THE IRREDUCIBLE BRAUER CHARACTERS OF THE AUTOMORPHISM GROUP OF THE CHEVALLEY GROUP $F_4(2)$

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Dedicated to the memory of our friend Richard Parker

ABSTRACT. Using computational methods, we determine the irreducible Brauer characters of the automorphism group of the Chevalley group $F_4(2)$, up to one parameter and one consistency issue.

1. INTRODUCTION AND RESULTS

The purpose of this work, envisaged as a series of two articles, is to compute the decomposition matrices of the groups of the form $2.F_4(2).2$, i.e. of the covering groups of the automorphism group $F_4(2).2$ of the simple Chevalley group $F_4(2)$. In this first part we concentrate on the group $F_4(2).2$. In the second part we will consider the covering groups of $F_4(2).2$ and settle the questions left over here. Since the decomposition matrices of the covering group $2.F_4(2)$ of $F_4(2)$ are known, our objective falls into the following general problem.

1.1. **The problem.** Let $\tilde{H}$ be a finite group containing the normal subgroup $H \trianglelefteq \tilde{H}$ of index 2. Choose $x \in \tilde{H} \setminus H$, and let σ denote the automorphism of H induced by conjugation with x. Further, let ε denote the irreducible character of $\tilde{H}$ with kernel H.

Let p be an odd prime and let $\tilde{B}$ be a p-block of $\tilde{H}$. The task is to compute the decomposition matrix of $\tilde{B}$, provided the decomposition matrices of H are known. For this purpose, let B be a p-block of H covered by $\tilde{B}$. Then there are three situations, for which we summarize some well known results.

Date: December 13, 2024.

2010 *Mathematics Subject Classification.* 20C20, 20C33, 20C40.

Key words and phrases. Modular characters, condensation, Chevalley group $F_4(2)$.

This is a contribution to Project-ID 286237555 – TRR 195 – by the German Research Foundation.

1.1.1. *The block B is not σ-invariant.* Then B and $\tilde{B}$ are Morita equivalent; see [11, Theorem 6.8.3]. In particular, the decomposition matrices of B and $\tilde{B}$ can be identified via restrictions of characters.

1.1.2. *The block B is σ-invariant, and there is a block $\tilde{B}' \neq \tilde{B}$ covering B.* Then multiplication by ε provides a Morita equivalence between $\tilde{B}$ and $\tilde{B}'$. Moreover, restriction of characters induces a Morita equivalence between $\tilde{B}$ and B. The first assertion is clear, for the second use [4, Théorème 2.4].

1.1.3. *The block B is σ-invariant, and $\tilde{B}$ is the unique block of $\tilde{H}$ covering B.* By assumption, we know the PIMs of B. Let Φ be one of these. If $\Phi \neq \Phi^\sigma$, then $\mathrm{Ind}_H^{\tilde{H}}(\Phi)$ is indecomposable. Suppose then that $\Phi = \Phi^\sigma$. Then

$$\mathrm{Ind}_H^{\tilde{H}}(\Phi) = \tilde{\Phi}^+ + \tilde{\Phi}^-,$$

with $\tilde{\Phi}^- = \varepsilon \otimes \tilde{\Phi}^+ \neq \tilde{\Phi}^+$. Let χ, ψ be irreducible constituents of Φ occurring with multiplicity a and b, respectively. Assume that $\chi^\sigma \neq \chi$ and that $\psi^\sigma = \psi$. Then $\mathrm{Ind}_H^{\tilde{H}}(\chi) =: \tilde{\chi}$ is irreducible and $\mathrm{Ind}_H^{\tilde{H}}(\psi) = \tilde{\psi}^+ + \tilde{\psi}^-$ with distinct irreducible constituents $\tilde{\psi}^- = \varepsilon \otimes \tilde{\psi}^+ \neq \tilde{\psi}^+$. These constituents occur in $\tilde{\Phi}^+$ and $\tilde{\Phi}^-$ according to the following scheme.

	$\tilde{\Phi}^+$	$\tilde{\Phi}^-$
$\tilde{\chi}$	a	a
$\tilde{\psi}^+$	c	$\tilde{c}$
$\tilde{\psi}^-$	$\tilde{c}$	c

where $c + \tilde{c} = b$. The assertions outlined above follow from Clifford theory; for details see [5, Subsection 2.2].

1.2. **The results.** In the present first part of our paper, we investigate the above problem for the group $\tilde{H} = \mathrm{Aut}(F_4(2))$. Let us write $G = F_4(2)$ for the simple Chevalley group of type F_4 over the field with two elements. Then G is a normal subgroup of $\mathrm{Aut}(G)$ of index 2, if we identify $\mathrm{Inn}(G)$ with G. Adopting Atlas notation, we write $\mathrm{Aut}(F_4(2)) = G.2$. The decomposition matrices for G are known; see [9, 3].

With this notation, we prove the following results, where the tables we refer to are collected in the appendix.

Theorem 1.1. *The decomposition matrix of the principal* 3*-block* B_1 *of* $G.2$ *can be determined from* Tables 2 *and* 1, *where the parameters in Table* 2 *are* $a = 3$ *and* $\tilde{a} = 1$.

Some comments on the statement of Theorem 1.1 are appropriate. Firstly, the body X of Table 2 gives the decomposition numbers for the characters of a special basic set BS_1 of B_1. Secondly, the body Y of Table 1 contains the expansions of the other irreducible characters of B_1 in terms of the basic set characters, on restriction to the 3-regular classes. The decomposition matrix of the irreducible characters of B_1 not in the basic set equals $Y^t X$.

Theorem 1.2. *The decomposition matrices of the* 3*-blocks* B_2 *and* B_9 *of* $G.2$ *are as given in* Tables 4 *and* 5*, with the unknown parameters* $b, \tilde{b} \in \{0, 1\}$ *and* $b + \tilde{b} = 1$.

Theorem 1.3. *There are two distinct blocks of* $G.2$ *covering the principal* 5*-block of* G*. The decomposition matrices of these two blocks thus agree with the decomposition matrix of the principal* 5*-block of* G*; see* 1.1.2*.*

Theorem 1.4. *The decomposition matrix of the principal* 7*-block of* $G.2$ *is given in* Table 6*, where the parameters take the values* $a = \tilde{d} \in \{0, 1\}$ *and* $b = c = 1$*. For* $x \in \{a, b, c, d\}$*, we have* $\tilde{x} = 1 - x$*.*

The proofs of Theorems 1.1–1.4 are given in Sections 3–5 below. Our investigations are based on ideas and concepts devised by Richard Parker. The results of Sections 3, 4 have been obtained with the GAP-package MOC [12]. This is built upon the original MOC system developed by Richard Parker, Klaus Lux and the first author in the 1980s, see [10]. Section 5 describes our application of the condensation techniques, which go back to Richard Parker and Jon Thackray [16], and the MeatAxe64 due to Richard Parker [14, 15].

Remark 1.5. (a) As $[G.2 : G] = 2$, the irreducible 2-modular characters of $G.2$ agree with those of G.

(b) The parameter b in Theorem 1.2 remains undetermined for the time being. The irreducible Brauer characters $5262894^{\pm}$ in the two cases differ by their values at class 4P, which equal 0, if $b = 1$, and $+/-28$, if $b = 0$. By condensing a suitable tensor product or an induced representation, and computing the trace at a 4P-element, one can determine b. We postpone this computation to the second paper of our series.

(c) The two cases $a = 1$, respectively $a = 0$ in Theorem 1.4 result in two pairs of PIMs for $1377^{\pm}$, which only differ at the classes 32A, 32B (in Atlas notation). By swapping these classes in the character table of $G.2$ and leaving the order of the irreducible characters fixed, the two cases are transformed into one another.

In fact there is an automorphism (32A, 32B) of the character table of $G.2$, which swaps the two ordinary characters $\chi_{44}^{\pm} = 947700^{\pm}$, and fixes all other irreducible characters. With regard to the p-modular characters for $p = 7$, we may thus assume that $a = 1$. This specification may lead to consistency issues, whenever the classes 32A, 32B are involved in a similar way in Brauer character tables of $G.2$ for other primes. This is indeed the case for $p = 17$; see [17, Theorem 2.5]. In order to solve this problem, one has to fix a representative y for class 32A, say, in terms of a specific word in a standard generating system for $G.2$. Then, the trace on y of the irreducible Brauer characters 5951582^{+} will decide the question. Again, this computation is deferred to our next paper.

(d) The blocks considered in Theorems 1.1–1.4 account for all blocks of $G.2$ with non-cyclic defect groups. The decomposition matrices of $G.2$ for blocks with cyclic defect groups have been computed by Donald White in [17].

As a preparation for this work, Thomas Breuer computed the character tables of all maximal subgroups of $G.2$, using Magma [1] and the permutation representation of $G.2$ of degree 139776, provided by the Atlas of Finite Group Representations [18], now part of the AtlasRep package of GAP [19].

2. Notation and preliminaries

Recall the notation introduced at the beginning of Subsection 1.2. Thus G denotes the simple Chevalley group $F_4(2)$. The ordinary character table of G is contained in the Atlas [6, Pages 167–169]. This table is also available in GAP [7], and serves as a basis for our computations.

We use the numbering of the irreducible characters of G given in the Atlas; this agrees with the numbering in GAP. Thus $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_{95}\}$. As usual, an irreducible character of G is sometimes denoted by its degree.

The group G is contained, as an index 2 subgroup, in the automorphism group $\mathrm{Aut}(F_4(2))$ of $F_4(2)$, and we write $G.2 = \mathrm{Aut}(F_4(2))$. Let $x \in G.2$ denote the exceptional automorphism of $F_4(2)$ of order 2.

2.1. **Labelling the blocks.** We label the blocks of G and $G.2$ as in GAP; this labelling is also used in the database of decomposition matrices of the Modular Atlas home page [2]. We write B_i for the p-block of G which is the block with number i in GAP.

2.2. **Labelling the ordinary irreducible characters of** $G.2$**.** Let σ denote the automorphism of G induced by conjugation with x, and let ε be the unique irreducible character of $G.2$ with kernel G.

Let $\chi := \chi_i \in \mathrm{Irr}(G)$. If $\chi^\sigma \neq \chi$, then $\mathrm{Ind}_G^{G.2}(\chi) \in \mathrm{Irr}(G.2)$, and we write

$$\chi_i^0 := \mathrm{Ind}_G^{G.2}(\chi),$$

so that $\chi_i^0 = \chi_j^0$ if $\chi_i^\sigma = \chi_j$.

Otherwise $\mathrm{Ind}_G^{G.2}(\chi)$ is the sum of two distinct irreducible extensions of χ to $G.2$, and we write

$$\mathrm{Ind}_G^{\hat{G}}(\chi) = \chi_i^+ + \chi_i^-.$$

Then $\chi_i^- = \varepsilon \otimes \chi_i^+$. In particular, $\chi_i^-(x) = -\chi_i^+(x)$. If these numbers are real and non-zero, we choose the notation such that $\chi_i^+(x) > 0$. This condition is not satisfied exactly for the two characters χ_{13} and χ_{44}. In these cases, we choose χ_i^+ as one of the two extension of χ_i, at random. The symbol $\chi_i^\pm$ denotes the pair (χ_i^+, χ_i^-).

The above conventions are expanded analogously to the case when $\chi \in \mathrm{Irr}(G)$ is denoted by its degree.

2.3. **Character table automorphisms.** In the course of our proofs we induce projective characters from the maximal subgroups $[2^{20}]:A_6.2^2$ and $[2^{22}]:(S_3 \times S_3):2$ of $G.2$. In each case, there is one orbit of class fusions under the character table automorphisms of the subgroup and those of $G.2$, and we choose one fusion out of its orbit. We have to make sure that our arguments are independent of the chosen fusion map. For this we check that the projective characters of the subgroups we induce are invariant under all table automorphisms of the subgroup. In turn we check that the induced characters are invariant under table automorphisms of $G.2$. Hence any other possible fusion will yield the same projective characters.

3. The prime 3

Here, we prove the elementary parts of Theorems 1.1 and 1.2.

3.1. **The principal block.** Let B_1 denote the principal 3-block of $G.2$. Then $k(B_1) = 56$ and $\ell(B_1) = 31$. The restrictions to the 3-regular classes of the following characters yields a special basic set BS_1 of B_1: $\chi_1^\pm$, $\chi_2^\pm$, χ_3^0, $\chi_5^\pm$, $\chi_7^\pm$, $\chi_8^\pm$, χ_9^0, $\chi_{13}^\pm$, χ_{14}^0, $\chi_{20}^\pm$, χ_{22}^0, $\chi_{24}^\pm$, $\chi_{26}^\pm$, χ_{33}^0, $\chi_{45}^\pm$, χ_{46}^0, χ_{50}^0, $\chi_{58}^\pm$, $\chi_{94}^\pm$. The relations displayed in Table 1 prove this assertion.

Table 3 gives some projective characters of B_1, restricted to BS_1. These either arise as induced PIMs of the subgroup $[2^{22}]:(S_3 \times S_3):2$

of $G.2$ or as products of defect 0 characters of $G.2$ with ordinary characters. The list below gives more details; in the latter case, the defect 0 character is the first factor in the displayed product, in the former case the PIMs of $[2^{22}]{:}(S_3 \times S_3){:}\,2$ are given as sums of the ordinary irreducible characters θ_i, $i = 1, \ldots, 384$, of $[2^{22}]{:}(S_3 \times S_3){:}\,2$ in the ordering of the GAP-character table. The superscript in square brackets refers to the fact that we are inducing characters from the fourth maximal subgroup.

Φ	Origin
32	$\theta_1^{[4]} + \theta_6^{[4]} + \theta_7^{[4]}$
33	$\chi_{27}^{+} \otimes 833^{+}$
34	$\theta_{11}^{[4]} + \theta_{31}^{[4]}$
35	$\theta_{41}^{[4]}$
36	$\theta_4^{[4]} + \theta_7^{[4]} + \theta_9^{[4]}$
37	$\theta_{92}^{[4]} + \theta_{98}^{[4]} + \theta_{100}^{[4]}$
38	$\theta_{15}^{[4]} + \theta_{32}^{[4]}$
39	$\chi_{27}^{+} \otimes 1326^{+}$
40	$\chi_{16}^{+} \otimes 1326^{+}$

We now derive the parametrized decomposition matrix given in Table 2, up to the parameters a and $\tilde{a}$. These are non-negative integers such that $a + \tilde{a} = 4$. The PIMs of B_1 are denoted by $\Phi_1, \ldots, \Phi_{31}$. First, we induce the PIMs of the principal block of G to $G.2$. The non-invariant PIMs of this block yield the PIMs $\Phi_5, \Phi_{12}, \Phi_{15}$, Φ_{18}, Φ_{23}, Φ_{26} and Φ_{27} of B_1. The invariant PIMs of the principal block of G induce to pairs of splitting PIMs, i.e. the two elements of such a pair differ by multiplication with ε. The splitting follows the rules noted in 1.1.3.

Observe that Φ_{40} is a PIM, which is one member of a pair of splitting PIMs, thus yielding Φ_{28} and Φ_{29}. Similarly, Φ_{39} and Φ_{38} determine the splitting of Φ_{24} and Φ_{25}, respectively Φ_{19} and Φ_{20}. Also, Φ_{37} yields the given form of Φ_{16} and Φ_{17}. The splitting of Φ_{13} and Φ_{14} follows from the fact that the two liftable Brauer characters of degree 63700 are complex conjugates, whereas all other constituents in the corresponding PIMs are real valued. Since Φ_{36} contains the PIM Φ_{21}, the former projective character determines the splitting of Φ_{10} and Φ_{11}, as well as that of Φ_{21} and Φ_{22}. The form of Φ_8 and Φ_9 is proved with the help of Φ_{35} and Φ_{29}. Similarly, Φ_{34} and Φ_{33} determine the splitting of Φ_6 and Φ_7, respectively Φ_3 and Φ_4. Finally, $\Phi_1 = \Phi_{32} - 2\Phi_5 - 3\Phi_{10} - 2\Phi_{11}$, where the multiplicities of the PIMs to be subtracted from Φ_{32} are implied by the triangular shape of the decomposition matrix, already proved. This completes the proof for the parametrized decomposition matrix for the principal 3-block of $G.2$. It remains to determine the parameter a. This will be achieved in Section 5.

3.2. **The non-principal blocks of non-cyclic defect.** These are the two blocks B_2 and B_9 in the numbering of GAP. Let B be one of the blocks B_2 or B_9. Then $k(B) = 9$ and $\ell(B) = 7$, and we number the PIMs of B by $\Phi_1, \ldots, \Phi_7$. There is a unique block of G covered by B, whose decomposition matrix is given in [9]. The PIM Φ_5 is obtained by inducing a non-invariant PIM of G.

Further PIMs of B are obtained according to the following lists.

B_2	
Φ	Origin
1	$\theta_6^{[2]}$
2	$\theta_8^{[2]}$
3	$\theta_{66}^{[4]}$
4	$\theta_{62}^{[4]}$
6	$\chi_{16}^{+} \otimes 833^{+}$
7	$\chi_{16}^{-} \otimes 833^{+}$

B_9	
Φ	Origin
3	$\theta_{18}^{[4]}$
4	$\theta_{28}^{[4]}$
6	$\theta_{26}^{[4]}$
7	$\theta_{22}^{[4]}$

Here, the irreducible characters of the maximal subgroup $[2^{20}]{:}A_6.2^2$ are denoted by $\theta_i^{[2]}$, $i = 1, \ldots, 331$, and those of the maximal subgroup $[2^{22}]{:}(S_3 \times S_3){:}2$ by $\theta_i^{[4]}$, $i = 1, \ldots, 384$. The characters induced are all of 3-defect 0. The splitting of the PIMs Φ_1, Φ_2 of block B_9 follows the rules formulated in 1.1.3.

4. The prime 7

The columns labelled by $\Phi_1, \ldots, \Phi_{24}$ of Table 6 constitute the parametrized decomposition matrix of the principal 7-block of $G.2$. The parameters a, b, c, d take the values 0 or 1, and $\tilde{x} = 1 - x$ for $x \in \{a, b, c, d\}$. The PIMs Φ_3, Φ_8, Φ_9, Φ_{16}, Φ_{19} and Φ_{20} are induced from PIMs of G. There are PIMs of the form

$$\Phi_i := (833^{+} \otimes \chi_j^{\pm})_{B_1},$$

with i, j and $\pm$ as in the following list.

i	1	2	4	5	14	15	21	22	23	24
j	2	2	7	7	24	24	16	16	26	26
$\pm$	+	−	+	−	+	−	+	−	+	−

The parametrized PIMs are pairs of splitting PIMs, i.e. the two elements of such a pair differ by multiplication with ε. There are two pairs of complex conjugate PIMs, (Φ_{10}, Φ_{12}) and (Φ_{11}, Φ_{13}). All other PIMs are real.

Finally, there are two projective characters

$$\Phi_{25} := (\chi_8^+ \otimes \chi_6^+)_{B_1}$$

and

$$\Phi_{26} := (\chi_7^+ \otimes \chi_{20}^+)_{B_1}.$$

Notice that

$$\Phi_{25} = \Phi_6 + \tilde{a}\Phi_{17} + a\Phi_{18}.$$

Since 5640192^- is not a constituent of Φ_{25}, we get $\tilde{a}\tilde{d} + ad = 0$, which implies that $d = \tilde{a}$. Also,

$$\Phi_{26} = \Phi_{10} + \Phi_{12} + \Phi_{19} + 4\Phi_{20} + (3-2b)\Phi_{21} + (1-2\tilde{b})\Phi_{22} + x\Phi_{23} + y\Phi_{24}$$

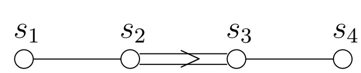
for some non-negative integers x, y. This implies $1 - 2\tilde{b} \geq 0$ and thus $b = 1$.

It remains to determine the parameter c. This is achieved in Section 5 below.

5. Condensation

To determine the parameters a in Theorem 1.1 and c in Theorem 1.4, we use condensation techniques. Let $\mathbb{F}$ be one of $\mathbb{F}_3$ or $\mathbb{F}_7$ and let p denote the characteristic of $\mathbb{F}$. For $\chi \in \mathrm{Irr}(G.2)$ we write χ° for the restriction of χ to the p-regular classes.

5.1. **The setup.** Assume that (B, N) is a split BN-pair for G. Then $U := B$ is a Sylow 2-subgroup of G and $W := N$ is the Weyl group of G. The corresponding Dynkin diagram of G equals



where $s_1, \ldots, s_4$ denote the Coxeter generators of W. The standard graph automorphism σ of G (see [8, Theorem 1.15.2(a), Definition 1.15.5(e)]) fixes U and W, and swaps s_1 with s_4 and s_2 with s_3. Recall that $x \in G.2$ is an involution which induces σ by conjugation. Thus x

normalizes U and W and $s_1^x = s_4$, $s_2^x = s_3$. In particular, x preserves the length of the elements of W. As in [3], let

$$e := \left(\sum_{u \in U} u\right)\left(\sum_{w \in W} (-1)^{\ell(w)} w\right) \in \mathbb{F}G$$

denote the Steinberg element. Then x centralizes e.

Put $\mathrm{St} := e\mathbb{F}G$. Then St is an $\mathbb{F}G$-module with character χ_{94}, the Steinberg character of G. Also, St has $\mathbb{F}$-basis $\{eu \mid u \in U\}$. Now $G.2 = \langle G, x\rangle$, and x centralizes e, so St is an $\mathbb{F}G.2$-module, denoted by St^+. As x permutes the basis $\{eu \mid u \in U\}$ with exactly 4096 fixed points, the character of St^+ equals χ_{94}^+.

Now a, respectively c is the multiplicity of the irreducible Brauer character $\chi_{20}^{+,\circ}$ in $\chi_{94}^{+,\circ}$. To determine this multiplicity, we condense St^+, using the same condensation subgroup V as in [3]; that is, V is the center of the unipotent radical of the standard parabolic subgroup P of type C_3 of G. In particular, $V \leq G$. The condensation idempotent is $\iota := 1/|V| \sum_{v \in V} v$. In this setup, simple $\mathbb{F}G.2$-modules with characters $\chi_{20}^{\pm,\circ}$ and $\chi_{21}^{\pm,\circ}$ condense to characters of degree 720, and no other constituent of St^+ condenses to a character of this degree.

Condensing the same elements of G as in [3, Subsection 2.4], we can use the formulas given in [3, Subsections 2.2, 2.3], with $\mathbb{F}_3$ replaced by $\mathbb{F}$. As remarked in [3, Subsection 2.4], these condensed elements are algebra generators of $\iota\mathbb{F}G\iota$. In addition, we condense

$$y_1 := x s_4 s_3 s_2 s_1 \in G.2 \setminus G$$

and

$$y_2 := x s_1 s_2 s_3 s_4 \in G.2 \setminus G.$$

The action of x on the basis $\{eu \mid u \in U\}$ of St^+ is given by applying σ to these basis elements, that is we need to know how σ permutes the elements of U. We first determine how σ permutes the root subgroups U_α for positive roots α (and so the non-trivial elements in these subgroups). We write the reflection w_α along a root α as word in the Coxeter generators s_i, $i = 1, 2, 3, 4$, of W. We know how x permutes the s_i and so we find the positive root β such that $w_\beta = w_\alpha^x$. Then σ maps U_α to U_β.

Using this map, symbolic descriptions of the elements of U as products of root elements in some fixed ordering of the roots and the commutator relations in U, we obtain a symbolic description of the permutation σ on U.

Thus the matrix coefficients for the action of $\iota y_i \iota$ on $\mathrm{St}^+\iota$ for $i = 1, 2$ can be computed as in the remarks of [3, Subsection 2.4].

5.2. **Applying the trace formula.** In order to determine the multiplicity of the composition factor M^+ of St^+ with Brauer character $\chi_{20}^{+,\circ}$, we need to find a suitable element $y \in G.2$, such that the trace of $\iota y \iota$ on $\iota M^+ \iota$ is non-zero. Let y_1, y_2 be the elements defined in Subsection 5.1.

Lemma 5.3. (a) *The set $Z_i := \{y_i v \mid v \in V\}$ distributes among the conjugacy classes of $G.2$ according to the following tables, where we use Atlas notation for the conjugacy classes.*

Z_1:

Class	2E	4P	4Q	8L/M	8R
No.	1	3	28	48	48

Z_2:

Class	2E	4P	4Q	8L/M
No.	2	22	40	64

(b) *Let χ^+ be one of χ_{20}^+ or χ_{21}^+. Then $\sum_{v \in V} \chi^+(y_1 v) = 128$ and $\sum_{v \in V} \chi^+(y_2 v) = 0$.*

(c) *Let M be a composition factor of St^+ with $\varphi_M \in \{\chi_{20}^{+,\circ}, \chi_{21}^{+,\circ}\}$. Denote the $\iota \mathbb{F}G.2\iota$-character of $M\iota$ by $\varphi_{M\iota}^{\iota}$. Then $\varphi_{M\iota}^{\iota}(\iota y_1 \iota) = 1$ and $\varphi_{M\iota}^{\iota}(\iota y_2 \iota) = 0$.*

Proof. (a) This is computed with GAP, using the permutation representation of $G.2$ on 139776 points, provided by the AtlasRep-package; see [19]. The first generator of $G.2$ in this representation is an involution in $G.2 \setminus G$. As all these involutions are conjugate by an element of G, we may set x equal to this first generator. The task now is to identify the elements $s_1, \dots, s_4$; in other words, we have to solve a constructive recognition problem.

In a first step, we construct an x-invariant Sylow 2-subgroup U' of G. As all x-invariant Sylow 2-subgroups of G are conjugate by an element of $C_G(x)$, we may assume that $U' = U = B$ is the first component of the BN-pair of G we are looking for. In our situation, $N = W$ is the corresponding x-invariant Weyl group, generated by the Coxeter generators $s_1, \dots, s_4$, with $s_1 = s_4^x$ and $s_2 = s_3^x$. The standard parabolic subgroup P of type C_3 corresponding to s_2, s_3, s_4 is the centralizer of a central element of U. We may thus construct P. Now $P = L \ltimes U_P$ with $L \cong \mathrm{Sp}_6(2)$. It is easily checked that $|C_L(s_4)| = 4608$. The character tables of P and G and the corresponding class fusions are available in GAP. Using these, we conclude that s_4 lies in the unique conjugacy class C of P of length 5040, which fuses into a conjugacy class of G with centralizer order 47563407360. The class C is easily located in P. Also, $s_3 \in C$, since s_4 and s_3 are conjugate in $\langle s_3, s_4 \rangle \leq P$. Thus $(s_4, s_3) \in \mathcal{C}$ with $\mathcal{C}$ the set of all elements $(s, t) \in C \times C$ satisfying the following conditions:

- $|st| = 3$;
- $|ss^x| = 2$;
- $|st^x| = 2$;
- $|tt^x| = 4$;
- $|U \cap U^s| = 2^{23}$;
- $|U \cap U^t| = 2^{23}$;
- $C_{C_U(x)}(\langle s, t\rangle) = \{1\}$.

The first four conditions arise from the presentation of W, the next two conditions from the BN-axioms. The last condition follows from $C_U(W) = \{1\}$. Using GAP, we find that $|\mathcal{C}| = 4096$. As $C_U(x)$, a group of order 4096, acts on $C \times C$ and preserves the above conditions, $\mathcal{C}$ is a regular $C_U(x)$-orbit on $C \times C$.

For $(s, t) \in \mathcal{C}$, put

$$Z_{s,t} := \{xstt^x s^x v \mid v \in V\} \tag{1}$$

and

$$Z'_{s,t} := \{xs^x t^x tsv \mid v \in V\} \tag{2}$$

As V is a normal subgroup of U, the sets $Z_{s,t}$ are conjugate by elements of $C_U(x)$ as (s, t) varies over $\mathcal{C}$, and an analogous remark applies to the sets $Z'_{s,t}$. By the order distribution of a subset of $G.2$, we understand the multiset of orders of the elements of this set. By what we have already said, the order distributions of the sets $Z_{s,t}$ are the same for all $(s, t) \in \mathcal{C}$, and likewise for the sets $Z'_{s,t}$.

We find the order distribution

$$\{2^1, 4^{31}, 8^{96}\}$$

for the sets $Z_{s,t}$ and

$$\{2^2, 4^{62}, 8^{64}\}$$

for the sets $Z'_{s,t}$, where the multiplicity of an element is indicated by a superscript.

The elements of order 4 of $G.2 \setminus G$ are distinguished by the orders of their centralizer in G. The same is true for the elements of order 8, except if the centralizer has order 1280 or 128, respectively. In the former case, the elements lie in one of the classes 8L or 8M. In the latter case, in one of the classes 8Q or 8R, which are, however, distinguished by the centralizer order of their squares. We find the multiset of centralizer orders

$$\{35942400^1, 20480^3, 3072^{28}, 1280^{48}, 128^{48}\}$$

for $Z_{s,t}$, respectively

$$\{35942400^2, 20480^{22}, 3072^{40}, 1280^{64}\}$$

for $Z'_{s,t}$. In the former case, all elements with centralizer order 128 square to elements with centralizer order 16384, so lie in the class 8R. A look into the Atlas completes the proof.

(b) This follows from (a) and the character values of χ^+ on the respective elements.

(c) This follows from (b) and the trace formula

$$\varphi^{\iota}_{M\iota}(\iota y_i \iota) = \frac{1}{|V|} \sum_{v \in V} \varphi_M(y_i v)$$

given in [13, Subsection 3.6]. □

5.4. **Results of the condensation.** We apply the MeatAxe with generators of the algebra $\langle \iota \mathbb{F} G \iota, \iota y_1 \iota, \iota y_2 \iota \rangle$. Although the latter might be strictly contained in $\iota \mathbb{F} G.2 \iota$, this suffices for our purposes. Indeed, the 720-dimensional $\iota \mathbb{F} G.2 \iota$-composition factors of $\mathrm{St}^+ \iota$ are irreducible as $\iota \mathbb{F} G \iota$-modules, and we can compute the traces of $\iota y_1 \iota$ and $\iota y_2 \iota$ on them.

Suppose first that $p = 3$. Then the module $\mathrm{St}^+ \iota$ has four composition factors of degrees 720, which come in two isomorphism types $720a$ and $720b$ and multiplicities 1 and 3, respectively. Moreover, the traces of $\iota y_1 \iota$ on $720a$ and $720b$ are -1 and 1, respectively. By Lemma 5.3(b) and Table 2, we obtain $a = 3$.

Now suppose that $p = 7$, Then $\mathrm{St}^+ \iota$ has composition factors of degrees

$$720, 720, 3711, 18555, 39900, 67466,$$

where the two composition factors of degree 720 are not isomorphic. In fact, these correspond to the composition factors of St^+ with Brauer characters $\chi_{20}^{+,\circ}$ and $\chi_{21}^{+,\circ}$, if $c = 1$, respectively $\chi_{20}^{-,\circ}$ and $\chi_{21}^{-,\circ}$, if $c = 0$. The trace of $\iota y_1 \iota$ on each of these two composition factors equals 1. By Lemma 5.3(b) and Table 6, we obtain $c = 1$.

Acknowledgements

It is our pleasure to thank Thomas Breuer for computing the character tables of the maximal subgroups of $\mathrm{Aut}(F_4(2))$, and Klaus Lux for his invaluable help with the MeatAxe64 and the MOC system. We also thank Frank Himstedt for independently checking the results of Lemma 5.3(a).

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Appendix A. Tables

The notation in the following tables is introduced in Subsections 2.1 and 2.2. The parameters in Tables 2, 5 and 6 are discussed in Subsection 1.2.

Table 1. The relations in the principal 3-block with respect to the basic set BS_1

χ	χ_{11}^0	χ_{18}^0	χ_{21}^-	χ_{22}^-	χ_{28}^0	χ_{32}^+	χ_{32}^-	χ_{35}^0	χ_{37}^+	χ_{37}^-	χ_{40}^0	χ_{52}^0	χ_{54}^0
1^+	1	−1	.	.	−2	.	1	−1	1	.	−1	1	−1
1^-	1	−1	.	.	−2	1	.	−1	.	1	−1	1	−1
833^+	−1	1	.	.	1	−1	−1	.	−1	−1	1	−2	.
833^-	−1	1	.	.	1	−1	−1	.	−1	−1	1	−2	.
2210^0	−1	.	.	.	2	−1	−1	.	−1	−1	.	−2	.
1326^+	.	.	.	.	1	.	.	1	.	−1	.	1	.
1326^-	.	.	.	.	1	.	.	1	−1	.	.	1	.
21658^+	.	1	.	.	.	.	.	.	.	.	.	.	1
21658^-	.	1	.	.	.	.	.	.	.	.	.	.	1
22932^+	1	.	.	.	−1	1	.	.	.	1	−1	2	.
22932^-	1	.	.	.	−1	.	1	.	1	.	−1	2	.
46410^0	1	−1	.	.	−2	1	1	−1	1	1	−1	2	.
63700^+	.	1	.	.	1	.	.	1	.	.	.	1	1
63700^-	.	1	.	.	1	.	.	1	.	.	.	1	1
198900^0	.	1	.	.	1	.	.	1	.	.	1	.	1
183600^+	.	.	1	.	.	1	.	1	.	.	.	2	.
183600^-	.	.	.	1	.	.	1	1	.	.	.	2	.
433160^0	.	.	.	.	.	.	.	1	.	.	1	1	.
249900^+	.	.	.	.	1	.	.	.	1	.	.	−1	.
249900^-	.	.	.	.	1	.	.	.	.	1	.	−1	.
270725^+	.	.	.	.	.	.	1	.	1	.	.	1	1
270725^-	.	.	.	.	.	1	.	.	.	1	.	1	1
1082900^0	.	.	.	.	.	.	.	.	.	.	1	−1	.
1082900^+	.	.	.	.	.	.	.	.	.	.	.	.	−1
1082900^-	.	.	.	.	.	.	.	.	.	.	.	.	−1
2598960^0	.	.	.	.	.	.	.	.	.	.	.	.	.
3898440^0	.	.	.	.	.	.	.	.	.	.	.	1	.
3411968^+	.	.	.	.	.	.	.	.	.	.	.	.	1
3411968^-	.	.	.	.	.	.	.	.	.	.	.	.	1
16777216^+	.	.	.	.	.	.	.	.	.	.	.	.	.
16777216^-	.	.	.	.	.	.	.	.	.	.	.	.	.

Table 1. (cont.) The relations in the principal 3-block with respect to the basic set BS_1

χ	χ_{56}^0	χ_{59}^0	χ_{61}^0	χ_{67}^0	χ_{72}^+	χ_{72}^-	χ_{73}^0	χ_{75}^0	χ_{88}^0	χ_{91}^0	χ_{95}^+	χ_{95}^-
1^+	.	1	.	1	.	1	1	1	.	.	1	1
1^-	.	1	.	1	1	.	1	1	.	.	1	1
833^+	−1	−2	.	−1	−1	−1	−2	−2	.	.	−1	−1
833^-	−1	−2	.	−1	−1	−1	−2	−2	.	.	−1	−1
2210^0	−1	−2	.	−1	−1	−1	−2	−2	.	.	−1	−1
1326^+	1	1	−1	.	1	.	1	.	.	−1	−1	.
1326^-	1	1	−1	.	.	1	1	.	.	−1	.	−1
21658^+	.	−1	1	.	−1	.	.	1	−1	−1	.	−1
21658^-	.	−1	1	.	.	−1	.	1	−1	−1	−1	.
22932^+	1	1	.	1	1	1	2	2	.	−1	1	.
22932^-	1	1	.	1	1	1	2	2	.	−1	.	1
46410^0	1	2	.	1	1	1	2	2	.	−1	1	1
63700^+	1	.	.	.	.	.	1	1	−1	−2	−1	−1
63700^-	1	.	.	.	.	.	1	1	−1	−2	−1	−1
198900^0	1	.	.	.	.	.	.	.	−1	−1	−1	−1
183600^+	1	2	−1	1	2	.	2	1	.	−1	−1	1
183600^-	1	2	−1	1	.	2	2	1	.	−1	1	−1
433160^0	1	2	−1	1	1	1	1	.	.	.	.	.
249900^+	.	−1	.	−1	.	−1	−1	−1	.	.	−1	.
249900^-	.	−1	.	−1	−1	.	−1	−1	.	.	.	−1
270725^+	1	1	.	.	.	1	1	.	.	−1	.	.
270725^-	1	1	.	.	1	.	1	.	.	−1	.	.
1082900^0	.	.	.	.	.	.	−1	−1	.	1	.	.
1082900^+	.	1	.	1	1	.	1	1	.	.	.	1
1082900^-	.	1	.	1	.	1	1	1	.	.	1	.
2598960^0	.	.	1	1	.	.	1	1	−1	.	.	.
3898440^0	1	1	.	1	1	1	2	1	.	−1	.	.
3411968^+	.	.	1	.	.	.	.	1	−1	.	.	.
3411968^-	.	.	1	.	.	.	.	1	−1	.	.	.
16777216^+	.	.	.	.	.	.	.	.	1	1	1	.
16777216^-	.	.	.	.	.	.	.	.	1	1	.	1

Table 2. The parametrized basic set decomposition matrix of the principal 3-block

Φ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1^+	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1^-	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
833^+	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
833^-	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
2210^0	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
1326^+	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
1326^-	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
21658^+	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
21658^-	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
22932^+	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.
22932^-	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.
46410^0	.	.	1	1	.	.	.	.	.	.	.	1	.	.	.
63700^+	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
63700^-	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
198900^0	1	1	.	.	.	.	.	.	.	.	.	1	.	.	1
183600^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
183600^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
433160^0	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.
249900^+	1	.	1	.	.	.	1	.	.	1	.	1	.	.	.
249900^-	.	1	.	1	.	1	.	.	.	.	1	1	.	.	.
270725^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
270725^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1082900^0	.	.	.	.	2	.	.	.	.	1	1	1	.	.	.
1082900^+	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
1082900^-	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
2598960^0	.	.	.	.	.	1	1	.	.	.	.	.	1	1	1
3898440^0	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
3411968^+	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
3411968^-	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
16777216^+	1	.	1	.	.	1	1	1	.	.	.	1	2	2	1
16777216^-	.	1	.	1	.	1	1	.	1	.	.	1	2	2	1

Table 2. (cont.) The parametrized basic set decomposition matrix of the principal 3-block

Φ	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
833^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
833^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2210^0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1326^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1326^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21658^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21658^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22932^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22932^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
46410^0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
63700^+	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
63700^-	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
198900^0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
183600^+	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
183600^-	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
433160^0	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
249900^+	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
249900^-	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
270725^+	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
270725^-	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
1082900^0	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
1082900^+	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
1082900^-	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
2598960^0	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
3898440^0	.	.	.	1	1	.	.	1	.	.	.	1	.	.	.	.
3411968^+	2	.	1	.	.	.	.	.	1	.	.	.	1	.	.	.
3411968^-	.	2	1	.	.	.	.	.	.	1	.	.	.	1	.	.
16777216^+	a	$\tilde{a}$	1	1	.	1	.	.	2	.	1	1	1	.	1	.
16777216^-	$\tilde{a}$	a	1	.	1	.	1	.	.	2	1	1	.	1	.	1

Table 3. Some projective characters of the principal 3-block

Φ	32	33	34	35	36	37	38	39	40
1^+	1	.	.	.	.	.	.	.	.
1^-	.	.	.	.	.	.	.	.	.
833^+	.	1	.	.	.	.	.	.	.
833^-	.	.	.	.	.	.	.	.	.
2210^0	2	.	.	.	.	.	.	.	.
1326^+	.	.	1	.	.	.	.	.	.
1326^-	.	.	.	.	.	.	.	.	.
21658^+	.	.	.	1	.	.	.	.	.
21658^-	.	.	.	.	.	.	.	.	.
22932^+	5	.	.	.	1	.	.	.	.
22932^-	4	.	.	.	.	.	.	.	.
46410^0	.	1	.	.	.	.	.	.	.
63700^+	.	.	.	.	.	.	.	.	.
63700^-	.	.	.	.	.	.	.	.	.
198900^0	1	.	.	.	.	.	.	.	.
183600^+	.	.	.	.	.	1	.	.	.
183600^-	.	.	.	.	.	.	.	.	.
433160^0	.	1	.	.	.	.	.	.	.
249900^+	4	1	.	.	1	.	1	.	.
249900^-	2	.	1	.	.	.	.	.	.
270725^+	.	.	.	.	1	.	.	.	.
270725^-	.	.	.	.	.	.	.	.	.
1082900^0	9	.	.	.	2	.	.	.	.
1082900^+	.	.	.	1	.	.	.	1	.
1082900^-	.	.	.	.	.	.	.	.	.
2598960^0	.	.	1	.	.	1	.	.	.
3898440^0	.	.	.	.	1	1	1	.	.
3411968^+	.	2	.	.	.	3	.	1	1
3411968^-	.	.	.	1	.	.	.	.	.
16777216^+	1	2	1	1	1	10	2	2	1
16777216^-	.	.	1	1	.	6	.	.	.

Table 4. The decomposition matrix of the 3-block B_2

Φ	1	2	3	4	5	6	7
1377^+	1	.	.	.	.	.	.
1377^-	.	1	.	.	.	.	.
269892^+	.	.	1	.	.	.	.
269892^-	.	.	.	1	.	.	.
716040^0	1	1	.	.	1	.	.
1253070^0	.	.	1	1	1	.	.
10024560^0	.	.	.	.	1	1	1
5640192^+	1	.	1	.	1	1	.
5640192^-	.	1	.	1	1	.	1

Table 5. The parametrized decomposition matrix of the 3-block B_9

Φ	1	2	3	4	5	6	7
877149^+	1	.	.	.	.	.	.
877149^-	.	1	.	.	.	.	.
877149^+	.	.	1	.	.	.	.
877149^-	.	.	.	1	.	.	.
8771490^0	1	1	.	.	1	.	.
8771490^0	.	.	1	1	1	.	.
17542980^0	.	.	.	.	1	1	1
14034384^+	b	$\tilde{b}$	1	.	1	1	.
14034384^-	$\tilde{b}$	b	.	1	1	.	1

TABLE 6. The parametrized decomposition matrix of the principal 7-block

Φ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1^+	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1^-	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2210^0	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1326^+	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
1326^-	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
1377^+	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
1377^-	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
88400^0	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.
198900^0	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.
183600^+	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.
183600^-	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
183600^+	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
183600^-	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.
322218^+	1	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.
322218^-	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.
716040^0	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1
947700^+	.	.	.	.	.	a	$\tilde{a}$	1	.	.	.	.	.	.	.	.
947700^-	.	.	.	.	.	$\tilde{a}$	a	1	.	.	.	.	.	.	.	.
2685150^0	.	.	.	1	1	.	.	.	1	.	.	.	.	.	.	.
5657600^0	.	.	.	.	.	.	.	1	.	1	1	1	1	.	.	1
9052160^0	.	.	1	.	.	.	.	.	1	.	.	.	.	1	1	.
5640192^+	.	.	.	1	.	.	.	.	.	b	$\tilde{b}$	b	$\tilde{b}$	.	.	.
5640192^-	.	.	.	.	1	.	.	.	.	$\tilde{b}$	b	$\tilde{b}$	b	.	.	.
16110900^0	.	.	.	.	.	.	.	.	1	1	1	1	1	.	.	1
21481200^0	.	.	.	.	.	.	.	.	.	1	1	1	1	.	.	1
16777216^+	.	.	.	.	.	.	.	.	.	c	$\tilde{c}$	c	$\tilde{c}$	1	.	.
16777216^-	.	.	.	.	.	.	.	.	.	$\tilde{c}$	c	$\tilde{c}$	c	.	1	.

Table 6. (cont.) The parametrized decomposition matrix of the principal 7-block

Φ	17	18	19	20	21	22	23	24	25	26
1^+	.	.	.	.	.	.	.	.	.	.
1^-	.	.	.	.	.	.	.	.	.	.
2210^0	.	.	.	.	.	.	.	.	.	.
1326^+	.	.	.	.	.	.	.	.	.	.
1326^-	.	.	.	.	.	.	.	.	.	.
1377^+	.	.	.	.	.	.	.	.	1	.
1377^-	.	.	.	.	.	.	.	.	.	.
88400^0	.	.	.	.	.	.	.	.	1	.
198900^0	.	.	.	.	.	.	.	.	.	.
183600^+	.	.	.	.	.	.	.	.	.	1
183600^-	.	.	.	.	.	.	.	.	.	.
183600^+	.	.	.	.	.	.	.	.	.	1
183600^-	.	.	.	.	.	.	.	.	.	.
322218^+	.	.	.	.	.	.	.	.	.	.
322218^-	.	.	.	.	.	.	.	.	.	.
716040^0	.	.	.	.	.	.	.	.	.	.
947700^+	1	.	.	.	.	.	.	.	1	.
947700^-	.	1	.	.	.	.	.	.	1	.
2685150^0	.	.	1	.	.	.	.	.	.	1
5657600^0	1	1	1	.	.	.	.	.	1	3
9052160^0	.	.	.	1	.	.	.	.	.	4
5640192^+	d	$\tilde{d}$	1	.	1	.	.	.	1	4
5640192^-	$\tilde{d}$	d	1	.	.	1	.	.	.	2
16110900^0	.	.	1	1	1	1	.	.	.	9
21481200^0	.	.	.	1	.	.	1	1	.	14
16777216^+	.	.	.	1	1	.	1	.	.	11
16777216^-	.	.	.	1	.	1	.	1	.	9