THE ORTHOGONAL CHARACTER TABLE OF $\text{SL}_2(q)$

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Abstract. The rational invariants of the $\text{SL}_2(q)$-invariant quadratic forms on the real irreducible representations are determined. There is still one open question (see Remark 6.5) if $q$ is an even square.

1. Introduction

Throughout the paper let $G$ be a finite group. The isomorphism classes of $\mathbb{C}G$-modules are parametrized by their characters. Our aim is to extend this connection in order to also determine the $G$-invariant quadratic forms from the character table of $G$. The ordinary character table displays the characters $\chi_V$ of the absolutely irreducible $\mathbb{C}G$-modules $V$. For each $\chi_V$ let $K$ be the maximal real subfield of the character field of $V$ and $W$ the irreducible $KG$-module such that $V$ occurs in $W \otimes_K \mathbb{C}$. Then the space

$$\mathcal{F}_G(W) := \left\{ F : W \times W \to K \mid F(v,w) = F(w,v) \text{ and } F(gw,gv) = F(w,v) \text{ for all } g \in G, v,w \in W \right\}$$

of $G$-invariant symmetric bilinear forms on $W$ is at least one-dimensional and every non-zero $F \in \mathcal{F}_G(W)$ is non-degenerate. The character $\chi_V$ also determines the $K$-isometry classes of the elements of $\mathcal{F}_G(W)$. The orthogonal character table additionally contains the invariants (see Section 2) that determine the $K$-isometry classes of $(W,F)$ for in the case where these are independent of the choice of the non-zero $F \in \mathcal{F}_G(W)$.

For

$$G = \text{SL}_2(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_q^{2 \times 2} \mid ad - bc = 1 \right\}$$

the ordinary character table was already known to Schur, [16]. This paper determines the orthogonal character tables of $\text{SL}_2(q)$ for all prime powers $q$. For $q = 2^n$ with $n$ even and the characters of degree $q + 1$ we could not specify which even primes ramify in the Clifford algebra (see Section 6).

This work grew out of the first author’s PhD thesis [2] written under the supervision of the second author. In this thesis, the first author also determines the ordinary orthogonal character tables for all (non-abelian) finite quasisimple groups of order up to 200,000.

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2. INVARIANTS OF QUADRATIC SPACES

Let $K$ be a field of characteristic 0, $V$ an $n$-dimensional vector space over $K$ and $F : V \times V \to K$ a non-degenerate symmetric bilinear form. The two most important invariants attached to such a space $(V, F)$ are the discriminant and the Clifford invariant.

The discriminant of $(V, F)$ is

$$d_\pm(V, F) := (-1)^{(n-1)/2} \det(V, F)$$

where the determinant $\det(V, F) \in K^\ast/(K^\ast)^2$ is defined as the square class of the determinant of a Gram matrix of $F$ with respect to any basis.

The Clifford algebra $C(V, F)$ is the quotient of the tensor algebra by the two-sided ideal $\langle v \otimes v - F(v, v) \cdot 1 \mid v \in V \rangle$. A $K$-basis of $C(V, F)$ is given by the ordered tensors $(b_{i_1} \otimes \ldots \otimes b_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n)$ of any basis $(b_1, \ldots, b_n)$ of $V$, in particular $\dim(C(V, F)) = 2^n$. Put

$$c(V, F) := \begin{cases} C(V, F) & \text{if } n \text{ is even,} \\ C_0(V, F) := (b_{i_1} \otimes \ldots \otimes b_{i_k} \mid k \text{ even}) & \text{if } n \text{ is odd.} \end{cases}$$

Then $c(V, F) \cong D^{\times r}$ is a central simple $K$-algebra with involution and therefore it has order 1 or 2 in the Brauer group. The Clifford invariant of $(V, F)$ is defined as the Brauer class of $c(V, F)$:

$$c(V, F) := [c(V, F)] = [D] \in \text{Br}(K).$$

A more detailed exposition of this material may be found e.g. in [15].

Our interest in these two isometry invariants of quadratic spaces is mainly due to the following classical result by Helmut Hasse.

**Theorem 2.1 ([7]).** Over a number field $K$ the isometry class of a quadratic space is uniquely determined by its dimension, its determinant, its Clifford invariant and its signature at all real places of $K$.

We are mainly interested in the case where $K$ is a number field. Then $D$ is either $K$ or a quaternion division algebra over $K$. We use two notations for these $D$, either as a symbol algebra or by giving all the local invariants of $D$:

**Definition 2.2.** For $a, b \in K$ let $(a, b) := \left(\frac{a, b}{K}\right) \in \text{Br}(K)$ where

$$\left(\frac{a, b}{K}\right) := \{1, i, j, k \mid i^2 = a, j^2 = b, ij = -ji = k\}.$$  

By the Theorem of Hasse, Brauer, Noether, Albert (see [14, Theorem (32.11)]) any quaternion algebra $D$ over $K$ is determined by the set of places $\wp_1, \ldots, \wp_s$ (the ramified places) of $K$, for which the completion of $D$ stays a division algebra. Therefore we also describe $D = Q_{\wp_1, \ldots, \wp_s}$ by its ramified places, where we assume that the center $K$ is clear from the context.
**Remark 2.3.** Let \((V, F)\) be a bilinear space and \(a \in K^\times\). Then the scaled space \((V, aF)\) has the following algebraic invariants (see [10, Chapter 5, (3.16)]) for the Clifford invariant:

\[
d_{\pm}(V, aF) = \begin{cases} 
d_{\pm}(V, F) & \text{if dim}(V) \text{ is even}, \\
ad_{\pm}(V, F) & \text{if dim}(V) \text{ is odd}. 
\end{cases}
\]

and

\[
c(V, aF) = \begin{cases} 
c(V, F)(a, d_{\pm}(V, F)) & \text{if dim}(V) \text{ is even}, \\
c(V, F) & \text{if dim}(V) \text{ is odd}. 
\end{cases}
\]

If \((V, F) = (V_1, F_1) \perp (V_2, F_2)\) is the orthogonal direct sum of two subspaces the determinant is just the product \(\det(V, F) = \det(V_1, F_1) \cdot \det(V_2, F_2)\).

The behavior of the Clifford invariant is more complicated, cf. [10, Chapter five (3.13)]: \(c(V, F) = \)

\[
\begin{cases} 
c(V_1, F_1)c(V_2, F_2)(d_{\pm}(V_1, F_1), d_{\pm}(V_2, F_2)), & \dim(V_1) \equiv \dim(V_2) \pmod{2}, \\
c(V_1, F_1)c(V_2, F_2)(-d_{\pm}(V_1, F_1), d_{\pm}(V_2, F_2)), & \dim(V) \equiv \dim(V_1) \equiv 1 \pmod{2}.
\end{cases}
\]

**Example 2.4.** Let \(I_n\) be the \(n\)-dimensional \(\mathbb{Q}\)-vector space that has an orthonormal basis \((e_1, \ldots, e_n)\). Then \(d_{\pm}(I_n) = (-1)^{n(n-1)/2}(\mathbb{Q}^\times)^2\) and

\[
c(I_n) = \begin{cases} 
(1, 1) & n \equiv 0, 1, 2, 7 \pmod{8}, \\
(-1, -1) & n \equiv 3, 4, 5, 6 \pmod{8}.
\end{cases}
\]

The space \(A_{n-1} := \langle \sum_{i=1}^n e_i \rangle^\perp \leq I_n\) is the orthogonal complement of a space of discriminant \(n\) in \(I_n\). This allows to compute the discriminant and Clifford invariant of \(A_{n-1}\) using the formulas from the previous example: \(d_{\pm}(A_{n-1}) = (-1)^{(n-1)(n-2)/2}n(\mathbb{Q}^\times)^2\) and \(c(A_{n-1})\) depends on the value of \(n\) modulo 8:

<table>
<thead>
<tr>
<th>(n \pmod{8})</th>
<th>0, 1</th>
<th>2, 3</th>
<th>4, 5</th>
<th>6, 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c(A_{n-1}))</td>
<td>1</td>
<td>((-1, n))</td>
<td>((-1, -1))</td>
<td>((-1, -n))</td>
</tr>
</tbody>
</table>

3. Methods

**3.1. Orthogonal character tables.** Let \(\chi\) be a complex irreducible character of the finite group \(G\) and let \(K = \mathbb{Q}(\chi)^+\) be the maximal real subfield of the character field \(\mathbb{Q}(\chi)\). Let \(V\) be the irreducible \(CG\)-module affording the character \(\chi\) and let \(W\) be the irreducible \(KG\)-module such that \(V\) is a constituent of \(W_G := C \otimes_K W\). Put

\[
\mathcal{F}_G(W) := \left\{ F : W \times W \to K \mid F(v, w) = F(w, v) \text{ and } F(gv, gw) = F(w, v) \text{ for all } g \in G, v, w \in W \right\}
\]

the space of \(G\)-invariant symmetric bilinear forms on \(W\). As \(W\) is irreducible, all non-zero elements of \(\mathcal{F}_G(W)\) are non-degenerate and an easy averaging argument shows that \(\mathcal{F}_G(W)\) always contains a totally positive definite form \(F_0\). We call \(W\) **uniform** if \(\mathcal{F}_G(W) = \{aF_0 \mid a \in K\}\) is one-dimensional over \(K\).

**Remark 3.1.** There are three different situations to be considered:
(a) \( K = \mathbb{Q}(\chi) \) and \( V = W_G \): Then \( W \) is an absolutely irreducible \( KG \)-module and hence uniform.

(b) \( K = \mathbb{Q}(\chi) \) and \( W_G \cong V \oplus V \): Then the Schur index of \( \chi \) over \( K \) is 2, \( \chi(1) \) is even, and [18] tells us that \( d_{\pm}(F) \in (K^\times)^2 \) for all non-zero \( F \in \mathcal{F}_G(W) \). If the real Schur index of \( \chi \) is one, then \( \dim(\mathcal{F}_G(W)) = 3 \). If the real Schur index of \( \chi \) is 2, \( W \) is uniform and [18, Theorem B] also gives the Clifford invariant of \( (W,F) \):

\[
\mathcal{c}(W,F) = \begin{cases} 
1 & \text{if } \dim_K(W) \equiv 0 \pmod{8} \\
\lfloor \text{End}_{KG}(W) \rfloor & \text{if } \dim_K(W) \equiv 4 \pmod{8}.
\end{cases}
\]

(c) \([\mathbb{Q}(\chi) : K] = 2\). Then \( \chi_W = m(\chi + \chi) \) for some \( m \in \mathbb{N} \) and \( W \) is uniform if and only if \( m = 1 \). Choose \( \delta \in K \) such that \( \mathbb{Q}(\chi) = K(\sqrt{\delta}) \), then \( d_{\pm}(F) = \delta^{m\chi(1)}(K^\times)^2 \) for all \( 0 \neq F \in \mathcal{F}_G(W) \) (see [15, Chapter 10, Remark 1.4], [2, Theorem 4.3.9]).

**Definition 3.2.** Let \( \chi, K := \mathbb{Q}(\chi)^+ \), \( W \) be as above and put \( n := \dim_K(W) \).
Assume that \( W \) is uniform and choose \( 0 \neq F \in \mathcal{F}_G(W) \).
If \( n \) is even then we define

\[
d_{\pm}(\chi) := d_{\pm}(W,F).
\]

If \( n \) is odd, or \( n \) is even and \( d_{\pm}(\chi) = 1 \), then we put

\[
\mathcal{c}(\chi) := \mathcal{c}(W,F).
\]

The orthogonal character table of \( G \) is the complex character table of \( G \) with this additional information added.

As \( \mathcal{F}_G(W) = \{aF \mid a \in K\} \) Remark 2.3 and Remark 3.1 show that the values \( d_{\pm}(\chi) \) and \( \mathcal{c}(\chi) \) are well defined, i.e. independent of the choice of the non-zero \( F \in \mathcal{F}_G(W) \).

3.2. **Clifford orders.** Let us now assume that \( K \) is a local or global field of characteristic 0, i.e. \( K \) is a finite extension of either \( \mathbb{Q}_p \) or \( \mathbb{Q} \), and let \( R \) denote the ring of integers in \( K \). Let \( V \) be a finite dimensional vector space over \( K \) and \( F : V \times V \to K \) a symmetric bilinear form with associated quadratic form

\[
Q_F : V \to K, v \mapsto Q_F(v) = \frac{1}{2} F(v,v).
\]

**Definition 3.3.** A lattice \( L \) in \( V \) is a finitely generated \( R \)-submodule of \( V \) that contains a \( K \)-basis of \( V \). The lattice \( L \) is called integral, if \( F(L,L) \subseteq R \) and even, if \( Q_F(L) \subseteq R \). The dual lattice of \( L \) is \( L^\# := \{v \in V \mid F(v,L) \subseteq R\} \) and \( L \) is called unimodular, if \( L = L^\# \).

Even unimodular lattices are called regular quadratic \( R \)-modules in [9]. If \( 2 \not\in R^\times \), then there are no regular \( R \)-modules \( L \) of odd dimension. Kneser calls an even lattice \( L \) of odd dimension such that \( L^\#/L \cong R/2R \) a semi-regular quadratic \( R \)-module.
Theorem 3.4 ([9, Satz 15.8]). Assume that $R$ is a complete discrete valuation ring (with finite residue class field) and let $L$ be a regular or semi-regular quadratic $R$-module in $(V, Q_F)$. If $\dim(V) \geq 3$ then $L \cong H(R) \perp M$ for some regular or semi-regular quadratic $R$-module $M$. Here $H(R)$ is the hyperbolic plane, the regular free $R$-lattice with basis $(e, f)$ such that $Q_F(e) = Q_F(f) = 0$ and $F(e, f) = 1$.

As both invariants, the Clifford invariant and the discriminant, of the hyperbolic plane $H(K) = K H(R)$ are trivial, we obtain the following corollary.

Corollary 3.5. Additionally to the assumptions of the theorem let $\dim(V)$ be odd. Then $c(V, F) = 1$.

Proof. We proceed by induction on the dimension of $V$. If $\dim(V) = 1$ then $c(V, F) = K$ and so $c(V, F) = 1$. So assume that $\dim(V) \geq 3$ and let $L$ be a semi-regular quadratic $R$-lattice in $(V, Q_F)$. Then $L \cong H(R) \perp M$ and hence $V \cong H(K) \perp KM$ for some semi-regular lattice $M$ in $KM$. By induction we have $c(KM, F^{KM}) = 1$. So
\[ c(V, F) = c(KM, F^{KM}) c(H(K)) (-d_\pm(KM, F^{KM}), d_\pm(H(K))) = 1. \]

Remark 3.6. Let $L$ be an even lattice in $V$. Then the Clifford order $C(L, F)$ of $L$ is defined to be the $R$-subalgebra of $C(V, F)$ generated by $L$. As $Q_F(L) \subseteq R$, the Clifford order is an $R$-lattice in $C(V, F)$, in particular finitely generated over $R$. If $L$ has an orthogonal basis $(b_1, \ldots, b_n)$, then the ordered tensors $(b_{i_1} \otimes \cdots \otimes b_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n)$ form an $R$-basis of $C(L, F)$. In this case it is easy to compute the determinant of $C(L, F)$ and of $C_0(L, F)$ with respect to the reduced trace bilinear form (see [2, Theorem 7.2.2]): Up to some power of $2$ they are both powers of $Q_F(b_1) \cdots Q_F(b_n)$.

Corollary 3.7. Assume that $K$ is a number field, $2 \neq p \in \mathbb{Z}$ is some odd prime and $\wp$ is a prime ideal of $K$ containing $p$. Denote the completion of $K$ at $\wp$ by $K_{\wp}$ and its valuation ring by $R_{\wp}$. Assume that there is a lattice $L$ in $V$ such that $L_{\wp} = R_{\wp} \otimes L$ is a unimodular $R_{\wp}$-lattice. Then
\[ [c(V, F) \otimes K_{\wp}] = 1 \in Br(K_{\wp}). \]

Proof. Since $2 \in R_{\wp}$, the lattice $L_{\wp}$ has an orthogonal basis and Remark 3.6 shows that the determinant of the Clifford order $C(L_{\wp}, F)$ and also of $C_0(L_{\wp}, F)$ is a unit in $R_{\wp}$. In particular the determinant of a maximal order in $c(V, F) \otimes K_{\wp}$ is a unit in $R_{\wp}$, which shows that this central simple $K_{\wp}$-algebra is a matrix ring over $K_{\wp}$ (see for instance [14, Theorem (20.3)]).

A bit more generally we may also compute the Clifford invariant of a bilinear space that contains a lattice of prime determinant:

Corollary 3.8. Keep the assumptions of Corollary 3.7 and let $(W_{\wp}, E_{\wp})$ be a 1-dimensional bilinear $K_{\wp}$-vector space such that the $\wp$-adic valuation of the discriminant of $E_{\wp}$ is odd. Then
\[ c((V \otimes K_{\wp}, F) \perp (W_{\wp}, E_{\wp})) = 1 \in Br(K_{\wp}) \text{ if and only if } d_\pm(V \otimes K_{\wp}, F) \in (K_{\wp}^\times)^2. \]
Proof. Clearly the Clifford invariant of the 1-dimensional space is trivial and also $c(V \otimes K_p, F)$ is trivial by Corollary 3.7. So the formulas in Remark 2.3 give us the Clifford invariant of the orthogonal sum as

$$c((V \otimes K_p, F) \perp (W_k, E_k)) = (d \pm (V \otimes K_p, F), u\pi)$$

where $u$ is a unit and $\pi$ is a prime element in the valuation ring $R_p$. As $d := d \pm (V \otimes K_p, F)$ has even valuation, this quaternion symbol is trivial if and only if $d$ is a square. $\square$

3.3. A Clifford theory of orthogonal representations. Let $N \triangleleft G$ be a normal subgroup. Clifford theory explains the interplay between irreducible representations of $N$ and $G$ (see for instance [4, Section 11.1]). We want to describe the behavior of invariant forms under this correspondence.

Let $K$ be a totally real number field and $V$ an irreducible $KG$-module with a non-degenerate invariant form $F$. We will then call $(V, F)$ an orthogonal representation of $G$. Let $U$ be an irreducible $KN$-module occurring as a direct summand of $V|_N$ with multiplicity $e$. Let $I$ be the inertia group of $U$, of index $t := [G : I]$ in $G$, and let $G = \bigsqcup_{i=1}^{t} g_i I$ be a decomposition of $G$ into left $I$-cosets. We then have the following decomposition of $V|_N$ into pairwise non-isomorphic irreducible $KN$-modules $g_i U$ ($i = 1, \ldots, t$):

(1) $V|_N \cong \bigoplus_{i=1}^{t} (g_i U)^e$,

In this situation we obtain the following lemma

Lemma 3.9. The decomposition (1) is orthogonal

$$(V|_N, F) = (g_1 U^e, F_1) \perp (g_2 U^e, F_2) \perp \ldots \perp (g_t U^e, F_t)$$

and the forms $F_i$ are non-degenerate and pairwise $K$-isometric.

Proof. Clearly, the restriction of $F$ to the direct summand $g_i U^e$ is $N$-invariant. For $i \neq j$ we have

$$g_i U \cong g_i U^e \ncong g_j U$$

so the summands $(g_i U)^e$ are orthogonal to each other and the $F_i$ are non-degenerate. The elements $g_j^{-1} g_i \in G \leq O(V, F)$ induce isometries between $F_i$ and $F_j$. $\square$

Example 3.10. Consider an odd prime $p$, a natural number $n$ and abbreviate $q := p^n$. Let $C_{(q-1)/2} \cong H \leq GL_n(F_p)$ be a subgroup acting with regular orbits on $F_p^n \setminus \{0\}$ in its natural action. Then the group $G := C_p^n \rtimes H$, which is isomorphic to the normalizer of a Sylow $p$-subgroup in $PSL_2(q)$ has $(q-1)/2$ linear characters and two non-linear characters $\psi_1, \psi_2$ of degree $(q-1)/2$ with Schur index 1 and character field

$$Q(\psi_1) = Q(\psi_2) = \begin{cases} Q(\sqrt{q}) & \text{if } q \equiv 1 \pmod{4}, \\ Q(\sqrt{-q}) & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$
Let $H_1$ be the unique subgroup of $H$ of order $\frac{p-1}{2}$ and put $N := C_p^m \rtimes H_1$. Then $N \triangleleft G$ and we will apply Lemma 3.9 to this normal subgroup in order to compute the discriminant $d_\pm(\psi_i)$ in the case $q \equiv 1 \pmod 4$.

Let $\psi \in \{\psi_1, \psi_2\}$, $K = Q(\psi) = Q(\sqrt{q})$ and $(V, F)$ an orthogonal $KG$-module whose character is $\psi$.

There is a character $1 \neq \chi \in \text{Irr}(C_p^n)$ such that $\psi = \text{ind}_{C_p^n}^G(\chi) = \text{ind}_N^G(\text{ind}_{C_p^n}^N(\chi))$.

Ordinary Clifford theory shows that $\chi$ is a faithful character of a group isomorphic to $C_p \times C_{p-1}$. As the trace forms of cyclotomic fields are well understood (cf. [11, Section 3.3.2]), we can find the determinants of the $(V, F_i)$ as

$$\det(V_i, F_i) = \det(V_1, F_1) = \begin{cases} p(Q^x)^2 & \text{if } n \text{ is even} \\ u\sqrt{p}(Q(\sqrt{p})^x)^2 & \text{if } n \text{ is odd} \end{cases}$$

for some unit $u$ of the ring of integers of $Q(\sqrt{p})$. In conclusion, we obtain

$$\det(\psi) = \begin{cases} 1(Q^x)^2 & \text{if } n \equiv 0 \pmod 4 \text{ or } p \equiv 3 \pmod 4, \\ p(Q^x)^2 & \text{if } n \equiv 2 \pmod 4 \text{ and } p \equiv 1 \pmod 4, \\ u\sqrt{p}(Q(\sqrt{p})^x)^2 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

In the case $q \equiv 3 \pmod 4$ the character $\psi$ has non-real values and we find $d_\pm(\psi) = -p(Q^x)^2$.

4. THE ORTHOGONAL CHARACTER TABLE OF $\text{SL}_2(q)$ FOR ODD $q$

Let $p$ be an odd prime, $n$ a natural number, put $q := p^n$ and let $G := \text{SL}_2(q)$ be the group of all $2 \times 2$ matrices of determinant 1 over the field with $q$ elements. A reference for the ordinary (and modular) representation theory of this group is, for example, [1]. We use the ordinary character table and the notation for the absolutely irreducible characters from [5]:

**Theorem 4.1** ([5, Theorem 38.1]). Let $\langle \nu \rangle = F_q^\times$. Consider

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

and let $b \in \text{SL}_2(q)$ be an element of order $q + 1$.

For $x \in \text{SL}_2(q)$, let $(x)$ denote the conjugacy class containing $x$. $\text{SL}_2(q)$ has the following $q + 4$ conjugacy classes of elements, listed together with the size of the classes.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1$</th>
<th>$z$</th>
<th>$c$</th>
<th>$d$</th>
<th>$zc$</th>
<th>$zd$</th>
<th>$a^\ell$</th>
<th>$b^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>(x)</td>
<td>$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\frac{1}{2}(q^2 - 1)$</td>
<td>$\frac{1}{2}(q^2 - 1)$</td>
<td>$\frac{1}{2}(q^2 - 1)$</td>
</tr>
</tbody>
</table>
where \( 1 \leq \ell \leq \frac{q-3}{2}, 1 \leq m \leq \frac{q-1}{2}. \)

Put

\[
\varepsilon := (-1)^{(q-1)/2}, \quad \zeta_r := \exp(2\pi i/r) \text{ and } \psi_r^{(s)} := \zeta_r^s + \zeta_r^{-s} \quad \text{for } r, s \in \mathbb{N}.
\]

Then the character table of \( SL_2(q) \) reads as

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\chi & z & c & d & a^\ell & b^m \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\psi & q & q & 0 & 0 & -1 \\
\chi_i & q + 1 & (-1)^i(q + 1) & 1 & 1 & \psi_{q-1}^{(i)} \\
\theta_j & q - 1 & (-1)^j(q - 1) & -1 & -1 & 0 \quad -\psi_{q+1}^{(j)} \\
\xi_1 & \frac{1}{2}(q + 1) & \frac{1}{2}(1 + \sqrt{\varepsilon q}) & \frac{1}{2}(1 - \sqrt{\varepsilon q}) & -1 & (1) \\
\xi_2 & \frac{1}{2}(q + 1) & \frac{1}{2}(1 - \sqrt{\varepsilon q}) & \frac{1}{2}(1 + \sqrt{\varepsilon q}) & 0 & (1) \\
\eta_1 & \frac{1}{2}(q - 1) & \frac{1}{2}(1 + \sqrt{\varepsilon q}) & \frac{1}{2}(1 + \sqrt{\varepsilon q}) & 0 & (1) \\
\eta_2 & \frac{1}{2}(q - 1) & \frac{1}{2}(1 - \sqrt{\varepsilon q}) & \frac{1}{2}(1 - \sqrt{\varepsilon q}) & 0 & (1) \\
\hline
\end{array}
\]

where \( 1 \leq i \leq \frac{q-3}{2}, 1 \leq j \leq \frac{q-1}{2}, 1 \leq \ell \leq \frac{q-3}{2}, 1 \leq m \leq \frac{q-1}{2}. \)

The columns for the classes \((zc)\) and \((zd)\) are omitted because for any irreducible character \(\chi\) the relation \(\chi(zc) = \frac{\chi(z)}{\chi(c)}\chi(c)\) holds.

**Theorem 4.2.** The following table gives the orthogonal character table of \( SL_2(q) \).
We use the abbreviations introduced in Theorem 4.1. As before $K$ is the maximal real subfield of the character field and $W$ the irreducible KG-module, whose character contains $\chi$.

5. The proof of Theorem 4.2

5.1. The faithful characters of $G$. The faithful irreducible characters of $\text{SL}_2(q)$ either have real Schur index 2 or they take values in an imaginary quadratic number field. Janusz [8, Section 2] contains an explicit description of the endomorphism rings $\text{End}_{KG}(W)$. In particular their discriminants and Clifford invariants can be read off from this information using Remark 3.1 (b) and (c).

5.2. The non-faithful characters $\eta_i$. If $q \equiv 3 \pmod{4}$ then the characters $\eta_1$ and $\eta_2$ of degree $(q-1)/2$ have character field $\mathbb{Q}(\sqrt{q}) = \mathbb{Q}(\sqrt{-p})$ and Schur index 1. So Remark 3.1 (c) yields their discriminant.

5.3. The Steinberg character. The character $\psi$ is a non-faithful character of degree $q$. As $1 + \psi$ is the character of a 2-transitive permutation representation of $G$, the invariants of $\psi$ are those of $\mathbb{A}_q$ as given in Example 2.4.

5.4. The characters $\theta_j$, $j$ even. For even $j$, the character $\theta_j$ is a non-faithful character of even degree $q-1$ with totally real character field $K$ and Schur index 1. Let $(W,F)$ be the orthogonal KG-module affording the character $\theta_j$. Then the restriction of $W$ to the Borel subgroup $B \cong (C_p)^n \rtimes C_{(q-1)/2}$ of $\text{PSL}_2(q)$ has character $\psi_1 + \psi_2$ from Example 3.10. As $d_{\pm}(\psi_1)$ and $d_{\pm}(\psi_2)$ are Galois conjugate, the formula for $d_{\pm}(\psi_1)$ in Example 3.10 yields

$$d_{\pm}(\theta_j) = \begin{cases} 1(K^\times)^2 & n \text{ even} \\ \varepsilon p(K^\times)^2 & n \text{ odd}. \end{cases}$$

If $n$ is even then we can also deduce the Clifford invariant of $(W,F)$: In this case $q \equiv 1 \pmod{4}$ so $-\zeta_{q+1}^2$ is a primitive $q+1$st root of unity and hence all characters of degree $q-1$ of the group $\text{PSL}_2(q)$ extend to characters of $\text{PGL}_2(q)$ with the same character field (see [17, Table III] for a character table) and of Schur index 1 (see [6]). So $(W,F)$ is also an orthogonal representation of $\text{PGL}_2(q)$ and restricting $(W,F)$ to $B$, we obtain the orthogonal sum of two isometric spaces $(W,F) \cong (V_1,F_1) \perp (V_2,F_2)$ because the normalizer of $B$ in $\text{PGL}_2(q)$ interchanges $V_1$ and $V_2$. By Example 3.10 we have $d_{\pm}(V_i,F_i) = p$ if $n \equiv 2 \pmod{4}$ and $p \equiv 1 \pmod{4}$ and $d_{\pm}(V_i,F_i) = 1$ otherwise ($i = 1,2$). In both cases $(d_{\pm}(V_1,F_1),d_{\pm}(V_2,F_2)) = 1 \in \text{Br}(\mathbb{Q})$ and so by Remark 2.3 $c(W,F) = c(V_1,F_1)c(V_2,F_2) = c(V_1,F_1)^2 = 1$.

5.5. The characters $\chi_i$, $i$ even. For even $i$, the character $\chi_i$ is a non-faithful character of even degree $q+1$ with totally real character field $K$ and Schur index 1. As before we restrict $\chi_i$ to the Borel subgroup and obtain

$$\chi_i|_B = \psi_1 + \psi_2 + \alpha + \overline{\alpha}$$

where $\psi_1, \psi_2$ are as in 5.4 and $\alpha$ is a complex linear character of $B$. Comparing character values we obtain that $\alpha(y) = \zeta_{q-1}^i$ for a suitably chosen generator $y$ of
Proposition 5.1. The Clifford invariant \( C(q^{(q-1)/2} \leq B \). In particular \( Q(\alpha) = Q(\zeta_{q-1}^{i}) = K(\sqrt{\psi_{q-1}^{(2i)}} - 2) \) and hence Remark 3.1 (c) tells us that \( d_{\pm}(\alpha) = \psi_{q-1}^{(2i)} - 2 \). The discriminant of \( \psi_{1} \) and \( \psi_{2} \) behave as in 5.4 and hence we compute the discriminant \( d_{\pm}(\chi_{i}) = \varepsilon(\psi_{q-1}^{(2i)} - 2)^{q/2} \).

5.6. The characters \( \xi_{1}, \xi_{2} \) for \( q \equiv 1 \) (mod 4). Assume that \( q = p^{n} \equiv 1 \) (mod 4). Then the two characters \( \xi_{1} \) and \( \xi_{2} \) of odd degree \( \frac{q+1}{2} \) factor through \( \text{PSL}_{2}(q) \) and have a totally real character field \( K = Q(\chi_{1}) = Q(\chi_{2}) = Q(\sqrt{q}) \).

Proposition 5.1. The Clifford invariant \( c(\xi_{1}) = c(\xi_{2}) \) is stable under the Galois group of \( K \) and the only primes that ramify in \( c(\xi_{1}) \) are the ones that divide 2p. More precisely there are the following two possibilities for \( c(\xi_{1}) = c(\xi_{2}) \):

<table>
<thead>
<tr>
<th>( q ) (mod 16)</th>
<th>( n ) even</th>
<th>( n ) odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 * ( c(\xi_{1}) = c(\xi_{2}) )</td>
<td>( 1 ) ( Q_{p,\infty} )</td>
<td>( 1 ) ( Q_{\psi_{1},\infty} )</td>
</tr>
<tr>
<td>( c(\xi_{1}) = c(\xi_{2}) )</td>
<td>( \frac{[\mathbb{Q}<em>{2},\mathbb{Q}</em>{\infty}]}{[\mathbb{Q}_{2,\infty}]} )</td>
<td>( \frac{[\mathbb{Q}<em>{\psi</em>{1},\psi_{2}}]}{[\mathbb{Q}<em>{\psi</em>{2},\infty}]} )</td>
</tr>
</tbody>
</table>

Here, for \( p \equiv 1 \) (mod 8) and \( n \) odd, \( \varphi_{1} \) and \( \varphi_{2} \) denote the two places of \( K = Q(\sqrt{p}) \) that divide 2.

Proof. Let \( \xi \) be one of \( \xi_{1} \) or \( \xi_{2} \), \( K = Q(\sqrt{q}) \) and \( W \) the \( KG \)-module affording the character \( \xi \). Since \( F_{G}(W) \) always contains a totally positive definite form, we know that \( c(\xi) \otimes \mathbb{R} = 1 \) if \( q \equiv 1 \), \( -3 \) (mod 16) and \( c(\xi) \otimes \mathbb{R} \neq 1 \) otherwise, for all real places of \( K \). If \( K \neq \mathbb{Q} \) then \( \xi_{1} \) and \( \xi_{2} \) are Galois conjugate and so are \( c(\xi_{1}) \) and \( c(\xi_{2}) \). The outer automorphism of \( G \) interchanges \( \xi_{1} \) and \( \xi_{2} \) which shows that \( c(\xi_{1}) = c(\xi_{2}) \), so this algebra is stable under the Galois group of \( K \). Moreover the only possible finite primes of \( K \) that ramify in \( c(\xi) \) are those dividing \( p \) or 2. This is seen as follows: The representation \( \xi \) is irreducible modulo all other primes \( \ell \) (see [1, Section 9.3]) so in particular there is a \( G \)-invariant lattice \( L \) in \( W \) whose determinant is not divisible by \( \ell \) and hence \( \ell \) does not ramify in \( c(W,F) \) by Remark 3.6. Noting that \( 2 \) is decomposed in \( K \) if and only if \( n \) is odd and \( p \equiv 1 \) (mod 8), we are left with the possibilities for \( c(\xi) \) as stated. \( \square \)

Lemma 5.2. The primes of \( K \) that divide 2 are not ramified in \( c(\xi_{1}) \), so \( c(\xi_{1}) = c(\xi_{2}) \) is given in line \( \star \) of Proposition 5.1.

Proof. Let \( \varphi \) be a prime ideal of \( K \) that contains 2 and let \( R_{\varphi} \) be the valuation ring in the completion \( K_{\varphi} \) (so \( R_{\varphi} \cong \mathbb{Z}_{2} \) if \( q \equiv 1 \) (mod 8) and \( R_{\varphi} \cong \mathbb{Z}_{2}[\zeta_{3}] \) if \( q \equiv 5 \) (mod 8)). By [13, Theorem VII.12 and Theorem VII.4] the image of \( R_{\varphi}G \) in \( \text{End}(K_{\varphi} \otimes W) \) is isomorphic to

\[
\Delta_{\xi_{1}}(R_{\varphi}G) = \begin{pmatrix}
R_{\varphi} & 2R_{\varphi}^{1/(q-1)/2} \\
R_{\varphi}^{(q-1)/2 \times 1} & R_{\varphi}^{(q-1)/2 \times (q-1)/2}
\end{pmatrix}
\]

In particular the \( R_{\varphi}G \)-lattices in \( K_{\varphi} \otimes W \) form a chain

\[
\ldots \supset L' \supset L \supset 2L' \supset 2L \ldots
\]

with \( L'/L \cong R_{\varphi}/2R_{\varphi} \). If \( F \in F_{G}(W) \) is non-degenerate and \( L \) is \( G \)-invariant, then also its dual lattice is \( G \)-invariant. This shows that there is some \( F \in F_{G}(W) \)
such that $L'$ is the dual lattice of $L$. But then $Q_F(L) \subseteq R_\nu$ because otherwise the even sublattice of $L$ would be a $G$-invariant sublattice of index 2 in $L$. So $L$ is a semi-regular quadratic $R_\nu$-module in $(K_\nu \otimes W, F)$ and by Corollary 3.5 this implies that $c(K_\nu \otimes W, F) = 1$. □

Note that for $n = 1$ and $n = 2$ it is also possible to find the Clifford invariant of $\xi_1$ and $\xi_2$ using the character theoretic method from [12] (see [2, Section 6.4]).

6. The orthogonal character table of $\text{SL}_2(2^n)$

We now assume that $q = 2^n$ with $n \geq 2$ and put $G := \text{SL}_2(q)$. Then the ordinary character table of $G$ is given in [5, Theorem 38.2]:

**Theorem 6.1** ([5, Theorem 38.2]). Let $\nu$ be a generator of $F_q^\times$ and consider the elements

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a := \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}
\]

of $G$. The group also contains an element $b$ of order $q + 1$. The character table of $G$ is

<table>
<thead>
<tr>
<th>Character</th>
<th>$1_G$</th>
<th>$c$</th>
<th>$a^\ell$</th>
<th>$b^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$q$</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_i$, $1 \leq i \leq \frac{q-2}{2}$</td>
<td>$q + 1$</td>
<td>1</td>
<td>$\zeta_{q-1}^i + \zeta_{q-1}^{-i}$</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_j$, $1 \leq j \leq \frac{q}{2}$</td>
<td>$q - 1$</td>
<td>$-1$</td>
<td>0</td>
<td>$-\zeta_{q+1}^{jm} - \zeta_{q+1}^{-jm}$</td>
</tr>
</tbody>
</table>

where $1 \leq i \leq \frac{q-2}{2}$, $1 \leq j \leq \frac{q}{2}$, $1 \leq \ell \leq \frac{q-2}{2}$, $1 \leq m \leq \frac{q}{2}$.

In contrast to the odd characteristic case all characters have totally real character field and Schur index 1.

**Theorem 6.2** (Orthogonal representations of $\text{SL}_2(2^n)$). Let $q = 2^n$, $n \geq 2$ and $G = \text{SL}_2(q)$. Then the non-trivial irreducible characters of $G$ have $G$-invariant bilinear forms with the following algebraic invariants.

<table>
<thead>
<tr>
<th>Character</th>
<th>Invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>$d_\pm(\psi) = q + 1$</td>
</tr>
<tr>
<td>$\chi_i$, $1 \leq i \leq \frac{q-2}{2}$</td>
<td>$c(\chi_i) = \begin{cases} 1 &amp; \text{if } n \text{ is odd, see Theorem 6.4} \ \text{see Theorem 6.3} &amp; \text{if } n \text{ is even} \end{cases}$</td>
</tr>
<tr>
<td>$\theta_j$, $1 \leq j \leq \frac{q}{2}$</td>
<td>$c(\theta_j) = \begin{cases} (-1, -1) \in \text{Br}(Q(\sqrt{5})) &amp; \text{if } q = 4, \ 1 \in \text{Br}(Q(\theta_j)) &amp; \text{if } q \geq 8. \end{cases}$</td>
</tr>
</tbody>
</table>

**Proof.** For the Steinberg character $\psi$ we again have that $\psi + 1$ is the character of a 2-transitive permutation representation. In particular $d_\pm(\psi) = d_\pm(\chi_q) = q + 1$. For the characters $\theta_j$ of degree $q - 1$ we note that the restriction of these characters to the normalizer $B \cong C_2^n \rtimes C_{q-1}$ of the Sylow-2-subgroup of $G$ is the character of an irreducible rational monomial representation $V$. So $V$ has an orthonormal basis and hence $c(\theta_j) = c(\mathcal{I}_q \otimes K)$ is given in Example 2.4. □
To describe the Clifford invariant of the characters $\chi_i$ of degree $q + 1$ note that for the infinite places of $K$ the invariant $[c(\chi_i) \otimes_K R] \in \text{Br}(R)$ is non-trivial if and only if $q = 4$, because in all other cases, the character degree is $1 \pmod{8}$.

For the odd finite primes of $K$, the Clifford invariant of $\chi_i$ is given in the next theorem:

**Theorem 6.3.** Let $1 \leq i \leq (q - 2)/2$, $K = \mathbb{Q}(\chi_i) = \mathbb{Q}[\psi_{q-1}]$, and let $\varphi$ be some maximal ideal of $\mathbb{Z}_K$ such that $\varphi \cap \mathbb{Z} = p\mathbb{Z}$ for some odd prime $p$. Then $[c(\chi_i) \otimes K] \in \text{Br}(K_p)$ is not trivial if and only if

\[(i) \ p \equiv \pm 3 \pmod{8}, \text{ and } (ii) \ (q - 1)/(\gcd(q - 1, i)) \text{ is a power of } p.\]

**Proof.** We first note that condition (ii) implies that $p$ divides $q - 1$. If condition (ii) is not fulfilled, then the reduction of $\chi_i$ modulo $\varphi$ is an irreducible Brauer character (see for instance [3]). In particular the orthogonal $K_p G$-module $V$ affording the character $\chi_i$ contains a unimodular $R_p$-lattice. So Corollary 3.7 tells us that $[c(\chi_i) \otimes K_p] \in \text{Br}(K_p)$.

If the condition (ii) is satisfied, then $\varphi$ is the unique prime ideal of $K$ that contains $p$, the extension $K_p/Q_p$ is totally ramified, and (again by [3]) the $\varphi$-modular Brauer tree of the block containing $\chi_i$ is given as

![Diagram](image)

where the multiplicity of the exceptional vertex $\chi$ is $\nu_a - 1$ with $a = \nu_p(q - 1)$. In particular [13, Theorem (VIII.3)] yields that the $R_\varphi$-order $R_\varphi G$ acts on $V$ as

$$\Delta_{\chi_i}(R_\varphi G) = \begin{pmatrix} R_\varphi^{1 \times 1} & \varphi R_\varphi^{1 \times q} \\ \varphi R_\varphi^{q \times 1} & R_\varphi^{q \times q} \end{pmatrix}.$$ 

As in the proof of Lemma 5.2 the $R_\varphi G$-invariant lattices in $V$ form a chain:

$$\ldots \supset L' \supset L \supset \varphi L' \supset \varphi L' \ldots$$

with $L'/L \cong R_\varphi/\varphi R_\varphi$. So there is a $G$-invariant form $F$ on $V$ such that $L' = L^\#$, in particular the $\varphi$-adic valuation of the determinant of $L$ is $1$. Choose $(b_1, \ldots, b_q) \in L^q$ such that the images form a basis $B$ of $L/\varphi L'$ and put $W := (b_1, \ldots, b_q)_{K_p} \leq V$.

The modular representation $L/\varphi L'$ is isomorphic to the $\varphi$-modular reduction of the Steinberg module $\psi$. In particular the determinant of the Gram matrix of $\overline{B}$ is $q + 1 \in \mathbb{Z}/p\mathbb{Z} \cong R_\varphi/\varphi R_\varphi$. As $\varphi$ is odd and $q + 1 \in R_\varphi^\times$ this gives the discriminant of the bilinear $K_p$-module

$$d_\pm(W, F|_W) = (q + 1)(K_p^\times)^2 = 2(K_p^\times)^2$$

because $q + 1 \equiv 2 \pmod{p}$ since $p$ divides $q - 1$. We can now apply Corollary 3.8 to conclude that the Clifford invariant of $(V, F)$ is non-trivial, if and only if $2$ is not a square in $K_p$, if and only if $2$ is not a square in $F_p = R_\varphi/\varphi$ which is equivalent to condition (i) by quadratic reciprocity. \qed

**Theorem 6.4.** If $q = 2^n$ and $n$ is odd then $c(\chi_i) = 1 \in \text{Br}(\mathbb{Q}(\chi_i))$ for all $1 \leq i \leq \frac{q - 2}{2}$. 

**Proof.** Let $M := \mathbb{Q}_2[\zeta_{2^n-1}]$ be the unramified extension of $\mathbb{Q}_2$ of degree $n$. Then $M$ is a splitting field for $G$. Moreover the $M$-representation $V_M$ affording the character $\chi_i$ is induced up from a linear $M$-representation of the normalizer $B = C_2^n \rtimes C_{2^n-1}$ of the Sylow-2-subgroup of $G$. In particular $V_M$ is an irreducible monomial representation and hence the standard form $F_M$ is $G$-invariant, so $(V_M, F_M) \cong I_{2^n+1} \otimes M$. For $n \geq 3$ the dimension of $V_M$ is $1 \mod 8$ and so by Example 2.4 the Clifford invariant of $(V_M, F_M)$ is trivial in $\text{Br}(M)$. Now let $K = \mathbb{Q}(\chi_i), (V, F)$ an orthogonal $KG$-module affording the character $\chi_i$, and let $\varphi$ be some prime ideal of $K$ dividing 2. As $K \subseteq \mathbb{Q}[\zeta_{2^n-1}]$ the completion of $K$ at $\varphi$ is contained in $M$ and, by the same argument as before, $(V \otimes M, F) \cong (V_M, aF_M)$ for some non-zero $a \in M$. In particular $c(V \otimes M, F) = 1$ in $\text{Br}(M)$. As $[M : \mathbb{Q}_2] = n$ is assumed to be odd, also $[M : K_\varphi]$ is odd and hence $c(V \otimes K_\varphi, F) = 1$ in $\text{Br}(K_\varphi)$. This argument shows that no even prime $\varphi$ of $K$ ramifies in $c(V, F)$. Also the real primes do not ramify because $\text{dim}(V) \equiv 1 \mod 8$. So by Theorem 6.3 there is at most one prime ideal of $K$ that ramifies in $c(V, F)$. But the number of ramified primes is even, which shows that $c(\chi_i) = 1$ in the Brauer group of $K$.\[\square\]

Note that Theorem 6.4 together with Theorem 6.3 implies the well known fact that if $n$ is odd then all primes $p$ dividing $2^n - 1$ satisfy $p \equiv \pm 1 \mod 8$ (because then $2^{(n+1)/2}$ is a square root of 2 modulo $p$).

**Remark 6.5.** In the situation of Theorem 6.3 if $[c(\chi_i) \otimes K_\varphi] \in \text{Br}(K_\varphi)$ is non-trivial and $q \neq 4$, then an odd number of even primes of $K$ also ramify in $c(\chi_i)$. However, we did not determine in general which even primes of $K$ ramify in $c(\chi_i)$ for the case that $n$ is even. Of course the same argument as in the proof of Theorem 6.4 works if the primes above 2 are decomposed in $\mathbb{Q}(\zeta_{q-1})/\mathbb{Q}(\sqrt[4]{q^{(i)}})$.

**References**


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