On the cokernel of the Witt decomposition map.

Gabriele Nebe *

Dedicated to Jacques Martinet.

1 Introduction

In [Dre 75], A. Dress defines Grothendieck-Witt groups $GW(R,G)$ for finite groups G and Dedekind domains R. If K is the field of fractions of R, then there is an exact sequence

$$(\star) \quad 0 \to GW(R,G) \to GW(K,G) \stackrel{\delta}{\to} \oplus_{\wp \trianglelefteq R} GW(R/\wp, G)$$

where $\wp$ runs through all maximal ideals of R. The map δ is called the Witt decomposition map. In the first section of this paper, the necessary terminology is introduced, to define these and more general Witt groups for orders with involution. In our terminology the groups $GW(R,G)$ are denoted by $W(RG, \ ^\circ)$, where $\ ^\circ$ is the R-linear involution on the group ring RG defined by $g^\circ = g^{-1}$ for all $g \in G$.

Dress asked to calculate the cokernel of δ. This paper is intended to answer this question in some cases. Section 4 shows that δ is surjective for all finite groups G, if K is a finite extension of the p-adic numbers. This can be used to show that in the case of number fields K, the composition $\delta_\wp$ of δ with the projection onto $W((R/\wp)G, \ ^\circ)$ is surjective for all prime ideals $\wp$ of R (Theorem 4.6). The example $R = \mathbb{Z}$, $G = C_4$ and $p = 5$ shows that this is not true for the classical decomposition map of Brauer.

J. Morales ([Mor 90]) investigates the sequence $(\star)$ for p-groups G, where p is an odd prime, and number fields K. He shows that in this situation the cokernel of δ is isomorphic to the exponent-2-subgroup of the ideal class group of K as in the classical case $G = 1$. This theorem can be easily generalized to nilpotent groups G of odd order (Theorem 5.2). Using an induction theorem [Dre 75, Theorem 2] (cf. Theorem 4.1), one immediately gets that δ is surjective for groups of odd order, if the class number of K is odd (see Theorem 5.3), which is shown in [Miy 90] for $K = \mathbb{Q}$. But in general δ is not surjective for $K = \mathbb{Q}$ as one sees by looking at dihedral groups of order $2p$ (see Section 5.2). The methods to investigate the sequence $(\star)$ in Section 5.1 heavily depend on Morita theory for hermitian forms. Therefore this theory is revisited in Section 3.

2 Hermitian and covariant forms

Throughout the paper let R be a Dedekind domain with field of fractions K. Let A be a K-algebra with K-linear involution $\ ^\circ$ and $\Lambda = \Lambda^\circ$ an R-order in A that is invariant

*Lehrstuhl B für Mathematik, RWTH Aachen, Templergraben 64, 52062 Aachen, Germany, gabi@math.rwth-aachen.de

under the involution $\circ$. Let V be a right A-module and $L \subseteq V$ be a Λ-lattice in V, i.e. a finitely generated R-module that spans V as a vector space over K such that $L\Lambda = L$.

Definition 2.1 *(h) An R-bilinear form $h : L \times L \to \Lambda$ is called* hermitian*, if $h(l_1, l_2) = h(l_2, l_1)^\circ$ and $h(l_1, l_2\lambda) = h(l_1, l_2)\lambda$ for all $l_1, l_2 \in L$, $\lambda \in \Lambda$.*

(c) An R-bilinear form $b : L \times L \to R$ is called covariant*, if $b(l_1, l_2) = b(l_2, l_1)$ and $b(l_1\lambda, l_2) = b(l_1, l_2\lambda^\circ)$ for all $l_1, l_2 \in L$, $\lambda \in \Lambda$.*

Let $M^* := \mathrm{Hom}_\Lambda(M, \Lambda)$. Then M^* is naturally a left Λ-module and becomes a right Λ-module by letting $(f \cdot \lambda)(m) := \lambda^\circ f(m)$ for all $m \in M, \lambda \in \Lambda$ and $f \in M^*$. The hermitian forms correspond bijectively to the symmetric Λ-homomorphisms $h \longrightarrow \tilde{h} \in \mathrm{Hom}_\Lambda(M, M^*)$ defined by $\tilde{h}(m)(m') := h(m, m')$. Similarly covariant forms correspond to symmetric elements of $\mathrm{Hom}_\Lambda(M, M^\#)$, where $M^\# := \mathrm{Hom}_R(M, R)$ is a right Λ-module by letting $(f \cdot \lambda)(m) := f(m\lambda^\circ)$, for all $m \in M, \lambda \in \Lambda$ and $f \in M^\#$.

In particular if there is a functorial isomorphism $M^\# \cong M^*$ for all Λ-lattices M, then the categories of hermitian and covariant forms are equivalent. One can show that this is true if $\Lambda \cong \mathrm{Hom}_R(\Lambda, R)$ as a bimodule (see [ARS 97, Proposition IV.3.8]). Since the isomorphism $M^\# \cong M^*$ is functorial, this is also a necessary condition. Here, the most important case is that $\Lambda = RG$ is a group ring of some finite group G. Then the concepts of hermitian and covariant forms are equivalent and are used simultaneously, according to which notion is more convenient to work with.

In [Dre 75] a sequence of equivariant Witt groups is investigated. To introduce this sequence naturally, one also needs hermitian and covariant Λ-torsion modules. If M is a Λ-torsion module, then define $M^* := \mathrm{Hom}_\Lambda(M, A/\Lambda)$ and $M^\# := \mathrm{Hom}_R(M, K/R)$. Also hermitian resp. covariant forms on a torsion module take values in A/Λ resp. K/R.

Definition 2.2 *Let M be either a Λ-lattice or a Λ-torsion module and let h resp. b be a hermitian resp. covariant form on M. Then h resp. b are called* regular*, if $\tilde{h} : M \to M^*$ resp. $\tilde{b} : M \to M^\#$ are isomorphisms. A regular hermitian or covariant module is called* metabolic*, if it contains a Λ-submodule U with $U = U^\perp$.*

The set of isometry-classes of regular hermitian resp. covariant forms a semi group with respect to orthogonal sums. Introducing the relations $[M, h] = 0$ for all metabolic hermitian resp. covariant modules, one obtains a group, called the equivariant Witt group of hermitian resp. covariant (torsion) modules, denoted by $Wh(\Lambda, \circ)$ resp. $W(\Lambda, \circ)$ ($WTh(\Lambda, \circ)$ resp. $WT(\Lambda, \circ)$ for torsion modules).

Let (V, h) be a hermitian A-module. For any Λ-lattice L in V, the hermitian dual lattice $L_h^* := \{v \in V \mid h(v, l) \in \Lambda \text{ for all } l \in L\}$ is a Λ-module isomorphic to L^*. Note that (L, h) is regular, if and only if $L_h^* = L$ (i.e. L is a unimodular R-lattice). The lattice (L, h) is called integral, if $L \subseteq L_h^*$. For any integral Λ-lattice (L, h), the hermitian form h induces a hermitian form $\overline{h}$ on the Λ-torsion module L_h^*/L by

$$\overline{h}(v + L, w + L) := h(v, w) + \Lambda \in A/\Lambda \text{ for all } v, w \in L_h^*.$$

Analogous notations are used for covariant A-modules (V, b). In particular $L_b^\# := \{v \in V \mid b(v, l) \in R \text{ for all } l \in L\}$ is the dual lattice of L with respect to b and the covariant form $\overline{b}$ induced on the Λ-torsion module $L_b^{\#}/L$ for any integral lattice L is $\overline{b}(v+L, w+L) := b(v,w) + R \in K/R$ for all $v, w \in L_b^{\#}$.

Lemma 2.3 *([Tho 84],[Dre 75], [Mor 88]) Let (V,h) resp. (V,b) be a hermitian resp. covariant A-module and L an integral Λ-lattice in V. Then $[L_h^*/L, \overline{h}] = [M_h^*/M, \overline{h}] \in WTh(\Lambda, \ ^\circ)$ resp. $[L_b^{\#}/L, \overline{b}] = [M_b^{\#}/M, \overline{b}] \in WT(\Lambda, \ ^\circ)$ for all integral Λ-lattice M in V.*

Since the mapping in the lemma maps metabolic modules to metabolic torsion modules, one obtains a well defined map

$$\delta : W(A, \ ^\circ) \to WT(\Lambda, \ ^\circ), [(V,b)] \mapsto [(L_b^{\#}/L, \overline{b})]$$

where L is any integral Λ-lattice in V.

Definition 2.4 *δ is called the* Witt decomposition map

Putting $\iota([L,b]) = [L \otimes K, b]$, for any regular Λ-lattice (L,b), it is clear that $\delta \circ \iota = 0$. One gets an exact sequence

$$(\star) \quad 0 \to W(\Lambda, \ ^\circ) \stackrel{\iota}{\to} W(A, \ ^\circ) \stackrel{\delta}{\to} WT(\Lambda, \ ^\circ)$$

(cf. [Dre 75, Theorem 5]). Working with hermitian modules, one obtains an analogous exact sequence $(\star_h)$.

3 Morita theory for hermitian modules

Let $\Lambda = \Lambda^\circ$ be an R-order in the separable K-algebra A with involution $\ ^\circ$. Let (L,h) be a regular hermitian Λ-lattice with endomorphism ring $O := \mathrm{End}_\Lambda(L)$. Then L is a left O-module and h induces an involution $\bar{\ }$ on O by

$$h(ov, w) = h(v, \bar{o}w) \text{ for all } v, w \in L, o \in O.$$

Lemma 3.1 *(cf. [Mor 90, Section 3], [Miy 90, Section 3], [Knu 91, Section I.9])*
Let (V,h) be a regular hermitian A-module and $L \subseteq V$ be a projective Λ-lattice such that (L,h) is regular. Let $D := End_A(V)$ and $O := End_\Lambda(L)$ and assume that $L^* \otimes_O L \cong \Lambda$ and $L \otimes_\Lambda L^* \cong O$ as bimodules. Then there are isomorphisms ϕ, ϕ', ϕ'' such that

$$\begin{array}{ccccccc} 0 & \to & Wh(\Lambda, \ ^\circ) & \to & Wh(A, \ ^\circ) & \to & WhT(\Lambda, \ ^\circ) \\ & & \phi \downarrow & & \phi' \downarrow & & \phi'' \downarrow \\ 0 & \to & Wh(O, \bar{\ }) & \to & Wh(D, \bar{\ }) & \to & WhT(O, \bar{\ }) \end{array}$$

commutes, where $\bar{\ }$ is the involution on D (and on O) induced by h.

Proof. Let ϕ, ϕ', ϕ'' be the mappings defined on [Mor 90, p. 214,215]: For any hermitian Λ-lattice (M,ψ) let $\phi(M,\psi) := (\phi(M), \phi(\psi))$ where $\phi(M) = \mathrm{Hom}_\Lambda(L,M)$ is an O-right module via $(f \cdot o)(l) := f(ol)$ for all $f \in \phi(M), l \in L, o \in O$. To define the hermitian form $\phi(\psi)$ let $(\phi(\psi))(f,g) \in \mathrm{End}_\Lambda(L) = O$ for $f,g \in \mathrm{Hom}_\Lambda(L,M)$ be the composition

$$L \xrightarrow{g} M \xrightarrow{\tilde{\psi}} M^* = \mathrm{Hom}_\Lambda(M,\Lambda) \xrightarrow{f^*} \mathrm{Hom}_\Lambda(L,\Lambda) = L^* \xrightarrow{\tilde{h}^{-1}} L$$

where $\tilde{\psi} : M \to M^*, m \mapsto \psi(m,\cdot)$ is the isomorphism induced by ψ. One easily checks that $\phi(\psi)$ is an O-hermitian form.

The inverse of ϕ is defined by $\phi^{-1}(N,\gamma) = (\phi^{-1}(N), \phi^{-1}(\gamma))$, where

$$\phi^{-1}(N) := N \otimes_O L \text{ and } \phi^{-1}(\gamma)(f_1 \otimes l_1, f_2 \otimes l_2) := h(l_1, \gamma(f_1,f_2)l_2).$$

If (M,ψ) is a hermitian Λ-lattice then by [CuR 81, Proposition (2.29)]

$$\phi^{-1}(\phi(M)) = \mathrm{Hom}_\Lambda(L,M) \otimes_O L \cong M \otimes_\Lambda L^* \otimes_O L \cong M,$$

because $L^* \otimes_O L \cong \Lambda$ by assumption. If $f_i \in \mathrm{Hom}_\Lambda(L,M)$, $l_i \in L$ $(i=1,2)$, then

$$\phi^{-1}(\phi(\psi))(f_1 \otimes l_1, f_2 \otimes l_2) = h(l_1, \phi(\psi)(f_1,f_2)l_2) = h(l_1, \tilde{h}^{-1}[(f_1^* \circ \tilde{\psi} \circ f_2)(l_2)]) =$$

$$(f_1^* \circ \tilde{\psi} \circ f_2)(l_2)(l_1))^\circ = \psi(f_2(l_2), f_1(l_1))^\circ = \psi(f_1(l_1), f_2(l_2))$$

Similarly one shows that $\phi \circ \phi^{-1} = id$.

The mapping ϕ' is defined analogously and also ϕ'' is similarly defined by: $\phi''(T,\beta) := (\phi''(T), \phi''(\beta))$, with $\phi''(T) := \mathrm{Hom}_\Lambda(L,T)$ and $(\phi''(\psi))(f,g) \in \mathrm{Hom}_\Lambda(L, V/L) \cong D/O$ for $f,g \in \mathrm{Hom}_\Lambda(L,T)$ is the composition

$$L \xrightarrow{g} T \xrightarrow{\tilde{\beta}} \hat{T} := \mathrm{Hom}_\Lambda(T, A/\Lambda) \xrightarrow{f^*} \mathrm{Hom}_\Lambda(L, A/\Lambda) = V^*/L^* \xrightarrow{\tilde{h}^{-1}} V/L.$$

One easily sees that these maps map metabolic modules to metabolic ones and that the diagram is commutative. □

The next (trivial) rule says that the sequence $(\star)$ (or $(\star_h)$) is a direkt sum, if the R-order Λ decomposes as a direct sum of two involution invariant orders:

Lemma 3.2 *Let $\epsilon = \epsilon^\circ \in \Lambda$ be a central idempotent. Then any (hermitian or covariant) Λ-module (L,h) decomposes as the orthogonal sum $(L\epsilon, h) \perp (L(1-\epsilon), h)$ yielding a direct sum decomposition*

$$W(\Lambda, \ ^\circ) \cong W(\Lambda\epsilon, \ ^\circ) \oplus W(\Lambda(1-\epsilon), \ ^\circ)$$

such that

$$\begin{array}{ccccccc} 0 & \to & W(\Lambda, \ ^\circ) & \to & W(A, \ ^\circ) & \to & WT(\Lambda, \ ^\circ) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & W(\Lambda\epsilon, \ ^\circ) & & W(A\epsilon, \ ^\circ) & & WT(\Lambda\epsilon, \ ^\circ) \\ 0 & \to & \oplus & \to & \oplus & \to & \oplus \\ & & W(\Lambda(1-\epsilon), \ ^\circ) & & W(A(1-\epsilon), \ ^\circ) & & WT(\Lambda(1-\epsilon), \ ^\circ) \end{array}$$

commutes.

Recall that a regular hermitian or covariant Λ-module M is called anisotropic, if the only Λ-submodule $U \leq M$ with $U \subseteq U^{\perp}$ is $U = \{0\}$. If $U \leq U^{\perp} \leq M$ is a submodule of M, then one easily sees that M is equivalent to $U^{\perp}/U$ (with the induced regular hermitian or covariant form) in the corresponding Witt group (see e.g. [Scha 85, Lemma 5.1.3]). Therefore each element of the Witt group has an anisotropic representative. Since the different primary components of hermitian or covariant Λ-torsion modules are orthogonal to each other and any $\wp$-primary component of an anisotropic Λ-torsion module is annihilated by the prime ideal $\wp \trianglelefteq R$, the anisotropic Λ-torsion modules are orthogonal sums of $R/\wp \otimes_R \Lambda$-modules with a hermitian or bilinear form that takes values in $\wp^{-1}\Lambda/\Lambda$ respectively $\wp^{-1}/R$. Identifying $\wp^{-1}/R$ with $R/\wp$ this reduces the study of the Witt group of torsion Λ-modules to the one of covariant or hermitian modules over Artinian algebras.

Remark 3.3 *There is a (non canonical) isomorphism*

$$WT(\Lambda, \,^\circ) \cong \oplus_\wp W(R/\wp \otimes_R \Lambda, \,^\circ)$$

$$WhT(\Lambda, \,^\circ) \cong \oplus_\wp Wh(R/\wp \otimes_R \Lambda, \,^\circ)$$

where $\wp$ runs through the maximal ideals of the Dedekind domain R.

For algebras (A or $R/\wp \otimes_R \Lambda$) over fields, the anisotropic modules are semisimple, because for every submodule U of an anisotropic module V one has $U \cap U^{\perp} = 0$ and hence $V = U \oplus U^{\perp}$. This shows the following lemma.

Lemma 3.4 *(see e.g. [Dre 75, Lemma 4.2]) Let $\wp \trianglelefteq R$ be a maximal ideal. Then any anisotropic $R/\wp \otimes_R \Lambda$-module is an orthogonal sum of simple regular hermitian or covariant $R/\wp \otimes_R \Lambda$-modules.*

4 The surjectivity of δ for local fields.

A. Dress proves in [Dre 75] an anologon to a theorem of Brauer on induced characters: Let G be a finite group and let

$$\mathcal{E} := \mathcal{E}(G) := \{U \leq G \mid U = U_1 \times U_2, \ U_1 \text{ cyclic}, U_2 \ p\text{-group } \}$$

and

$$\mathcal{H}_2 := \mathcal{H}_2 := \{U \leq G \mid U \text{ has a cyclic normal subgroup of 2-power index}\}.$$

Theorem 4.1 *([Dre 75, Theorem 2]) For any Dedekind domain R the induction yields a surjective mapping*

$$\oplus_{U \in \mathcal{E} \cup \mathcal{H}_2} W(RU, \,^\circ) \to W(RG, \,^\circ).$$

The theorem of Brauer can be used to show that for a finite extension K of $\mathbb{Q}_p$ with valuation ring R and residue class field $k := R/\wp$, where $\wp = \pi R$ is the maximal ideal of R, the decomposition map from the ring of generalized characters of G over K to that over k is surjective (see [Ser 77, Chapter 17]).

The same method, using Theorem 4.1 also shows that the sequence

$$(\star) \quad 0 \to W(RG, \circ) \to W(KG, \circ) \xrightarrow{\delta_\pi} W(kG, \circ) \to 0$$

is exact. Most of the exactness is already shown in [Dre 75, Theorem 5]. The only missing ingredient is the surjectivity of the composition $\delta_\pi : W(KG, \circ) \to W(kG, \circ)$ of δ with the isomorphism $WT(RG, \circ) \cong W(kG, \circ)$ (Remark 3.3) given by multiplication with π, i.e. $\delta_\pi[(V,B)] = [(L_B^\#/L, \tilde{B})]$ for any maximal integral Λ-lattice L in V, where

$$\tilde{B} : L_B^\#/L \times L_B^\#/L \to R/\wp; \tilde{B}(v+L, w+L) := \pi B(v,w) + \pi R \in R/\pi R = k.$$

If the group order is invertible in R, then this surjectivity follows by the general Morita theory in the last section. But it is easy to establish a slightly stronger result:

Lemma 4.2 *Assume that $|G|$ is invertible in R. Let M be a simple kG-module and $(b_1, \ldots, b_n)$ a k-basis of the space of covariant forms on M. Then there is a simple KG-module V, a lattice $L \subset V$, and an R-basis $(B_1, \ldots, B_n)$ of the space of integral covariant forms on L, such that $(\pi^{-1}L/L, \widetilde{\pi B_i}) \cong (M, b_i)$ for $i = 1, \ldots, n$.*

Proof. [Ser 77, 15.5] shows that there is a KG-module V such that $M \cong L/\pi L$ for any RG-lattice L in V. Let B_i' be any symmetric bilinear form on L such that $B_i' \equiv b_i \pmod{\pi}$ and define $B_i := \frac{1}{|G|}\sum_{g\in G} gB_i'g^{tr}$ $(i = 1, \ldots, n)$. Since $gB_i'g^{tr} \equiv b_i \pmod{\pi}$ for all $g \in G$, the forms B_i are G-invariant forms lifting b_i $(i = 1, \ldots, n)$. They form an R-basis of the lattice of all integral covariant forms on L since their reductions modulo π form such a k-basis for $L/\pi L$. Moreover $L_{B_i}^\# = L$ for all the forms B_i. Therefore $L_{\pi B_i}^\# = \pi^{-1}L$ and $(L_{\pi B_i}^\#/L, \tilde{\pi B_i}) \cong (M, b_i)$ for $i = 1, \ldots, n$. □

For elementary subgroups the surjectivity of δ in $(\star)$ can be seen by number theoretical arguments:

Theorem 4.3 *([Neb 99, Satz 4.3.6]) $G := C : P$ be the semidirect product of a cyclic group C of order not divisible by the prime l and an l-group P. Then δ_π is surjective.*

More precisely, for every simple regular kG-module (M, b) there is a regular RG-lattice (L, B) such that $(\pi^{-1}L/L, \tilde{\pi B}) \cong (M, b)$.

Proof. The first part of the proof follows closely the one of [Ser 77, Theorem 41]. Let $p := \mathrm{char}(k)$.

By Remark 3.4 it suffices to show that all simple kG-modules M that have a regular G-invariant symmetric bilinear form b are in the image of δ. So let (M, b) be such a simple orthogonal kG-module.

- Assume first that $l \neq p$. Then the Sylow-p-subgroup S of G is normal in G and therefore acts trivially on M, so M can be viewed as a kG/S-module. Since $l \not| |G/S|$ the theorem follows from Lemma 4.2.

• We now assume that $l = p$. By induction we assume that M is a faithful kG-module. Since the centralizer of C in P is a normal p-subgroup of G and hence acts trivially on M, we assume that $C_P(C) = 1$ so P acts faithfully on C. Now $\mathrm{char}(k) \not| |C|$ implies that M is a semisimple kC-module. Let $M = \bigoplus_\alpha M_\alpha$ be a decomposition of M into kC-isotypic components. Since M is an irreducible kG-module, G permutes the M_α transitively. Let $G_\alpha = C : P_\alpha$ be the stabilizer of M_α. Then $M = \mathrm{Ind}_{G_\alpha}^G(M_\alpha)$ and M_α is an irreducible G_α-module. Since M_α is a homogeneous kC-module, the image of the representation $kC \to \mathrm{End}(M_\alpha)$ is a field $\tilde{k} \cong k[\zeta]$, where ζ is a primitive $|C|$-th root of unity. Since $\mathrm{char}(k) = p$ and P_α is a p-group, there is $0 \neq v \in M_\alpha$ such that $vg = v$ for all $g \in P_\alpha$. Then $\tilde{k}v \leq M_\alpha$ is a G_α-invariant subspace of M_α, because C is normal in G_α, and therefore $M_\alpha = \tilde{k}v$. Identifying v with $1 \in \tilde{k}$, we identify M_α with $\tilde{k}$. Then P_α acts as Galois automorphisms on M_α. Let $\tilde{K} = K[\zeta]$ be the unramified extension of K with residue class field $\tilde{k} \cong \tilde{R}/\tilde{\wp}$ where $\tilde{R} = R[\zeta]$ is the ring of integers in $\tilde{K}$ and $\tilde{\wp} = \tilde{R}\wp$ the maximal ideal of $\tilde{R}$. The homomorphism $C \to \tilde{k}^*$ lifts uniquely to a homomorphism $C \to \tilde{R}^*$, which makes $\tilde{R}$ into a RC-module. Since the Galois groups $\mathrm{Gal}(\tilde{K}/K)$ and $\mathrm{Gal}(\tilde{k}/k)$ are canonically isomorphic, the group P_α acts naturally on $\tilde{R}$ as Galois automorphisms. This make $\tilde{R}$ into an RG_α-lattice, with $\tilde{R}/\tilde{\wp} \cong M_\alpha$.

We now consider the invariant form b on M. To this purpose let $M_\alpha^\# = \mathrm{Hom}_k(M_\alpha, k)$ be the dual kC-module. Distinguish two cases:

a) $M_\alpha \cong M_\alpha^\#$ as kC-modules.

b) $M_\alpha \not\cong M_\alpha^\#$ as kC-modules.

If one also identifies $M_\alpha^\#$ with $\tilde{k}$, then $^- : \tilde{k} \to \tilde{k}$, $\zeta \mapsto \zeta^{-1}$ is a k-linear Galois automorphism of $\tilde{k}$ in case a) but not in case b).

In case a) the module M_α has a kG_α-invariant regular symmetric bilinear form $b' : M_\alpha \times M_\alpha \to k$, $b'(x,y) := \mathrm{trace}_{\tilde{k}/k}(x\overline{y})$. Since the different isotypic components are orthogonal in this case, the module (M, b) is induced from $(M_\alpha, b_{|M_\alpha})$. By induction on $|G|$ we may assume that $M = M_\alpha$ and $G = G_\alpha$. Let $\tilde{k}^+ := \mathrm{Fix}(^-)$ be the fixed field of $^-$ in $\tilde{k}$. Then the symmetric C-invariant bilinear forms are the forms $b_z : (x, y) \mapsto \mathrm{trace}_{\tilde{k}/k}(xz\overline{y})$ with $z \in \tilde{k}^+$. Clearly b_z is P-invariant, if and only if $z \in k^+$ lies in the fixed field k^+ of P in $\tilde{k}^+$. In particular the form $b = b_{z'}$ for some $z' \in k^+$.

Since $^-$ is a Galois automorphism of $\tilde{k}$ fixing k, the map $^- : \tilde{K} \to \tilde{K}$, $\zeta \mapsto \zeta^{-1}$ defines a Galois automorphsim of $\tilde{K}/K$. Let $K^+ \leq \tilde{K}$ be the fixed field of $\langle P, ^- \rangle \leq \mathrm{Gal}(\tilde{K}/K)$ with ring of integers R^+ and maximal ideal $\wp R^+ =: \wp^+$. Then the G-invariant symmetric bilinear forms on $\tilde{K}$ are the forms $B_z : (x, y) \mapsto \mathrm{trace}_{\tilde{K}/K}(xz\overline{y})$ with $z \in K^+$. Let $Z' \in R^+$ be a preimage of $z' \in R^+/\wp^+ = k^+$. Then $Z' \in (R^+)^*$ is a unit and $(\tilde{R}, B_{Z'})$ is a regular covariant RG-module with $(\pi^{-1}\tilde{R}/\tilde{R}, \tilde{B}_{Z'}) \cong (M, b)$.

Now consider the case b), that $M_\alpha \not\cong M_\alpha^\#$. Since M is self dual, the module $M_\alpha^\#$ is isomorphic to some other isotypic component $M_{\alpha'}$ of M. As above we may assume by induction that $M = M_\alpha + M_{\alpha'}$. Then $M \cong \tilde{k} + \tilde{k}$ where the action of a generator g of C is $(x + y)g := x\zeta + y\zeta^{-1}$. Now the C-invariant symmetric bilinear forms on M are of the form $b_z : (x_1 + y_1, x_2 + y_2) \mapsto \mathrm{trace}(x_1 z y_2 + y_1 z x_2)$ with $z \in \tilde{k}$. One also checks that b_z is G-invariant, if and only if $z \in k^+ := \mathrm{Fix}(P_\alpha)$. Similarly as in the case a), these invariant forms can be lifted to invariant forms on $\tilde{R} + \tilde{R}$ and one finds a preimage of (M, b). □

With Theorem 4.1 this allows to conclude the surjectivity of δ for arbitrary groups G.

Corollary 4.4 *Let R be the valuation ring in a finite extension K of $\mathbb{Q}_p$ with maximal ideal πR and residue class field $k = R/\pi R$. Let G be a finite group. Then there is an exact sequence*

$$(\star) \quad 0 \to W(RG, \,^\circ) \to W(KG, \,^\circ) \to W(kG, \,^\circ) \to 0$$

Proof. The proof is based on the following commutative diagram:

$$\begin{array}{ccccccc} 0\to & \oplus_{U\in\mathcal{E}\cup\mathcal{H}_2} W(RU, \,^\circ) & \to & \oplus_{U\in\mathcal{E}\cup\mathcal{H}_2} W(KU, \,^\circ) & \to & \oplus_{U\in\mathcal{E}\cup\mathcal{H}_2} W(kU, \,^\circ) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & W(RG, \,^\circ) & \to & W(KG, \,^\circ) & \to & W(kG, \,^\circ) & \to 0 \end{array}$$

The vertical arrows are surjective by Theorem 4.1, so it is enough to show the claim for the elementary subgroups $U \in \mathcal{E} \cup \mathcal{H}_2$. In particular it suffices to prove the corollary for such groups G that contain a cyclic normal subgroup of l-power index for some prime l. But such a group is isomorphic to $C : P$ for an l-group P. Therefore the corollary follows from Theorem 4.3. □

Here it is essential that Λ is a group ring. For arbitrary symmetric orders one easily constructs counterexamples to the surjectivity of δ_π:

Example 4.5 *Let* $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $a := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $b := \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, $c := \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \in R^{2\times 2}$, *where $R := \mathbb{Z}_2$. Let Λ be the sublattice of $R^{2\times 2} \oplus R^{2\times 2}$ with R-basis*

$$I + I, c + c, 2I + 0, 0 + 2c, a + 0, 0 + a, b + 0, 0 + b.$$

Then Λ is symmetric with respect to $\frac{1}{4}(Tr_1 + Tr_2)$ where Tr_i is the reduced trace of the i-th component $R^{2\times 2}$. Taking the transpose in each component defines an involution $^\circ$ on Λ. If (V, B) is a simple regular covariant $\mathbb{Q}_2 \otimes_R \Lambda$-module, then $\delta_2(V, B) = 0$ or $\delta_2(V, B) = (S, 1) \perp (S, 1)$, where S is the simple Λ-module. Therefore δ_2 is not surjective.

The surjectivity of δ for p-adic fields has the important consequence, that for number fields K, the composition of δ with the projection on one component $W(KG, \,^\circ) \to W(R/\wp G, \,^\circ)$ is surjective. This is in general not true for the classical decomposition map: Let $G \cong C_4$. Then G has only 3 irreducible representations over $\mathbb{Q}$, but 4 irreducible representations over $\mathbb{F}_5$. Therefore the 5-modular decomposition map over the rationals is not surjective.

Theorem 4.6 *Let G be an arbitrary group and R the ring of integers in a number field K. Let $\wp \trianglelefteq R$ be a prime ideal of R. Then the composition $\pi_\wp \circ \delta =: \delta_\wp : W(KG, \,^\circ) \to W((R/\wp)G, \,^\circ)$ is surjective.*

Proof. As above it suffices to prove the theorem for the groups $G \cong C : P$ as in Theorem 4.3. Let $k := R/\wp$ and G be such a group and (M, b) a simple regular kG-module. Let $R_\wp$ be the completion of R at $\wp$ with maximal ideal $\pi R_\wp = R_\wp \otimes_R \wp$. Then Theorem 4.3 yields a regular $R_\wp G$-lattice $(L'_\wp, B'_\wp)$, such that $(M, b) \cong ((L'_\wp)^{\#}_{\pi B'_\wp}/L'_\wp, \widetilde{\pi B'_\wp})$. Let V be the irreducible KG-module such that $W_\wp := K_\wp \otimes_{R_\wp} L'_\wp$ is a constituent of $V_\wp = K_\wp \otimes_K V$. Then there is a regular G-invariant form $B_\wp : V_\wp \times V_\wp \to K_\wp$ and an $R_\wp G$-lattice $L_\wp \subseteq V_\wp$ such that $(M, b) \cong ((L_\wp)^{\#}_{B_\wp}/L_\wp, \tilde{B}_\wp)$. Let $L' := L_\wp \cap V$. Then L' is a lattice for the localization $R_{(\wp)}$ of R at $\wp$ (cf. [Rei 75, Theorem (5.2)(ii)]). Let L be any RG-lattice in V such that $R_{(\wp)} \otimes_R L = L'$. Then $R_\wp \otimes_R L = L_\wp$. Let $|G| = p^a q$ with $p \not| q$ and $r \in \mathbb{Z}$ with $rq \equiv 1 \pmod p$. Choose any symmetric bilinear form $B' : L \times L \to R$, such that $B' \equiv B_\wp \pmod{p^a}$ and let $B(v, w) := rp^{-a} \sum_{g \in G} B'(vg, wg)$ for all $v, w \in V$. Then B is G-invariant and integral on L and $B \equiv B_\wp \pmod \wp$. Using the same $\tilde{}$-construction for R with any element $\pi_0 \in R$ with $\pi_0 \equiv \pi \pmod{\wp^2}$, one finds $(k \otimes_R (L^{\#}_B/L), \tilde{B}) \cong (M, b)$. □

5 The cokernel of δ for number fields.

In this section let K be a number field and R its ring of integers. If G is a finite group then one has a forgetful map: $W(RG, \circ) \to W(R)$, mapping an orthogonal RG-lattice (L, b) onto the underlying R-lattice (L, b). Let $W_0(RG, \circ)$ be the kernel of this map. Analogously one defines $W_0(KG, \circ)$ and $WT_0(RG, \circ)$. Then one has an exact diagramm

$$(**)\quad \begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W_0(RG, \circ) & \to & W_0(KG, \circ) & \stackrel{\delta_0}{\to} & WT_0(RG, \circ) & \to & C_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W(RG, \circ) & \to & W(KG, \circ) & \stackrel{\delta}{\to} & WT(RG, \circ) & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W(R) & \to & W(K) & \to & WT(R) & \to & C(K)/C(K)^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where C_0 and C are the respective cokernels, $W(R)$, $W(K)$, and $WT(R)$ are the classical Witt groups of regular symmetric bilinear forms. The exactness of the last row is shown in [Scha 85, Theorem 6.6.11] (cf. also [MiH 73, Example IV.3.4]).

Remark 5.1 *For all finite groups G and number fields K, the cokernel of δ has an epimorphic image $C(K)/C(K)^2$.*

5.1 Groups of odd order.

This section mainly intends to give a survey on known results, though most of them are stated more generally as the ones in the literature.

Theorem 5.2 *(cf. [Mor 90, Corollary 3.10] for p-groups G) Let G be a nilpotent group of odd order and let K be a number field with ring of integers R. Then the cokernel of δ is isomorphic to the exponent-2-factor group $C(K)/C(K)^2$ of the class group $C(K)$ of K.*

Proof. Let $G = P_1 \times \ldots \times P_m$ where P_i is the largest normal p_i-subgroup of G and $p_1, \ldots, p_m$ are distinct primes. We proceed by induction on m to show that the restriction δ_0 of δ to $W_0(KG, \circ)$ in diagram $(**)$ is surjective.

If K is not totally real, then $W_0(KG, \circ) = 0$ (cf. [Mor 90, Proposition 3.3]) and we are done. So assume that K is a totally real number field.

If $m = 0$ then $G = 1$ and the statement is trivial. For $m = 1$, the theorem is [Mor 90, Theorem 3.9].

Assume that $m > 0$. Let $S := \{\wp \trianglelefteq R \mid p_1 \ldots p_m \in \wp, \wp \text{ prime ideal }\}$. First we show that $\oplus_{\wp \in S} W_0((R/\wp)G, \circ)$ is in the image of δ_0. To this purpose let $\wp \in S$. Then $p_i \in \wp$ for some $1 \leq i \leq m$. Since P_i is normal in G, it acts trivially on all simple $(R/\wp)G$-modules. So the simple orthogonal $R/\wp G$-modules are modules for G/P_i. By the induction hypotheses these modules are in the image of δ_0. Hence the cokernel of δ_0 is an epimorphic image of the factor group

$$WT_0(RG, \circ)/\bigoplus_{\wp \in S} W_0((R/\wp)G, \circ) \cong \bigoplus_{\wp \notin S} W_0((R/\wp)G, \circ) = WT_0(R[\frac{1}{|G|}]G, \circ).$$

Therefore the cokernel of δ_0 is isomorphic to the corresponding cokernel C_S in the localized situation defined by the exact sequence

$$0 \to W_0(R[\frac{1}{|G|}]G, \circ) \to W_0(KG, \circ) \to WT_0(R[\frac{1}{|G|}]G, \circ) \to C_S \to 0.$$

The $R[\frac{1}{|G|}]$-order $R[\frac{1}{|G|}]G \cong R[\frac{1}{|G|}] \oplus \bigoplus_{i=1}^m \Lambda_i$ is a maximal order in KG where $\Lambda_1, \ldots, \Lambda_m$ are maximal orders in the simple constituent of KG, that do not correspond to the trivial representation of KG. Moreover $\Lambda_i^\circ = \Lambda_i$ since K is totally real. By Lemma 3.2 $W_0(R[\frac{1}{|G|}]G, \circ) \cong \oplus_{i=1}^m W(\Lambda_i, \circ)$. As in the proof of [Mor 90, Theorem 3.4] one finds for every simple self dual KG-module V an G-covariant form on V and an $R[\frac{1}{|G|}]G$-lattice that is self dual with respect to this form. The endomorphism ring of this lattice is the maximal order in the totally complex CM-field $\mathrm{End}_{KG}(V)$. Applying the Morita theory of Section 3, one proves $C_S = 0$ as in [Mor 90, Theorem 3.9]. □

To deduce the surjectivity of δ_0 for arbitrary groups of odd order, one has to show the surjectivity of the induction map (Theorem 4.1) also for W_0 and WT_0, which I could not establish. So we have to restrict to the case that the class number of K is odd to show:

Theorem 5.3 *(cf. [Miy 90, Theorem C] for $K = \mathbb{Q}$) Let G be a group of odd order and assume that $|C(K)|$ is odd. Then δ is surjective.*

Proof. This follows immediately from the surjectivity of δ for the elementary subgroups of G shown in Theorem 5.2 and Theorem 4.1. □

5.2 A counterexample to the surjectivity of δ for $K = \mathbb{Q}$: Dihedral groups

Proposition 5.4 *Let $p > 2$ be a prime, $G := \langle x, y \mid x^p = y^2 = (xy)^2 = 1\rangle \cong D_{2p}$ the dihedral group of order $2p$ and $K := \mathbb{Q}[\zeta_p + \zeta_p^{-1}]$ the maximal real subfield of the p-th cyclotomic field. Then*

$$(\star) \quad 0 \to W(\mathbb{Z}G,\ ^\circ) \to W(\mathbb{Q}G,\ ^\circ) \xrightarrow{\delta} WT(\mathbb{Z}G,\ ^\circ) \to C(K)/C(K)^2 \to 0$$

is exact.

Proof. It remains to show that $\operatorname{coker}(\delta) \cong C(K)/C(K)^2$.

Using the argumentation in [Scha 85, p 176/177] that shows the surjectivity of the Witt-decomposition map for $G = 1$ and $K = \mathbb{Q}$, one sees that $GW(\mathbb{F}_p, G) \subset \bigoplus_{r\in\mathbb{P}} GW(\mathbb{F}_r, G)$ is in the image of δ (this follows also from the surjectivity of δ for C_2 [Mor 90, Theorem 2.3]). Let $S := \mathbb{Z}[\frac{1}{p}]$. Then $(\star)$ is exact if and only if

$$(\star)_p \quad 0 \to W(SG,\ ^\circ) \to W(\mathbb{Q}G,\ ^\circ) \xrightarrow{\delta} \bigoplus_{p\neq r\in\mathbb{P}} W(\mathbb{F}_r G,\ ^\circ) \to C(K)/C(K)^2 \to 0$$

is exact. The group ring SG is isomorphic to $SC_2 \oplus R[\frac{1}{p}]^{2\times 2}$, where $R := \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ is the ring of integers in K. Hence by Lemma 3.2, the seqeunce $(\star)_p$ is a direct sum of two sequences. One easily sees that the sequence

$$0 \to W(SC_2,\ ^\circ) \to W(\mathbb{Q}C_2,\ ^\circ) \to \bigoplus_{p\neq r\in\mathbb{P}} GW(\mathbb{F}_r C_2,\ ^\circ) \to 0$$

is exact. So we only have to deal with the other direct summand of SG, which is Morita equivalent to $R[\frac{1}{p}]$. To apply Lemma 3.1 one has to construct a unimodular hermitian SG-lattice in the irreducible $\mathbb{Q}G$-module V of dimension $p-1$. But V can be identified with $\mathbb{Q}[\zeta_p]$, where x acts as multiplication by the primitive p-th root of unity ζ_p and y as the Galois automorphism $\zeta_p \mapsto \zeta_p^{-1}$. Then $h: V\times V \to G$ defined by $h(\zeta_p^i, \zeta_p^j) := x^{i-j} \in G$ is an G-hermitian form on V. Let $L := S[\zeta_p]$. Then (L, h) is a regular SG-lattice. By Lemma 3.1 one can replace $R[\frac{1}{p}]^{2\times 2}$ by $R[\frac{1}{p}] = \operatorname{End}_{\mathbb{Z}[\frac{1}{p}]G}(L)$ if one considers Witt groups of hermitian forms. But the involuton on $R[\frac{1}{p}]$ is trivial, so by a classical result (see e.g. [Scha 85, Theorem 6.6.11], [MiH 73, Example IV.3.4]), the following sequence is exact:

$$0 \to Wh(R[\frac{1}{p}], ^-) \to Wh(K, ^-) \to \bigoplus_{\gamma} WTh(R[\frac{1}{p}]/\gamma, ^-) \to C(R[\frac{1}{p}])/C(R[\frac{1}{p}])^2 \to 0,$$

where γ runs through the prime ideals of the Dedekind domain $R[\frac{1}{p}]$. The prime ideal of R over p is generated by $(\zeta_p - \zeta_p^{-1})^2$ and hence principal. Therefore the class group $C(K)$ of fractional R-ideals in K is isomorphic to the class group $C(R[\frac{1}{p}])$ of fractional $R[\frac{1}{p}]$-ideals and the cokernel of δ is $C(R[\frac{1}{p}])/C(R[\frac{1}{p}])^2 \cong C(K)/C(K)^2$. $\square$

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