Low dimensional strongly perfect lattices.
III: Dual strongly perfect lattices of dimension 14.

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ABSTRACT: The extremal 3-modular lattice $[\pm G_2(3)]_{14}$ with automorphism group $C_2 \times G_2(\mathbb{F}_3)$ is the unique dual strongly perfect lattice of dimension 14.

1 Introduction.

This paper continues the classification of strongly perfect lattices in [24], [17], [18], [19]. A lattice L in Euclidean space $(\mathbb{R}^n, (,))$ is called strongly perfect, if the set of its minimal vectors forms a spherical 4-design. The most important property of such lattices is that they provide nice examples of extreme lattices, where the density of the associated lattice sphere packing attains a local maximum. In fact [22] shows that the density of a strongly perfect lattice is even a local maximum on the space of all periodic packings, so these lattice packings are periodic extreme.

All strongly perfect lattices are known up to dimension 12. They all share the property of being dual strongly perfect, which means that both lattices L and its dual lattice L^* are strongly perfect (see Definition 2.4). The only known strongly perfect lattice for which the dual is not strongly perfect is K'_{21} (see [24, Tableau 19.2]) in dimension 21.

One method to show that a lattice L is strongly perfect is to use its automorphism group $G = \mathrm{Aut}(L)$, the stabilizer of L in the orthogonal group. If this group has no harmonic invariant of degree ≤ 4, then all G-orbits are spherical 4-designs, and so is

$$\mathrm{Min}(L) := \{\ell \in L \mid (\ell, \ell) = \min(L)\}.$$

Since $\mathrm{Aut}(L) = \mathrm{Aut}(L^*)$ such lattices are also dual strongly perfect. Moreover all non empty layers

$$L_a := \{\ell \in L \mid (\ell, \ell) = a\}$$

are 4-designs, since they are also disjoint unions of G-orbits. We call such lattices universally perfect (see Definition 2.4). Though being universally perfect involves infinitely many layers of the lattice it can be checked with a finite computation using the theory of modular forms. For any harmonic polynomial p of degree d and a lattice $L \subset (\mathbb{R}^n, (,))$ the theta series

$$\theta_{L,p} := \sum_{\ell \in L} p(\ell) q^{(\ell,\ell)}$$

is a modular form of weight $k := \frac{n}{2} + d$ and hence lies in a finite dimensional vector space $M_k(\ell)$ (depending on the level ℓ of L). A lattice is universally perfect, if and only if this theta series $\theta_{L,p}$ vanishes for all harmonic polynomials p of degree 2 and 4, a condition that can be tested from the first few Fourier coefficients of $\theta_{L,p}$.

Under certain conditions, the theory of modular forms allows to show that $\theta_{L,p} = 0$ for all harmonic polynomials p of degree 2 and 4 which again shows that all layers of L are spherical 4-designs (see for instance [5], [24, Theorème 16.4]). In fact this is the only known way to conclude that all even unimodular lattices of dimension 32 and minimum 4 provide locally densest lattice sphere packings see [24, Section 16]. By [13] there are more than 10^6 such lattices and a complete classification is unknown.

If L is a universally perfect lattice, then the theta-transformation formula shows that also $\theta_{L^*,p} = 0$ for all harmonic polynomials of degree 2 and 4 and hence L is dual strongly perfect. So a classification of all dual strongly perfect lattices includes those that are universally perfect.

Universally perfect lattices also play a role in Riemannian geometry. If L is a universally perfect lattice (of fixed covolume $\mathrm{vol}(\mathbb{R}^n/L)^2 = \det(L) = 1$, say) then the torus $\mathbb{R}^n/L^*$ defined by the dual lattice L^* provides a strict local minimum of the height function on the set of all n-dimensional flat tori of volume 1 ([10, Theorem 1.2]). R. Coulangeon also shows that universally perfect lattices L achieve local minima of Epstein's zeta function, they are so called ζ-extreme lattices. The question to find ζ-extreme lattices has a long history going back to Sobolev's work [23] on numerical integration and [12].

Restricting to dual strongly perfect lattices gives us additional powerful means for classification and non-existence proofs: Our general method to classify all strongly perfect lattices in a given dimension usually starts with a finite list of possible pairs (s, γ), where $s = s(L) = \frac{1}{2}|\mathrm{Min}(L)|$ is half of the kissing number of L and

$$\gamma = \gamma'(L)^2 = \gamma'(L^*)^2 = \min(L)\min(L^*)$$

the Bergé-Martinet invariant of L. For both quantities there are good upper bounds given in [4] resp. [7]. Note that γ is just the product of the values of the Hermite function on L and L^*. Using the general equations for designs given in Section 2.4 a case by case analysis allows either to exclude certain of the possibilities (s, γ) or to factor $\gamma = m \cdot r$ such that rescaled to minimum $\min(L^*) = m$, the lattice L^* is integral (or even) and in particular contained in its dual lattice L (which is then of minimum r). For dual strongly perfect lattices we can use a similar argumentation to obtain a finite list of possibilities (s', γ) for $s' = s(L^*)$ and in each case a factorization $\gamma = m' \cdot r'$ such that L is integral (or even) if rescaled to $\min(L) = m'$. But this allows to obtain the exponent (in the latter scaling)

$$\exp(L^*/L) = \frac{m}{r'}$$

which either allows a direct classification of all such lattices L or the use of modular forms to exclude the existence of a modular form $\theta_L = \theta_{L,1}$ of level $\frac{m}{r'}$ and weight $\frac{n}{2}$ starting with $1 + 2sq^{m'} + \ldots$, such that its image under the Fricke involution starts with $1 + 2s'q^m + \ldots$ and both q-expansions have nonnegative integral coefficients.

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2 Some general equations.

2.1 General notation.

For a lattice Λ in n-dimensional Euclidean space $(\mathbb{R}^n, (\,,\,))$ we denote by

$$\Lambda^* := \{v \in \mathbb{R}^n \mid (v, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$$

its dual lattice and by

$$\Lambda_a := \{\lambda \in \Lambda \mid (\lambda, \lambda) = a\}$$

the vectors of square length a. In particular, if $\min := \min(\Lambda) := \min\{(\lambda, \lambda) \mid 0 \neq \lambda \in \Lambda\}$ denotes the minimum of the lattice then $\Lambda_{\min} = \mathrm{Min}(\Lambda)$ is the set of minimal vectors in Λ, its cardinality is known as the kissing number $2s(\Lambda)$ of the lattice Λ. There are general bounds on the kissing number, of an n-dimensional lattice. For $n = 14$ the bound is $|\Lambda_{\min}| \leq 2 \cdot 1746$ and hence $s(\Lambda) \leq 1746$ (see [4]).

Let

$$\gamma_n := \max\{\frac{\min(\Lambda)}{\det(\Lambda)^{1/n}} \mid \Lambda \text{ is an } n\text{-dimensional lattice } \}$$

denote the Hermite constant. The precise value for γ_n is known for $n \leq 8$ and $n = 24$. However upper bounds are given in [7]. In particular $\gamma_{14} \leq 2.776$. This gives upper and lower bounds on the determinant of a lattice Γ if one knows $\min(\Gamma)$ and $\min(\Gamma^*)$:

Lemma 2.1 *Let Γ be an n-dimensional lattice of minimum m and let $r := \min(\Gamma^*)$. Then for any $b \geq \gamma_n$*

$$\left(\frac{m}{b}\right)^n \leq \det(\Gamma) \leq \left(\frac{b}{r}\right)^n.$$

Proof. We have

$$\frac{m}{\det(\Gamma)^n} \leq b \text{ and } r(\det(\Gamma)^n) \leq b$$

since $\det(\Gamma^*) = \det(\Gamma)^{-1}$. □

Lemma 2.2 *Let Γ be an integral lattice and $X \subset \Gamma^*$ be a linearly independent set with Grammatrix $F := ((x, y))_{x,y \in X}$. Assume that the elementary divisors of F are $(\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k})$ for coprime pairs of integers (a_i, b_i). Then the product $b_1 \dots b_k$ divides $\det(\Gamma)$.*

Proof. Let $X = \{x_1, \dots, x_k\}$ and $U := \langle \overline{x}_1, \dots, \overline{x}_k \rangle \leq \Gamma^*/\Gamma$. Then U is a finite abelian group and any relation matrix $A \in \mathbb{Z}^{k \times k}$ such that $\sum_{j=1}^k A_{ij}\overline{x}_j = 0$ for all i satisfies that $\sum_{j=1}^k A_{ij}(x_j, x_t) \in \mathbb{Z}$ for all i, t, hence $AF \in \mathbb{Z}^{k \times k}$. Therefore $\det(A)$ is a multiple of $b_1, \dots, b_k$. □

2.2 Genus symbols and mass formulas.

We will often encounter the problem to enumerate all even lattices L of a given determinant $\det(L) = |L^*/L|$ or of given invariants of the abelian group L^*/L (which are the elementary divisors of the Grammatrix of L).

To list all such genera we implemented a SAGE program which uses the conditions of Section 7.7 [9, Chapter 15] in particular [9, Theorem 15.11]. The python code is available in SAGE and also from [16]. Also the genus symbol that we use in the classification is the one given in [9, Chapter 15]. For odd primes p the lattice $L \otimes \mathbb{Z}_p$ has a unique Jordan decomposition $\perp_{i=a}^{b} p^i f_i$ where the forms f_i are regular p-adic forms. The determinant $\det(f_i) = d_i$ is a unit in $\mathbb{Z}_p$ and uniquely determined up to squares. Let $\epsilon_i := 1$ if d_i is a square and $\epsilon_i := -1$ if not. Then the p-adic symbol of L is $((p^i)^{\epsilon_i \dim(f_i)})_{i=a..b}$ where 0-dimensional forms are omitted. For $p = 2$ we obtain a similar Jordan decomposition which is in general not unique. To distinguish even and odd forms f_i we add the oddity, the trace of a diagonal matrix representing f_i, as an index, if f_i is odd. There are also 4 square-classes in $\mathbb{Z}_2^*$, so we put $\epsilon_i := 1$ if $\det(f_i) \equiv \pm 1 \pmod 8$ and $\epsilon_i := -1$ if $\det(f_i) \equiv \pm 3 \pmod 8$. Since we only use the symbol for even primitive lattices the symbol of the first form in the Jordan decomposition is omitted, as it can be obtained from the others. So the symbol for the root lattice A_5 is $2_3^{-1}3^1$.

Any genus

$$G(L_1) = \{L \subset (\mathbb{R}^n, (,)) \mid L \otimes \mathbb{Z}_p \cong L_1 \otimes \mathbb{Z}_p \text{ for all primes } p\}$$

of positive definite lattices only consists of finitely many isomorphism classes

$$G(L_1) = [L_1] \cup [L_2] \cup \ldots \cup [L_h].$$

The sum

$$\sum_{i=1}^{h} |\mathrm{Aut}(L_i)|^{-1} = \mathrm{mass}(G(L_1))$$

of the reciprocals of the orders of the automorphism groups of representatives L_i of the isomorphism class $[L_i]$ in the genus is known as the mass of the genus an may be computed a priori from the genus symbol. A formula is for instance given in [8] and a Magma program to calculate its value from the genus symbol is available from [16]. This is used to check whether a list of pairwise non isometric lattices in the genus is complete.

To find all lattices in a genus we use the Kneser neighbouring method [14] (see also [20]).

2.3 The maximal even lattices of level dividing 12.

The following table lists all genera of maximal even lattices L such that $\sqrt{12}L^*$ is again an even lattice. The first column gives the genus symbol as explained in [9, Chapter 15], followed by the class number h. Then we give one representative of the genus which is usually a root lattice. $\perp$ denotes the orthogonal sum and (a) a 1-dimensional lattice with Grammatrix (a). The last column gives the mass of the genus.

genus	level	h	repr.	mass
3^1	3	2	$E_6 \perp E_8$	$691/(2^{23}3^9 5^3 7 \cdot 11 \cdot 13)$
2_6^2	4	4	D_{14}	$42151/(2^{25}3^8 5^3 7^2 11 \cdot 13)$
$2_0^2 3^1$	12	8	$E_8 \perp A_5 \perp (2)$	$29713/(2^{17}3^8 5^3 7 \cdot 11 \cdot 13$
$2^{-2}3^{-1}$	6	6	$A_2 \perp D_{12}$	$29713/(2^{24}3^9 5^2 7 \cdot 11)$
$2_2^2 3^2$	12	28	$A_2 \perp A_2 \perp E_8 \perp (2) \perp (2)$	$1683131581/(2^{25}3^8 5^2 7^2 11 \cdot 13)$

2.4 Designs and strongly perfect lattices

For $m \in \mathbb{R}$, $m > 0$ denote by

$$S^{n-1}(m) := \{y \in \mathbb{R}^n \mid (y, y) = m\}$$

the $(n-1)$-dimensional sphere of radius $\sqrt{m}$.

Definition 2.3 *A finite nonempty set $X \subset S^{n-1}(m)$ is called a* spherical t-design*, if*

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \int_{S^{n-1}(m)} f(x) d\mu(x)$$

for all polynomials of $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $\leq t$, where μ is the $O(n)$-invariant measure on the sphere, normalized such that $\int_{S^{n-1}(m)} 1 d\mu(x) = 1$.

Since the condition is trivially satisfied for constant polynomials f, and the harmonic polynomials generate the orthogonal complement $\langle 1 \rangle^\perp$ with respect to the $O(n)$-invariant scalar product $< f, g > := \int_{S^{n-1}(m)} f(x)g(x)d\mu(x)$ on $\mathbb{R}[x_1, \ldots, x_n]$, it is equivalent to ask that

$$\sum_{x \in X} f(x) = 0$$

for all harmonic polynomials f of degree $\leq t$.

Definition 2.4 *A lattice $\Lambda \subset \mathbb{R}^n$ is called* strongly perfect*, if its minimal vectors $\Lambda_{\min}$ form a spherical 4-design.*
We call Λ dual strongly perfect*, if both lattices, Λ and its* dual lattice

$$\Lambda^* := \{v \in \mathbb{R}^n \mid (v, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$$

are strongly perfect.
Λ is called universally perfect*, if $\theta_{\Lambda,p} = 0$ for all harmonic polynomials p of degree 2 and 4.*

Let Λ be a strongly perfect lattice of dimension n, $m := \min(\Lambda)$ and choose $X \subset \Lambda_m$ such that $X \cup -X = \Lambda_m$ and $X \cap -X = \emptyset$. Put $s := |X| = s(\Lambda)$.

By [24] the condition that $\sum_{x\in X} f(x) = 0$ for all harmonic polynomials f of degree 2 and 4 may be reformulated to the condition that for all $\alpha \in \mathbb{R}^n$

$$(D4)(\alpha): \quad \sum_{x\in X}(x,\alpha)^4 = \frac{3sm^2}{n(n+2)}(\alpha,\alpha)^2.$$

Applying the Laplace operator

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial \alpha_i^2}$$

to $(D4)(\alpha)$ one obtains

$$(D2)(\alpha): \quad \sum_{x\in X}(x,\alpha)^2 = \frac{sm}{n}(\alpha,\alpha).$$

Since $(x,\alpha)^2((x,\alpha)^2-1)$ is divisible by 12 if $(x,\alpha)\in\mathbb{Z}$

$$\frac{1}{12}\sum_{x\in X}(x,\alpha)^2((x,\alpha)^2-1) = \frac{sm}{12n}(\alpha,\alpha)(\frac{3m}{n+2}(\alpha,\alpha)-1) \in \mathbb{Z} \text{ for all } \alpha \in \Lambda^*$$

Substituting $\alpha := \xi_1\alpha_1 + \xi_2\alpha_2$ in $(D4)$ and comparing coefficients, one finds

$$(D22)(\alpha_1,\alpha_2): \quad \sum_{x\in X}(x,\alpha_1)^2(x,\alpha_2)^2 = \frac{sm^2}{n(n+2)}(2(\alpha_1,\alpha_2)^2 + (\alpha_1,\alpha_1)(\alpha_2,\alpha_2))$$

Also homogeneous polynomials of higher degrees can be used to obtain linear inequalities. To this aim we use the Euclidean inner product on the space of all homogeneous polynomials in n variables introduced in [24]. For $i := (i_1,\ldots,i_n) \in \mathbb{Z}^n_{\geq 0}$ let $x^i := x_1^{i_1}\cdots x_n^{i_n}$ and let $c(i) := \frac{(i_1+\ldots+i_n)!}{i_1!\cdots i_n!}$ denote the multinomial coefficient. For two polynomials

$$f := \sum_i c(i)a(i)x^i, \; g := \sum_i c(i)b(i)x^i \text{ put } [f,g] := \sum a(i)b(i).$$

Let $\omega(x) := (x,x)$ denote the quadratic form. For $\alpha \in \mathbb{R}^n$ the polynomial $\rho_\alpha^m(x) := (x,\alpha)^m$ is homogeneous of degree m. It is shown in [24, Proposition 1.1] that

$$[\rho_\alpha^m, f] = f(\alpha) \text{ for all homogeneous } f \text{ of degree } m$$

and by [24, Proposition 1.2]

$$[\omega^{m/2}, f] = m!\Delta^{m/2}(f).$$

Lemma 2.5 *Let $X \subset \mathbb{R}^n$ be a finite set of vectors of equal norm $m = (x,x)$ such that $X \cap -X = \emptyset$. For $t \in \mathbb{N}$ let $c_t := (1\cdot 3\cdot 5\cdots(2t-1))/(n(n+2)(n+4)\cdots(n+2t-2))$. Then*

$$\sum_{x_1,x_2\in X}(x_1,x_2)^{2t} \geq c_t|X|^2m^t$$

with equality if and only if $X \dot\cup -X$ is a spherical $2t$-design.

Proof. Define the homogeneous polynomial

$$p(\alpha) := \sum_{x\in X}(x,\alpha)^{2t} - c_t|X|(\alpha,\alpha)^t = \sum_{x\in X}\rho_x(\alpha)^{2t} - c_t|X|\omega(\alpha)^t.$$

Then the formulas in [24, Exemple 1.5] show that $\Delta^t(p) = 0$. We calculate the norm

$$\begin{aligned}[p,p] &= \textstyle\sum_{x\in X}[p,\rho_x^{2t}] - c_t|X|[p,\omega^t] = \sum_{x\in X} p(x) - c_t|X|\Delta^t(p) \\ &= \textstyle\sum_{x_1,x_2\in X}(x_1,x_2)^{2t} - c_t|X|^2m^t\end{aligned}$$

which is ≥ 0 as a norm of a vector in an Euclidean space. Moreover this sum equals 0 if $p = 0$ which is equivalent to the $2t$-design property of the antipodal set $X \mathbin{\dot\cup} -X$. □

Lemma 2.6 *(see [17, Lemma 2.1]) Let $\alpha \in \mathbb{R}^n$ be such that $(x,\alpha) \in \{0,\pm1,\pm2\}$ for all $x \in X$. Let $N_2(\alpha) := \{x \in X \cup -X \mid (x,\alpha) = 2\}$ and put*

$$c := \frac{sm}{6n}\left(\frac{3m}{n+2}(\alpha,\alpha) - 1\right).$$

Then $|N_2(\alpha)| = c(\alpha,\alpha)/2$ and

$$\sum_{x\in N_2(\alpha)} x = c\alpha.$$

Lemma 2.6 will be often applied to $\alpha \in \Lambda^*$. Rescale Λ such that $\min(\Lambda) = m = 1$ and let $r := \min(\Lambda^*)$. Since $\gamma(\Lambda)\gamma(\Lambda^*) = \min(\Lambda)\min(\Lambda^*) \leq \gamma_n^2$, we get $r \leq \gamma_n^2$ and for $\alpha \in \Lambda_r^*$ we have $(\alpha,x)^2 \leq r$ for all $x \in \Lambda_1$. Hence if $r < 9$ then $(\alpha,x) \in \{0,\pm1,\pm2\}$ for all $x \in X$ and Lemma 2.6 may be applied.

The next lemma yields good bounds on $n_2(\alpha)$.

Lemma 2.7 *(see [18, Lemma 2.4]) Let $m := \min(\Lambda)$ and choose $\alpha \in \Lambda_r^*$. If $r\cdot m < 8$, then*

$$|N_2(\alpha)| \leq \frac{rm}{8-rm}.$$

We also need the case when $r \cdot m = 8$.

Lemma 2.8 *Let $m := \min(\Lambda)$ and choose $\alpha \in \Lambda_r^*$ such that $r \cdot m = 8$.*

$$|N_2(\alpha)| \leq 2(n-1).$$

Proof. Let $N_2(\alpha) = \{x_1,\ldots,x_k\}$ with $k = |N_2(\alpha)|$. Consider the projections

$$\overline{x}_i := x_i - \frac{2}{r}\alpha$$

of the x_i onto $\alpha^\perp$. Then for all $i \neq j$

$$(\overline{x}_i,\overline{x}_j) = (x_i,x_j) - \frac{4}{r} = (x_i,x_j) - \frac{m}{2} \leq 0$$

since $|(x_i, x_j)| \leq \frac{m}{2}$ as x_i, x_j are minimal vectors of a lattice. So the $\overline{x}_i$ form a set of k distinct vectors of equal length in $(n-1)$-dimensional space having pairwise non-positive inner products. Moreover $\sum_{i=1}^k \overline{x}_i = 0$ since $\sum_{i=1}^k x_i = c\alpha$ by Lemma 2.6.
We **claim** that the set

$$\overline{N}_2(\alpha) := \{\overline{x}_1, \ldots, \overline{x}_k\} = E_1 \mathbin{\dot\cup} E_2 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} E_\ell$$

partitions into ℓ disjoint sets such that $\sum_{x \in E_i} x = 0$ for all i and that this is the only relation. So $\operatorname{rank}(\langle E_i \rangle) = |E_i| - 1$ and in total

$$n - 1 \geq \operatorname{rank}(\langle \overline{N}_2(\alpha) \rangle) = k - \ell \geq \frac{k}{2}$$

since $|E_i| \geq 2$ for all i.
The proof of the claim is standard but for convenience of the reader we sketch it here. Any relation between the $\overline{x}_i$ can be written as $y := \sum a_i \overline{x}_i = \sum b_j \overline{x}_j$ for nonnegative a_i, b_j. But then

$$(y, y) = (\sum a_i \overline{x}_i, \sum b_j \overline{x}_j) = \sum a_i b_j (\overline{x}_i, \overline{x}_j) \leq 0$$

and hence $y = 0$ so any minimal relation between the $\overline{x}_i$ has only positive coefficients. Subtracting the relations $\sum_{i=1}^k \overline{x}_i = 0$ we see that all these coefficients have to be equal. □

Lemma 2.9 *([24, Théorème 10.4]) Let L be a strongly perfect lattice of dimension n. Then*

$$\gamma(L)\gamma(L^*) = \min(L)\min(L^*) \geq \frac{n+2}{3}.$$

A strongly perfect lattice L is called of minimal type, *if* $\min(L)\min(L^*) = \frac{n+2}{3}$ *and* of general type *otherwise.*

2.5 Certain sublattices.

The next lemma about indices of sublattices are used quite often in the argumentation below. Since we are dealing with norms modulo some prime number p, we may pass to the localization $\mathbb{Z}_p := (\mathbb{Z} - p\mathbb{Z})^{-1}\mathbb{Z} \subset \mathbb{Q}$ of $\mathbb{Z}$ at p.

Lemma 2.10 *(see [18, Lemma 2.8, 2.9])*
a) *Let Γ be a $\mathbb{Z}_2$-lattice such that $(\gamma, \gamma) \in \mathbb{Z}_2$ for all $\gamma \in \Gamma$. Let $\Gamma^{(e)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 2\mathbb{Z}_2\}$. If $\Gamma^{(e)}$ is a sublattice of Γ, then $[\Gamma : \Gamma^{(e)}] \in \{1, 2, 4\}$.*
b) *Let Γ be a $\mathbb{Z}_3$-lattice such that $(\gamma, \gamma) \in \mathbb{Z}_3$ for all $\gamma \in \Gamma$. Let $\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 3\mathbb{Z}_3\}$. Assume that*

$$(\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta) \in 3\mathbb{Z}_3 \text{ for all } \alpha, \beta \in \Gamma.$$

Then $\Gamma^{(t)}$ is a sublattice of Γ and $[\Gamma : \Gamma^{(t)}] \in \{1, 3\}$.

Note that in both cases of Lemma 2.10, if Γ is universally perfect, then so is $\Gamma^{(e)}$ resp. $\Gamma^{(t)}$ since both lattices consist of certain layers of Γ.

3 General type.

3.1 Kissing numbers.

Let Λ be a strongly perfect lattice in dimension 14, rescaled such that $\min(\Lambda) = 1$. Then by Lemma 2.9 and the Cohn-Elkies bound that $\gamma_{14} \leq 2.776$ in [7] we find

$$16/3 \leq r := \min(\Lambda^*) \leq \gamma_{14}^2 \leq 7.71.$$

Hence $\alpha \in \Lambda_r^*$ satisfies the hypothesis of Lemma 2.6 and Lemma 2.7 yields

$$n_2(\alpha) \leq \frac{r}{8-r} \leq \frac{\gamma_{14}^2}{8-\gamma_{14}^2} \leq 26.3 < 27.$$

Let $s := |X|$ where $X \dot{\cup} -X = \Lambda_1$ be half the kissing number of Λ. Then by the bound given in [4] $\frac{15 \cdot 14}{2} = 105 \leq s \leq 1746$ and $r < 7.71$ is a rational solution of

$$1 \neq n_2(\alpha) = \frac{sr}{12 \cdot 14 \cdot 16}(3r - 16) \leq \frac{r}{8-r}$$

and $\frac{s \cdot r}{14} \in \mathbb{Z}$, $\frac{3sr^2}{14 \cdot 16} \in \mathbb{Z}$. Going through all possibilities by a computer we find:

Proposition 3.1 *With the notation above, one of the following holds:*

n_2	2	2	2	3	3	4	4	4	4
s	324	448	1200	486	672	225	343	363	525
r	56/9	6	28/5	56/9	6	112/15	48/7	224/33	32/5

n_2	5	5	8	8	11	12	19	20
s	384	504	450	567	672	675	968	1029
r	7	20/3	112/15	64/9	22/3	112/15	84/11	160/21

or Λ is of minimal type, i.e. $r = 16/3$.

Most of the cases in Proposition 3.1 are ruled out quite easily. More precisely we will prove the following

Theorem 3.2 *Let Λ be a strongly perfect lattice of dimension 14, with $\min(\Lambda) := 1$. Let $s := \frac{1}{2}|\Lambda_1|$ and $r := \min(\Lambda^*)$. Then we have the following three possibilities*

(a) $s = 672$ and $r = 6$ or $r = 22/3$.

(b) $s = 504$ and $r = 20/3$.

(c) $r = 16/3$, which means that Λ is of minimal type.

3.2 Proof of Theorem 3.2 for $s \neq 450$.

For the proof we scale Λ, such that $\min(\Lambda) = 1$. For $\alpha \in \Lambda^*$ write $(\alpha, \alpha) = \frac{p}{q}$ with coprime integers p and q. Then

$$(\star) \quad \frac{1}{12}(D4 - D2)(\alpha) = \frac{s}{2^7 3 \cdot 7} \frac{p}{q^2}(3p - 16q) \in \mathbb{Z}.$$

Moreover

$$\begin{aligned} D2(\alpha) &= \tfrac{s}{14}\tfrac{p}{q} \in \mathbb{Z} \\ D4(\alpha) &= \tfrac{3s}{14 \cdot 16}\tfrac{p^2}{q^2} \in \mathbb{Z}. \end{aligned}$$

Lemma 3.3 $(s, r) \neq (324, 56/9)$.

Proof. Assume that $s = 324 = 2^2 3^4$ and $r = 56/9$. Then $D4$ yields that $\frac{3^5}{2^3 7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 3^2 and 7 divides p. Moreover by $(\star)$ $\frac{3^3}{2^5 7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ yields that p is divisible by 2^3. Therefore $\Gamma := \sqrt{\frac{9}{28}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $\frac{28}{9} > 2$ which is a contradiction. □

Lemma 3.4 $(s, r) \neq (448, 6)$.

Proof. Assume that $s = 448 = 2^6 7$ and $r = 6$. Then $D4$ yields that $6\frac{p^2}{q^2} \in \mathbb{Z}$ hence all norms in Λ^* are integral $(q = 1)$. Moreover by $(\star)$ $\frac{1}{6}p(3p - 16) \in \mathbb{Z}$ yields that p is divisible by 6. Therefore $\Gamma := \sqrt{\frac{1}{3}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $3 > 2$ which is a contradiction. □

Lemma 3.5 $(s, r) \neq (1200, 28/5)$.

Proof. Assume that $s = 1200 = 2^4 3 \cdot 5^2$ and $r = 28/5$. Then $D4$ yields that $\frac{3^2 5^2}{14}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 15 and 14 divides p. Moreover by $(\star)$ $\frac{5^2}{2^3 7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ shows that p is a multiple of 4. Put $\Gamma := \sqrt{\frac{5}{14}}\Lambda^*$. Then $\min(\Gamma^*) = \frac{14}{5}$ and equation (D22) shows that

$$3 \cdot 14(2(\alpha, \beta)^2 + (\alpha, \alpha)(\beta, \beta)) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Gamma$$

Therefore

$$\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in \mathbb{Z}\} = \Gamma \cap \Gamma^*$$

is an even sublattice of Γ of index 1 or 3. Moreover $\min(\Gamma^{(t)}) = 2$ and $\min(\Lambda) = \frac{14}{5} > 2$ contradicting the fact that $\Gamma^{(t)} \subset \Lambda$. □

Lemma 3.6 $(s, r) \neq (486, 56/9)$.

Proof. Assume that $s = 486 = 2 \cdot 3^5$ and $r = 56/9$. Then $D4$ yields that $\frac{3^6}{2^4 7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 3^3 and 28 divides p. Moreover by $(\star)$ $\frac{3^4}{2^6 7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ yields that q divides 9 and p is divisible by 2^3. Therefore $\Gamma := \sqrt{\frac{9}{28}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $\frac{28}{9} > 2$ which is a contradiction. □

Lemma 3.7 $(s,r) \neq (225, 112/15)$.

Proof. Assume that $s = 225 = 3^2 5^2$ and $r = 112/15$. Then $D4$ yields that $\frac{3^3 5^2}{2^5 7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 15 and 56 divides p. Moreover by $(\star)$ $\frac{3 \cdot 5^2}{2^7 7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ yields that p is divisible by 2^4. Therefore $\Gamma := \sqrt{\frac{15}{56}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $\frac{56}{15} > 2$ which is a contradiction. $\square$

Lemma 3.8 $(s,r) \neq (343, 48/7)$.

Proof. Assume that $s = 343 = 7^3$ and $r = 48/7$. Then $D4$ yields that $\frac{3 \cdot 7^2}{2^5}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 7 and 8 divides p. Moreover by $(\star)$ $\frac{7^2}{2^7 3}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ yields that p is divisible by $2^4 3$. Therefore $\Gamma := \sqrt{\frac{7}{24}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $\frac{24}{7} > 2$ which is a contradiction. $\square$

Lemma 3.9 $(s,r) \neq (363, 224/33)$.

Proof. Assume that $s = 363 = 3 \cdot 11^2$ and $r = 224/33$. Then $D4$ yields that $\frac{3^2 \cdot 11^2}{2^5 7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 33 and $2^3 7$ divides p. Moreover by $(\star)$ $\frac{11^2}{2^7 7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ yields that p is divisible by 2^4. Therefore $\Gamma := \sqrt{\frac{33}{56}}\Lambda^*$ is an even lattice of minimum 4 and Γ^* has minimum $\frac{56}{33}$. In the new scaling the equation $D22$ reads as $\frac{14}{3}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z}$ for all $\alpha, \beta \in \Gamma$. Therefore

$$\Gamma^{(t)} := \{\gamma \in \Gamma \mid (\gamma,\gamma) \in 3\mathbb{Z}\}$$

is a sublattice of index 3 in Γ and hence 3^{12} divides $\det(\Gamma)$. This yields

$$531441 = 3^{12} \leq \det(\Gamma) \leq (\frac{\gamma_{14}}{\min(\Gamma^*)})^{14} < 982$$

which is a contradiction. $\square$

Lemma 3.10 $(s,r) \neq (525, 32/5)$.

Proof. Assume that $s = 525 = 3 \cdot 5^2 7$ and $r = 32/5$. Then $D4$ yields that $\frac{3^2 \cdot 5^2}{2^5}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 15 and 2^3 divides p. Moreover by $(\star)$ $\frac{5^2}{2^7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ shows that p is a multiple of 2^4. Put $\Gamma := \sqrt{\frac{5}{8}}\Lambda^*$. Then $\min(\Gamma) = 4$ and $\Gamma^* = \frac{\frac{8}{5}}{\Lambda}$ has minimum $\frac{8}{5}$. The equality (D22) yields that

$$12((\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Gamma$$

hence

$$\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha,\alpha) \in \mathbb{Z}\} = \Gamma \cap \Gamma^*$$

is a sublattice of Γ of index 3 or 1.

Let $\alpha \in \mathrm{Min}(\Gamma)$. Then $(\alpha,\alpha) = 4$ and hence $\alpha \in \Gamma^{(t)}$. Moreover

$$|N_2(\alpha)| = \frac{1}{12}(D4 - D2)(\alpha) = \frac{1}{2}(\alpha,\alpha)(3(\alpha,\alpha) - 10) = 4$$

and $N_2(\alpha) =: \{x_1, x_2 := \alpha - x_1, x_3, x_4 := \alpha - x_3\}$ for certain $x_1, x_3 \in \mathrm{Min}(\Gamma^*)$. Since these are minimal vectors of a lattice, we have $(x_1,x_3) \leq \frac{1}{2}(x_1,x_1) = \frac{4}{5}$ and hence $(x_1,x_4) = (x_1, \alpha - x_3) = 2 - (x_1,x_3) \geq 2 - \frac{4}{5} = \frac{6}{5}$ yielding the vector $x_1 - x_4 \in \Gamma^*$ of norm $< \min(\Gamma^*)$, a contradiction. □

Lemma 3.11 $(s,r) \neq (384, 7)$.

Proof. Assume that $s = 384 = 2^7 3$ and $r = 7$. Then $D4$ yields that $\frac{2\cdot 3^2}{7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 3 and 7 divides p and therefore $\Gamma := \sqrt{\frac{6}{7}}\Lambda^*$ is an even lattice of minimum 6 and Γ^* has minimum $\frac{7}{6}$. In the new scaling the equation $D22$ reads as $\frac{7}{3}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z}$ for all $\alpha, \beta \in \Gamma$. Therefore

$$\Gamma^{(t)} := \{\gamma \in \Gamma \mid (\gamma,\gamma) \in 3\mathbb{Z}\}$$

is a sublattice of index 3 in Γ and hence 3^{12} divides $\det(\Gamma)$. This yields

$$531441 = 3^{12} \leq \det(\Gamma) \leq \left(\frac{\gamma_{14}}{\min(\Gamma^*)}\right)^{14} < 186474$$

which is a contradiction. □

Lemma 3.12 $(s,r) \neq (567, 64/9)$.

Proof. Assume that $s = 567 = 3^4 7$ and $r = 64/9$. Then $D4$ yields that $\frac{3^5}{2^5}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 3^2 and 2^3 divides p. Moreover by $(\star)$ $\frac{3^3}{2^7}\frac{p}{q^2}(3p - 16q) \in \mathbb{Z}$ yields that p is divisible by 2^4. Therefore $\Gamma := \sqrt{\frac{9}{8}}\Lambda^*$ is an even lattice of minimum 8 and Γ^* has minimum $\frac{8}{9}$. Choose $\alpha \in \mathrm{Min}(\Gamma)$. Then $N_2(\alpha) = \{x_1, \ldots, x_8\}$ with $\sum_{i=1}^{8} x_i = 2\alpha$. Moreover

$$4 = (x_1, 2\alpha) = (x_1, \sum_{i=1}^{8} x_i) = (x_1,x_1) + \sum_{i=2}^{8}(x_1,x_i) \leq \frac{8}{9} + 7\frac{4}{9} = 4$$

yields that $N_2(\alpha) = \frac{2}{3}A_8$. In particular the denominator of the determinant of the Grammatrix of $N_2(\alpha)$ is 9^7. By Lemma 2.2 this implies that 9^7 divides the order $|\Gamma^*/\Gamma| = \det(\Gamma) \leq (\gamma_{14}/(8/9))^{14} < 2 \cdot 9^7$. Hence $\det(\Gamma) = 9^7$ and

$$L := \langle \Gamma, 3x_1, \ldots, 3x_7 \rangle$$

is an even overlattice containing Γ of index 3^7. Therefore L is unimodular, a contradiction since there is no even unimodular lattice in dimension 14. □

Lemma 3.13 $(s,r) \neq (675, 112/15)$.

Proof. Assume that $s = 675 = 3^3 5^2$ and $r = 112/15$. Then $D4$ yields that $\frac{3^4 5^2}{2^5 7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides $3^2 5$ and $2^3 7$ divides p. Moreover by $(\star)$ $\frac{3^2 5^2}{2^7 7}\frac{p}{q^2}(3p-16q) \in \mathbb{Z}$ yields that q divides 15 and p is divisible by 2^4. Therefore $\Gamma := \sqrt{\frac{15}{56}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $\frac{56}{15} > 2$ which is a contradiction. □

Lemma 3.14 $(s,r) \neq (968, 84/11)$.

Proof. Assume that $s = 968 = 2^3 11^2$ and $r = 84/11$. Then $D4$ yields that $\frac{3 \cdot 11^2}{2^2 7}\frac{p^2}{q^2} \in \mathbb{Z}$ hence q divides 11 and $2 \cdot 7$ divides p. Moreover by $(\star)$ $\frac{11^2}{2^4 3 \cdot 7}\frac{p}{q^2}(3p-16q) \in \mathbb{Z}$ yields that p is divisible by $2^2 3 \cdot 7$. Therefore $\Gamma := \sqrt{\frac{11}{42}}\Lambda^*$ is an even lattice of minimum 2 and Γ^* has minimum $\frac{42}{11} > 2$ which is a contradiction. □

Lemma 3.15 $(s,r) \neq (1029, 160/21)$.

Proof. Assume that $(s,r) = (1029, 160/21)$ and choose $\alpha \in \mathrm{Min}(\Lambda^*)$. Then $N_2(\alpha)$ generates a rescaled version of the root lattice A_{20}, which is of course impossible in dimension 14. □

3.3 The case $s = 450$.

To show Theorem 3.2 it suffices to treat the case $s = 450$ which is done in this subsection because it involves a bit more calculations than the cases treated in the previous section.

Let Λ be a strongly perfect lattice of dimension 14 and assume that $s(\Lambda) = \frac{1}{2}|\Lambda_{min}| = 450$. Without loss of generality we assume that Λ is generated by its minimal vectors. Rescale such that $\min(\Lambda) = \frac{28}{15}$ and let $\Gamma := \Lambda^*$. Then $\min(\Gamma) = 4$ and the equations $(D4)$ and $\frac{1}{12}((D4) - (D2))$ yield that

$$21(\alpha,\alpha)^2 \in \mathbb{Z} \text{ and } \frac{1}{4}(\alpha,\alpha)(7(\alpha,\alpha) - 20) \in \mathbb{Z}$$

hence Γ is an even lattice. Since Λ is generated by elements of norm $\frac{28}{15}$, we have that 15 divides $\det(\Gamma)$ and hence by Lemma 2.1

$$(+) \quad 11 \cdot 15 \leq \det(\Gamma) \leq 17 \cdot 15.$$

Moreover for any prime $p \neq 3, 5$ that divides $\det(\Gamma)$ the Sylow p-subgroup of Γ^*/Γ is generated by isotropic elements and hence cannot be cyclic. In particular p^2 divides $\det(\Gamma)$ and therefore $\det(\Gamma) = 2^a 3^b 5^c$ for some $a, b, c \in \mathbb{N}_0$. Let $M \supseteq \Gamma$ be a maximal even overlattice of Γ. Then also $\min(M^*) \geq \frac{28}{15}$. We have the following possibilities for the genus of M:

genus	h	mass
3^1	2	$691/(2^{23}3^95^37\cdot 11\cdot 13)\sim 3.344666\cdot 10^{-14}$
$3^{-1}5^1$	8	$650231/(2^{21}3^85^37^213)\sim 5.9349666\cdot 10^{-10}$
3^15^{-1}	9	$650231/(2^{21}3^85^37^213)\sim 5.9349666\cdot 10^{-10}$
$3^{-1}5^{-2}$	48	$5407504111/(2^{23}3^95^37\cdot 11)\sim 3.4026316\cdot 10^{-6}$
$2_0^23^15^{-1}$	93	$82579337/(2^{15}3^85^37^213)\sim 4.823941\cdot 10^{-6}$
$2_0^23^{-1}5^1$	91	$82579337/(2^{15}3^85^37^213)\sim 4.823941\cdot 10^{-6}$
$2_0^{-2}3^15^1$	46	$82579337/(2^{22}3^75^37\cdot 13)\sim 7.914278\cdot 10^{-7}$
$2_0^{-2}3^{-1}5^{-1}$	48	$82579337/(2^{22}3^75^37\cdot 13)\sim 7.914278\cdot 10^{-7}$

The table gives the genus symbol as explained in Section 2.2, the class number h of the genus followed by the mass.

Proof. We show that the list of possible elementary divisors of M is complete, then the list of possible genus symbols is obtained with the SAGE program mentioned in Section 2.2.

• Assume first that the determinant of M is 3^b5^c. Then by Milgram's formula (see [21, Cor. 5.8.2 and 5.8.3], [17, Lemma 2.3, 2.4]) b is odd and hence $\det(M)=3$, $3\cdot 5$, or $3\cdot 5^2$.

• Assume now that $\det(M)$ is even. Then the Sylow 2-subgroup of M^*/M has rank 2 and is anisotropic. Since the Sylow 2-subgroup of Γ^*/Γ is generated by isotropic elements, the index of Γ in M is even. By the bound in (+), we obtain $\det(\Gamma)=2^415$ and hence $\det(M)=2^2\cdot 3\cdot 5$. □

For none of the 345 relevant maximal even lattices M the dual lattice has minimum $\geq \frac{28}{15}$ so there is no strongly perfect lattice Λ of dimension 14 with kissing number $2\cdot 450$ and $\min(\Lambda)\min(\Lambda^*)=112/15$.

4 Dual strongly perfect lattices of general type.

In this section we show the following theorem.

Theorem 4.1 *Let Λ be a 14-dimensional dual strongly perfect lattice. Then Λ is of minimal type.*

To prove this theorem it is enough to consider the three remaining cases ($s=672$ and $\gamma=6$ or $\gamma=22/3$ resp. $s=504$ and $\gamma=20/3$) of Theorem 3.2.

So let Λ be a dual strongly perfect lattice that is not of minimal Type.

Lemma 4.2 $(s,\gamma)\neq(672,22/3)$.

Proof. Let $\Gamma:=\Lambda^*$ and scale such that

$$\min(\Gamma)=22,\min(\Lambda)=\frac{1}{3}.$$

Then for all $\alpha \in \Gamma$

$$\frac{1}{12}(D4 - D2) = \frac{1}{12}(\alpha,\alpha)((\alpha,\alpha) - 16) \in \mathbb{Z}$$

implies that Γ is an even lattice in which all norms are $\equiv 1, 0 \pmod 3$. The equation $(D22)$ implies that for all $\alpha, \beta \in \Gamma$

$$\frac{1}{3}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) \in \mathbb{Z}$$

and hence by Lemma 2.10 the lattice

$$\Gamma^{(t)} = \{\alpha \in \Gamma \mid 3 \mid (\alpha,\alpha)\}$$

is a sublattice of Γ of index 3. In particular 3^{12} divides the determinant of Γ so write $\det(\Gamma) = 3^{12}d$.

Now Λ is dual strongly perfect, so also Γ is a strongly perfect lattice of dimension 14. By Theorem 3.2 the lattice Γ has the same parameters as Λ. In particular $L := \sqrt{66}\Lambda$ is again an even lattices of minimum 22 possessing a sublattice $L^{(t)}$ of index 3 in which all inner products are multiples of 3. Hence 3^{12} divides

$$\det(L) = \frac{3^{14}}{22^{14}} \det(\Lambda) = 3^2 \cdot 22^{14} \cdot \frac{1}{d}$$

since $\Lambda = \Gamma^*$. But d is an integer and therefore we obtain a contradiction. □

Lemma 4.3 $(s, \gamma) \neq (504, 20/3)$.

Proof. Let $\Gamma := \Lambda^*$ and scale such that

$$\min(\Gamma) = 10, \min(\Lambda) = \frac{2}{3}.$$

Then $\frac{1}{12}(D4 - D2)$ shows that Γ is an even lattice. For $\alpha \in \Gamma_{10}$ consider

$$N_2(\alpha) = \{x \in \Lambda \mid (x,x) = \frac{2}{3}, (x,\alpha) = 2\}.$$

Then $|N_2(\alpha)| = 5$ and $\sum_{x \in N_2(\alpha)} x = \alpha$ which implies that $(x,y) = \frac{1}{3}$ for all $x \neq y \in N_2(\alpha)$ and hence $\langle N_2(\alpha)\rangle \cong \frac{1}{\sqrt{3}}A_5$. In particular 3^4 divides $\det(\Gamma)$.

Moreover by Theorem 3.2, the lattice Γ has the same parameters as Λ and therefore $\sqrt{15}\Lambda$ is again even of minimum 10 and hence $\det(\Gamma) = 3^a 5^b$ for some $a \geq 4$. Also by Milgram's formula (see [21, Cor. 5.8.2 and 5.8.3]) we know that the exponent a of 3 is odd. Interchanging the role of Λ and Γ if necessary we may assume that $b \geq 7$. By Lemma 2.1

$$61962301 \leq \det(\Gamma) \leq 471140124$$

so (a, b) is one of $(7, 7)$, $(5, 8)$. For both pairs (a, b) we have 2 possible genera of even lattices.

genus	mass
$3^7 5^7$	> 1043713033837
$3^{-7} 5^{-7}$	> 1043713033837
$3^5 5^8$	> 52297468940
$3^{-5} 5^{-8}$	> 52130384375

Note that the first two genera are dual to each other. In view of the masses of the genera given in the table, it is infeasible to list all lattices in these genera. Instead we will construct one lattice L in each of these genera as a sublattice of a suitable maximal even lattice and then use explicit calculations in the spaces of modular forms generated by the theta series of L and the cusp forms for $\Gamma_0(12)$ to the relevant character $\left(\frac{-\det(L)}{\cdot}\right)$:

Representatives of the genera of maximal even lattices of exponent dividing 15 are A_{14} $(3^1 5^{-1})$, $A_2 \perp A_4 \perp E_8$ $(3^{-1} 5^1)$, $E_6 \perp E_8$ (3^1), $A_2 \perp E_8 \perp L_4$ $(3^{-1} 5^{-2})$ where L_4 is the 4-dimensional maximal even lattice of determinant 5^2 with Grammatrix $F = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}$. One constructs representatives $L_1, \ldots, L_4$ of the 4 relevant genera of lattices of determinant $3^7 5^7$ respectively $3^5 5^8$ as sublattices of these 4 lattices above.

We now use the explicit knowledge of the spaces of modular forms for $\Gamma_0(15)$ with the character given by the Kronecker symbol of the discriminant of L_i. Since the strategy is the same in all cases, we explain it in the case where Γ is a sublattice of index $3^3 5^3$ of A_{14}. The relevant space of modular forms $\mathcal{M}_7(\Gamma_0(15), \left(\frac{-15}{\cdot}\right))$ has dimension 14 and its cuspidal subspace $\mathcal{S}$ has dimension 10. We construct the theta series T of some lattice in the genus of Γ. Then we know that $\theta_\Gamma = T + f$ for some $f \in \mathcal{S}$. Moreover the q-expansion of θ_Γ starts with

$$1 + 0q^2 + 0q^4 + 0q^6 + 0q^8 + 2 \cdot 21 \cdot 24 q^{10} + \ldots .$$

Applying the Atkin-Lehner involution W_{15} we obtain the same conditions on $W_{15}(\theta_\Gamma)$. To do this in Magma, we have to multiply θ_Γ with the theta series f_1 of the lattice with Grammatrix $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ to obtain a modular form for $\Gamma_0(15)$ of Haupttypus (without a character), that is then of weight 8. To such forms, we may apply the Atkin-Lehner operator in Magma. Then

$$W_{15}(\theta_\Gamma) = W_{15}(\theta_\Gamma f_1)/f_1$$

(since all lattices of dimension 2 are similar to their dual lattices). This way we obtain 12 linear conditions on the cusp form $f \in \mathcal{S}$, of which only 9 are linearly independent. So there is a 1-parametric space of solutions $\{\theta + a f_0 \mid a \in \mathbb{Z}\}$ where

$$\begin{array}{lccccccccc} \theta = & 1 & +1008q^{10} & +1896q^{12} & +43124q^{14} & -210044q^{16} & +340244q^{18} & +755692q^{20} & - \ldots \\ f_0 = & & & q^{12} & -11q^{14} & +44q^{16} & -51q^{18} & -154q^{20} & + \ldots \end{array}$$

One immediately sees that this space does not contain a function with positive coefficients, since $210044/44 = 52511/11 > 43124/11$. So no lattice in the genus of L has minimum 10, kissing number 1008 such that its rescaled dual lattice $\sqrt{15}L^*$ also has minimum 10 and kissing number 1008.

Similarly the other three genera are treated, where one gets a unique solution (with negative coefficient at q^{14}) in the two cases where the determinant of Γ is $3^5 5^8$. $\square$

Lemma 4.4 *If Λ is a dual strongly perfect lattice of dimension 14 then* $(s(\Lambda), \gamma'(\Lambda)^2) \neq (672, 6)$.

Proof. Let Λ be such a lattice and let $\Gamma := \Lambda^*$ rescaled such that $\min(\Gamma) = 6$, $\min(\Lambda) = 1$. Then the norms of the elements in Γ are in $\frac{2}{3}\mathbb{Z}$ and the equation $D22$ shows that Γ contains a sublattice

$$\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in \mathbb{Z}\} = \Lambda \cap \Gamma$$

of index 1 or 3. Since also Γ is strongly perfect, we may rescale to see that an analogous property is hold by $\sqrt{6}\Lambda$. Let $Z \dot{\cup} -Z := \mathrm{Min}(\Gamma)$ and $X \dot{\cup} -X = \sqrt{6}\mathrm{Min}(\Lambda)$. Then $T := \pm X \cup Z$ is a spherical 5-design consisting of $2 \cdot 2 \cdot 672$ vectors of norm 6. Moreover the inner products in T are

$$0, \pm 1, \pm 2, \pm 3, \pm 6, \pm\sqrt{6}, \pm 2\sqrt{6}.$$

We now choose some $\alpha \in Z$ and choose signs of the elements in Z such that $(z, \alpha) \geq 0$ for all $z \in Z$. Put

$$M_i(\alpha) := \{z \in Z \mid (z, \alpha) = i\} \text{ and } m_i := |M_i(\alpha)| \text{ for } i = 0, 1, 2, 3, 6.$$

Then $m_6 = 1$, $m_0 + m_1 + m_2 + m_3 + 1 = 672$ and $D2(\alpha)$ and $D4(\alpha)$ show that

$$(\star) \quad m_0 = 1148 - 10m_3, m_1 = 15m_3 - 1200, m_2 = 723 - 6m_3$$

so in particular $80 \leq m_3 \leq 114$. Moreover for $z \in M_3(\alpha)$ also $\alpha - z \in M_3(\alpha)$ so m_3 is even.

Claim 1: For all $\alpha \in Z$ we have $m_3 \neq 114$.

Assume that there is some $\alpha \in Z$ with $m_3 = 114$ and consider the set

$$\overline{M}_3 := \{\overline{z} := z - \frac{1}{2}\alpha \mid z \in M_3(\alpha)\} \subset \alpha^{\perp}.$$

Then $\overline{M}_3 = -\overline{M}_3$ is antipodal and for $\overline{z}_1, \overline{z}_2 \in \overline{M}_3$ we find

$$(\overline{z}_1, \overline{z}_2) = (z_1, z_2) - \frac{3}{2} = \frac{\pm a}{2} \text{ with } a \in \{9, 3, 1\}.$$

Choose some $y_0 \in \sqrt{2}\overline{M}_3 = Y \dot{\cup} -Y$ and let $a_i(y_0) := |\{y \in Y \mid (y, y_0) = \pm i\}|$ $(i = 1, 3)$. Then

$$a_3(y_0) + a_1(y_0) = 56 = |Y| - 1 \text{ and } \sum_{y_0 \in Y} 9^2 + a_1(y_0) + 9a_3(y_0) \geq 9^2 |Y| \frac{57}{13}.$$

In particular there is some $y_0 \in Y$ such that $a_3(y_0) \geq 27.2$, so $a_3(y_0) \geq 28$. Now $y_0 = \sqrt{2}(z_0 - \frac{1}{2}\alpha)$ for some $z_0 \in M_3(\alpha)$ and $(y, y_0) = 3$ is equivalent to $(z, z_0) = 3$ hence there are at least 28 vectors $z \in M_3(\alpha)$ with $(z, z_0) = 3$. This yields 28 vectors $z - z_0 \in M_0(\alpha)$ contradicting the fact that $m_0 = 1148 - 10m_3 = 8$ if $m_3 = 114$. This proves Claim 1. Of course by interchanging the roles of the two lattices Λ and Γ we similarly obtain
Claim 1': For all $\beta \in X$ $m_3'(\beta) := |\{x \in X \mid (x, \beta) = \pm 3\}| \leq 112$.
Claim 2: There is some $\alpha \in Z$ or some $\beta \in X$ with $m_3(\alpha) = 114$ resp. $m_3'(\beta) = 114$.
To see this we use the set $T = \mathrm{Min}(\Gamma) \cup \sqrt{6}\mathrm{Min}(Lambda) = \pm(X \cup Z)$ defined above and the positivity property (see Lemma 2.5)

$$\sum_{t_1, t_2 \in T} (t_1, t_2)^6 \geq 6^6 |T|^2 \frac{3 \cdot 5}{14 \cdot 16 \cdot 18}$$

In particular there is some $t \in T$ such that $\sum_{t_1 \in T} (t_1, t)^6 \geq 6^6 |T| \frac{3 \cdot 5}{14 \cdot 16 \cdot 18}$ and we may assume without loss of generality that $t = \alpha \in Z$. We obtain $n_i := |\{x \in X \mid (\alpha, x) = \pm i\sqrt{6}\}|$ as $n_2 = 3$ and $n_1 = 2^2 \cdot 3 \cdot 23$. Using $(\star)$ again we calculate

$$\sum_{t_1 \in T} (t_1, t)^6 = 6^6 + 2^6 6^3 n_2 + 6^3 n_1 + 3^6 m_3 + (723 - 6m_3) 2^6 (15m_3 - 1200) \geq 6^6 5$$

which implies that $m_3 \geq 112.4$ so $m_3 = 114$.
Of course Claim 2 contradicts Claim 1 or Claim 1'. □

5 Dual strongly perfect lattices of minimal type.

In this section we treat the 14-dimensional strongly perfect lattices of minimal type, where we focus on the dual strongly perfect lattices. Nevertheless some of the results can be used for a later classification of all strongly perfect lattices. The first two lemmata are of general interest and particularly useful in this section.

Lemma 5.1 *Let $Y = \mathrm{Min}(\Gamma)$ be a 4-design consisting of minimal vectors in a lattice $\Gamma \leq \mathbb{R}^n$ and $a := \min(\Gamma) = (y, y)$ for all $y \in Y$. Assume that $(y_1, y_2) \neq 0$ for all $y_1, y_2 \in Y$. Fix $y_0 \in Y$ and consider $N(y_0) := \{y \in Y \mid (y, y_0) = a/2\}$. Then $|N(y_0)| \leq 2(n-1)$ and $|N(y_0)|$ is even.*

Proof. For $y \in N(y_0)$ also $y_0 - y \in N(y_0)$ so $|N(y_0)|$ is even. To get the upper bound on $|N(y_0)|$ consider the projection

$$\overline{N} := \{\overline{y} := y - \frac{1}{2} y_0 \mid y \in N(y_0)\}$$

of $N(y_0)$ to $y_0^\perp$. For $y_1, y_2 \in N(y_0)$ one has $(y_1, y_2) \neq a/2$ since otherwise $(y_0, y_1 - y_2) = 0$. Therefore

$$(\overline{y}_1, \overline{y}_2) = (y_1, y_2) - a/4 \begin{cases} = 3a/4 & y_1 = y_2 \\ \leq 0 & y_1 \neq y_2 \end{cases}$$

hence distinct elements in $\overline{N}$ have non positive inner products. Therefore $|\overline{N}| = |N(y_0)| \leq 2(n-1)$. □

Let Λ be a 14-dimensional strongly perfect lattice of minimal type scaled such that $\min(\Lambda) = 1$. Then $\min(\Lambda^*) = \frac{16}{3}$. As above we choose $X \subset \Lambda_1$ such that $\Lambda_1 = X \mathbin{\dot\cup} -X$ and put $s := |X|$. Then by the bound for antipodal spherical codes given in [4], we have $s \leq 1746$. From equality $(D2)$ we know that $8\frac{s}{21} \in \mathbb{Z}$ and hence

$$s = 21 s_1 \text{ with } 5 \leq s_1 \leq 83 \ .$$

Lemma 5.2 *With the notation above let*

$$Z := \mathrm{Min}(\Lambda) \cup \frac{\sqrt{3}}{4}\mathrm{Min}(\Lambda^*) \subset \mathcal{S}^{(13)}(1).$$

Then Z is an antipodal kissing configuration and hence by [4] $|Z| \leq 2 \cdot 1746$.

Proof. Let $x \in \mathrm{Min}(\Lambda)$ and $y \in \mathrm{Min}(\Lambda^*)$. Since Λ is a strongly perfect lattice of minimal Type we have $|(x,y)| \leq 1$. In particular

$$|(x, \frac{\sqrt{3}}{4}y)| \leq \frac{\sqrt{3}}{4} < \frac{1}{2}.$$

Since $\mathrm{Min}(\Lambda)$ and $\mathrm{Min}(\Lambda^*)$ consist of minimal vectors of a lattice, we find $|(z_1,z_2)| \leq \frac{1}{2}$ for all $z_1 \neq \pm z_2 \in Z$. □

Remark 5.3 *Assume that there are $\alpha, \beta \in \Lambda^*_{16/3}$ with $0 \leq (\alpha,\beta) < \frac{8}{3}$. Then $\gamma := \alpha - \beta \in \Gamma$ satisfies $\frac{16}{3} < (\gamma,\gamma) \leq \frac{32}{3}$. Moreover for all $x \in X$ we have $|(x,\gamma)| \leq 2$ hence γ satisfies the conditions of Lemma 2.6.*

Using the Theorem by Minkowski on the successive minima of a lattice we get:

Remark 5.4 *If $|(\alpha,\beta)| = \frac{8}{3}$ for all $\alpha \neq \pm\beta \in \Lambda^*_5$ then $\Lambda^*_{16/3} = \sqrt{8/3}A_1$ or $\Lambda^*_{16/3} = \sqrt{8/3}A_2$. In particular there is some $\gamma \in \Lambda^*$ with $\frac{16}{3} < (\gamma,\gamma) < 8.2 < 9$.*

Corollary 5.5 *There is $\gamma \in \Lambda^*$ with $\frac{16}{3} < r := (\gamma,\gamma) \leq \frac{32}{3}$ that satisfies the conditions of Lemma 2.6. In particular the equation*

$$(\star) \qquad |N_2(\gamma)| =: n_2 = \frac{s_1 r(3r-16)}{2^7}$$

has a solution (n_2, s_1, r) with natural numbers $5 \leq s_1 \leq 83, n_2$ and some rational number $\frac{16}{3} < r \leq \frac{32}{3}$.

Remark 5.6 *Searching for such solutions of $(\star)$ by computer and using the bound that $n_2 \leq \frac{r}{8-r}$ for $r < 8$ (Lemma 2.7) resp. $n_2 \leq 26$ for $r = 8$ (Lemma 2.8) we find the following possibilities:*

(a) $r = 8$ or $r = 32/3$ with $s_1 = 6, 12, 18, 30, 36, 42$

(b) $r = 8$ *with* $s_1 = 10, 14, 20, 22, 26, 28, 34, 38, 44, 46, 52$. *(5.9)*

(c) $r = 32/3$ *with* $s_1 = 9, 15, 21, 33, 39, 45, 51, 57, 60, 63, 66, 69, 78$. *(5.8)*

(d) $r = 8$ *or* $r = 28/3$ *with* $s_1 = 8, 16, 40, 56, 80$. *(5.10)*

(e) $s_1 = 24$, $r = 20/3, 8, 28/3, 32/3$

(f) $s_1 = 25$, $r = 32/5, 128/15, 48/5$. *(5.7)*

(g) $s_1 = 27$, $r = 64/9, 80/9, 32/3$. *(5.14)*

(h) $s_1 = 32$, $r = 6, 22/3, 8, 28/3, 10$

(i) $s_1 = 48$, $r = 8, 28/3, 32/3$. *(5.17)*

(j) $s_1 = 49$, $r = 64/7, 208/21$. *(5.11)*

(k) $s_1 = 50$, $r = 8, 128/15, 48/5, 152/15$. *(5.16)*

(l) $s_1 = 54$, $r = 80/9, 88/9, 32/3$. *(5.13)*

(m) $s_1 = 64$, $r = 28/3, 10$. *(5.11)*

(n) $s_1 = 72$, $r = 28/3, 32/3$. *(5.11)*

(o) $s_1 = 75$, $r = 128/15, 48/5, 32/3$. *(5.12)*

(p) $s_1 = 81$, $r = 80/9, 32/3$. *(5.11)*

In brackets are the references, where we exclude the corresponding case. The cases (a), (e), and (h) are treated in the end of this section.

Lemma 5.7 $s_1 \neq 25$.

Proof. Assume that $s_1 = 25$ and rescale such that $\min(\Gamma) = 10$ and $\min(\Lambda) = 8/15$ and let $X \,\dot{\cup}\, -X = \mathrm{Min}(\Lambda)$. Then for all $\alpha, \beta \in \Gamma$

$$\begin{array}{lll} \sum_{x\in X}(x,\alpha)^2 & = 20(\alpha,\alpha) & \in \mathbb{Z} \\ \sum_{x\in X}(x,\alpha)^4 & = 2(\alpha,\alpha)^2 & \in \mathbb{Z} \\ \sum_{x\in X}(x,\alpha)^2(x,\beta)^2 & = \frac{2}{3}(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) & \in \mathbb{Z} \\ \frac{1}{12}(D4 - D2)(\alpha) & = \frac{1}{6}(\alpha,\alpha)((\alpha,\alpha) - 10) & \in \mathbb{Z} \end{array}$$

which shows that Γ is an even lattice with a sublattice

$$\Gamma^{(t)} := \{\gamma \in \Gamma \mid (\gamma,\gamma) \in 3\mathbb{Z}\}$$

of index 3. Then $\min(\Gamma^{(t)}) = 12$. Otherwise $\min(\Gamma^{(t)}) \geq 18$ and hence

$$9\det(\Gamma) = \det(\Gamma^{(t)}) \geq (\frac{18}{\gamma_{14}})^{14} \geq 232242985896 =: u.$$

On the other hand

$$\det(\Gamma) \le (\frac{15\gamma_{14}}{8})^{14} \le 10712486160 =: o.$$

Since $9o \le u$ this is a contradiction. In particular there is some $\beta \in \Gamma$ such that $(\beta, \beta) = 12$. Then

$$N_2(\beta) = \{x \in \mathrm{Min}(\Lambda) \mid (x, \beta) = 2\} = \{x_1, x_2, x_3, x_4\}$$

is of cardinality 4, $(x_i, x_j) = \frac{4}{15}$ for all $i \neq j$ and $x_1+x_2+x_3+x_4 = \frac{2}{3}\beta$. This implies that $\xi := \frac{1}{3}\beta \in \Lambda$, $(\xi, \xi) = \frac{4}{3}$ and $(x_1, \xi) = \frac{2}{3}$. Therefore $x := \xi - x_1 \in N_2(\beta)$, say $\xi - x_1 = x_2$. But then we have the relation $\frac{1}{3}\beta = x_1 + x_2$ implying that $(x_1, x_2) = \frac{2}{3} - \frac{8}{15} = \frac{2}{15}$, a contradiction. □

Lemma 5.8 *If* $r = 32/3$ *is the only possible norm of an element* $\alpha \in \Lambda^*$ *with* $16/3 < (\alpha, \alpha) \le 32/3$ *(so we are in case (c) of Remark 5.6) then* $Y := \mathrm{Min}(\Lambda^*)$ *is not a 4-design.*

Proof. In this case the inner products of $y_1, y_2 \in Y$ are $(y_1, y_2) \in \{\pm 16/3, \pm 8/3, 0\}$ so Y is a rescaled root system of rank 14. By [24, Theorem 6.11] the only root systems that form 4-designs are A_1, A_2, D_4, E_6, E_7 and E_8. □

Lemma 5.9 *If* $r = 8$ *is the only possible norm of an element* $\alpha \in \Lambda^*$ *with* $16/3 < (\alpha, \alpha) \le 32/3$ *(so we are in case (b) of Remark 5.6) then* $Y := \mathrm{Min}(\Lambda^*)$ *is not a 4-design.*

Proof. Rescale Λ^* such that $(y, y) = 4$ for all $y \in Y$. Then the inner products of $y_1, y_2 \in Y$ are $(y_1, y_2) \in \{\pm 4, \pm 2, \pm 1\}$. Assume that Y is a 4-design and let $a := |Y|/2$. For $y_0 \in Y$ and $i = 1, 2$ let $a_i := |\{y \in Y \mid (y_0, y) = i\}|$. Then the 4-design conditions yield that

$$1 + a_1 + a_2 = a, 4^2 + a_1 + 2^2 a_2 = 4^2 a/14, 4^4 + a_1 + 2^4 a_2 = 4^4 3a/(14 \cdot 16)$$

having the unique solution

$$a_2 = 0, a = 105, a_1 = 104.$$

So in this case Y is an antipodal tight 5-design, but no such tight design exists in dimension 14 (see for instance [6]). □

Lemma 5.10 *If* $r = 8$ *and* $r = 28/3$ *are the only possible norms of elements* $\alpha \in \Lambda^*$ *with* $16/3 < (\alpha, \alpha) \le 32/3$ *(so we are in case (d) of Remark 5.6) then* $Y := \mathrm{Min}(\Lambda^*)$ *is not a 4-design.*

Proof. Rescale Λ^* such that $(y, y) = 8$ for all $y \in Y$. Then the inner products of $y_1, y_2 \in Y$ are $(y_1, y_2) \in \{\pm 8, \pm 4, \pm 2, \pm 1\}$. Assume that Y is a 4-design and let $a := |Y|/2$. For $y_0 \in Y$ and $i = 1, 2, 4$ let $a_i := |\{y \in Y \mid (y_0, y) = i\}|$. Then by the 4-design conditions

$$a = 105 + 5a_4, a_1 = \frac{64}{21}a_4, a_2 = 104 + \frac{20}{21}a_4;$$

in particular a_4 is divisible by 21. Moreover $a_4 > 0$ by Lemma 5.9 and a_4 is even and $a_4 \leq 26$ by Lemma 5.1 which is a contradiction. □

Lemma 5.11 *In the cases (j), (m), (n), (p) of Remark 5.6* $\mathrm{Min}(\Gamma)$ *is not a 4-design.*

Proof. In these cases only three inner products $i, j, 8/3$ (up to sign) between distinct $y_1, y_2 \in Y$ are possible. For a fixed $y_0 \in Y$ let

$$\begin{array}{ll} a := |Y|/2, & b := \frac{1}{2}|\{y \in Y \mid (y, y_0) = \pm i\}| \\ c := \frac{1}{2}|\{y \in Y \mid (y, y_0) = \pm j\}|, & d := \frac{1}{2}|\{y \in Y \mid (y, y_0) = \pm 8/3\}|. \end{array}$$

Assuming that Y is a 4-design one obtains the system

$$1 + b + c + d = a,\ d\frac{64}{9} + cj^2 + bi^2 = (\frac{a}{14} - 1)\frac{16^2}{3^2},\ d\frac{8^4}{3^4} + cj^4 + bi^4 = (\frac{3a}{14 \cdot 16} - 1)\frac{16^4}{3^4}$$

of which the solution depends on one parameter. In the different situations we find:

(j) $s_1 = 49$, $i = 16/21, j = 8/21$

$$a = (24960 + 1440d)/299, b = (6825 + 112d)/23, c = -(64064 + 315d)/299$$

so c is negative here, a contradiction.

(m) $s_1 = 64$, $i = 2/3, j = 1/3$

$$a = (37585 + 2205d)/461, b = (176800 + 3024d)/461, c = -(139776 + 1280d)/461$$

so c is negative here, a contradiction.

(n) $s_1 = 72$, $i = 2/3, j = 0$

$$a = (1764 + 105d)/22, b = (3328 + 64d)/11, c = -(4914 + 45d)/22$$

so c is negative here, a contradiction.

(p) $s_1 = 81$, $i = 8/9, j = 0$.

$$a = 1/23(1960 + 112d), b = 1/23(4212 + 81d), c = 1/23(-2275 + 8d)$$

For c to be a nonnegative integer, d has to be of the form $293 + 23x$. Since d is even, $x = 1, 3, 5\ldots$ is odd. In particular $a \geq 1624$. Since $s = \frac{1}{2}|\mathrm{Min}(\Lambda)| = 21 \cdot 81 = 1701$ this is a contradiction to Lemma 5.2.

□

From now on we assume that Λ and Λ^* are both strongly perfect. In particular $Y \dot\cup -Y := \mathrm{Min}(\Lambda^*)$ is also a 4-design. We put $|Y| = 21t_1$ and $|X| = 21s_1$ (where $X \dot\cup -X = \mathrm{Min}(\Lambda)$). By the discussion above we know that both, s_1 and t_1, belong to the set

$$S := \{6, 12, 18, 24, 27, 30, 32, 36, 42, 48, 50, 54, 75\}$$

and $s_1 + t_1 \leq 83$ by Lemma 5.2. Moreover Remark 5.6 gives all possible inner products

$$i := |(y_1, y_2)| = (32/3 - r)/2$$

for $y_1, y_2 \in Y$ according to the different values of s_1. Since also Y is a 4-design, we obtain a system of equations for the values $a_i := |\{y \in Y \mid |(y, y_0)| = i\}|$ for fixed $y_0 \in Y$. The values a_i are nonnegative integers. Going through the different possibilities we find the next lemmata:

Lemma 5.12 $s_1 \neq 75$.

Proof. Assume that $s_1 = 75$. Rescale the lattice such that $\min(\Lambda) = \frac{8}{15}$. Then $\min(\Gamma) = 10$ and for $y_1, y_2 \in \Gamma_{10}$ we have

$$|(y_1, y_2)| \in \{10, 5, 2, 1, 0\}.$$

Fix $y_0 \in Y$ and let $a_i := |\{y \in Y \mid (y, y_0) = \pm i\}|$. Then the solution of the system of equations

$$\begin{aligned} 1 + a_5 + a_2 + a_1 + a_0 &= 21t_1 \\ 10^2 + 5^2 a_5 + 2^2 a_2 + a_1 &= 150t_1 \\ 10^4 + 5^4 a_5 + 2^4 a_2 + a_1 &= 5^4 3^2 t_1/2 \end{aligned}$$

has the following solution depending on the two parameters a_5 and $t_1 = 8t_2$:

$$a_2 = 1775t_2 - 825 - 50a_5,\ a_1 = 3200 + 175a_5 - 5900t_2,\ a_0 = 4293t_2 - 2376 - 126a_5.$$

In particular t_1 is divisible by 8, which leaves the possibilities $t_1 = 24, 32, 48$ contradicting the fact that $s_1 + t_1 \leq 83$ hence $t_1 \leq 8$. □

Lemma 5.13 $s_1 \neq 54$.

Proof. Rescale such that $\min(\Gamma) = 12, \min(\Lambda) = \frac{4}{9}$. Then the inner products between minimal vectors of Γ are

$$\{\pm 12, \pm 6, \pm 2, \pm 1, 0\}.$$

The solution of the respective system of equations for the a_i depends on two parameters a_6 and t_1:

$$a_2 = -1716 - 105a_6 + 468t_1, a_0 = -5005 - 280a_6 + 1209t_1, a_1 = 6720 + 384a_6 - 1656t_1.$$

In particular $a_0 \geq 0$ and $a_1 \geq 0$ yields

$$(1656t_1 - 6720)/384 \leq a_6 \leq (1209t_1 - 5005)/280$$

which implies that $t_1 \geq 70$, a contradiction. □

Lemma 5.14 *$s_1 \neq 27$.*

Proof. Rescale such that $\min(\Gamma) = 6, \min(\Lambda) = \frac{8}{9}$. Then Γ is an even lattice and the inner products between minimal vectors of Γ are

$$\{\pm 6, \pm 3, \pm 2, \pm 1, 0\}.$$

There is some pair $\alpha_1, \alpha_2 \in \Gamma_6$ such that $(\alpha_1, \alpha_2) = 2$, otherwise the fact that $\Gamma_6 = \pm Y$ is a 4-design, in particular the equations $D2(\alpha)$ and $D4(\alpha)$ would imply that for any fixed $\alpha \in \Gamma_6$

$$|Y| = 1470 + 14|Y \cap \alpha^{\perp}| \geq 1470$$

contradicting Lemma 5.2. So choose $\alpha_1, \alpha_2 \in \Gamma_6$ such that $(\alpha_1, \alpha_2) = -2$ and put $\beta := \alpha_1 + \alpha_2 \in \Gamma_8$. From the fact that $\Lambda_{8/9}$ is a 4-design, we get that

$$N_2(\beta) := \{x \in \Lambda_{8/9} \mid (x, \beta) = 2\} = \{x_1, \ldots, x_8\}$$

such that $\sum_{i=1}^{8} x_i = 2\beta$ and $(x_i, x_j) = \frac{4}{9}$ for all $i \neq j$. Hence $N_2(\beta)$ generates a rescaled version of the root lattice A_8 in Λ and the residue classes $x_i + \Gamma \in \Lambda/\Gamma$ generate a subgroup of Λ/Γ of order at least 9^7. Since $\min(\Lambda) = \frac{8}{9}$ we get

$$\det(\Gamma) \leq (9 \cdot 2.776/8)^{14} < 2 \cdot 9^7$$

and therefore $\det(\Gamma) = 9^7$. But then the lattice

$$\Delta := \langle 3x_1, \ldots, 3x_7, \Gamma \rangle \leq \Lambda$$

is an even unimodular sublattice of Λ of rank 14, which is a contradiction. □

We now rescale the lattice such that $\min(\Gamma) = 4$, $\min(\Lambda) = 4/3$. Then for all $\alpha, \beta \in \Gamma$

$$\begin{array}{lll}
\sum_{x\in X}(x,\alpha)^2 & = 2s_1(\alpha,\alpha) & \in \mathbb{Z} \\
\sum_{x\in X}(x,\alpha)^4 & = s_1/2(\alpha,\alpha) & \in \mathbb{Z} \\
\sum_{x\in X}(x,\alpha)^2(x,\beta)^2 & = s_1/6(2(\alpha,\beta)^2 + (\alpha,\alpha)(\beta,\beta)) & \in \mathbb{Z} \\
\frac{1}{12}\sum_{x\in X}(x,\alpha)^4 - (x,\alpha)^2 & = s_1/8(\alpha,\alpha)((\alpha,\alpha)-4) & \in \mathbb{Z}
\end{array}$$

In particular, if s_1 is not a multiple of 3 then Γ is 3-integral and the set

$$\Gamma^{(t)} := \{\alpha \in \Gamma \mid (\alpha, \alpha) \in 3\mathbb{Z}_{(3)}\}$$

is a sublattice of Γ of index 3.

Lemma 5.15 *$s_1 t_1$ is a multiple of 3.*

Proof. Assume that both, s_1 and t_1 are not divisible by 3. Then rescaled to $\min(\Gamma) = 4, \min(\Lambda) = 4/3$ the lattice Γ is 3-integral and also $\sqrt{3}\Lambda$ is 3-integral and both lattices have a sublattice of index 3 in which all inner products are divisible by 3. In particular the 3-part of the determinant of Γ is 3^a with $a \geq 12$ and the one of $\sqrt{3}\Lambda$ is 3^{14-a} again with $14 - a \geq 12$, which is a contradiction. □

Lemma 5.16 $s_1 \neq 50$.

Proof. Rescale such that $\min(\Gamma) = 20, \min(\Lambda) = \frac{4}{15}$. Then the inner products between minimal vectors of Γ are

$$\{\pm 20, \pm 10, \pm 5, \pm 4, \pm 2, \pm 1\}.$$

First assume that there is some $\beta \in \Gamma_{30}$ (this is certainly the case if there are $\alpha_1, \alpha_2 \in \Gamma_{20}$ with $(\alpha_1, \alpha_2) = 5$). Then

$$N_2(\beta) := \{x \in \Lambda_{4/15} \mid (x, \beta) = 2\} = \{x_1, \ldots, x_{25}\}$$

and for $x \in N_2(\beta)$, $\overline{x} := x - \frac{1}{15}\beta$ is perpendicular to β. Moreover for $x_1, x_2 \in N_2(\beta)$

$$(\overline{x}_1, \overline{x}_2) = (x_1, x_2) - \frac{2}{15} \begin{cases} = 2/15 & \text{if } x_1 = x_2 \\ \leq 0 & \text{if } x_1 \neq x_2 \end{cases}.$$

So the set $\overline{N}_2$ consists of 25 vectors of equal length in a 13 dimensional space having non positive inner products. Therefore there are two elements $x_1, x_2 \in N_2(\beta)$ with $\overline{x}_1 + \overline{x}_2 = 0$ i.e.

$$x_1 + x_2 = \frac{2}{15}\beta.$$

Hence $\beta \in 15\Gamma^*$ and therefore

$$(\alpha, \beta) \in \{0, \pm 15\} \text{ for all } \alpha \in \Gamma_{20}.$$

Since Γ_{20} is a 4-design of cardinality $42t_1$ the numbers $n_i := |\{\alpha \in \Gamma_{20} \mid (\alpha, \beta) = \pm i\}|$ satisfy

$$n_0 + n_{15} = 42t_1, 15^2 n_{15} = 1800t_1, 15^4 n_{15} = 202500t_1$$

which has 0 as only solution. So this case is impossible.
Now we fix $\alpha \in \Gamma_{20} = Y \,\dot{\cup}\, -Y$ and put

$$a_i := |\{y \in Y \mid (y, \alpha) = \pm i\}| \text{ for } i = 20, 10, 5, 4, 2, 1.$$

Then we have $a_{20} = 1$ and $a_5 = 0$ by the above consideration. Moreover $|Y| = 21t_1$ with $t_1 \leq 32$. The solutions of the 4-design equations for Y in the a_i depend on two parameters a_{10} and t_1 and one finds

$$5a_4 = -4389 + 1169t_1 - 264a_{10},\ 5a_1 = -16896 + 3816t_1 - 896a_{10},\ a_2 = 4256 - 976t_1 + 231a_{10}.$$

Going through the different possibilities for $t_1 \leq 32$ the only even nonnegative integer a_{10} such that a_1, a_2, a_4 are all nonnegative integers is

$$t_1 = 27, a_{10} = 96, a_1 = 24, a_2 = 80, a_4 = 366.$$

But this case is impossible by Lemma 5.14. □

Lemma 5.17 $s_1 \neq 48$.

Proof. Rescale Γ such that $\min(\Gamma) = 8$ and $\min(\Lambda) = 2/3$. Then Γ is an even lattice and the inner products between minimal vectors in Γ are

$$\{\pm 8, \pm 4, \pm 2, \pm 1, 0\}.$$

Assume that there is some $\alpha \in \Gamma_8$ such that $(\alpha, \alpha') \neq \pm 2$ for all $\alpha' \in \Gamma_8$. Since Γ_8 is a 4-design, the equations $D4(\alpha)$ and $D2(\alpha)$ imply that

$$|\Gamma_8 \cap \alpha^{\perp}| = -2(189 + 9t_1) < 0$$

which is a contradiction.

Choose such a pair $(\alpha_1, \alpha_2) \in \Gamma_8^2$ such that $(\alpha_1, \alpha_2) = -2$ and put $\beta := \alpha_1 + \alpha_2 \in \Gamma_{12}$. Then $N_2(\beta) = \{x \in \Lambda_{2/3} \mid (x, \beta) = 2\}$ has cardinality 24. Put

$$\overline{N}_2 := \{x - \frac{1}{6}\beta \mid x \in N_2(\beta)\}.$$

Then $\overline{N}_2 \subset \langle \alpha_1, \alpha_2 \rangle^{\perp}$ and any two distinct vectors in $\overline{N}_2$ have non positive inner product. In particular we may apply [18, Lemma 2.10] to write $\overline{N}_2 = \dot{\bigcup}_{i=1}^{k} E_i$ as a union of disjoint indecomposable components E_i (i.e. $\sum_{x \in E_i} x = 0, E_i \perp E_j$ if $i \neq j$) with

$$k = |\overline{N}_2| - \dim(\langle \overline{N}_2 \rangle) \geq 24 - 12 = 12.$$

So $k = 12$ and $E_i = \{\overline{x}_i, -\overline{x}_i\}$ for all i. In particular there are $x_1, x_{13} \in N_2(\beta)$ such that $x_1 + x_{13} = \frac{1}{3}\beta$. So $\beta \in 3\Gamma^*$ and hence $(\beta, \alpha) \in \{0, \pm 3, \pm 6\}$ for all $\alpha \in \Gamma_8$. For $i = 0, 3, 6$ put $n_i := |\{\alpha \in Y \mid (\alpha, \beta) = \pm i\}|$. Then from the fact that $\Gamma_8 = Y \,\dot\cup\, -Y$ is a 4-design of cardinality $42t_1$ one obtains a unique solution

$$n_0 = 9t_1, \ n_3 = 32/3t_1, \ n_6 = 4/3t_1.$$

Now consider the set

$$M_6(\beta) := \{\alpha \in \Gamma_8 \mid (\alpha, \beta) = 6\} = \{\alpha_1, \ldots, \alpha_{n_6}\}$$

of cardinality $n_6 = 4/3t_1$ and choose $\alpha \in M_6(\beta)$. The $x_i \in N_2(\beta) = \{x \in \Lambda_{2/3} \mid (x, \beta) = 2\}$ satisfy $\sum_{i=1}^{n2} x_i = \frac{1}{6} n_2 \beta$ and therefore $\sum_{i=1}^{n2} (\alpha, x_i) = n_2$ hence $(\alpha, x_i) = 1$ for all $x_i \in N_2(\beta)$. Hence the orthogonal projection $\overline{\alpha} := \alpha - \frac{1}{2}\beta \in \beta^{\perp}$ and $\overline{x} := x - \frac{1}{6}\beta \in \overline{N}_2$ are perpendicular for all $x \in N_2(\beta)$ since

$$(\alpha - \frac{1}{2}\beta, x - \frac{1}{6}\beta) = (\alpha - \frac{1}{2}\beta, x) = (\alpha, x) - \frac{1}{2}(\beta, x) = 1 - 1 = 0.$$

Therefore $\overline{N}_2 \perp \overline{M}_6 \perp \langle \beta \rangle$ and $\dim(\overline{N}_2) = 12$ implies that $\dim(\overline{M}_6(\beta)) = 1$. Since the elements in $\overline{M}_6(\beta)$ have all equal norm, this leaves $4/3t_1 = n_6 = |M_6(\beta)| = 2$ which is a contradiction. □

Lemma 5.18 *If $s_1 = 32$ then $t_1 = 6$.*

Proof. Assume that $s_1 = 32$ and scale such that $\min(\Lambda) = 1/3$, $\min(\Gamma) = 16$. Then Γ is an even lattice containing the sublattice

$$\Gamma^{(t)} := \{\gamma \in \Gamma \mid (\gamma, \gamma) \in 3\mathbb{Z}\}$$

of index 3. By Lemma 5.15 and Lemma 5.14 we know that $t_1 = 6t_2$ is a multiple of 6. Moreover the inner products (α_1, α_2) for two elements $\alpha_i \in \Gamma_{16}$ are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 7, \pm 8, \pm 16$. Fix $\alpha_0 \in \Gamma_{16} = \pm Y$ and let $a_j := |\{\alpha \in Y \mid (\alpha, \alpha_0) = j\}|$. Then a_8 is even.
Since Γ_{16} is a 4-design of cardinality $21 \cdot 6 \cdot t_2$ we may solve the corresponding system of equations to obtain

$$a_4 = 539 + 65/3 a_8 + 2/3 a_1 + 10 a_7 - 522 t_2.$$

In particular a_4 is odd and hence nonzero.
So there are $\alpha_0, \alpha_0' \in \Gamma_{16}$ such that $\beta := \alpha_0 + \alpha_0' \in \Gamma_{24}$ has norm 24. Consider the set

$$N_2(\beta) := \{x \in \mathrm{Min}(\Lambda) \mid (x, \beta) = 2\}.$$

Note that for all $x \in N_2(\beta)$ the inner products $(x, \alpha_0) = (x, \alpha_0') = 1$. Then $|N_2(\beta)| = 16$ and $\sum_{x \in N_2(\beta)} x = 4/3\beta$. The set

$$\overline{N}_2 := \{\overline{x} = x - \frac{1}{12}\beta \mid x \in N_2(\beta)\} \subset \langle \alpha_0, \alpha_0' \rangle^{\perp}$$

is a set of vectors of equal norm living in a 12-dimensional space such that distinct vectors have non positive inner product and satisfying the relation $\sum_{\overline{x} \in \overline{N}_2} \overline{x} = 0$. Write

$$\overline{N}_2 = E_1 \dot\cup \ldots \dot\cup E_k$$

as a disjoint union of indecomposable sets. Then by [18, Lemma 2.10] $\dim(\overline{N}_2) = 16 - k$ and hence $k \geq 4$. Moreover for all i

$$\sum_{x \in E_i} x = \frac{|E_i|}{12}\beta.$$

Distinguish two cases:
(a) There is some i such that $|E_i|$ is not a multiple of 4.
(b) $k = 4$ and $|E_i| = 4$ for all $i = 1, \ldots, 4$.
In case (a) the vector $\beta \in 6\Lambda \cap \Gamma$. In particular $(\alpha, \beta) \in \{0, \pm 6, \pm 12\}$ for all $\alpha \in \Gamma_{16} = Y \dot\cup -Y$. Since Γ_{16} is a 4-design of cardinality $6 \cdot 21 \cdot t_2$, the cardinalities

$$m_j := |\{\alpha \in Y \mid (\beta, \alpha) = \pm j\}|$$

can be calculated as

$$m_0 = 54 t_2, m_6 = 64 t_2, m_{12} = 8 t_2.$$

Now consider the set

$$M_{12}(\beta) := \{\alpha \in \Gamma_{16} \mid (\alpha,\beta) = 12\}$$

and its orthogonal projection

$$\overline{M}_{12} := \{\overline{\alpha} := \alpha - \frac{1}{2}\beta \mid \alpha \in M_{12}(\beta)\}$$

into $\beta^{\perp}$. Note that for $\alpha \in M_{12}(\beta)$ we have

$$16 = (\alpha, \frac{4}{3}\beta) = \sum_{x\in N_2(\beta)} (\alpha, x)$$

and hence $(\alpha, x) = 1$ for all $x \in N_2(\beta)$. In particular

$$\overline{M}_{12} \perp \overline{N}_2 \perp \langle\beta\rangle.$$

Moreover $\overline{M}_{12} = -\overline{M}_{12}$ is an antipodal set. For $\alpha_1, \alpha_2 \in M_{12}(\beta)$ we calculate

$$(\overline{\alpha}_1, \overline{\alpha}_2) = (\alpha_1, \alpha_2) - 6 \in \{10, 2, 1, -1, -2, -10\}.$$

Fix some $z_0 \in \overline{M}_{12}$ and let $n_i := |\{z \in \overline{M}_{12} \mid (z, z_0) = i\}|$ $(i = 10, 2, 1)$. Then $\dim(\overline{M}_{12}) \le 5$ and $|\overline{M}_{12}| = 8t_2$ and hence

$$10^2 + n_1 + 2^2 n_2 \ge \frac{4t_2}{5} 10^2 = 80t_2,\ n_1 + n_2 = 4t_2 - 1$$

which yields $3n_2 \ge 76t_2 - 99$. Now $n_2 \le 4t_2 - 1$ so $4t_2 - 1 \ge 76t_2 - 99$ yielding the inequality $t_2 \le \frac{98}{72} < 2$. So the only solution is $t_2 = 1$.
Assume now that we are in case (b). Let $E_1 = \{x_1, x_2, x_3, x_4\}$ with $\sum_{i=1}^{4} x_i = \frac{1}{3}\beta$. Assume that $t_1 \ne 24$. then $t_1 \in A$ and

$$(x_i, x_j) \in \{0, \pm\frac{1}{12}, \pm\frac{1}{6}, \pm\frac{1}{3}\}.$$

If $(x_i, x_j) = 0$, then $\frac{1}{6}\beta = x_i + x_j$ and E_1 is not indecomposable. Hence $(x_i, x_j) \ne 0$ for all $x_i, x_j \in E_1$. Moreover

$$\frac{1}{3} = (\frac{1}{3}\beta, x_1) - (x_1, x_1) = \sum_{i=2}^{4} (x_i, x_1) = \frac{1}{6} + \frac{1}{12} + \frac{1}{12}$$

has only this solution. So the Gram matrix is

$$A := ((x_i, x_j)_{i,j=1}^4) = \frac{1}{12}\begin{pmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 4 & 1 \\ 2 & 1 & 1 & 4 \end{pmatrix}.$$

The vectors in $N_2(\beta)$ together with α_0 (from above with $\beta = \alpha_0 + \alpha_0'$) hence generate a lattice in $\mathbb{Q}\Gamma$ with Gram matrix

$$M = \frac{1}{12}\begin{pmatrix} 4 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 1 & 4 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 1 & 2 & 4 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 2 & 1 & 1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 4 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 1 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 1 & 1 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 4 & 2 & 2 & 2 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 1 & 1 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 4 & 2 & 12 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 4 & 12 \\ 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 192 \end{pmatrix}.$$

The determinant of M is in $15(\mathbb{Q}^*)^2$. In particular this means that the determinant of Γ is a multiple of 5 which is impossible since the exponent of Λ/Γ divides 12.

For $t_1 = 24$ we can do the same considerations. But here we get more possibilities for the inner products of the elements in one component E_i. The possible Gram matrices are

$$A, B := \frac{1}{24}\begin{pmatrix} 8 & 1 & 3 & 4 \\ 1 & 8 & 4 & 3 \\ 3 & 4 & 8 & 1 \\ 4 & 3 & 1 & 8 \end{pmatrix}, \text{ or } C := \frac{1}{24}\begin{pmatrix} 8 & 3 & 3 & 2 \\ 3 & 8 & 2 & 3 \\ 3 & 2 & 8 & 3 \\ 2 & 3 & 3 & 8 \end{pmatrix}.$$

So there are 15 possibilities for the Gram matrix M. All these matrices have determinant $5d$ for some rational d with denominator and numerator not divisible by 5.
□

The following table lists the possible values of $s_1 = \frac{1}{42}|\mathrm{Min}(\Lambda)|$ together with a value $m := \mathrm{Min}(\Lambda)$ for which $\Gamma = \Lambda^*$ is an even lattice of minimum $r = \frac{16}{3m}$. For each possible s_1 we list the possibilities for $t_1 = \frac{1}{42}|\mathrm{Min}(\Gamma)|$ and give numbers $e = e(s_1, t_1)$ such that $e\Lambda \subset \Gamma$. A stands for the set $\{6, 12, 18, 30, 36, 42\}$.

		t_1	A	24	32
r	m	s_1			
4	4/3	A	3	6	$12(s_1 = 6)$
8	2/3	24	6	12	(5.18)
16	1/3	32	$12(t_1 = 6)$	(5.18)	(5.15)

There are two different strategies with which we treat the remaining cases: If $\exp(\Lambda/\Gamma)$ is 3 or 6 then we explicitly construct all possible lattices as sublattices of one of the maximal even lattices M given in Section 2.3. If this exponent is 12, then this explicit calculation becomes too involved and there are too many lattices to be constructed. In this case we use the theory of modular forms as explained in the proof of Lemma 4.3 to exclude the possible theta series.

The most interesting but also easy case is of course that both s_1 and t_1 lie in A, since the unique dual strongly perfect lattice $[\pm G_2(3)]_{14}$ of dimension 14 satisfies $s_1 = t_1 = 18$.

Theorem 5.19 *Assume that Γ is an even dual strongly perfect lattice of dimension 14 of minimal type such that its discriminant group Λ/Γ and also $\Gamma/3\Lambda$ are 3-elementary. Then $\Gamma = [\pm G_2(3)]_{14}$.*

Proof. The lattice Γ is a sublattice of one of the two maximal even lattices M of determinant 3 given in Section 2.3. It has minimum 4 and its dual lattice has minimum $\frac{4}{3}$. Successively constructing all sublattices L of M of 3-power index such that L^*/L is of exponent 3 and $\min(L^*) \geq 4/3$ we find up to isometry a unique such L that is of minimum 4. This is the extremal 3-modular lattice with kissing number $756 = 2\cdot 21\cdot 18$ and automorphism group $G_2(3) \times C_2$ as stated in the theorem. □

Lemma 5.20 *If $s_1 = 24$ then $t_1 \notin A$.*

Proof. Assume that $s(\Lambda) = 21\cdot t_1$ for some t_1 in A. Then $\Gamma = \Lambda^*$ rescaled to $\min(\Gamma) = 4$ is an even lattices and $\min(\Lambda) = \frac{4}{3}$. If $s(\Gamma) = 21 \cdot 24$, then $\sqrt{6}\Lambda$ is even, hence the exponent of Λ/Γ is 6, and Γ is contained in one of the maximal even lattices M from Section 2.3 for which $\sqrt{6}M^*$ is again even. These are the two lattices in the genus of $E_6 \perp E_8$ and the 6 lattices in the genus of $A_2 \perp D_{12}$. From the latter genus only 2 lattices M satisfy $\min(M^*) \geq \frac{4}{3}$. Going through all possible sublattices L of the relevant 4 maximal even lattices for which $\sqrt{6}L^*$ is even and has minimum ≥ 8 $(= 6 \cdot \frac{4}{3})$ one finds only the lattice from Theorem 5.19. □

Lemma 5.21 *If $s_1 = 24$ then $t_1 \neq 24$.*

Proof. Assume that $t_1 = s_1 = 24$ and rescale such that $\min(\Gamma) = 8$, $\Gamma = \Lambda^*$ with $\min(\Lambda) = \frac{2}{3}$. Then Γ is even and so is $\sqrt{12}\Lambda$. We proceed as in the proof of Lemma 4.3. From Lemma 2.1 we find that

$$2725130 \leq \det(\Gamma) = 2^a 3^b \leq 471140123.$$

Moreover $1 \leq a \leq 27$ and $1 \leq b \leq 13$ and the Jordan decomposition of $\mathbb{Z}_2 \otimes \Gamma$ is $f_0 \perp 2f_1 \perp 4f_2$ such that f_0 and f_2 are even unimodular forms. Without loss of generality we assume that $b \leq 7$ (otherwise we may interchange the roles of Λ and Γ). In total this leaves 176 possibilities for the genus of Γ. We construct one lattice from each of the 176 genera. Most of them were obtained as random sublattices of suitable index of the maximal even lattices given in Section 2.3, representatives for the last 20 genera had to be constructed as sublattices of orthogonal sums of scaled root lattices and their duals. The mass of most of the genera is too big to enumerate all lattices in them. For each genus we calculate the theta series T of the chosen representative. The determinant of Γ is either a square or 3 times a square since the dimensions of both forms (f_0 and f_2) in the Jordan decomposition above is even and so is $\dim(f_1)$ and hence a. Then we know that $\theta_\Gamma = T + f$ for some $f \in \mathcal{S}_\epsilon$, where for $\epsilon = -1$ or $\epsilon = -3$ the space $\mathcal{S}_\epsilon$ is the cuspidal subspace of $\mathcal{M}_7(\Gamma_0(12), (\frac{\epsilon}{\cdot}))$. The condition that

$$\begin{array}{lll} \theta_\Gamma = & 1 + 0q^2 + 0q^4 + 0q^6 + 2 \cdot 21 \cdot 24q^8 + \dots & \text{and} \\ \theta_{\sqrt{12}\Lambda^*} = W_{12}(\theta_\Gamma) = & 1 + 0q^2 + 0q^4 + 0q^6 + 2 \cdot 21 \cdot 24q^8 + \dots & \end{array}$$

has a 1-parametric space of solutions if $\epsilon = -3$ and a 2-parametric space of solutions if $\epsilon = -1$. Using the package [3] to search for a solution in this space of which the first 40 coefficients are nonnegative, we find no such solution in all 176 cases. This shows that there is no suitable theta series and hence no such lattice Γ. □

Lemma 5.22 *If $s_1 = 6$ then $t_1 \neq 32$.*

Proof. Assume that $s_1 = 6$ and $t_1 = 32$ and rescale such that $\min(\Gamma) = 4$, $\Gamma = \Lambda^*$ with $\min(\Lambda) = \frac{4}{3}$. Then Γ is even and so is 12Λ. We proceed as in the proof of Lemma 4.3. From Lemma 2.1 we find that

$$166 \leq \det(\Gamma) = 2^a 3^b \leq 28756.$$

Moreover $1 \leq a \leq 27$ and $1 \leq b \leq 13$ and the Jordan decomposition of $\mathbb{Z}_2 \otimes \Gamma$ is $f_0 \perp 2f_1 \perp 4f_2$ such that f_0 and f_2 are even unimodular forms. In total this leaves 149 possibilities for the genus of Γ. We construct one lattice from each of the 149 genera. For each genus we calculate the theta series T of the chosen representative. Again the determinant of Γ is either a square or 3 times a square and $\theta_\Gamma = T + f$ for some $f \in \mathcal{S}_\epsilon$, where for $\epsilon = -1$ or $\epsilon = -3$ the space $\mathcal{S}_\epsilon$ is the cuspidal subspace of $\mathcal{M}_7(\Gamma_0(12), (\frac{\epsilon}{\cdot}))$. The condition that

$$\begin{array}{lll} \theta_\Gamma = & 1 + 0q^2 + 2 \cdot 21 \cdot 32q^4 + \dots & \text{and} \\ \theta_{\sqrt{12}\Lambda^*} = W_{12}(\theta_\Gamma) = & 1 + 0q^2 + 0q^4 + 0q^6 + 0q^8 + 0q^{10} + 0q^{12} + 0q^{14} + 2 \cdot 21 \cdot 6q^{16} + \dots & \end{array}$$

determines θ_Γ uniquely. One finds no solution in all cases where $\epsilon = -3$ and a unique solution if $\epsilon = -1$. This unique modular form has negative coefficients in all cases. This shows that there is no suitable theta series and hence no such lattice Γ. □

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