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Algorithms of Representation Theory

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2.8 Algorithms of Representation Theory

Soon after the first applications of computers in group theory—to coset enumeration—algorithmic methods have been introduced in representation theory, beginning with the character theory of finite groups.

2.8.1 Ordinary Representation Theory

Characters. Early implementations, by S. Comet, of algorithms for the computation of characters of symmetric groups, date back to the late fifties [7]. Questions arising during the classification project of the finite simple groups lead to the demand for computing character tables of specific finite groups from incomplete knowledge of some of its characters. For this purpose interactive methods were implemented at various places (B. Fischer and T. Gabrysch (Bielefeld), D. Livingstone (Birmingham), J. H. Conway (Cambridge)). In the late seventies these methods were collected, extended and enhanced by an arithmetic for cyclotomic fields in the Aachen CAS system [26]. This included routines to compute tensor products, inner products, and symmetrizations of characters, induced characters and fusions of subgroups. The CAS system as well as its new implementation in GAP (see also [12] and 4.2.18) have lead to the computation of numerous character tables, some of them included in the *Atlas of Finite Groups* [9]. This contains the character tables of the sporadic groups, their covering groups and automorphism groups (in compound form). All of these tables, and many more, are now also available in GAP. It contains, for example, the character tables of most of the maximal subgroups of the sporadic simple groups.

In the sixties J. D. Dixon suggested an algorithm for computing the character table of a finite group from its class multiplication coefficients. A substantially improved version of *Dixon's algorithm*, due to G. Schneider, is included in GAP and MAGMA [4]. These implementations are capable of calculating the character tables of groups with up to several hundreds of conjugacy classes, provided the degrees of the irreducible characters are not too large.

More powerful algorithms exist for computing character tables of groups of special types, e.g., p-groups (Conlon [8]). For other classes of groups e.g., the symmetric groups, the rows and columns of the character table have natural labelings in terms of certain combinatorial objects, e.g., partitions. Moreover, there are algorithms, the *Murnaghan-Nakayama rules* and generalizations thereof, for computing the entries of the character tables in terms of the labels for the rows and columns. Such algorithms are known for all Weyl groups of classical types and have been implemented in GAP.

Representations. The irreducible matrix representations of a finite group over a field of characteristic 0 are considerably more difficult to construct than the corresponding characters. Nevertheless, some methods have emerged over the past few years.

Baum and Clausen [3] have described an algorithm to compute the irreducible matrix representations of a supersolvable group from a power commutator presentation.

Labonté [20] and Linton [21] (independently) described a method—analogous to the Todd-Coxeter coset enumeration discussed in Sims' article 2.7—to construct representations for finitely presented algebras. While this *Vector Enumerator*, as it came to be called, works most efficiently over finite fields, it is in principle applicable to algebras over fields of characteristic 0. One major application has been the construction of representations of Iwahori-Hecke algebras.

Methods for constructing rational irreducible representations of finite groups have been introduced by Plesken. One such method, for soluble groups, is used in the *Soluble Quotient Algorithm* of Plesken [30], implemented by Brückner [5]. Another method, based on liftings of representations, has recently been applied by Plesken and Souvignier to prove the infiniteness of certain finitely presented groups. A survey of these ideas and their applications is given in [32, Section 2].

Richard Parker has suggested an integral version of his Meat-Axe (see also 2.8.2 below) [29]. Rational and integral representations of finite groups play an essential role in the investigation of integral lattices and their automorphism groups. This information is used in the study of finite subgroups of the general linear groups over the integers or the rational numbers (see [31] for a survey). The monumental work of Plesken and Nebe [25] has lead to the classification of all finite subgroups of the groups $GL_n(\mathbf{Q})$ for $n \leq 31$.

Since crystallographic groups are constructed from integral representations of finite groups, these are of great importance in crystallography. The CARAT package (see [27] and 4.2.7) contains tables and implementations of various algorithms, including the *Zassenhaus algorithm* for computing extension groups, for handling enumeration and recognition problems for crystallographic groups.

The GAP share package AREP by Egner and Püschel (see [11] and 4.3.2) computes, symbolically, with structured representations of finite groups. Examples for structured representations are induced representations or tensor products of representations. Applications of AREP include the automatic construction of fast algorithms for discrete linear signal transforms.

2.8.2 Modular Representation Theory

The computation of Brauer character tables of finite groups was begun in the seventies with the work of Gordon James on the Mathieu groups. While James was still working by hand, Richard Parker soon applied his Meat-Axe [28] (see also 4.2.10), originally designed to construct the largest Janko group, to this kind of problems.

Since the Meat-Axe is of fundamental importance in computational representation theory, I shall sketch its main ideas. Given $d \times d$-matrices $a_1, \dots, a_n$ over a field F, let A denote the (unitary) F-algebra generated by $a_1, \dots, a_n$. The Meat-Axe aims to find a composition series of the natural left A-module $F^d := F^{d \times 1}$. Inductively, it suffices to find a non-trivial A-invariant subspace of F^d or to prove that A acts irreducibly on F^d. Let $v \in F^d$. By using a variation of the orbit algorithm for permutation groups (see 2.7), and the Gauß algorithm, the Meat-Axe computes the smallest A-invariant subspace Av of F^d containing v, and matrices for the actions of $a_1, \dots, a_n$ on the subspace Av and the quotient

F^d/Av. The search for vectors lying in proper A-invariant subspaces of F^d (if there are any) is guided by the following result of S. Norton. (We denote the transpose of a matrix b by b^t and by A^t the F-algebra generated by $a_1^t, \ldots, a_n^t$.)

Proposition. (Norton's irreducibility criterion) *Let $b \in A$. Then at least one of the following occurs.*

(1) *b is invertible.*
(2) *Av is a proper subspace of F^d for at least one non-zero v in the nullspace of b.*
(3) *$A^t v$ is a proper subspace of F^d for all non-zero v in the nullspace of b^t.*
(4) *A acts irreducibly on F^d.*

Thus one has to find a non-invertible element $b \in A$ with nullspace of small dimension (preferably 1). If F is a (small) finite field a random choice of elements of A is a reasonable strategy to find such a b. For larger fields more sophisticated methods, suggested by Holt and Rees [16], have to be applied. If all non-zero vectors of the nullspace of b fail to lie in a proper A-invariant subspace, chose a non-zero vector v in the nullspace of b^t and compute $A^t v$. If this is all of F^d, then F^d is an irreducible A-module. On the other hand, if F^d is a reducible A-module, one finds a proper invariant subspace this way.

A large number of Brauer character tables of sporadic groups have been computed by Parker and others with the help of the Meat-Axe. This method came to its limits with the degrees of the representations to be considered growing larger and larger.

The applicability of the Meat-Axe is greatly extended by *condensation* techniques, where the original algebra is replaced by a Morita equivalent one of considerably smaller dimension. Additional support for the Meat-Axe is provided by the MOC system, developed by Parker, Lux, Jansen and Hiss (see also 4.2.28). This system works with Brauer characters, rather than representations, so that there are no degree constraints. Computations with these systems lead to the construction of the Brauer character tables of all the Atlas groups of order at most 10^9 which were published in the [17]. Many more tables are now known (see `http://www.math.rwth-aachen.de/~MOC/`). The modular character tables for the symmetric groups S_n can also be computed with SPECHT, a GAP share package written by Andrew Mathas. They are known completely for $n \leq 16$.

The Meat-Axe is also used, of course, in the investigation of module structures, e.g., submodule lattices or Loewy series, and also for computing endomorphism rings of modules, direct decompositions of modules and Green correspondents. The matrix group recognition project provides a further field of applications for the Meat-Axe. Moreover, it is still used according to its original design, namely to construct specific finite groups via some of their linear representations. A spectacular example is provided by the matrix representation of the Monster sporadic group, constructed by Linton, Parker, Walsh, and Wilson, [22]. Robert A. Wilson has initiated and maintains a data base of representations of finite groups (`http://www.mat.bham.ac.uk/atlas/index.html`).

Explicit modular representations of finite groups were used in a substantial way in the work of Holt and Plesken in the construction of perfect groups [15].

In addition, this work also used the algorithms and implementations by Derek Holt [14] for computing first and second cohomology groups of finite groups. For example, there are programs to determine the Schur multiplier of a given finite group, or to find the extension classes of a finite group with a finite module.

New directions of algorithmic research aim to compute projective resolutions of modules and cohomology rings of finite groups (Adem, Milgram, Jon Carlson, David Green, Ed Green, Schneider, see, e.g., [1, 6]). A recent approach uses non-commutative Gröbner bases. These are also used in the Virginia Tech Hopf project for the algorithmic investigation of finite dimensional algebras, in particular Hopf algebras. Finite dimensional algebras and their representations are conveniently studied via a directed graph, a so-called quiver, which is a purely combinatorial object. The Bielefeld CREP system (see also 4.2.12) provides algorithms for using the quiver approach to finite dimensional algebras for research and teaching.

We close by pointing out the various recent applications of computer algebra in invariant theory, mainly due to Kemper (see [18] for a survey).

2.8.3 Generic Character Tables

It is often possible to encode the character tables of an infinite series of groups in a single table or in a program. Such a table or program is then called a *Generic Character Table*. The first example of a generic character table was computed by Frobenius in 1897: the generic table for the series of Chevalley groups $SL(2, 2^n)$, n a positive integer. This example already has all the features common to a generic character table for a series of Chevalley groups. The conjugacy classes and the irreducible characters of the groups in the series are parameterized in a suitable way, and the character values of the generic table are given as functions of these parameters. Here, a series is a set of groups arising from one particular Dynkin diagram with a fixed symmetry, when the underlying field is allowed to vary. Usually, there are a finite number of tables for a fixed Dynkin diagram with symmetry. For example, there are two generic tables for the series $SL_2(q)$, one for even and one for odd q.

Many computations with characters can be performed symbolically on such a generic table. One can compute scalar products of character types, calculate tensor products of characters or compute class multiplication coefficients. Such computations are valid for all groups in the series. The CHEVIE system (see also [13] and 4.2.9) contains a library of generic tables for Chevalley groups and a collection of MAPLE routines to perform such computations. As an application the 6-dimensional symplectic groups were shown to be Galois groups over abelian number fields [23].

Generic tables in form of programs have been implemented for Weyl groups of type A_n (i.e., the symmetric groups), B_n and D_n and are available in CHEVIE. The tables for the symmetric groups are also available in ACE, a MAPLE share package written by Sebastian Veigneau and in SYMMETRICA [19]. We are not aware of any implementation of generic character tables for the covering groups of the symmetric groups.

Related to the generic character tables of the Weyl groups are the generic character tables of the corresponding Iwahori-Hecke algebras, also available in CHEVIE, and, for type A, in ACE.

2.8.4 Summary of Systems

In this section I summarize the various systems presented above as well as some other packages not mentioned there. The most comprehensive systems for computational group and representation theory are GAP [12] and MAGMA [4].

Special purpose systems are AREP (Eindhoven, Pittsburgh) for computing symbolically with structured representations, CARAT (Aachen), for working with crystallographic groups, CHEVIE (Aachen, Kassel, Paris) for computing with generic character tables of Chevalley groups, Iwahori-Hecke algebras and Weyl groups, CREP (Bielefeld, see also 4.2.12) for the investigation of finite dimensional algebras, LIE (Eindhoven, see also 4.2.23) for computations with Lie algebras, Coxeter groups, and their representations, QUOTPIC (Warwick, see also 4.2.32), for the construction of quotients of finitely presented groups, SISYPHOS (Stuttgart), for computing in modular group algebras of finite p-groups, and SYMMETRICA (Bayreuth, see also 4.2.41), for combinatorics related to, and applications of the symmetric groups.

Combinatorics related to Lie algebras, Weyl groups, and symmetric functions are also contained in the two MAPLE share packages ACE by Sebastian Veigneau (see also 4.2.1) and SF by John Stembridge. Another MAPLE package, INVAR by Gregor Kemper, computes invariant rings of finite groups. The GAP share package SPECHT by Andrew Mathas contains algorithms for computing decomposition numbers of symmetric groups and Iwahori-Hecke algebras.

Finally, there is a large collection of stand alone programs related to the Meat-Axe, its *Condensation* enhancements, the *Vector-Enumerator* or MOC. Particular versions of these are available in GAP and MAGMA, as system commands as well as external packages.

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