The group ring of $SL_2(p^2)$ over the p-adic integers.

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Abstract

This paper describes the ring theoretic structure of the group rings of $SL_2(p^2)$ over the p-adic integers.

1 Introduction

The special linear group $SL_2(p^s)$ is the group of 2×2-matrices of determinant 1 over the field $\mathbb{F}_{p^s}$ with p^s-elements. Its representations over fields of characteristic 0 have been already investigated by I. Schur [11]. Also its modular representation theory is well understood, the irreducible $\overline{\mathbb{F}_p}SL_2(p^s)$ modules are described in [2]. Even the Loewy series of the projective indecomposable $\overline{\mathbb{F}_p}SL_2(p^s)$ modules are known (cf. [1]) so one might hope to be able to describe the ring theoretic structure of the group ring of $SL_2(p^s)$ over p-adic integers. For odd primes p, the principal 2-block of $\mathbb{Z}_2 SL_2(p^s)$ is investigated in [10, Chapter VII]. For the odd primes $q \neq p$ the blocks of $\mathbb{Z}_q SL_2(p^s)$ have cyclic defect. Therefore they are described by the general theory of blocks with cyclic defect groups (cf. [10, Chapter VIII], [8]). So the open case is the group ring $\mathbb{Z}_p SL_2(p^s)$. Its blocks of defect > 0 have an elementary abelian defect group of order p^s. By [5] the p-decomposition numbers of $SL_2(p^s)$ are 0 or 1, so the group ring is strongly related to a graduated order in the sense of [10, Definition II.1]. Here we treat the first non-trivial case $s = 2$. In this case the Cartan invariants are relatively small and the methods developed in [10] essentially suffice to describe the graduated hull of the group ring, which allows to read off the irreducible lattices. Here we give an explicit description of the basic order which is Morita equivalent to the group ring.

So let $G := SL_2(p^2)$. Let K be the unramified quadratic extension of the field $\mathbb{Q}_p$ of p-adic numbers and R the ring of integers in K. Using the decomposition numbers as given in [5] one finds the interesting fact, that the graduated hull of the blocks of RG only depend on one parameter each (cf. Theorem 4.1). This constant can be determined investigating the subring RB, where B is the normalizer of a Sylow-p-subgroup in G (cf. Propositions 4.2 and 4.3). The subring RG is a

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subdirect product of the components of the graduated hull Γ of RG. Let e and e' be orthogonal idempotents of RG that map onto two different central primitive idempotents of $RG/J(RG)$. Then the order $eRGe$ is already determined by the fact that RG is a symmetric order (cf. Lemma 4.7). The possibilities for the $eRGe$-$e'RGe'$-bimodule $eRGe'$ are less restricted if it belongs to more than two components of Γ. In this case the amalgamations can be obtained by looking at a subgroup $SL_2(p)$ of G (cf. Propositions 4.10 and 4.11). A list summarizing the used notation may be found at the end of the paper.

2 Decomposition numbers

Throughout the paper let p be an odd prime. A useful description of the decomposition numbers and the Cartan invariants of the group $SL_2(p^s)$ in characteristic p can be found in [5] (cf. also [3]). Let F be the Frobenius automorphism $x \mapsto x^p$ of $k := \mathbb{F}_{p^s}$. For a k-vector space V let $V^{(i)}$ be the vector space obtained by twisting V i-times with F, hence $V^{(i)} = V$ with scalar multiplication $x \cdot v := x^{p^i} v$ ($x \in k, v \in V$). The group $SL_2(k)$ acts as group of automorphisms on the algebra $k[X,Y]$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends X and Y to $dX - bY$, $-cX + aY$, respectively. For $0 \leq \lambda < p$ let $M_\lambda \subseteq k[X,Y]$ the subspace of homogenous polynomials of degree λ and for $\lambda = \sum_{i=0}^{s-1} \lambda_i p^i$ with $0 \leq \lambda_i < p$ let

$$M_\lambda := M_{\lambda_0} \otimes_k M_{\lambda_1}^{(1)} \otimes_k \ldots \otimes_k M_{\lambda_{s-1}}^{(s-1)}.$$

The modules $\bar{k}M_\lambda$ form a system of representatives of the isomorphism classes of simple $SL_2(p^s)$-modules over the algebraic closure $\bar{k}$ of k..

The characters of the absolutely irreducible $\mathbb{C}SL_2(p^s)$-modules are 1, δ, δ', δ_1, ..., $\delta_{(p^s-1)/2}$, St, η, η', η_1, ..., $\eta_{(p^s-3)/2}$, where $\eta(1) = \eta'(1) = (p^s+1)/2$, $\delta(1) = \delta'(1) = (p^s-1)/2$, $\delta_i(1) = p^s - 1$, and $\eta_i(1) = p^s + 1$. The character η_i is the induction of the character of the Borel subgroup B, the normalizer of a Sylow-p-subgroup of $SL_2(p^s)$, defined by $\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \mapsto \zeta_{p^s-1}^i$ for all $u \in k$ and a fixed generator t of the multiplicative group of k. One has $\eta_{(p^s-1)/2} = \eta + \eta'$ and $\eta_0 = 1 + St$. The characters δ_i can be distinguished by restricting them to the non split torus $S \cong C_{p^s+1} \leq SL_2(p^s)$. Let α be a generator of the group of irreducible characters of S. Then one can numerate the δ_i such that

$$\delta_{i|S} = 2\sum\{\alpha^j \mid j \equiv i \pmod 2, 0 \leq j \leq p^s\} - \alpha^i - \alpha^{-i}$$

for $1 \leq i \leq (p^s+1)/2$ where $\delta_{(p^s+1)/2} = \delta + \delta'$.

To describe the decomposition numbers define for $\lambda = \sum_{i=0}^{s-1} \lambda_i p^i$ the sets

$$W(\lambda) = \{0 \leq \nu \leq p^s - 1 \mid \nu = \sum_{i=0}^{s-1} \epsilon_i \tilde{\lambda}_i p^i \text{ for some } \epsilon_0, \ldots, \epsilon_{s-1} = \pm 1\}$$

where $\tilde{}$ is the bijection of the set $\{0, \ldots, p-1\}$ defined by $\tilde{x} = p - 1 - x$. Moreover let $V^\epsilon(\lambda) = W(\lambda) \cup (p^s - \epsilon - W(\lambda))$ for $\epsilon = \pm 1$.

For $j = 1, \ldots, (p^s-3)/2$ let $d_{\lambda,j}^{(1)} := d_{\lambda,\eta_j}$ be the multiplicity of the Brauer character belonging to M_λ in the restriction of η_j to the p-regular classes of $SL_2(p^s)$. Analogously let $d_{\lambda,j}^{(-1)} := d_{\lambda,\delta_j}$ $(j = 1, \ldots, (p^s-1)/2)$, $d_{\lambda,(p^s-1)/2}^{(1)} := d_{\lambda,\eta} = d_{\lambda,\eta'}$, and $d_{\lambda,(p^s+1)/2}^{(-1)} := d_{\lambda,\delta} = d_{\lambda,\delta'}$. Then one gets

Theorem 2.1 *([5, Theorem 2.7])*

a) $d_{\lambda,j}^{(\epsilon)} = 1$ *if* $j \in V^\epsilon(\lambda)$ *and 0 otherwise.*

b) $d_{\lambda,St} = 1$ *if* $\lambda = p^s - 1$ *and 0 otherwise.*
$d_{\lambda,1} = 1$ *if* $\lambda = 0$ *and 0 otherwise.*

3 The case $s = 2$.

Now let $s = 2$ and $G := SL_2(p^2)$. Let K be the unramified extension of degree 2 of $\mathbb{Q}_p$ and R the ring of integers in K. Then $(K, R, \mathbb{F}_{p^2})$ is a p-modular splitting system for G. The ring RG has 3 blocks, one of which is of defect 0, the other two are of defect 2: the principal block and the one containing the faithful irreducible characters of G.

The irreducible p-modular characters $\lambda = \lambda_0 + \lambda_1 p$ fall into 4 categories according to the parity of the λ_i. The Brauer character λ is called *even* if λ_0 and λ_1 are even and *odd* if both λ_0 and λ_1 are odd. Note that the even and odd characters $\neq 0$ are the (non faithful) irreducible Brauer characters that belong to the principal block of RG. The faithful irreducible Brauer characters satisfy $\lambda_0 \not\equiv \lambda_1 \pmod 2$. They are called *of type eo* (respectively *oe*) if λ_0 is even (respectively odd).

Definition 3.1 *Let* $\lambda = \lambda_0 + \lambda_1 p$ *and* $\lambda' = \lambda'_0 + \lambda'_1 p$, *be two irreducible p-Brauer characters belonging to the same block of G.* λ *and* λ' *are called* equivalent modulo 2 *if* $\lambda_0 - \lambda'_0$ *(and hence also* $\lambda_1 - \lambda'_1$*) is even.*

For two irreducible p-Brauer characters λ, λ' of a finite group H the Cartan invariant is defined as $c_{\lambda,\lambda'} = \sum_\chi d_{\lambda,\chi} d_{\lambda',\chi}$ where χ runs over all complex irreducible characters of H.

Lemma 3.2 *Let λ, λ' be two different irreducible Brauer characters belonging to the same block of RG. If λ and λ' are equivalent modulo 2 then $W(\lambda) \cap W(\lambda') = \emptyset$ and the Cartan invariant $c_{\lambda,\lambda'} \leq 1$.*

Proof: Let $\lambda = \tilde{\lambda}_0 + \tilde{\lambda}_1 p$ and $\lambda' = \tilde{\lambda}'_0 + \tilde{\lambda}'_1 p$ be equivalent modulo 2. Assume that $\epsilon_0\lambda_0 + \epsilon_1\lambda_1 p = \epsilon'_0\lambda'_0 + \epsilon'_1\lambda'_1 p \in W(\lambda) \cap W(\lambda')$. Without loss of generality we assume that $\lambda_0 \geq \lambda'_0$ and $\epsilon_0 = 1$. Then $0 \leq \lambda_0 - \epsilon'_0\lambda'_0 = p(\epsilon'_1\lambda'_1 - \epsilon_1\lambda_1) \leq 2(p-1) < 2p$. Hence either $\lambda_0 = \epsilon'_0\lambda'_0$ and then $\lambda = \lambda'$ or $\lambda_0 - \epsilon'_0\lambda'_0 = p$ is odd. Hence if λ and λ' are equivalent modulo 2 then $W(\lambda) \cap W(\lambda') = \emptyset$ and otherwise the intersection contains at most one element. Assume that λ and λ' are equivalent modulo 2 and let $\nu \in W(\lambda) \cap (p^2 - \epsilon - W(\lambda'))$. Since $0 \leq \nu \leq p^2 - \epsilon$ one has $\nu = \epsilon_0\lambda_0 + p\lambda_1 = p^2 - \epsilon - \epsilon'_0\lambda'_0 - p\lambda'_1$. Hence $\epsilon_0\lambda_0 + \epsilon'_0\lambda'_0 = p^2 - \epsilon - (\lambda_1 + \lambda'_1)p$. This determines ϵ_0 and ϵ'_0 since the right hand side is non zero. The only possibility to have $c_{\lambda,\lambda'} = 2$ is that $\lambda_1 + \lambda'_1 = p$ and $\epsilon_0\lambda_0 + \epsilon'_0\lambda'_0 = -\epsilon = \pm 1$. This contradicts the assumption that λ and λ' are equivalent modulo 2. □

From [5, Theorem 3.10] one gets the following

Lemma 3.3 *Let $\lambda \neq \lambda'$.*

a) If $(p^2 - \epsilon)/2 \notin W(\lambda) \cap W(\lambda')$ for $\epsilon = \pm 1$ then $c_{\lambda,\lambda'} \leq 2$.

b) If $(p^2 - \epsilon)/2 \in W(\lambda) \cap W(\lambda')$ for $\epsilon = \pm 1$ then $c_{\lambda,\lambda'} = 4$.

There are exactly two pairs λ, λ' of p-Brauer characters of G with $c_{\lambda,\lambda'} = 4$ namely the two p-modular constituents $\lambda_\Box := (p-1)/2 + p(p-1)/2$ and $\lambda'_\Box := (p-3)/2 + p(p-3)/2$ of η and η' and the two p-modular constituents $\gamma := (p-1)/2 + p(p-3)/2$ and $\gamma^F =: \gamma' = (p-3)/2 + p(p-1)/2$ of δ and δ'.

If χ is an irreducible Frobenius character of G, then let $\hat{\chi}$ be the Brauer character obtained by restricting χ to the p-regular classes of G.

Lemma 3.4 *i) If χ is an irreducible Frobenius character of G then $\hat{\chi}$ has at most four irreducible constituents.*

ii) If χ is not faithful, then at most 2 of these p-modular constituents are even respectively odd.

iii) If χ is faithful, then at most 2 of these p-modular constituents have type oe respectively eo.

Proof: With Theorem 2.1 one gets that $\hat{1}$ and $\hat{St}$ are irreducible and $\hat{\eta} = \hat{\eta'} = \lambda_\Box + \lambda'_\Box$ and $\hat{\delta} = \hat{\delta'} = \gamma + \gamma'$. Now assume that χ is not one of these 6 characters. Let $\epsilon = 1$ if $\chi = \eta_j$ and $\epsilon = -1$ if $\chi = \delta_j$ for some $0 < j < (p^2 - \epsilon)/2$. Writing $j = \lambda_1 + p\lambda_2 = -(p-\lambda_1) + p(\lambda_2+1)$ and $p^2 - \epsilon - j = \gamma_1 + p\gamma_2 = -(p-\gamma_1) + p(\gamma_2+1)$ with $0 \leq \gamma_1, \gamma_2, \lambda_1, \lambda_2 < p$ one gets the modular constituents of χ and the lemma follows immediately. □

Corollary 3.5 *Let $\chi = \delta_j$ for some $j \in \{1, \ldots, (p^2-1)/2\}$ and $\epsilon := -1$ or $\chi = \eta_j$ for some $j \in \{1, \ldots, (p^2-3)/2\}$ and $\epsilon := 1$. Then $\hat{\chi}$ has 3 constituents if $j \equiv 0$ or $-\epsilon \pmod p$ or $j \leq p - \epsilon$ and $(j, \epsilon) \neq (p, -1)$. The characters δ_p and δ_1 have only two p-modular constituents. In all other cases $\hat{\chi}$ has 4 constituents.*

Let Λ be a block of RG of defect > 0. Then one may define a graph whose vertices are the non trivial irreducible complex characters $\chi \neq 1$ of G that belong to the block Λ. Two characters χ and χ' are connected by an edge, if they have two p-modular constituents in common. The following Lemma (together with Lemma 4.9 below) shows that the quotient of this graph by the action of the Frobenius automorphism is connected.

Lemma 3.6 *i) If $0 < j < (p^2-1)/2$ then δ_j and η_j have two common p-modular constituents.*

ii) If $p < j \leq (p^2-1)/2$ then δ_j and η_{j-2} have two common p-modular constituents.

iii) Let $F \in Out(G)$ denote the automorphism that maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$. If $(p^2+3)/(2p) < j \leq p-1$ (resp. $2 \leq j \leq (p^2+3)/(2p)$ and $p > 3$) then $(\delta_j)^F = \delta_{j'}$ where $j' = (p^2+1) - pj \geq p+1$ (resp. $j' = pj \geq p+1$).

Proof: i) Let $j = \lambda_1 + p\lambda_2$. Since $j < (p^2-1)/2$ one gets $\lambda_2 < p-1$. So if $\lambda_1 > 0$ then the decomposition $j = -(p-\lambda_1) + p(\lambda_2+1)$ yields the second common constituent of $\hat{\delta}_j$ and $\hat{\eta}_j$. If $\lambda_1 = 0$ then $j = p\lambda_2$ with $\lambda_2 > 0$. But then $p^2+1-j = 1+p(p-\lambda_2)$ and $p^2-1-j = -1+p(p-\lambda_2)$ so p^2+1-j yields the second common constituent of $\hat{\delta}_j$ and $\hat{\eta}_j$. If $\lambda_1 < 0$ then $\lambda_2 \geq 1$ and $j = (p+\lambda_1) + p(\lambda_2-1)$ yields the second common constituent of $\hat{\delta}_j$ and $\hat{\eta}_j$.
ii) Clearly $p^2+1-j = p^2-1-(j-2) = \gamma_1 + p\gamma_2$ gives rise to a common constituent of $\hat{\delta}_j$ and $\hat{\eta}_{j-2}$. If $j \equiv 1 \pmod p$ then $j = 1 + \lambda_2 p$ and $j-2 = -1+\lambda_2 p$ hence j yields a second common modular constituent. If $j > p+1$ and $j \not\equiv 1 \pmod p$ then $p^2+1-j < p(p-1)$ and $p^2+1-j = p^2-1-(j-2) = \gamma_1+p\gamma_2 = -(p-\gamma_1)+p(\gamma_2+1)$ gives the 2 common p-modular constituents of δ_j and η_{j-2}.
iii) Clear. □

4 The ring $RSL_2(p^2)$

To describe the R-order $RSL_2(p^2)$ the language of exponent matrices developed in [10] is used. For simplicity we assume that $p > 3$.

Let K' be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers R' the ring of integers in K'. An R'-order Λ in a semisimple K'-algebra A is called *graduated*, if there are orthogonal idempotents $e_1, \ldots, e_r$ in Λ such that $e_i \Lambda e_i$ is a maximal order in $e_i A e_i$. Let $\mathcal{D}$ be a finite dimensional K'-division algebra with maximal order Ω and prime element π. Let Λ be a graduated R'-order in $\mathcal{D}^{n \times n}$. Let $J(\Lambda)$ be the Jacobson radical of Λ and $e_1, \ldots, e_r$ be orthogonal idempotents of Λ that map onto the central primitive idempotents of $\Lambda/J(\Lambda)$. Then the orders $e_i \Lambda e_i$ are maximal orders ($\cong \Omega^{n_i \times n_i}$) and $e_i \Lambda e_j$ is a $e_i \Lambda e_i$-$e_j \Lambda e_j$ bimodule, hence isomorphic to $\pi^{m_{ij}} \Omega^{n_i \times n_j}$ for some $m_{ij} \in \mathbb{Z}$. If one puts $M := (m_{ij})$ (where $m_{ii} = 0$) then

$$\Lambda \cong \Lambda(\Omega, n_1, \ldots, n_r, M) := \{X := (X_{ij}) \in \mathcal{D}^{n \times n} \mid X_{ij} \in \pi^{m_{ij}} \Omega^{n_i \times n_j}\}.$$

M is called the *exponent matrix* of Λ. The entries of M satisfy $m_{ij} + m_{jk} \geq m_{ik}$ and $m_{ij} + m_{ji} > 0$ for $i \neq j$.

Now let H be a finite group and $\Lambda = R'H$ be a group ring and let $f_1, \ldots, f_t$ be the central primitive idempotents of $K'H$. Assume that the decomposition numbers of H are ≤ 1 and that for all $1 \leq i \leq t$ the center $Z(\Lambda f_i)$ is the maximal order in $Z(K'Hf_i)$. Then the orders $f_i \Lambda$ are graduated orders in $f_i K'H$ and $\tilde{\Lambda} := \oplus_{i=1}^t f_i \Lambda$ is the unique graduated hull of Λ. Automorphisms and anti-automorphisms of $f_i \Lambda$ give rise to linear equalities between the entries of the exponent matrices of $\tilde{\Lambda}$ in an obvious way (cf. [10, Proposition (IV.1)]).

Another important tool to determine these entries are the amalgamation matrices of the projective indecomposable Λ-lattices (cf. [10, Definition (IV.6)]). For $j = 1, \ldots, t$ let $u_j := (d_j/|H|) f_j$ where d_j is the degree of an absolutely irreducible constituent of the K'-irreducible character that belongs to f_j. Then Λ is a symmetric order in $K'H$ with respect to the generalized trace map $T_u : a \mapsto \sum_{j=1}^t trace_j(u_j a)$ where $trace_j$ is the reduced trace of the K'-algebra $K'Hf_j$. The conductor of $\tilde{\Lambda}$ in Λ is the maximal $\tilde{\Lambda}$-ideal $\oplus_{j=1}^t (\Lambda \cap f_j \Lambda)$ contained in Λ. Since Λ is symmetric, this conductor coincides with the dual of $\tilde{\Lambda}$ with respect to T_u which can be calculated with the conductor formula [10, Theorem (III.8)]. Let $e_1, \ldots, e_r$ be orthogonal idempotents in Λ that map onto the central primitive idempotents in $\Lambda/J(\Lambda)$. For $i = 1, \ldots, r$ let S_i be the simple Λ-module corresponding to the idempotent e_i and P_i be its projective cover. Then the conductor formula also gives the multiplicity of S_i in $(\oplus_{j=1}^t f_j P_k)/(\oplus_{j=1}^t (f_j P_k \cap P_k))$ $(1 \leq k \leq r)$ in terms of the corresponding entries in the exponent matrices of $\tilde{\Lambda}$ (cf. [10, Theorem (IV.4)]).

Let $G = SL_2(p^2)$ and R be as above. For $\epsilon = \pm 1$ and $0 < j < (p^2 - \epsilon)/2$ let $\Lambda_j^{(\epsilon)}$ be the graduated order $f_j^{(\epsilon)} RG$ where $f_j^{(\epsilon)}$ is the central primitive idempotent of KG that belongs to δ_j, respectively η_j, if $\epsilon = -1$, respectively 1. Let $\Lambda^{(-1)}, \Lambda^{(-1)'}$ and $f^{(-1)}, f^{(-1)'}$ (respectively $\Lambda^{(1)}, \Lambda^{(1)'}$ and $f^{(1)}, f^{(1)'}$) be the corresponding orders and central primitive idempotents that belong to δ, δ' (respectively η, η'). Finally let f_0 (resp. f_{St}) denote the idempotent of KG that corresponds to the trivial character 1 (resp. to the Steinberg character St).

Notation. For the graduated orders we use the following notation: The rows and columns of the exponent matrices are indexed by irreducible Brauer characters of G (and not by their degree) and $\Omega = R$ is omitted. Hence $\Lambda(\lambda_1, \ldots, \lambda_s, M) := \Lambda(R, \lambda_1(1), \ldots, \lambda_s(1), M)$.

Theorem 4.1 *There are $c_o, c_e \in \{1, 2\}$ such that for all $\epsilon = \pm 1$ and $0 < j < (p^2 - \epsilon)/2$ the orders $\Lambda_j^{(\epsilon)}$ are as follows, where $c = c_e$ if $\Lambda_j^{(\epsilon)}$ belongs to the principal block of RG and $c = c_o$ if the order belongs to the faithful block of RG:*

$$(\Lambda_p^{(-1)})^F = \Lambda_1^{(-1)} \cong \Lambda(p(p-2), (p-2) + p(p-1), \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}).$$

If $j \neq -\epsilon, -\epsilon p$ and either $j \equiv 0 \pmod{p}$ or $j \equiv -\epsilon \pmod{p}$ or $j \leq p - \epsilon$ then

$$\Lambda_j^{(\epsilon)} \cong \Lambda(\lambda_1, \lambda_2, \lambda_3, \begin{pmatrix} 0 & c & 2 \\ 0 & 0 & 1 \\ 0 & c-1 & 0 \end{pmatrix})$$

where the irreducible Brauer characters λ_1 and λ_3 are equivalent modulo 2. In all other cases

$$\Lambda_j^{(\epsilon)} \cong \Lambda(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \begin{pmatrix} 0 & c & c & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & c-1 & c-1 & 0 \end{pmatrix})$$

where the irreducible Brauer characters λ_1 and λ_4 are equivalent modulo 2.

Proof: The rows and columns of the exponent matrix of $\Lambda_j^{(\epsilon)}$ are indexed by the p-modular constituents of the corresponding Frobenius character δ_j or η_j, the number of which follows from Corollary 3.5. One may always normalize the exponent matrix in such a way that the first column contains zeroes only by writing the order with respect to a suitable basis of the projective $\Lambda_j^{(\epsilon)}$-lattice whose head corresponds to the first constituent. This also has the consequence that the entries of the exponent matrix are nonnegative. Let $m^{(\epsilon j)}_{\lambda,\lambda'}$ denote the entry of the exponent matrix of $\Lambda^{(\epsilon)}_j$ that corresponds to the 2 constituents λ, λ'. Then for $\lambda \neq \lambda'$ one gets $0 < m^{(\epsilon j)}_{\lambda,\lambda'} + m^{(\epsilon j)}_{\lambda',\lambda} \leq 2$. By [10, Corollary (IV.7)] one has $m^{(\epsilon j)}_{\lambda,\lambda'} + m^{(\epsilon j)}_{\lambda',\lambda} = 2$ if the Cartan invariant $c_{\lambda,\lambda'} = 1$.

Assume that the irreducible ordinary character belonging to $\Lambda^{(\epsilon)}_j$ has 4 p-modular constituents, $\lambda_1, \ldots, \lambda_4$ ordered in such a way that λ_1 and λ_4 resp. λ_2 and λ_3 are equivalent modulo 2. Then $c_{\lambda_1,\lambda_4} = c_{\lambda_2,\lambda_3} = 1$ and the exponent matrix of $\Lambda^{(\epsilon)}_j$ is

$$\begin{pmatrix} 0 & y & x & 2 \\ 0 & 0 & 2-a & u_2 \\ 0 & a & 0 & z_2 \\ 0 & u_1 & z_1 & 0 \end{pmatrix}.$$

Since the character δ_j resp. η_j is selfdual, the order $\Lambda^{(\epsilon)}_j$ is invariant under the involution $\lambda \mapsto Q\lambda^{tr}Q^{-1}$ ($\lambda \in \Lambda^{(\epsilon)}_j$) for a non-singular G-invariant symmetric bilinear form Q on the corresponding irreducible KG-module. Since all Brauer characters of G are selfdual, this yields the following equalities:

$$z_1 = z_2 + x - 2,\ u_1 = u_2 + y - 2,\ a = 2 - a + y - x.$$

The last equation implies that $x \equiv y \pmod 2$ and therefore $x = y \in \{1, 2\}$ and $a = 1$. If $x = y = 2$ one gets $z_1 = z_2 = 1$ and $u_1 = u_2 = 1$. If $x = y = 1$ then $z_1 = z_2 - 1$ and $u_1 = u_2 - 1$ imply that $z_2 = 1 = u_2$ and $z_1 = u_1 = 0$. Hence the exponent matrix of $\Lambda^{(\epsilon)}_j$ is as in the proposition with c replaced by some constant $c^{(\epsilon)}_j \in \{1, 2\}$. The other (easier) cases are dealt with analogously.

Now assume that $c_{\lambda,\lambda'} = 2 = d^{(\epsilon_1)}_{j,\lambda} d^{(\epsilon_1)}_{j,\lambda'} + d^{(\epsilon_2)}_{i,\lambda} d^{(\epsilon_2)}_{i,\lambda'}$. Then by [10, Corollary (IV.7)] $m^{(\epsilon_1 j)}_{\lambda,\lambda'} + m^{(\epsilon_1 j)}_{\lambda',\lambda} = m^{(\epsilon_2 i)}_{\lambda,\lambda'} + m^{(\epsilon_2 i)}_{\lambda',\lambda}$. Hence by Lemma 3.6 one gets that $c^{(1)}_j = c^{(-1)}_j$ for $0 < j < (p^2-1)/2$ and $c^{(1)}_{j-2} = c^{(-1)}_j$ for $p < j \leq (p^2-1)/2$. Using the notation of Lemma 3.6 (iii) it holds that $c^{(-1)}_j = c^{(-1)}_{j'}$ for $2 \leq j \leq p-1$, because the Frobenius automorphism induces a $\mathbb{Z}_p$-automorphism of RG and therefore an isomorphism between $\Lambda^{(-1)}_j$ and $\Lambda^{(-1)}_{j'}$. Therefore one gets the same constant $c_e = c^{(\epsilon)}_j$ for all even j and $c_o = c^{(\epsilon)}_j$ for all odd j. □

Proposition 4.2 $c_e = 1$.

Proof: We only deal with the principal block of RG, so let $B \cong C_p \times C_p : C_{(p^2-1)/2}$ be the normalizer of the Sylow-p-subgroup of $PSL_2(p^2)$. B is isomorphic to a subgroup of index 2 in the affine group $A := Aff(1,p^2) = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_{p^2}^*, b \in \mathbb{F}_{p^2}\}$. The group A has one faithful character χ of degree (p^2-1) and (p^2-1) linear characters $\chi_0, \chi_1, \ldots, \chi_{p^2-2}$ indexed in such a way that χ_i is the i-th power of χ_1. The Brauer characters $\hat{\chi}_i$ are the characters of the (p^2-1) different simple $\mathbb{F}_{p^2}A$-modules S_i and $\hat{\chi} = \sum \hat{\chi}_i$. Hence the group ring RA consists of one block with multiplicity 1 (cf. [10, Chapter IV]). Let P_0 be the projective indecomposable RA-module with head S_0. The isomorphism $A \cong Aff(1,p^2)$ shows that $Ext^1(S_0,S_1)$ is non trivial. Applying the Frobenius automorphism, one also gets $Ext^1(S_0,S_p) \neq 0$. Tensoring with S_i and applying Jennings theorem (cf. [6]) one finds that the Loewy series of P_0/pP_0 is

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & p & & & \\
 & & 2 & & p+1 & & 2p & & \\
 & \cdot^{\cdot^{\cdot}} & & & & & & \ddots & \\
p-1 & & 2(p-1) & & \cdots & & & & p(p-1) \\
 & 2p-1 & & \cdots & & \cdots & & p^2-p+1 & \\
 & & \ddots & & & & \cdot^{\cdot^{\cdot}} & & \\
 & & & p^2-p-1 & & p^2-2 & & & \\
 & & & & 0 & & & &
\end{array}$$

where the simple module with character $\hat{\chi}_i$ is simply denoted by i $(0 \leq i \leq p^2-2)$.

The restriction of the character χ to the subgroup B of A is the sum of two characters ψ and ψ'. ψ and ψ' have the same p-modular constituents $(\hat{\chi}_i)_{|B}$ $(i = 0,\ldots,(p^2-1)/2-1)$. Restricting P_0 to the subgroup B of A, one obtains a projective indecomposable B-module, the subdirect sum of the projective lattices f_0P_0, $f_\psi P_0 =: L$ and $f_{\psi'}P_0 =: L'$. The amalgamating factor $(f_\psi + f_{\psi'})P_0/f_\psi P_0 \oplus f_{\psi'}P_0$ has to contain all those simple $\mathbb{F}_{p^2}B$-modules that do not occur twice in the same layer of the Loewy series of $(P_0/pP_0)/soc(P_0/pP_0) \cong (f_\psi + f_{\psi'})P_0/p(f_\psi + f_{\psi'})P_0$. In particular it contains the module with character $(\hat{\chi}_{(p-1)/2})_{|B}$.

Now let $\lambda = \tilde{\lambda}_1 + p\tilde{\lambda}_2$ be an irreducible p-modular character of $PSL_2(p^2)$. The eigenvalues of an element of order p^2-1 in $SL_2(p^2)$ on the module V_λ are $\zeta^{\lambda_1-2k_1}\zeta^{p(\lambda_2-2k_2)}$ where $\zeta \in \mathbb{C}$ is a primitive (p^2-1)-th root of unity and k_i runs from 0 to λ_i $(i=1,2)$. Hence the trivial character $(\hat{\chi}_0)_{|B}$ occurs in the restriction of λ to B, if and only if λ is even. On the other hand, if λ is even and $\lambda_1, \lambda_2 < p-1$, then $(\hat{\chi}_{(p-1)/2})_{|B}$ is not a constituent of $\lambda_{|B}$. Namely assume that $(\lambda_1 - 2k_1) + p(\lambda_2 - 2k_2) \equiv (p-1) \pmod{p^2-1}$. Since $-(p-1) < \lambda_i - 2k_i < (p-1)$ one has $p^2-1 = (p-1)+p(p-1) > \lambda_1 - 2k_1 + p(\lambda_2 - 2k_2) > -(p^2-1)$ and therefore $(\lambda_1 - 2k_1) + p(\lambda_2 - 2k_2) = p-1$ or $p-1-(p^2-1) = p(1-p)$. This is a contradiction.

For even $j \in \{2, \dots (p^2-1)/2\}$ the restriction of δ_j to B is $\psi+\psi'$. Hence $(\hat{\delta}_j)_B = 2\sum_{i=0}^{(p^2-1)/2}(\hat{\chi}_i)_{|B}$. To prove $c = 1$ is suffices to find a character δ_j of $PSL_2(p^2)$ such that the degrees of the two even modular constituents of δ_j are not divisible by p. For instance for $j = (p^2-1)/2$, these constituents are $(p-1)/2+p(p-1)/2$, $(p-3)/2+p(p-3)/2$, $(p+1)/2+p(p-1)/2$, and $(p-5)/2+p(p-3)/2$. This implies the proposition since $p > 3$. □

Proposition 4.3 $c_o = c_e$

Proof: Let P be the tensor product of the projective hull P_1 of the natural $\mathbb{F}_{p^2}SL_2(p^2)$-module $V_1 \otimes V_0^{(1)} \cong V_1$ with the natural $\mathbb{F}_{p^2}SL_2(p^2)$-module V_1. Then P is a projective $\mathbb{F}_{p^2}PSL_2(p^2)$-module. The composition factors of the tensor products $V_\lambda \otimes V_{\lambda'}$ can easily be calculated looking at the eigenvalues of the elements of a split torus on this module. So $V_1 \otimes V_1 = V_0 \oplus V_2$, because $Ext^1(V_2, V_0) = Ext^1(V_0, V_2) = 0$. Hence $P = P_0 \oplus P_2 \oplus V_{St}$. The character of P_1 is $\delta_3 + \delta_{2p-1} + \eta_{2p-3} + \eta_1$. If $c_o = 2$ then the second layer of the Loewy series of P_1 is $(p-4+p(p-1)) \oplus (2+p(p-2))^2 \oplus (p-2+p(p-1))$
$\oplus\ (p(p-2))^2 \oplus (p-4+p(p-3)) \oplus (p-3+p)^2 \oplus (p-2+p(p-3))$ where the simple module V_λ is simply denoted by λ and λ^2 stands for $V_\lambda \oplus V_\lambda$. Now $V_1 \otimes V_{p^2-4} = V_{p^2-3} \oplus V_{p^2-5}$ (again the Ext-groups are 0). But none of these two modules occurs in the second layer of the Loewy series of P. So one gets a contradiction and $c_o = 1$. □

Note that the last two propositions can also be obtained from [1].

The following proposition is shown in [9, Theorem 3.1].

Proposition 4.4 $\Lambda^{(1)} \cong \Lambda^{(1')} \cong \Lambda(\lambda_\square, \lambda'_\square, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

Proposition 4.5 $\Lambda^{(-1)} \cong \Lambda^{(-1)'} \cong \Lambda(\gamma, \gamma', \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

Proof: In [4] it is shown that the p-adic Schur index of η is 2. Hence $\Lambda^{(-1)} = R \otimes \Lambda$ where Λ is a graduable order in a matrix ring over the central division algebra D of degree 4 over $\mathbb{Q}_p$. The graduated hull of Λ is a maximal order in this algebra. In particular the maximal separable suborder Λ_0 of Λ is isomorphic to a matrix ring over R. Note that R is isomorphic to the ring of integers in a maximal unramified subfield of D. Hence $\Lambda = \Lambda_0 \oplus p^j \pi \Lambda_0$ for some j, where π is a prime element in D. In particular the sum of the two non diagonal entries of the exponent matrix of $R \otimes \Lambda$ is odd. Since the defect of the whole block is two, this implies that $R \otimes \Lambda$ is as in the proposition. $\square$

The consideration above describes the graduated hull $\Gamma = \oplus f_j^{(\epsilon)} RSL_2(p^2)$ of RG. To describe the group ring (resp. the two blocks of defect 2) itself, we have to describe the amalgamations between the idempotents of Γ that map onto the central primitive idempotents of $\Gamma/J(\Gamma)$ in RG.

Definition 4.6 *Let Λ be an R-order. Let P be the direct sum of representatives of the isomorphism classes of projective indecomposable Λ-modules. Then the order $\Delta := End_\Lambda(P)$ is called the* basic order *of Λ.*

The module P induces a Morita equivalence between Λ and its basic order Δ which also gives rise to a Morita equivalence of $K\Lambda$ and $K\Delta$ respectively $\Lambda/J(\Lambda)$ and the commutative ring $\Delta/J(\Delta)$. In particular there is a natural bijection of the central primitive idempotents of $\Lambda/J(\Lambda)$ and $\Delta/J(\Delta)$ (respectively $K\Lambda$ and $K\Delta$).

The order RG is a symmetric order with respect to $T_u : (a,b) \mapsto trace(aub)$ where $u = p^{-2}(\sum \epsilon f_j^{(\epsilon)} + f_0 + \frac{1}{2}(-f^{(-1)} - f^{(-1)'} + f^{(1)} + f^{(1)'})) + f_{St}$ (cf. [10, (III.2)]). For $0 \leq \lambda < p^2$ let e_λ be an idempotent of RG that maps onto the central primitive idempotent of $RG/J(RG)$ belonging to the Brauer character λ. Then [12, Proposition (6.4)] also the orders $e_\lambda RG e_\lambda$ are symmetric with respect to the restriction of T_u. Let Δ be the basic order of RG. For each irreducible Brauer character λ of RG let e'_λ be an idempotent of Δ that maps onto the central primitive idempotent of $\Delta/J(\Delta)$ corresponding to $e_\lambda + J(RG)$ and for a central primitive idempotent f of KG, the corresponding central primitive idempotent of $K\Delta$ is denoted by f'. Then the order $e'_\lambda \Delta e'_\lambda =: O_\lambda$ is a symmetric R-order in R^n where $n = c_{\lambda,\lambda}$. Moreover $O_\lambda/J(O_\lambda) \cong R/pR$ and the conductor of R^n in O_λ, which is defined as the maximal R^n-ideal that is contained in O_λ, is p^2R^n. These conditions determine the orders $e_\lambda RG e_\lambda$ up to Morita equivalence as shown in the next lemma:

Lemma 4.7 *Let R' be a discrete valuation ring with prime element π and O an R'-order in R'^n that is symmetric with respect to $((a_1,\dots,a_n),(b_1,\dots,b_n)) \mapsto \pi^{-2}\sum d_i a_i b_i$. Assume that $O/J(O) \cong R'/\pi R'$ is simple and that the conductor of R'^n in O is $\pi^2 R'^n$. Then O has an R'-basis $((1,\dots,1)$, $(0,\pi,0,\dots,0,-d_2/d_n\pi)$, $\dots$, $(0,\dots,0,\pi,-d_{n-1}/d_n\pi)$, $(0,\dots,0,\pi^2))$.*

Proof: Since O is symmetric, the conductor of R'^n in O is the dual $(R'^n)^*$ and therefore $\nu_\pi(d_i) = 0$ for all $1 \leq i \leq n$. Moreover $|R'^n/O| = |R'/\pi R'|^n$. Since $O/J(O) = O/(\pi R'^n \cap O)$ is simple, it follows that $R'^n/O \cong (R'/\pi R')^{n-2} \oplus R'/(\pi^2)R'$. Now $1 \in O$ implies that O has a basis $((1,\dots,1)$, $(0,\pi,0,\dots,0,a_2\pi)$, $\dots$, $(0,\dots,0,\pi,a_{n-1}\pi)$, $(0,\dots,0,\pi^2))$ for some $a_i \in R'$. Taking scalar products of these elements with 1 it follows that $a_i \equiv -d_i/d_n \pmod{\pi}$. □

Since $RG = \oplus_{\lambda,\lambda'} e_\lambda RG e_{\lambda'}$, we now only have to describe the $e_\lambda RG e_\lambda - e_{\lambda'} RG e_{\lambda'}$-bimodules $e_\lambda RG e_{\lambda'}$ for $\lambda \neq \lambda'$. If $c_{\lambda,\lambda'} \leq 2$ the module is already determined by the corresponding entries in the exponent matrix of the relevant $\Lambda_j^{(\epsilon)}$. Namely then $e'_\lambda \Delta e'_{\lambda'}$ is of the form $RR(x) := \{(a,b) \in R \oplus R \mid a \equiv bx \pmod{p}\}$ for some $x \in R$ and hence isomorphic to $RR(1)$. But even in this easy situation one may have the problem that not every system of bimodule automorphisms lifts to a ring automorphism because there may be multiplicative relations between the relevant matrix units in the graduated overorder. But here one can show using Lemma 3.6 that every bimodule automorphism is indeed induced by conjugation with diagonal matrices.

So we only have to deal with $e_\lambda RG e_{\lambda'}$ for $\lambda \neq \lambda'$ with $c_{\lambda,\lambda'} = 4$. Only the two p-modular constituents $\lambda_\square$ and $\lambda'_\square$ of η and η' and the two p-modular constituents γ and γ' of δ and δ' satisfy $c_{\lambda 2,\lambda' 2} = c_{\gamma,\gamma'} = 4$.

Lemma 4.8 *Let $\epsilon = \pm 1$ and L (respectively L') be an irreducible $\Lambda^{(\epsilon)}$-lattice (respectively $\Lambda^{(\epsilon')}$-lattice). Then L/pL and L'/pL' are not isomorphic as RG-modules.*

Proof: Assume that $L/pL \cong L'/pL'$ are isomorphic RG-modules. Let $SL_2(p) \cong U \leq G$ be a subgroup of G isomorphic to $SL_2(p)$. Choose η, η', δ, δ' such that the elements x of order p of U satisfy $\eta(x) = (1+p)/2$, $\eta'(x) = (1-p)/2$, $\delta(x) = (-1+p)/2$, and $\delta'(x) = (-1-p)/2$. Let $s := (-1)^{(p-1)/2}$. Calculating scalar products with help of the character tables of G and U in [11], one gets that $\eta_{|U} = 1 + 1 + p + ((1-s)/2)((p+1)/2 + (p+1)/2') + 2\sum(p+1)$ is the sum of 2 times the trivial character plus the Steinberg character plus the two conjugate characters of degree $(p+1)/2$ if they are not faithful plus two times the sum over all non faithful characters of degree $p+1$ of U and $\eta'_{|U} = p + ((1+s)/2)((p-1)/2 + (p-1)/2') + 2\sum(p-1)$ is the Steinberg character plus the two conjugate characters of degree $(p-1)/2$ if they are not faithful plus two times the sum over all non faithful characters of degree $p-1$ of U. So the ordinary characters in the principal block of RU that belong to η correspond to the first, third, … vertex of the Brauer tree of RU and the ones of η' to the remaining vertices. In particular L and L' are direct sums of homogenous RU-lattices, i.e. preserved by the central primitive idempotents of KU.

Let M (resp. M') be the maximal RG-sublattice of L (resp. L'). Since $\lambda_\Box$ and $\lambda'_\Box$ are distinct selfdual Brauer characters of G, the lattice M is isomorphic to the dual lattice of L. Hence the RU-constituents in the head of L/M are the duals of the ones in the socle of L/M Since the RU-constituents of L/M are selfdual and occur with multiplicity 1, L/M is a semi simple RU-module. Analogously one gets that L'/M', M/pL, and M'/pL' are semi simple RU-modules.

Therefore one finds that L and L' are direct sums of irreducible RU-lattices. In particular the trivial constituent occurs twice in the head of $L_{|RU}$ and only once in the head of $L'_{|RU}$.

Analogously $\delta_{|U} = ((1+s)/2)((p+1)/2 + (p+1)/2') + 2\sum(p+1)$ is the sum of the two characters of degree $(p+1)/2$ if they are faithful plus two times the sum over all faithful characters of degree $p+1$ of U and $\delta'_{|U} = ((1-s)/2)((p-1)/2 + (p-1)/2') + 2\sum(p-1)$ is the sum of the two characters of degree $(p-1)/2$ if they are faithful plus two times the sum over all faithful characters of degree $p-1$ of U.

As above one finds that as RU-lattices L and L' are distinct direct sums of irreducible RU-lattices and L/pL is not isomorphic to L'/pL' as RU-module.
$\Box$

From [5] one gets

Lemma 4.9 *The irreducible characters having modular constituents $\lambda_\Box$ and $\lambda'_\Box$ are $\eta, \eta', \delta_{(p^2-1)/2}$, and $\delta^F_{(p^2-1)/2} = \delta_{(p-1)^2/2}$. The irreducible characters having modular constituents γ and γ' are $\delta, \delta', \eta_{(p^2-3)/2}$, and $\eta^F_{(p^2-3)/2} = \eta_{(p^2-2p-1)/2}$.*

Let $e := e'_{\lambda 2}$ and $f := e'_{\lambda 2}$. Write the matrices of Δ and its graduated hull Γ' with respect to a suitable basis of the projective indecomposable Δ-module Δe. Then the order $(e+f)\Gamma'(e+f)$ is the direct sum of 4 hereditary orders isomorphic to $\Lambda(\lambda_\Box, \lambda'_\Box, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, corresponding to the central primitive idempotents of $K\Delta$ in the order of the lemma above, plus the direct sum of two copies of R. In particular $f\Gamma'e$ is the direct sum of four copies of R (belonging to the zeroes in the lower left corner of the four 2×2 exponent matrices).

Proposition 4.10 *View the order Δ embedded in its graduated hull Γ'. Then $f\Gamma'e$ is a direct sum of 4 copies of R corresponding to the central primitive idempotents in the order of Lemma 4.9.*

i) $f\Delta e$ is spanned by the rows of the matrix $M := \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$.

ii) $e\Delta f$ *is the dual of* $f\Delta e$ *spanned by the rows of* $\begin{pmatrix} 2p^2 & 0 & 0 & 0 \\ 0 & 2p^2 & 0 & 0 \\ 2p & 2p & p & 0 \\ 2p & -2p & 0 & p \end{pmatrix}$.

Proof: Because of Lemma 4.8 the projections onto the first two columns of M are not equivalent modulo p. By [1], the second layer of the Loewy series of the projective indecomposable RG-module $P_{\lambda 2}$ with head $\lambda_\square$ has only two direct summands isomorphic to $\lambda'_\square$. This implies that the elementary divisors of M are $1, 1, p, p$. Therefore M is of the form $M := \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & a & d \\ 0 & 0 & p & g \\ 0 & 0 & 0 & p \end{pmatrix}$ for some $a, b, c, d \in R^*$, $g \in R$. The dual of $f\Delta e$ is contained in $e\Gamma f$ which implies that $g = 0$.

The Frobenius automorphism F fixes the first two components of $e\Gamma' f$ and interchanges the last two ones, whereas the outer automorphism α induced by the elements in $PGL_2(p^2) \setminus PSL_2(p^2)$ interchanges the first two components of $f\Gamma' e$ and fixes the last two ones. Therefore α and F act as right multiplication with

$$\begin{pmatrix} 0 & \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix} \text{resp.} \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & \beta_3 \\ 0 & 0 & \beta_4 & 0 \end{pmatrix}$$

for some $\alpha_i, \beta_i \in R^*(1 \leq i \leq 4)$. This gives the equalities $c = \alpha_3/\alpha_1 a = \alpha_2/\alpha_3 a$, $d = \alpha_4/\alpha_1 b = \alpha_2/\alpha_4 b$, $b = \beta_1/\beta_4 a = \beta_3/\beta_1 a$, $d = \beta_2/\beta_4 c = \beta_3/\beta_2 c$.

Multiplying α with $\alpha_3^{-1} Id$ and F with $\beta_1^{-1} Id$ we may assume that $\alpha_3 = \beta_1 = 1$. Then $\beta_2 = \alpha_4 = \pm 1$ and $\beta_4 = 1/\beta_3$ and $\alpha_2 = 1/\alpha_1$.

With respect to a suitable basis one may assume that $a = b = c = 1$, $d = \beta_2 = \pm 1$.

The order $e\Delta f$ is the dual of $f\Delta e$ and hence spanned by the rows of $\begin{pmatrix} 2p^2 & 0 & 0 & 0 \\ 0 & 2p^2 & 0 & 0 \\ 2p & 2p & p & 0 \\ 2p & 2dp & 0 & p \end{pmatrix}$.

By Lemma 4.8 also here the projections onto the first two columns are not congruent modulo p^2 and therefore $d = -1$. □

The discussion in the case of γ is similar. So let $e := e'_\gamma$ and $f := e'_{\gamma'}$.

Proposition 4.11 *View the order* Δ *embedded in its graduated hull* Γ'. *Then* $f\Gamma' e$ *is a direct sum of 3 copies of* R *and one copy of* pR *corresponding to the central primitive idempotents in the order of Lemma 4.9.*

i) $f\Delta e$ is spanned by the rows of the matrix $M := \begin{pmatrix} 1 & 0 & 1 & p \\ 0 & 1 & 1 & -p \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}$.

ii) $e\Delta f$ is the dual of $f\Delta e$ spanned by the rows of $\begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p^2 & 0 & 0 \\ 2p & 2p & p & 0 \\ 2p & -2p & 0 & 1 \end{pmatrix}$.

Proof: Because of Lemma 4.8 the projections onto the first two columns of M are not equivalent modulo p. By [1] the second layer of the Loewy series of the projective indecomposable RG-module P_γ with head γ has only two direct summands isomorphic to γ'. This implies that the elementary divisors of M are $1, 1, p, p^2$. So $M = \begin{pmatrix} 1 & 0 & a & pb \\ 0 & 1 & c & pd \\ 0 & 0 & p & g \\ 0 & 0 & 0 & p^2 \end{pmatrix}$. Dualizing yields $g = 0$.

The outer automorphism α induced by the elements in $PGL_2(p^2) \setminus PSL_2(p^2)$ interchanges the first two components of $f\Gamma' e$ and fixes the last two ones. As in the proof of Proposition 4.10 one gets that there is $\alpha_1 \in R^*$ and $\epsilon \in \pm 1$ such that $c = \alpha_1 a$ and $d = \epsilon\alpha_1 b$.

Now the Frobenius automorphism F maps $f\Delta e$ onto its dual $e\Delta f$. One has $M^F = \begin{pmatrix} px & 0 & pbt & az \\ 0 & py & pdt & cz \\ 0 & 0 & 0 & pz \\ 0 & 0 & p^2 t & 0 \end{pmatrix}$ for some $x, y, z, t \in R^*$. Since $e\Delta f$ is the dual of $f\Delta e$ one gets $tbc + zad \equiv tad + zbc \equiv cd(t+z) - y/2 \equiv ab(t+z) - x/2 \equiv 0 \pmod{p}$. Therefore $t + \epsilon z = 0$ but $t + z \neq 0$ and hence $\epsilon = -1$. After multiplying the basis vectors in Γ' by some elements of R^* one may achieve $a = c = b = -d = 1$. □

Remark 4.12 *One easily concludes that the description of the group ring $RSL_2(p^2)$ above is also valid for $p = 3$.*

5 The ring $\mathbb{Z}_p SL_2(p^2)$.

Definition 5.1 *Let K' be a finite extension of $\mathbb{Q}_p$ with ring of integers R' and A a finite dimensional separable K'-algebra. An R'-order O in A is called* nearly graduated, *if O contains a full system of orthogonal primitive idempotents of A.*

Remark 5.2 *Let O be a nearly graduated order in $A \cong \oplus A_i$. Then O is isomorphic to a direct sum of nearly graduated orders O_i in the simple algebras A_i.*
If D is a division algebra and $A \cong D^{n\times n}$ is simple, then O is isomorphic to an order of the form

$$(O_{ij}^{n_i\times n_j})_{1\le i,j\le s} =: \Lambda(n_1,\ldots,n_s,(O_{ij})_{1\le i,j\le s})$$

for some s and $n_1+\ldots+n_s = n$ where the O_{ii} are orders in D and O_{ij} is an O_{ii}-O_{jj}-bimodule in D, $1\le i,j\le s$.

For the nearly graduated orders, the dual can be calculated almost as easily as for graduated orders. If O is an overorder of a symmetric order Λ, then the dual coincides with the (right or left) conductor of O in Λ (cf. [7], [10, Proposition (III.7)]). If $O = \oplus f_i\Lambda$ is obtained from Λ by adjoining some central idempotents f_i of $K\Lambda$ to Λ then the conductor of O in Λ is $\oplus(\Lambda\cap f_i\Lambda)$.

The next lemma can be shown as [10, Theorem (III.8)].

Lemma 5.3 *(Conductor formula for nearly graduated orders) Let K' be a finite extension of $\mathbb{Q}_p$ and R' the ring of integers in K'. Let Λ be a symmetric R'-order in a separable K'-algebra $A = \oplus_{k=1}^h A_k \cong D_k^{n_k\times n_k}$ with respect to the generalized trace map T_u, $u=\sum_{k=1}^h u_k$ $(0\ne u_k\in Z(A_k))$, and O an order in A containing Λ. Then the conductor of O in Λ is the dual $O^\#$ of O. If O is of the form $\oplus_{k=1}^h O^{(k)}$ and $O^{(k_0)}$ is a nearly graduated order of the form $(O_{ij}^{n_i\times n_j})_{1\le i,j\le s}$ for some s for some $1\le k_0\le h$ then $O^\# = \oplus_{k=1}^h O^{(k)\#}$ with $O^{(k_0)\#} = u_{k_0}^{-1}((O_{ji}^\#)^{n_i\times n_j})_{1\le i,j\le s}$ where $O_{ji}^\#$ is the dual of O_{ji} with respect to the reduced trace: $D_{k_0}\to K'$.*

Remark 5.4 *Let O be a $\mathbb{Z}_p$-order in K such that $R\otimes_{\mathbb{Z}_p} O \cong \{(x,y)\in R\oplus R \mid x\equiv y \pmod{p^aR}\}$ for some $a\in\mathbb{Z}_{\ge 0}$. Then $O = \mathbb{Z}_p + p^aR =: R(a)$.*
The $R(a)$-modules in K are of the form $M(j,i) = p^j\mathbb{Z}_p u + p^i R$ for some unit $u\in R$ and $i\ge j\in\mathbb{Z}$ with $i-j\le a$.

From the previous section we know the ring $RG = R\otimes_{\mathbb{Z}_p}\mathbb{Z}_pSL_2(p^2)$. Let Γ_0 be the order in $\mathbb{Q}_pG$ generated by $\mathbb{Z}_pG$ and the central primitive idempotents of $\mathbb{Q}_pG$. Then $\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\Gamma_0$ is contained in the graduated hull Γ of RG and one has $\mathbb{Z}_pG\cong(\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\Gamma_0)\cap RG$. Therefore it suffices to describe Γ_0.

Lemma 5.5 *Assume the $\lambda\ne\lambda^F$ are p-modular constituents of some irreducible character χ of $SL_2(p^2)$.*

a) If χ is not faithful then $\chi=\chi^F$.

b) If χ is faithful then $\lambda = \gamma = (p-1)/2 + p(p-3)/2$ or $\lambda = \gamma^F = \gamma'$.

Proof: a) If χ (and therefore λ) is not faithful, then λ and λ^F are equivalent modulo 2. By Lemma 3.2 the Cartan invariant $c_{\lambda,\lambda^F} = 1$. Since λ and λ^F are also constituents of χ^F, this implies $\chi = \chi^F$.
b) Let $\lambda = \tilde{\lambda_1} + p\tilde{\lambda_2}$. Then exactly one of λ_1 and λ_2 is odd. Assume that there are $\epsilon_1, \epsilon_2 \in \pm 1$ with $\epsilon_1\lambda_1 + p\lambda_2 = \epsilon_2\lambda_2 + p\lambda_1$. Hence $(p-\epsilon_1)/(p-\epsilon_2)\lambda_1 = \lambda_2$. Therefore either $\epsilon_1 = \epsilon_2$ and $\lambda^F = \lambda$ or $\epsilon_1 = -\epsilon_2$ and $\lambda = \gamma$ or γ'.
Assume now that there are $\epsilon, \epsilon_1, \epsilon_2 \in \pm 1$ with $\epsilon_1\lambda_1 + p\lambda_2 = p^2 - \epsilon - (\epsilon_2\lambda_2 + p\lambda_1)$. If $\epsilon = 1$ then either $\epsilon_1 = \epsilon_2$ and $\lambda_1 \equiv \lambda_2 \pmod 2$ which is a contradiction, or $\epsilon_1 = -\epsilon_2$ and $\lambda_1 = (p+\epsilon_2)/2$, $\lambda_2 = (p-\epsilon_2)/2$, and λ is one of γ or γ^F. Hence $\epsilon = -1$ and $p(\lambda_1+\lambda_2) + \epsilon_2\lambda_2 + \epsilon_1\lambda_1 = p^2 + 1$. If $\lambda_1 + \lambda_2 \leq p-2$ then the left hand side is $\leq p(p-2) + (p-2) = p^2 - p - 2 < p^2 + 1$ and if $\lambda_1 + \lambda_2 \geq p+2$ then the left hand side is $\geq p(p+2) - (p+2) = p^2 + p - 2 > p^2 + 1$ ($p = 3$ is here impossible since the $\lambda_i \leq p-1$ $(i = 1,2)$. Since $\lambda_1 + \lambda_2$ is odd, this implies that $\lambda_1 + \lambda_2 = p$ and $\epsilon_2\lambda_2 + \epsilon_1\lambda_1 = 1$. This again implies that $\lambda = \gamma$ or $\lambda = \gamma'$. □

Theorem 5.6 *Let Γ_0 be the $\mathbb{Z}_p$-order in $\mathbb{Q}_pG$ generated by $\mathbb{Z}_pG$ and the central primitive idempotents of $\mathbb{Q}_pG$ and let f be a central primitive idempotent of KG. Let $f\Gamma = \Lambda(\lambda_1, \ldots, \lambda_s, M)$ be the direct summand of the graduated hull Γ of RG corresponding to f. Assume that the action of the Frobenius automorphism F on the irreducible Brauer characters λ_i is such that $\lambda_i^F = \lambda_i$ for $1 \leq i \leq a$ and $\lambda_i^F \neq \lambda_i$ for $a < i \leq s$ for some $1 \leq a \leq s$.*

a) If $f \neq f^F$ then $f_0 := f + f^F$ is a central primitive idempotent in $\mathbb{Q}_pG$. If f belongs to $\eta_{(p^2-3)/2}$ or $\eta_{(p^2-2p-1)/2} = \eta^F_{(p^2-3)/2}$ then $f_0\Gamma_0$ is isomorphic to the suborder of the graduated order

$$X := \Lambda(R, (p^2-4p+3)/4, (p^2+4p+3)/4, (p^2-1)/4, (p^2-1)/4, \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix})$$

having the last two blockdiagonal matrices congruent modulo p.
In the other cases $f_0\Gamma_0$ is a nearly graduated $\mathbb{Z}_p$-order in $f_0\mathbb{Q}_pG$ and $f_0\Gamma_0 \cong \Lambda(\lambda_1(1), \ldots, \lambda_s(1), (p^{m_{ij}}R_{ij})_{1\leq i,j\leq s})$ where

$$R_{ij} = \begin{cases} R(1) & \text{if } i = j \leq a \text{ or } i \neq j \leq a \text{ and } c_{\lambda_i,\lambda_j} = 2 \\ R & \text{else} \end{cases} .$$

b) If $f = f^F$ then $f \in \mathbb{Q}_pSL_2(p^2)$ let O be the suborder $O := R + p\mathbb{Z}_p^{2\times 2}$ of $\mathbb{Z}_p^{2\times 2}$ where $R \leq \mathbb{Z}_p^{2\times 2}$ via the regular representation and $\mathcal{O}$ be the maximal *order in the central $\mathbb{Q}_p$-division algebra of index 2. Assume that $\lambda_{a+i}^F = \lambda_{(a+s)/2+i}$ $(1 \leq i \leq (s-a)/2)$ and that the exponent matrix of $f\Gamma$ is written with respect to a basis of an F-invariant irreducible $f\Gamma$-lattice. Then $f\Gamma_0 \cong (p^{m_{ij}} R_{ij}^{\lambda_i(1)\times\lambda_j(1)})_{1\leq i,j\leq (a+s)/2}$ where*

$$R_{ij} = \begin{cases} \mathbb{Z}_p & \text{if } i,j \leq a \\ \mathbb{Z}_p^{1\times 2} & \text{if } i \leq a, j > a \\ \mathbb{Z}_p^{2\times 1} & \text{if } j \leq a, i > a \\ \mathbb{Z}_p^{2\times 2} & \text{if } i \neq j > a \\ O & \text{if } i = j > a \text{ and } m_{i,(s-a)/2+i} + m_{(s-a)/2+i,i} = 2 \\ \mathcal{O} & \text{if } i = j > a \text{ and } m_{i,(s-a)/2+i} + m_{(s-a)/2+i,i} = 1 \end{cases}.$$

If the last case occurs for some i then $\chi = \delta$ or δ' and $f\Gamma_0 \cong \mathcal{O}^{(p^2-1)/2\times(p^2-1)/2}$.

<u>Proof:</u> a) Then $f_0\Gamma_0$ is an order in $f_0\mathbb{Q}_pSL_2(p^2) \cong K^{m\times m}$ for some $m \in \mathbb{N}$ with $R \otimes f_0\Gamma_0 = (f+f^F)RG$. Since fRG is a graduated order, there are orthogonal primitive idempotents $x_j \in fRG$ $(1 \leq j \leq m)$.

Assume first that f does not belong to $\eta_{(p^2-3)/2}$ or $\eta_{(p^2-2p-1)/2} = \eta^F_{(p^2-3)/2}$ With Lemma 5.5 and Lemma 4.9 one finds that then $x_j + x_j^F \in f_0\Gamma_0$ $(1 \leq j \leq m)$ is a system of orthogonal primitive idempotents of $f_0\mathbb{Q}_pSL_2(p^2)$. Therefore $f_0\Gamma_0$ is a nearly graduated order $f_0\Gamma_0 \cong \Lambda(\lambda_1(1),\ldots,\lambda_s(1),(O_{ij})_{1\leq i,j\leq s})$ where O_{ij} are $\mathbb{Z}_p$-submodules of K. By Remark 5.4 one has $O_{ii} = R(1)$ if $i \leq a$ and $O_{ii} = R$ if $i > a$. Moreover O_{ij} is an $O_{ii} - O_{jj}$-bimodule in K and therefore $O_{ij} = p^{m_{ij}}R$ if $i > a$ or $j > a$ and $O_{ij} = p^{m_{ij}}R$ or $O_{ij} \cong p^{m_{ij}}R(1)$ if $i,j \leq a$. The dual of $R(1)$ with respect to the *trace* : $K \to \mathbb{Q}_p$ is $\frac{1}{p}\mathbb{Z}_pv + R$ for any unit $v \in R$ with $trace(v) = 0$.

If $O_{ij} \cong p^{m_{ij}}R(1)$ then the conductor formula Lemma 5.3 says that the (ji) position of $f_0\Gamma_0^{\#}$ is $p^{2-m_{ij}}\frac{1}{p}\mathbb{Z}_pv + R \cong p^{2-m_{ij}-1}R(1)$ Therefore $m_{ij} + m_{ji} = 1$ and $fe_{\lambda_i}\mathbb{Z}_pGe_{\lambda_j} = fe_{\lambda_i}\Gamma_0e_{\lambda_j}$ and $fe_{\lambda_j}\mathbb{Z}_pGe_{\lambda_i} = fe_{\lambda_j}\Gamma_0e_{\lambda_i}$.

If $c_{\lambda_i,\lambda_j} = 2$ then tensoring with R shows that $O_{ij} \cong p^{m_{ij}}R(1)$ but if $c_{\lambda_i,\lambda_j} > 2$ then Proposition 4.10 implies that $O_{ij} \cong p^{m_{ij}}R$. This implies a) in the second case.

Now assume that f belongs to $\eta_{(p^2-3)/2}$ or $\eta_{(p^2-2p-1)/2} = \eta^F_{(p^2-3)/2}$ Using [5] one finds that the p-modular constituents of $\eta_{(p^2-3)/2}$ are $(p-5)/2 + p(p-3)/2$, $(p+1)/2 + p(p-1)/2$, γ, and γ'. As above one finds that $f_0\Gamma_0$ is a suborder of the graduated order X. The amalgamations between the idempotents of X in $f\Gamma_0$ are the projections (via f_0) of the ones in Lemma 4.7. By Proposition 4.11 there are no amalgamations between the off diagonal entries.

b) Now one has to make a Galois descent for the graduated order $f\Gamma$. The semisimple $\mathbb{F}_p$-algebra $f\Gamma_0/J(f\Gamma_0)$ is isomorphic to $\oplus_{i=1}^a \mathbb{F}_p^{\lambda_i(1)\times\lambda_i(1)} \oplus \oplus_{j=1}^{(a+s)/2} \mathbb{F}_{p^2}^{\lambda_j(1)\times\lambda_j(1)}$

as it can be read off from the action of F on the modular constituents of the Frobenius character χ that belongs to f. Lifting the central primitive idempotents of $f\Gamma_0/J(f\Gamma_0)$ to orthogonal idempotents $e_1, \ldots, e_{(a+s)/2}$ of $f\Gamma_0$ one has $p^{m_{ij}} R_{ij} \cong e_i f\Gamma_0 e_j$. Since the center of $\mathbb{Q}_p f\Gamma_0$ is $\mathbb{Q}_p$ one gets that $R_{ij} \cong \mathbb{Z}_p$ if $i,j \leq a$. If $i > a$ then R_{ii} is a $\mathbb{Z}_p$-order in a central simple algebra of dimension 4 over $\mathbb{Q}_p$. If $m_{i,(s-a)/2+i} + m_{(s-a)/2+i,i} = 1$ then λ_i and λ_i^F are not equivalent modulo 2. Hence λ_i is faithful and Lemma 5.5 says that $\lambda_i = \gamma$ or γ^F. Lemma 4.9 now yields that $\chi = \delta$ or δ' and $f\Gamma_0 \cong \mathcal{O}^{(p^2-1)/2 \times (p^2-1)/2}$.

As in the proof of Proposition 4.5 one finds that $R_{ii} = R + p\mathbb{Z}_p^{2\times 2}$ if $m_{i,(s-a)/2+i} + m_{(s-a)/2+i,i} = 2$. Since the exponent matrix is written with respect to an F-invariant lattice, one has $m_{j,i} = m_{j,(s-a)/2+i}$ for all $j \leq a$. Therefore $R_{ji} \cong \mathbb{Z}_p^{1\times 2}$ and $R_{ij} \cong \mathbb{Z}_p^{2\times 1}$ for $j \leq a < i$. If $s - a > 2$ then $a = 0$ and $s = 4$ and the exponent matrix of $f\Gamma$ is of the shape $\begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & c & d \\ 1-a & 1-c & 0 & 1 \\ 1-b & 1-d & 1 & 0 \end{pmatrix}$ for some $a,b,c,d \in \{0,1\}$. Since all Brauer characters of G are self dual, one finds $a = b = c = d$. Hence $R_{12} \cong R_{21} \cong \mathbb{Z}_p^{2\times 2}$ in this case. □

Frequently used notations:

p: an odd prime, $p > 3$ in Chapter 4
R, K: $K/\mathbb{Q}_p$ unramified of degree 2 with ring of integers R.
K', R': arbitrary p-adic fields resp. valuation rings.
$G := SL_2(p^2)$.
Γ: graduated hull of RG.
Δ: basic order of RG.
η_a $(1 \leq a \leq (p^2-3)/2)$, η, η': Frobenius characters of G of degree p^s+1, $(p^s+1)/2$
δ_a $(1 \leq a \leq (p^2-1)/2)$, δ, δ': Frobenius characters of G of degree p^2-1, $(p^2-1)/2$
$f_a^{(\epsilon)}$: central primitive idempotent of KG corresponding to δ_a if $\epsilon = -1$ resp. η_a if $\epsilon = 1$.
$\Lambda_a^{(\epsilon)} := f_a^{(\epsilon)} RG$.
χ: arbitrary Frobenius characters.
F: Frobenius automorphism of $\mathbb{F}_{p^2}/\mathbb{F}_p$ or $K/\mathbb{Q}_p$.
$\lambda = \lambda_1 + p\lambda_2$: Brauer character of G.
e_λ: an idempotent in RG that maps onto the central primitive idempotent of $RG/J(RG)$ corresponding to λ.
$\lambda_\square := (p-1)/2 + p(p-1)/2$: the p-modular constituent of η of degree $((p+1)/2)^2$.
$\lambda'_\square := (p-3)/2 + p(p-3)/2$: the p-modular constituent of η of degree $((p-1)/2)^2$.
$\gamma := (p-1)/2 + p(p-3)/2$, $\gamma' := (p-3)/2 + p(p-1)/2$: the p-modular constituents

of δ (of degree $(p^2-1)/4$).

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