogonal character table of* $\mathrm{SL}_2(q)$.

χ	K	$\dim_K(W)$	$\mathfrak{c}(\chi)$	$\mathrm{d}_\pm(\chi)$	q
$\mathbb{1}$	$\mathbb{Q}$	1	1	$-$	*all*
ψ	$\mathbb{Q}$	q	$\mathfrak{c}(\mathbb{A}_q)$	$-$	*all*
χ_i *i even*	$\mathbb{Q}(\vartheta_{q-1}^{(i)})$	$q+1$	$-$	$\varepsilon(\vartheta_{q-1}^{(2i)}-2)q$	*all*
χ_i *i odd*	$\mathbb{Q}(\vartheta_{q-1}^{(i)})$	$2(q+1)$	$[\mathrm{End}_{KG}(W)]$	1	$1 \pmod 4$
			1	1	$3 \pmod 4$
θ_j *j even*	$\mathbb{Q}(\vartheta_{q+1}^{(j)})$	$q-1$	1 *if* $q = \square$	εq	*all*
θ_j *j odd*	$\mathbb{Q}(\vartheta_{q+1}^{(j)})$	$2(q-1)$	1	1	$1 \pmod 4$
			$[\mathrm{End}_{KG}(W)]$	1	$3 \pmod 4$
ξ_1, ξ_2	$\mathbb{Q}(\sqrt{q})$	$\frac{q+1}{2}$	1	$-$	$q \equiv 1, -3 \pmod{16}$
			$[\mathcal{Q}_{p,\infty} \otimes K]$	$-$	$q \equiv 5, 9 \pmod{16}$
$\xi_1 = \overline{\xi_2}$	$\mathbb{Q}$	$q+1$	1	1	$3 \pmod 8$
			$(-1,-1)$	1	$7 \pmod 8$
η_1, η_2	$\mathbb{Q}(\sqrt{q})$	$q-1$	$[\mathcal{Q}_{p,\infty} \otimes K]$	1	$1 \pmod 4$
$\eta_1 = \overline{\eta_2}$	$\mathbb{Q}$	$q-1$	$-$	$-q$	$3 \pmod 4$

We use the abbreviations introduced in Theorem 4.1. As before K is the maximal real subfield of the character field and W the irreducible KG-module, whose character contains χ.

5. The proof of Theorem 4.2

5.1. **The faithful characters of G.** The faithful irreducible characters of $\mathrm{SL}_2(q)$ either have real Schur index 2 or they take values in an imaginary quadratic number field. Janusz [8, Section 2] contains an explicit description of the endomorphism rings $\mathrm{End}_{KG}(W)$. In particular their discriminants and Clifford invariants can be read off from this information using Remark 3.1 (b) and (c).

5.2. **The non-faithful characters η_i.** If $q \equiv 3 \pmod 4$ then the characters η_1 and η_2 of degree $(q-1)/2$ have character field $\mathbb{Q}(\sqrt{\epsilon q}) = \mathbb{Q}(\sqrt{-p})$ and Schur index 1. So Remark 3.1 (c) yields their discriminant.

5.3. **The Steinberg character.** The character ψ is a non-faithful character of degree q. As $\mathbb{1} + \psi$ is the character of a 2-transitive permutation representation of G, the invariants of ψ are those of $\mathbb{A}_q$ as given in Example 2.4.

5.4. **The characters θ_j, j even.** For even j, the character θ_j is a non-faithful character of even degree $q-1$ with totally real character field K and Schur index 1. Let (W,F) be the orthogonal KG-module affording the character θ_j. Then the restriction of W to the Borel subgroup $B \cong (C_p)^n \rtimes C_{(q-1)/2}$ of $\mathrm{PSL}_2(q)$ has character $\psi_1+\psi_2$ from Example 3.10. As $\mathrm{d}_\pm(\psi_1)$ and $\mathrm{d}_\pm(\psi_2)$ are Galois conjugate, the formula for $\mathrm{d}_\pm(\psi_1)$ in Example 3.10 yields

$$\mathrm{d}_\pm(\theta_j) = \begin{cases} 1(K^\times)^2 & n \text{ even} \\ \varepsilon p(K^\times)^2 & n \text{ odd.} \end{cases}$$

If n is even then we can also deduce the Clifford invariant of (W,F): In this case $q \equiv 1 \pmod 4$ so $-\zeta_{q+1}^2$ is a primitive $q+1$st root of unity and hence all characters of degree $q-1$ of the group $\mathrm{PSL}_2(q)$ extend to characters of $\mathrm{PGL}_2(q)$ with the same character field (see [17, Table III] for a character table) and of Schur index 1 (see [6]). So (W,F) is also an orthogonal representation of $\mathrm{PGL}_2(q)$ and restricting (W,F) to B, we obtain the orthogonal sum of two isometric spaces $(W,F) \cong (V_1,F_1) \perp (V_2,F_2)$ because the normalizer of B in $\mathrm{PGL}_2(q)$ interchanges V_1 and V_2. By Example 3.10 we have $\mathrm{d}_\pm(V_i,F_i) = p$ if $n \equiv 2 \pmod 4$ and $p \equiv 1 \pmod 4$ and $\mathrm{d}_\pm(V_i,F_i) = 1$ otherwise $(i = 1,2)$. In both cases $(\mathrm{d}_\pm(V_1,F_1), \mathrm{d}_\pm(V_2,F_2)) = 1 \in \mathrm{Br}(\mathbb{Q})$ and so by Remark 2.3 $\mathfrak{c}(W,F) = \mathfrak{c}(V_1,F_1)\mathfrak{c}(V_2,F_2) = \mathfrak{c}(V_1,F_1)^2 = 1$.

5.5. **The characters χ_i, i even.** For even i, the character χ_i is a non-faithful character of even degree $q+1$ with totally real character field K and Schur index 1. As before we restrict χ_i to the Borel subgroup and obtain

$$\chi_i|_B = \psi_1 + \psi_2 + \alpha + \overline{\alpha}$$

where ψ_1, ψ_2 are as in 5.4 and α is a complex linear character of B. Comparing character values we obtain that $\alpha(y) = \zeta_{q-1}^i$ for a suitably chosen generator y of

$C_{(q-1)/2} \leq B$. In particular $\mathbb{Q}(\alpha) = \mathbb{Q}(\zeta_{q-1}^i) = K(\sqrt{\vartheta_{q-1}^{(2i)} - 2})$ and hence Remark 3.1 (c) tells us that $\mathrm{d}_\pm(\alpha) = \vartheta_{q-1}^{(2i)} - 2$. The discriminant of ψ_1 and ψ_2 behave as in 5.4 and hence we compute the discriminant $\mathrm{d}_\pm(\chi_i) = \varepsilon(\vartheta_{q-1}^{(2i)} - 2)q$.

5.6. **The characters ξ_1, ξ_2 for $q \equiv 1 \pmod 4$.** Assume that $q = p^n \equiv 1 \pmod 4$. Then the two characters ξ_1 and ξ_2 of odd degree $\frac{q+1}{2}$ factor through $\mathrm{PSL}_2(q)$ and have a totally real character field $K = \mathbb{Q}(\chi_1) = \mathbb{Q}(\chi_2) = \mathbb{Q}(\sqrt{q})$.

Proposition 5.1. *The Clifford invariant $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$ is stable under the Galois group of K and the only primes that ramify in $\mathfrak{c}(\xi_1)$ are the ones that divide $2p$. More precisely there are the following two possibilities for $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$:*

	n even		*n odd*			
$q \pmod{16}$	1	9	1	-3	9	5
$\star$ $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$	1	$[\mathcal{Q}_{p,\infty}]$	1	1	$[\mathcal{Q}_{\infty_1,\infty_2}]$	$[\mathcal{Q}_{\infty_1,\infty_2}]$
$\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$	$[\mathcal{Q}_{2,p}]$	$[\mathcal{Q}_{2,\infty}]$	$[\mathcal{Q}_{\wp_1,\wp_2}]$	$[\mathcal{Q}_{2,\sqrt{p}}]$	$[\mathcal{Q}_{\infty_1,\infty_2,\wp_1,\wp_2}]$	$[\mathcal{Q}_{\infty_1,\infty_2,2,\sqrt{p}}]$

Here, for $p \equiv 1 \pmod 8$ and n odd, $\wp_1$ and $\wp_2$ denote the two places of $K = \mathbb{Q}(\sqrt{p})$ that divide 2.

Proof. Let ξ be one of ξ_1 or ξ_2, $K = \mathbb{Q}(\sqrt{q})$ and W the KG-module affording the character ξ. Since $\mathcal{F}_G(W)$ always contains a totally positive definite form, we know that $\mathfrak{c}(\xi) \otimes \mathbb{R} = 1$ if $q \equiv 1, -3 \pmod{16}$ and $\mathfrak{c}(\xi) \otimes \mathbb{R} \neq 1$ otherwise, for all real places of K. If $K \neq \mathbb{Q}$ then ξ_1 and ξ_2 are Galois conjugate and so are $\mathfrak{c}(\xi_1)$ and $\mathfrak{c}(\xi_2)$. The outer automorphism of G interchanges ξ_1 and ξ_2 which shows that $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$, so this algebra is stable under the Galois group of K. Moreover the only possible finite primes of K that ramify in $c(\xi)$ are those dividing p or 2. This is seen as follows: The representation ξ is irreducible modulo all other primes ℓ (see [1, Section 9.3]) so in particular there is a G-invariant lattice L in W whose determinant is not divisible by ℓ and hence ℓ does not ramify in $\mathfrak{c}(W, F)$ by Remark 3.6. Noting that 2 is decomposed in K if and only if n is odd and $p \equiv 1 \pmod 8$, we are left with the possibilities for $\mathfrak{c}(\xi)$ as stated. □

Lemma 5.2. *The primes of K that divide 2 are not ramified in $\mathfrak{c}(\xi_1)$, so $\mathfrak{c}(\xi_1) = \mathfrak{c}(\xi_2)$ is given in line $\star$ of Proposition 5.1.*

Proof. Let $\wp$ be a prime ideal of K that contains 2 and let $R_\wp$ be the valuation ring in the completion $K_\wp$ (so $R_\wp \cong \mathbb{Z}_2$ if $q \equiv 1 \pmod 8$ and $R_\wp \cong \mathbb{Z}_2[\zeta_3]$ if $q \equiv 5 \pmod 8$). By [13, Theorem VII.12 and Theorem VII.4] the image of $R_\wp G$ in $\mathrm{End}(K_\wp \otimes W)$ is isomorphic to

$$\Delta_{\xi_1}(R_\wp G) = \begin{pmatrix} R_\wp & 2R_\wp^{1\times(q-1)/2} \\ R_\wp^{(q-1)/2\times 1} & R_\wp^{(q-1)/2\times(q-1)/2} \end{pmatrix}.$$

In particular the $R_\wp G$-lattices in $K_\wp \otimes W$ form a chain

$$\ldots \supset L' \supset L \supset 2L' \supset 2L \ldots$$

with $L'/L \cong R_\wp/2R_\wp$. If $F \in \mathcal{F}_G(W)$ is non-degenerate and L is G-invariant, then also its dual lattice is G-invariant. This shows that there is some $F \in \mathcal{F}_G(W)$

such that L' is the dual lattice of L. But then $Q_F(L) \subseteq R_\wp$ because otherwise the even sublattice of L would be a G-invariant sublattice of index 2 in L. So L is a semi-regular quadratic $R_\wp$-module in $(K_\wp \otimes W, F)$ and by Corollary 3.5 this implies that $\mathfrak{c}(K_\wp \otimes W, F) = 1$. □

Note that for $n = 1$ and $n = 2$ it is also possible to find the Clifford invariant of ξ_1 and ξ_2 using the character theoretic method from [12] (see [2, Section 6.4]).

6. The orthogonal character table of $\mathrm{SL}_2(2^n)$

We now assume that $q = 2^n$ with $n \geq 2$ and put $G := \mathrm{SL}_2(q)$. Then the ordinary character table of G is given in [5, Theorem 38.2]:

Theorem 6.1 ([5, Theorem 38.2])**.** *Let ν be a generator of $\mathbb{F}_q^\times$ and consider the elements*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ c := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ a := \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

of G. The group also contains an element b of order $q+1$. The character table of G is

	1_G	c	a^ℓ	b^m
$\mathbb{1}$	1	1	1	1
ψ	q	0	1	-1
χ_i	$q+1$	1	$\zeta_{q-1}^{i\ell} + \zeta_{q-1}^{-i\ell}$	0
θ_j	$q-1$	-1	0	$-\zeta_{q+1}^{jm} - \zeta_{q+1}^{-jm}$

where $1 \leq i \leq \frac{q-2}{2}$, $1 \leq j \leq \frac{q}{2}$, $1 \leq \ell \leq \frac{q-2}{2}$, $1 \leq m \leq \frac{q}{2}$.

In contrast to the odd characteristic case all characters have totally real character field and Schur index 1.

Theorem 6.2 (Orthogonal representations of $\mathrm{SL}_2(2^n)$)**.** *Let $q = 2^n$, $n \geq 2$ and $G = \mathrm{SL}_2(q)$. Then the non-trivial irreducible characters of G have G-invariant bilinear forms with the following algebraic invariants.*

Character	*Invariant*
ψ	$\mathrm{d}_\pm(\psi) = q+1$
χ_i, $1 \leq i \leq \frac{q-2}{2}$	$\mathfrak{c}(\chi_i) = \begin{cases} 1 \in \mathrm{Br}(\mathbb{Q}(\chi_i)) & \text{if } n \text{ is odd, see Theorem 6.4} \\ \text{see Theorem 6.3} & \text{if } n \text{ is even} \end{cases}$
θ_j, $1 \leq j \leq \frac{q}{2}$	$\mathfrak{c}(\theta_j) = \begin{cases} (-1,-1) \in \mathrm{Br}(\mathbb{Q}(\sqrt{5})) & \text{if } q = 4, \\ 1 \in \mathrm{Br}(\mathbb{Q}(\theta_j)) & \text{if } q \geq 8. \end{cases}$

Proof. For the Steinberg character ψ we again have that $\psi + \mathbb{1}$ is the character of a 2-transitive permutation representation. In particular $\mathrm{d}_\pm(\psi) = \mathrm{d}_\pm(\mathbb{A}_q) = q+1$. For the characters θ_j of degree $q-1$ we note that the restriction of these characters to the normalizer $B \cong C_2^n \rtimes C_{q-1}$ of the Sylow-2-subgroup of G is the character of an irreducible rational monomial representation V. So V has an orthonormal basis and hence $\mathfrak{c}(\theta_j) = \mathfrak{c}(\mathbb{I}_q \otimes K)$ is given in Example 2.4. □

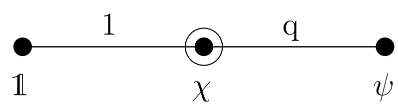
To describe the Clifford invariant of the characters χ_i of degree $q+1$ note that for the infinite places of K the invariant $[c(\chi_i)\otimes_K \mathbb{R}] \in \mathrm{Br}(\mathbb{R})$ is non-trivial if and only if $q=4$, because in all other cases, the character degree is $1 \pmod 8$.

For the odd finite primes of K, the Clifford invariant of χ_i is given in the next theorem:

Theorem 6.3. *Let $1 \leq i \leq (q-2)/2$, $K = \mathbb{Q}(\chi_i) = \mathbb{Q}[\vartheta_{q-1}^{(i)}]$, and let $\wp$ be some maximal ideal of $\mathbb{Z}_K$ such that $\wp \cap \mathbb{Z} = p\mathbb{Z}$ for some odd prime p. Then $[c(\chi_i) \otimes K_\wp] \in \mathrm{Br}(K_\wp)$ is not trivial if and only if*

(i) $p \equiv \pm 3 \pmod 8$, and (ii) $(q-1)/(\gcd(q-1,i))$ is a power of p.

Proof. We first note that condition (ii) implies that p divides $q-1$. If condition (ii) is not fulfilled, then the reduction of χ_i modulo $\wp$ is an irreducible Brauer character (see for instance [3]). In particular the orthogonal $K_\wp G$-module V affording the character χ_i contains a unimodular $R_\wp$-lattice. So Corollary 3.7 tells us that $[c(\chi_i)\otimes K_\wp] = 1 \in \mathrm{Br}(K_\wp)$. If the condition (ii) is satisfied, then $\wp$ is the unique prime ideal of K that contains p, the extension $K_\wp/\mathbb{Q}_p$ is totally ramified, and (again by [3]) the $\wp$-modular Brauer tree of the block containing χ_i is given as



where the multiplicity of the exceptional vertex χ is $\frac{p^a-1}{2}$ with $a = \nu_p(q-1)$. In particular [13, Theorem (VIII.3)] yields that the $R_\wp$-order $R_\wp G$ acts on V as

$$\Delta_{\chi_i}(R_\wp G) = \begin{pmatrix} R_\wp & \wp R_\wp^{1\times q} \\ R_\wp^{q\times 1} & R_\wp^{q\times q} \end{pmatrix}.$$

As in the proof of Lemma 5.2 the $R_\wp G$-invariant lattices in V form a chain:

$$\ldots \supset L' \supset L \supset \wp L' \supset \wp L \ldots$$

with $L'/L \cong R_\wp/\wp R_\wp$. So there is a G-invariant form F on V such that $L' = L^{\#}$, in particular the $\wp$-adic valuation of the determinant of L is 1. Choose $(b_1,\ldots,b_q) \in L^q$ such that the images form a basis $\overline{B}$ of $L/\wp L'$ and put $W := \langle b_1,\ldots,b_q\rangle_{K_\wp} \leq V$. The modular representation $L/\wp L'$ is isomorphic to the $\wp$-modular reduction of the Steinberg module ψ. In particular the determinant of the Gram matrix of $\overline{B}$ is $\overline{q+1} \in \mathbb{Z}/p\mathbb{Z} \cong R_\wp/\wp R_\wp$. As $\wp$ is odd and $q+1 \in R_\wp^\times$ this gives the discriminant of the bilinear $K_\wp$-module

$$\mathrm{d}_\pm(W, F_{|W}) = (q+1)(K_\wp^\times)^2 = 2(K_\wp^\times)^2$$

because $q+1 \equiv 2 \pmod p$ since p divides $q-1$. We can now apply Corollary 3.8 to conclude that the Clifford invariant of (V,F) is non-trivial, if and only if 2 is not a square in $K_\wp$, if and only if 2 is not a square in $\mathbb{F}_p = R_\wp/\wp$ which is equivalent to condition (i) by quadratic reciprocity. ☐

Theorem 6.4. *If $q = 2^n$ and n is odd then $\mathfrak{c}(\chi_i) = 1 \in \mathrm{Br}(\mathbb{Q}(\chi_i))$ for all $1 \leq i \leq \frac{q-2}{2}$.*

Proof. Let $M := \mathbb{Q}_2[\zeta_{2^n-1}]$ be the unramified extension of $\mathbb{Q}_2$ of degree n. Then M is a splitting field for G. Moreover the M-representation V_M affording the character χ_i is induced up from a linear M-representation of the normalizer $B = C_2^n \rtimes C_{2^n-1}$ of the Sylow-2-subgroup of G. In particular V_M is an irreducible monomial representation and hence the standard form F_M is G-invariant, so $(V_M, F_M) \cong \mathbb{I}_{2^n+1} \otimes M$. For $n \geq 3$ the dimension of V_M is $\equiv 1 \pmod 8$ and so by Example 2.4 the Clifford invariant of (V_M, F_M) is trivial in $\mathrm{Br}(M)$. Now let $K = \mathbb{Q}(\chi_i)$, (V, F) an orthogonal KG-module affording the character χ_i, and let $\wp$ be some prime ideal of K dividing 2. As $K \subseteq \mathbb{Q}[\zeta_{2^n-1}]$ the completion of K at $\wp$ is contained in M and, by the same argument as before, $(V \otimes M, F) \cong (V_M, aF_M)$ for some non-zero $a \in M$. In particular $\mathfrak{c}(V \otimes M, F) = 1$ in $\mathrm{Br}(M)$. As $[M : \mathbb{Q}_2] = n$ is assumed to be odd, also $[M : K_\wp]$ is odd and hence $\mathfrak{c}(V \otimes K_\wp, F) = 1$ in $\mathrm{Br}(K_\wp)$. This argument shows that no even prime $\wp$ of K ramifies in $c(V, F)$. Also the real primes do not ramify because $\dim(V) \equiv 1 \pmod 8$. So by Theorem 6.3 there is at most one prime ideal of K that ramifies in $c(V, F)$. But the number of ramified primes is even, which shows that $\mathfrak{c}(\chi_i) = 1$ in the Brauer group of K. □

Note that Theorem 6.4 together with Theorem 6.3 implies the well known fact that if n is odd then all primes p dividing $2^n - 1$ satisfy $p \equiv \pm 1 \pmod 8$ (because then $2^{(n+1)/2}$ is a square root of 2 modulo p).

Remark 6.5. *In the situation of Theorem 6.3 if $[c(\chi_i) \otimes K_\wp] \in \mathrm{Br}(K_\wp)$ is non-trivial and $q \neq 4$, then an odd number of even primes of K also ramify in $c(\chi_i)$. However, we did not determine in general which even primes of K ramify in $c(\chi_i)$ for the case that n is even. Of course the same argument as in the proof of Theorem 6.4 works if the primes above 2 are decomposed in $\mathbb{Q}(\zeta_{q-1}^i)/\mathbb{Q}(\vartheta_{q-1}^{(i)})$.*

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