THE INDUCTIVE BLOCKWISE ALPERIN WEIGHT CONDITION FOR THE CHEVALLEY GROUPS $F_4(q)$

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Abstract. We verify the inductive blockwise Alperin weight condition in odd characteristic ℓ for the finite exceptional Chevalley groups $F_4(q)$ for q not divisible by ℓ.

Contents

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1. Introduction

Alperin's weight conjecture, published in [1] in 1987, is one of the famous intriguing conjectures in the representation theory of finite groups. It postulates the coincidence of two invariants of a finite group, which are defined from global, respectively local data, and which seem unrelated at first sight. Given a prime ℓ and a finite group G, the global invariant in Alperin's weight conjecture is the number of conjugacy classes of G of elements of order prime to ℓ. This is the same as the number of absolutely irreducible ℓ-modular characters of G. The local invariant is the number of conjugacy classes of ℓ-weights of G. An ℓ-weight is a pair (Q, χ), where Q is a finite ℓ-subgroup of G, and χ is an irreducible character of $N_G(Q)/Q$ with the property that ℓ does not divide $|N_G(Q)/Q|/\chi(1)$. If (Q, χ) is an ℓ-weight, we also call Q a weight subgroup. In his original paper [1], Alperin proved the truth of his conjecture in many instances, for example in case G is a finite group of Lie type of characteristic ℓ. Soon after its appearance, the Alperin weight conjecture has been verified for various series of finite groups, in particular the alternating and symmetric groups and their Schur covering groups, as well as the general linear groups; see [3] and [65].

Beginning with a paper by Knörr and Robinson [52], the Alperin weight conjecture has generated a huge new field of research, including various reformulations, generalizations and reductions. The number of contributions in this direction is too large to be listed here.

Since the classification of the finite simple groups, it has become a common approach to reduce questions on general finite groups to related, often more complex questions on finite simple groups or their universal covering groups. A reduction theorem to Alperin's weight conjecture in this spirit has been established by Navarro and Tiep [67]. The authors formulated a set of conditions on a non-abelian finite simple group and all of its perfect covering groups. If this set of conditions is satisfied, the corresponding finite simple group is called AWC-good. The main theorem of [67] states that if all non-abelian finite simple groups are AWC-good, then the Alperin weight conjecture holds for all finite groups.

There is a blockwise version of Alperin's weight conjecture, already present in [1], and a reduction theorem for this blockwise version was proved by Späth in [75]. Refining the conditions for AWC-goodness, Späth introduced a collection of properties for a non-abelian finite simple group, called the *inductive blockwise Alperin weight condition* (inductive BAW condition for short). Naturally, this set of conditions, which is formulated for a fixed prime ℓ and a fixed ℓ-block, is even more intricate than the one of [67]. If the condition is satisfied for a prime ℓ and all ℓ-blocks of a non-abelian finite simple group G and its covering groups, we say the inductive BAW condition holds for G at the prime ℓ. According to the main theorem of [75], the blockwise Alperin weight conjecture for the prime ℓ holds for all finite groups if the inductive BAW condition is satisfied for all non-abelian finite simple groups at the prime ℓ. A simplified version of the inductive BAW conditions, adapted to our purposes, is presented in Hypothesis 2.13 below.

The inductive BAW condition has been verified for various series of simple groups, for example some groups of Lie type of small rank already by Späth in [75]. Schulte shows in [70] that the inductive BAW condition holds for the exceptional Chevalley groups $G_2(q)$ and the Steinberg triality groups. Malle verified the inductive BAW condition for the Suzuki and Ree groups in [62, Theorem 5.1]. For blocks with cyclic defect groups the inductive Alperin Weight condition holds by a theorem of Koshitani and Späth; see [54, Theorem 1.1]. Also, if G is a finite simple group of Lie type of characteristic ℓ, then the inductive BAW condition for G holds at the prime ℓ by a result of Späth; see [75, Theorem C]. Our article is a contribution to the programme of verifying the inductive BAW condition for the non-abelian finite simple groups.

Main Theorem. *Let ℓ be an odd prime and let $G = F_4(q)$ for a prime power q coprime to ℓ. Then the inductive blockwise Alperin weight condition holds for every ℓ-block of G.*

Let G and ℓ be as in the main theorem. The proof of this relies on a careful analysis of the ℓ-blocks of G and their invariants, as well as the candidates for the weight subgroups. The major part of our paper is devoted to the case $\ell = 3$. If $\ell > 3$, the Sylow ℓ-subgroups of G are abelian, and substantial results towards the main theorem are already contained in [62] by exhibiting natural bijections between weights and absolutely irreducible ℓ-modular characters, and it remains to establish equivariance of these bijections with respect to outer automorphisms of G. The classification of the ℓ-blocks follows the route laid out by the paper [14] by Broué and Michel, i.e. it is based on the classification of the semisimple conjugacy classes of elements of order prime to ℓ. The semisimple conjugacy classes are grouped into finitely many class types, and our results can be proved uniformly for all elements inside each class type. The class types and properties of the corresponding elements, in particular their centralizers are determined and enumerated in [58].

Our task is simplified to some extent as the Schur multiplier of G, with one exception, is trivial, as the outer automorphism group of G is cyclic and as all proper Levi subgroups of G are of classical type. The case $\ell = 3$ presents an interesting example for the verification of the inductive BAW condition, as the Sylow 3-subgroups of G are non-abelian. The results for the principal 3-block reveal a distinctive different behavior in the cases when 9 divides or does not divide q^2-1. Our investigations for $\ell = 3$ are largely supported by the fact that 3 is good for all proper Levi subgroups of G, and that the radical 3-subgroups of G, which are the candidates for the weight subgroups have been classified in [6] and [4]. The missing classification of the radical 2-subgroups of G prevents us from extending our results to the case $\ell = 2$. Many of our investigations are highly assisted by deep results of Bonnafé, Dat and Rouquier [9], as well as by recent work of Boltje and Perepelitsky [7].

Let us now comment on the contents of the individual sections of our article. Section 2 is devoted to the introduction of notation and background material on groups and representations, occasionally refined and extended for our purpose. Subsection 2.12 contains the version of the inductive BAW condition relevant to our investigations. In Section 3 we recall the principal concepts and results on finite groups of Lie type needed later on. In particular, we summarize the above mentioned theorems of Bonnafé, Dat and Rouquier and various others in Theorem 3.9 to have a convenient reference. Some consequences of this major theorem are derived. We also collect some auxiliary results useful in our later study. Section 4 introduces the group $F_4(q)$ as the group of fixed points under a Steinberg morphism of a simple algebraic group $\mathbf{G}$ of type F_4 over the algebraic closure of the field with q elements. We establish our notation for the corresponding root system and the Weyl group. We also introduce class types, and a duality of Levi subgroups of G arising from the fact that G is isomorphic to its dual group. We then investigate in great detail the structure of some Levi subgroups of G. This yields a first new result in our paper, Corollary 4.20, which states that the e-split Levi subgroups of G satisfy the maximal extendibility condition. This is a crucial ingredient in the proof of our main theorem and might be of independent interest. Section 5 is devoted to the description of the ℓ-blocks of G and some of their invariants for primes ℓ not dividing q. The description for the primes $\ell > 3$ was known before and is due to Broué, Malle and Michel [13], as well as Broué and Michel [14]; see also [62]. The main effort here is spent to handle the prime $\ell = 3$ for the non-unipotent blocks. The unipotent blocks have been treated by Enguehard in [27]. Although not pursued furthermore, we also include a subsection for the prime $\ell = 2$. The results are presented in form of tables in the Appendix; see Tables 1–19 and 21. The action of the outer automorphisms of G on the set of absolutely irreducible ℓ-modular characters of an invariant block is determined in Proposition 5.13.

Section 6 is dedicated solely to the prime 3. We recall the construction of the radical 3-subgroups of G established in [4] and [6] to some detail, as this will be important later on. The defect groups of the 3-blocks of G are, in particular, radical 3-subgroups. We describe these defect groups by identifying the corresponding conjugacy class of radical 3-subgroups. A preliminary result states that the Sylow 3-subgroups of the centralizers of semisimple elements are all radical 3-subgroups. It would be interesting to find an a priori reason for this observation, which might extend to other groups of Lie type. The weight subgroups are radical 3-subgroups, and we determine the candidates for the weight subgroups among the radical 3-subgroups.

Section 7 contains a proof of one half of the inductive BAW condition by giving a bijection between the absolutely irreducible ℓ-modular characters of a block and the conjugacy classes of weights associated to the block; see Theorem 7.1. It is worth remarking that the blocks in G are split in the sense that the associated canonical characters extend to their inertia subgroups.

Finally, Section 8 proves the equivariance of the bijections established in Theorem 7.1 with respect to outer automorphisms. This equivariance is established in Theorem 8.24 after a long series of rather technical and involved preparations. It is unfortunate that we were not able to find a uniform approach to these results. Instead, we develop numerous ad hoc methods for specific situations. On the other hand, the variety of the methods introduced here might facilitate analogous investigation for other exceptional groups of Lie type.

The referee of the first version of this article has pointed us to the preprint [30]. This reduces the proof of the equivariance condition for the non-quasi-isolated blocks of G to the verification of the inductive blockwise Alperin weight condition for quasi-isolated blocks of simple groups involved in G. At this time, to the best of our knowledge, this verification has not been established completely for all the cases relevant to our work, which includes groups such as $\mathrm{Sp}_6(q)$ or $\mathrm{Spin}_7(q)$. We have therefore decided to retain the original presentation of Section 8, as this is comprehensive and self-contained.

2. Notation and preliminaries

Throughout this section G denotes a group, which is assumed to be finite, except in Subsection 2.1 below.

2.1. **Groups.** Most of our notation for groups is standard. If $g, x \in G$ we write $g^x := x^{-1}gx$ for the right conjugation of g by x. This notation is extended to subsets of G. If $K, H \leq G$, then we write $K \leq_G H$

whenever there exists $x \in G$ with $K^x \leq H$. Analogously, we define $H =_G K$. For $H \leq G$, we write

$$\mathrm{Out}_G(H) := N_G(H)/HC_G(H).$$

Notice that if $H \leq M \leq G$, then $\mathrm{Out}_M(H)$ naturally embeds into $\mathrm{Out}_G(H)$. The commutator subgroup of G is denoted by $[G, G]$.

If p is a prime and G is a finite abelian p-group, we write $\Omega_1(G)$ for the subgroup of G generated by the elements of order p. Then $\Omega_1(G)$ is an elementary abelian p-group.

If n, m are positive integers, then $[n]$ denotes a cyclic group of order n, and $[n]^m$ the direct product of m copies of $[n]$. If $n \in \{2, 3, 6\}$, we also omit the outer brackets for simplicity of notation.

Our notation for simple groups and related groups follows one of the standard conventions from the literature.

Recall the notation $\mathrm{SL}_n^\varepsilon(q)$ and $\mathrm{GL}_n^\varepsilon(q)$ for $\varepsilon \in \{1, -1\}$: if $\varepsilon = 1$, then these are the special linear and general linear groups of degree n over the field $\mathbb{F}_q$; if $\varepsilon = -1$, then these are the corresponding special unitary and unitary groups, respectively, defined over $\mathbb{F}_{q^2}$. Recall that $\mathrm{SL}_2(q) = \mathrm{SL}_2^{-1}(q) = \mathrm{Sp}_2(q)$.

The notation for group extensions follows the Atlas [23] convention, i.e. $A.B$ is a group with a normal subgroup isomorphic to A and corresponding factor group isomorphic to B; see [23, Page xx]. If not indicated by brackets, we read group extensions $A.B.C$ from left to right, that is, $A.B.C = (A.B).C$.

2.2. **Characters and modular systems.** Let ℓ be a prime. Fix an ℓ-modular system $(\mathcal{K}, \mathcal{O}, \overline{\mathcal{O}})$ for G, where $\mathcal{O}$ is a complete discrete valuation ring of characteristic 0 with residue class field $\overline{\mathcal{O}}$ of characteristic ℓ and field of fractions $\mathcal{K}$, which is large enough for G, i.e. $\mathcal{K}$ contains a $|G|$th root of unity. In the following, the term ℓ-block refers to a block of $\overline{\mathcal{O}}G$ or its lift to $\mathcal{O}G$. If θ is a $\mathcal{K}$-valued class function of G, we write $\check{\theta}$ for the restriction of θ to the set of ℓ-regular elements of G. The set of ordinary irreducible characters (i.e. irreducible $\mathcal{K}$-characters) of G is denoted by $\mathrm{Irr}(G)$, and the subset of $\mathrm{Irr}(G)$ of ℓ-defect zero characters by $\mathrm{Irr}^0(G)$. We also write $\mathrm{IBr}_\ell(G)$ for the set of irreducible Brauer characters of G (with respect to $(\mathcal{K}, \mathcal{O}, \overline{\mathcal{O}})$). If B is a union of ℓ-blocks, we use the notation $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ for the sets of irreducible ordinary, respectively Brauer characters of B, and we write $\mathbb{Z}[\mathrm{Irr}(B)]$, respectively $\mathbb{Z}[\mathrm{IBr}(B)]$ for the corresponding sets of generalized characters.

Let $H \leq G$, and let χ and ψ denote $\mathcal{K}$-valued class functions on G, respectively H. Then $\mathrm{Res}_H^G(\chi)$ and $\mathrm{Ind}_H^G(\psi)$ denote the restriction of χ

to H, respectively the class function of G obtained by inducing ψ to G. If H is a finite group and χ and ψ class function of G, respectively H, with values in $\mathcal{K}$, we write $\chi \boxtimes \psi$ for the outer product of χ and ψ. This is a $\mathcal{K}$-valued class function of $G \times H$.

2.3. **Actions of automorphisms.** We collect some miscellaneous results on the action of automorphisms on characters.

Lemma 2.4. *Let $N \trianglelefteq G$ and let $\sigma \in \mathrm{Aut}(G)$ stabilize N. Assume that every coset of N in G contains a σ-stable element.*

Let $\lambda \in \mathrm{Irr}(G)$ with $\lambda(1) = 1$ such that $\mathrm{Res}^G_N(\lambda)$ is σ-stable. Then λ is σ-stable.

Proof. This is a straightforward computation using the fact that λ is a homomorphism. □

Lemma 2.5. *Let $N \trianglelefteq G$ with G/N a group of prime order. Further, let $\sigma \in \mathrm{Aut}(G)$ stabilize N and let $\chi \in \mathrm{Irr}(G)$. Suppose that $\mathrm{Res}^G_N(\chi)$ is σ-stable. If $\mathrm{Res}^G_N(\chi)$ is irreducible, assume that $0 \neq \chi(g) = {}^\sigma\chi(g)$ for some $g \in G \setminus N$. Then σ fixes χ.*

Proof. Suppose first that $\mathrm{Res}^G_N(\chi)$ is reducible, and let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of $\mathrm{Res}^G_N(\chi)$. By our assumption, $\chi = \mathrm{Ind}^G_N(\psi)$, and ${}^\sigma\chi = \mathrm{Ind}^G_N({}^\sigma\psi) = \chi$, as ${}^\sigma\psi$ is an irreducible constituent of $\mathrm{Res}^G_N(\chi)$.

Suppose next that $\mathrm{Res}^G_N(\chi)$ is irreducible. Then ${}^\sigma\chi$ is an extension of $\mathrm{Res}^G_N(\chi)$ to G. All such extensions are of the form $\chi\lambda$, where λ is an irreducible character of G/N. Hence $\chi(g) = {}^\sigma\chi(g) = \chi(g)\lambda(g)$ for some $\lambda \in \mathrm{Irr}(G/N)$. As $\chi(g) \neq 0$, this implies $\lambda(g) = 1$ and hence λ is the trivial character, as G/N has prime order. □

Let us record a corollary, which is relevant in our applications.

Corollary 2.6. *Let $M, N \trianglelefteq G$ such that G/N is a group of prime order and such that $G = MN$. Let $\sigma \in \mathrm{Aut}(G)$ stabilize M, N and let $\chi \in \mathrm{Irr}(G)$. Assume that $\mathrm{Res}^G_N(\chi)$ is reducible or that $\mathrm{Res}^G_{M\cap N}(\chi)$ is irreducible. Assume finally that $\mathrm{Res}^G_M(\chi)$ and $\mathrm{Res}^G_N(\chi)$ are σ-stable. Then χ is σ-stable.*

Proof. By Lemma 2.5 we may assume that $\mathrm{Res}^G_{M\cap N}(\chi)$ is irreducible. As $G/N \cong M/(M\cap N)$, there is $g \in M \setminus N$ such that $\chi(g) \neq 0$. Since $\mathrm{Res}^G_M(\chi)$ is σ-invariant, the claim follows from Lemma 2.5. □

We will need the following variant of Brauer's permutation lemma.

Lemma 2.7. *Let $m \leq n$ be positive integers, $U \in \mathcal{K}^{m\times n}$ of full rank, $M \in \mathrm{GL}_n(\mathcal{K})$. Suppose that there are permutation matrices P, P', Q of* *the appropriate sizes such that $UQ = PU$ and $MUQ = P'MU$. Then $P' = MPM^{-1}$, and thus P and P' have the same trace.*

Proof. We have $P'MU = MUQ = MPU$, and hence $P'M = MP$, as U has full rank. □

We indicate an application of this lemma.

Lemma 2.8. *Let σ be an automorphism of G and let B be a σ-stable union of ℓ-blocks of G. Suppose that $\mathcal{U}$ is a σ-stable set of $\mathcal{K}$-valued class functions of G with $|\mathcal{U}| = |\mathrm{IBr}(B)|$, such that $\check{\mathcal{U}} := \{\check{\theta} \mid \theta \in \mathcal{U}\}$ is a $\mathcal{K}$-basis of $\mathcal{K} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{IBr}(B)]$. Then the number of σ-stable elements of $\mathrm{IBr}(B)$ equals the number of σ-stable elements of $\mathcal{U}$.*

Proof. Write m and n for the number of irreducible Brauer characters in B and the number of ℓ-regular classes of G, respectively. Let $U \in \mathcal{K}^{m\times n}$ denote the character table of $\check{\mathcal{U}}$ and let Q denote the $(n \times n)$-permutation matrix arising from the permutation of σ on the set of ℓ-regular classes. As σ fixes $\mathcal{U}$, hence $\check{\mathcal{U}}$, there is a permutation matrix P such that $PU = UQ$. By assumption, there is $M \in \mathrm{GL}_m(\mathcal{K})$ such that MU is the Brauer character table of B. In particular, U has full rank. As σ fixes B, it stabilizes $\mathrm{IBr}(B)$, and thus there is a permutation matrix P' such that $P'MU = MUQ$. By Lemma 2.7, the permutation matrices P and P' have the same trace, and hence σ has the same number of fixed points on $\check{U}$ and on $\mathrm{IBr}(B)$. As U has full rank, it does not contain duplicate rows. Hence $\theta \in \mathcal{U}$ is fixed by σ, if and only if $\check{\theta}$ is fixed by σ. This proves our claim. □

2.9. **Central products.** Let $H_1, H_2 \leq G$ with $[H_1, H_2] = 1$, put $Z := H_1 \cap H_2$ and $H := H_1H_2$. Then $Z \leq Z(H)$ and H is a central product of H_1 and H_2 over Z, written as $H = H_1 \circ_Z H_2$. Let $U_i \leq H_i$ with $Z \leq U_i$ for $i = 1, 2$. Then $U := U_1U_2 = U_1 \circ_Z U_2$ and $N_H(U) = N_{H_1}(U_1)N_{H_2}(U_2) = N_{H_1}(U_1) \circ_Z N_{H_2}(U_2)$. If H_1 is abelian, we also have $C_H(U) = H_1C_{H_2}(U_2) = H_1 \circ_Z C_{H_2}(U_2)$ and thus $\mathrm{Out}_H(U) \cong \mathrm{Out}_{H_2}(U_2)$.

Every $\chi \in \mathrm{Irr}(H)$ can uniquely be written as $\chi = \chi_1\chi_2$ for $\chi_i \in \mathrm{Irr}(H_i)$, $i = 1, 2$ with $\chi_2(1)\mathrm{Res}^{H_1}_Z(\chi_1) = \chi_1(1)\mathrm{Res}^{H_2}_Z(\chi_2)$. (Under the latter assumption, the product $\chi_1\chi_2 : H \to \mathcal{K}, h_1h_2 \mapsto \chi_1(h_1)\chi_2(h_2)$ is well defined.) If $U_i \trianglelefteq H_i$ with $Z \leq U_i$ for $i = 1, 2$, and $\vartheta_i \in \mathrm{Irr}(U_i)$ for $i = 1, 2$ such that $\vartheta := \vartheta_1\vartheta_2 \in \mathrm{Irr}(U_1 \circ_Z U_2)$ is invariant in $H = H_1 \circ_Z H_2$, then each ϑ_i is invariant in H_i for $i = 1, 2$; moreover, ϑ extends to H, if and only if ϑ_i extends to H_i for $i = 1, 2$.

2.10. **Blocks and weights.** For easier reference we summarize a few well known results from modular representation theory of finite groups, thereby introducing our notation.

An ℓ-subgroup R of G is *radical* if $O_\ell(N_G(R)) = R$. We denote by $\mathcal{R}_\ell(G)$ the set of all radical ℓ-subgroups of G, and by $\mathcal{R}_\ell(G)/G$ the set of G-conjugacy classes of $\mathcal{R}_\ell(G)$.

Let b be an ℓ-block of G. We write $D(b)$ for the set of defect groups of b. We will use the concept of Brauer pairs in the following. An excellent reference for this notion, originally introduced by Alperin and Broué under the name of subpairs in [2], is [50, Sections 2, 3]. We will always implicitly assume that the Brauer pairs are defined with respect to ℓ, or, more precisely, with respect to the field $\overline{\mathcal{O}}$. Let (R, b_R) be a *Brauer pair*, i.e. R is an ℓ-subgroup of G and b_R is an ℓ-block of $C_G(R)$. For every subgroup $Q \leq R$, there is a unique Brauer pair (Q, b_Q) with $(Q, b_Q) \leq (R, b_R)$. Recall that (R, b_R) is called a b-Brauer pair, if $(\{1\}, b) \leq (R, b_R)$. Let (D, b_D) denote a maximal b-Brauer pair. Then $Z(D) \in D(b_D)$ and $D \in D(b)$. We call (R, b_R) *centric*, if $Z(R) \in D(b_R)$.

Let (Q, φ) be a *weight* of G, i.e. Q is an ℓ-subgroup of G and φ is an ℓ-defect 0 character of $N_G(Q)/Q$. Let b_Q be a block of $C_G(Q)$ covered by the block of $N_G(Q)$ containing (the inflation of) φ. Then (Q, b_Q) is a centric Brauer pair and $Q \in \mathcal{R}_\ell(G)$. If (Q, b_Q) is a b-Brauer pair, we say that (Q, φ) is a b-weight. In this case, we also have

$$Z(D) \leq Z(R) \leq R \leq D \tag{1}$$

for some conjugate R of Q; see, e.g. [66, Chapter 5, Theorem 5.21]. This fact will be used frequently in the following. If Q is an ℓ-subgroup such that there exists a (b-)weight (Q, φ), we call Q a (b-)*weight subgroup*.

Let $R \in \mathcal{R}_\ell(G)$ and let b_R be an ℓ-block of $C_G(R)$. We write $N_G(R, b_R)$ for the stabilizer of b_R in $N_G(R)$ and put

$$\mathrm{Out}_G(R, b_R) := N_G(R, b_R)/C_G(R)R.$$

Recall that $\mathrm{Out}_G(D, b_D)$ is an ℓ'-group if, as above, D is a defect group of the ℓ-block b of G and (D, b_D) is a maximal b-Brauer pair.

Now assume that (R, b_R) is centric. In this case, we denote by $\theta_R \in \mathrm{Irr}(C_G(R))$ the canonical character of b_R, i.e. the unique ordinary irreducible character in b_R with $Z(R)$ in its kernel. We may and will also view θ_R as a character of $C_G(R)R$ via inflation over R. Write $N_G(R, \theta_R)$ for the stabilizer of θ_R in $N_G(R)$. This notion is independent of whether we view θ_R as a character of $C_G(R)$ or as one of $C_G(R)R$. Clearly, $N_G(R, \theta_R) = N_G(R, b_R)$ and thus $\mathrm{Out}_G(R, b_R) = N_G(R, \theta_R)/C_G(R)R$.

Put

$$\mathrm{Irr}^0(N_G(R,\theta_R) \mid \theta_R) :=$$
$$\{\zeta \in \mathrm{Irr}(N_G(R,\theta_R)) \mid \zeta(1)_\ell = |N_G(R,\theta_R){:}R|_\ell, \text{ and } \zeta \text{ covers } \theta_R\},$$

and

$$\mathcal{W}(R, b_R) := |\mathrm{Irr}^0(N_G(R,\theta_R) \mid \theta_R)|. \tag{2}$$

By Schur's theory of projective characters, the fact that θ_R is invariant under $N_G(R,\theta_R)$ yields an element $\alpha \in H^2(\mathrm{Out}_G(R,\theta_R), \mathcal{K}^\times)$, called the *Külshammer-Puig* class associated to the centric Brauer pair (R, b_R). If $\mathcal{K}_\alpha(\mathrm{Out}_G(R,\theta_R))$ denotes the corresponding twisted group algebra, we have

$$\mathcal{W}(R, b_R) = |\mathrm{Irr}^0(\mathcal{K}_\alpha \mathrm{Out}_G(R,\theta_R))|.$$

In particular, if θ_R, viewed as a character of $C_G(R)R$, extends to $N_G(R,\theta_R)$, then

$$\mathcal{W}(R, b_R) = |\mathrm{Irr}^0(\mathrm{Out}_G(R, b_R))| \tag{3}$$

by Clifford theory.

If (R, b_R) is a b-Brauer pair and $\zeta \in \mathrm{Irr}^0(N_G(R,\theta_R) \mid \theta_R)$, then $\mathrm{Ind}_{N_G(R,\theta_R)}^{N_G(R)}(\zeta)$ is a b-weight. By choosing a maximal b-Brauer pair (D, b_D) and applying this construction to all centric b-Brauer pairs (R, b_R) with $(R, b_R) \leq (D, b_D)$, we obtain a set of representatives for the G-conjugacy classes of b-weights, the number of which is denoted by $\mathcal{W}(b)$ in the following.

Proposition 2.11. *Fix an ℓ-block b of G and a maximal b-Brauer pair (D, b_D). Then*

$$\mathcal{W}(b) = \sum_R \mathcal{W}(R, b_R), \tag{4}$$

where $R \in \mathcal{R}_\ell(G)$ runs through a set of G-conjugacy class representatives satisfying $R \leq D$ and the unique Brauer pair (R, b_R) with $(R, b_R) \leq (D, b_D)$ is centric.

Suppose that D is abelian. Then the only summand in Equation (4) *is the one for $R = D$, and thus $\mathcal{W}(b) = \mathcal{W}(D, b_D)$.*

Proof. The first statement is due to Alperin and Fong; see the discussion in [3, Page 3]. If D is abelian, then $R = D$ by Equation (1), and the second statement follows. □

Assume the situation and notation of Proposition 2.11 and that D is abelian. Then θ_D extends to $N_G(D,\theta_D)$ if $\mathrm{Out}_G(D, b_D)$ is cyclic, or if θ_D is linear and $C_G(D)$ has a complement in $N_G(D,\theta_D)$.

Alperin's weight conjecture for b postulates the equality $\mathcal{W}(b) = |\mathrm{IBr}(b)|$.

2.12. **The inductive blockwise Alperin weight condition.** The following hypotheses constitute a simplified version of Conditions (i) and (ii) of [54, Definition 3.2], adapted to our purpose. The latter, in turn, is a specialization of [75, Definitions 4.1, 5.17].

Hypothesis 2.13. Let ℓ be a prime, G a finite non-abelian simple group with trivial Schur multiplier, and let b be an ℓ-block of G. Write $N_{\mathrm{Aut}(G)}(b)$ for the stabilizer in $\mathrm{Aut}(G)$ of b. Assume that the following conditions are satisfied.

(i) For every $Q \in \mathcal{R}_\ell(G)$, there exists a subset $\mathrm{IBr}(b \mid Q)$ of $\mathrm{IBr}(b)$ satisfying the following conditions.

(1) If $\alpha \in N_{\mathrm{Aut}(G)}(b)$, then $\mathrm{IBr}(b \mid Q)^\alpha = \mathrm{IBr}(b \mid Q^\alpha)$.

(2) $\mathrm{IBr}(b)$ is the disjoint union of all $\mathrm{IBr}(b \mid Q)$, where Q runs through some set of representatives of $\mathcal{R}_\ell(G)/G$.

(ii) For every $Q \in \mathcal{R}_\ell(G)$, there is a bijection

$$\Omega_Q^G \colon \mathrm{IBr}(b \mid Q) \to \mathrm{Irr}^0(N_G(Q)/Q, b),$$

such that for all $\alpha \in N_{\mathrm{Aut}(G)}(b)$ and $\varphi \in \mathrm{IBr}(b \mid Q)$ we have that $\Omega_Q^G(\varphi)^\alpha = \Omega_{Q^\alpha}^G(\varphi^\alpha)$. Here, $\mathrm{Irr}^0(N_G(Q)/Q, b)$ consists of all characters $\varphi \in \mathrm{Irr}^0(N_G(Q)/Q)$ such that (Q, φ) is a b-weight.

Remark 2.14. Let G and ℓ be as in Hypothesis 2.13. Suppose in addition that $\mathrm{Out}(G)$ cyclic. If Hypothesis 2.13 is satisfied for every ℓ-block b of G, then G satisfies the *inductive blockwise Alperin weight condition at the prime* ℓ. This is proved in [75, Lemma 6.1] under the stronger condition that G is AWC-good, but the proof of this lemma only uses Hypothesis 2.13. We are grateful to Britta Späth for pointing out the relevance of [75, Lemma 6.1] for this reduction, and for clarifying remarks regarding this issue. □

We formulate a set of conditions which simplifies the verification of Hypothesis 2.13.

Hypothesis 2.15. Let G, ℓ and b be as in Hypothesis 2.13. Put $A := N_{\mathrm{Aut}(G)}(b)$. Let $Q_1, \ldots, Q_n$ denote a set of representatives for the G-conjugacy classes in the set $\{Q \in \mathcal{R}_\ell(G) \mid \mathrm{Irr}^0(N_G(Q)/Q, b) \neq \emptyset\}$.

(1) There are pairwise disjoint subsets $\mathrm{IBr}(b \mid Q_i)$, $1 \leq i \leq n$ of $\mathrm{IBr}(b)$ such that

$$\bigcup_{i=1}^n \mathrm{IBr}(b \mid Q_i) = \mathrm{IBr}(b),$$

and there are bijections

$$\Omega_i : \mathrm{IBr}(b \mid Q_i) \to \mathrm{Irr}^0(N_G(Q_i)/Q_i, b)$$

for $1 \leq i \leq n$.

(2) Suppose that (1) holds and, in addition, that

$$\mathrm{IBr}(b \mid Q_i)^\alpha = \mathrm{IBr}(b \mid Q_j) \quad \text{and } \Omega_j(\varphi^\alpha) = \Omega_i(\varphi)^\alpha$$

for all $1 \leq i, j \leq n$, all $\alpha \in A$ with $Q_i^\alpha = Q_j$ and all $\varphi \in \mathrm{IBr}(b \mid Q_i)$. □

Remark 2.16. Let G, ℓ and b be as in Hypothesis 2.13, and put $A := N_{\mathrm{Aut}(G)}(b)$.

(a) Assume that both conditions of Hypothesis 2.15 hold. Let $Q \in \mathcal{R}_\ell(G)$ be G-conjugate to Q_i for some $1 \leq i \leq n$, say $Q = Q_i^g$ for some $g \in G$. Define $\mathrm{IBr}(b \mid Q) := \mathrm{IBr}(b \mid Q_i)^g$, and $\Omega_Q^G : \mathrm{IBr}(b \mid Q) \to \mathrm{Irr}^0(N_G(Q)/Q, b), \varphi^g \mapsto \Omega_i(\varphi)^g$ for $\varphi \in \mathrm{IBr}(b \mid Q_i)$. Then $\mathrm{IBr}(b \mid Q)$ and Ω_Q^G are well-defined, and the collection of theses sets and maps satisfies Hypothesis 2.13.

(b) Condition (1) of 2.15 is satisfied if and only if the Alperin weight conjecture holds for b. Suppose this is the case. If the G-orbit of some Q_i equals its A-orbit, e.g. if $n = 1$ which occurs if b has abelian defect, then Condition (2) amounts to an equivariance condition with respect to the action of $N_A(Q_i)$. □

In the case of our interest, $G = F_4(q)$, every G-orbit of every Q_i is A-invariant, except for one instance where the A-orbit of some Q_i splits into two G-orbits.

3. Finite reductive groups

We continue by recalling some basic concepts and results from the theory of finite reductive groups and their representations, to the extent needed later on.

3.1. **Notation.** Let p be a prime number and let $\mathbb{F}$ denote an algebraic closure of the finite field with p elements. Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{F}$. We also let F denote a Steinberg endomorphism of $\mathbf{G}$. Let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$. We then write $\mathbf{H}^\circ$ for the connected component of $\mathbf{H}$ containing 1, and if $\mathbf{H}$ is F-stable we write $H := \mathbf{H}^F$ for the finite group of F-fixed points of $\mathbf{H}$. Thus $G = \mathbf{G}^F$ is a finite reductive group. We also let $\mathbf{G}^*$ denote a group dual to $\mathbf{G}$ (with respect to a fixed F-stable maximally split torus of $\mathbf{G}$), endowed with a dual Steinberg endomorphism F^*.

We say that $\mathbf{G}$ is of classical type, if every minimal F-stable semisimple component $\mathbf{H}$ of $[\mathbf{G},\mathbf{G}]$ is of Dynkin type A, B, C or D, and if $\mathbf{H}^F$ is not isomorphic to ${}^3D_4(q)$ for some power q of p.

Finally, let ℓ be a prime different from p.

3.2. **Recollections and preliminary results.** If $\mathbf{G}$ has connected center, the centralizers of semisimple elements in $\mathbf{G}^*$ are connected. In this case, two semisimple elements of G^* are conjugate in G^*, if and only if they are conjugate in $\mathbf{G}^*$.

An F-stable Levi subgroup of $\mathbf{G}$ is called a *regular subgroup* of $\mathbf{G}$. The regular subgroups of $\mathbf{G}$ are exactly the centralizers of F-stable tori. If $\mathbf{G}$ has connected center, so has any regular subgroup of $\mathbf{G}$; see [22, Proposition 8.1.4]. If $\mathbf{L} \leq \mathbf{G}$ is regular in $\mathbf{G}$, and $\mathbf{M} \leq \mathbf{L}$ is regular in $\mathbf{L}$, then $\mathbf{M}$ is regular in $\mathbf{G}$. If $\mathbf{M} \leq \mathbf{G}$ is a regular subgroup, we write $W_{\mathbf{G}}(\mathbf{M}) := N_{\mathbf{G}}(\mathbf{M})/\mathbf{M}$ for the relative Weyl group of $\mathbf{M}$ in $\mathbf{G}$. Then $W_{\mathbf{G}}(\mathbf{M})^F = N_G(\mathbf{M})/M$.

The following lemma will be used to identify centralizers of 3-elements in $F_4(q)$.

Lemma 3.3. *Assume that $Z(\mathbf{G})$ is connected. Let $\mathbf{H}$ be an F-stable closed subgroup of $\mathbf{G}$ such that $\mathbf{H}$ is reductive and $Z(\mathbf{H})$ is not connected. Let $s \in Z(\mathbf{H})^F$ be of order coprime to $|(Z(\mathbf{H})/Z(\mathbf{H})^\circ)^F|$. Then $\mathbf{H} \lneq C_{\mathbf{G}}(s)$.*

Proof. Our assumption implies that $s \in (Z(\mathbf{H})^\circ)^F \leq Z(\mathbf{H})^\circ$. As $\mathbf{H}^\circ$ is reductive, $Z(\mathbf{H})^\circ$ is a torus, and thus $C_{\mathbf{G}}(Z(\mathbf{H})^\circ)$ is a regular subgroup of $\mathbf{G}$, hence has connected center. It follows that $\mathbf{H} \lneq C_{\mathbf{G}}(Z(\mathbf{H})^\circ) \leq C_{\mathbf{G}}(s)$, proving our claim. □

We will also need the following slight generalization of [39, Proposition 4.2].

Proposition 3.4. *Let $\mathcal{S}_\ell(G)$ and $\mathcal{S}_\ell(G^*)$ denote sets of representatives for the conjugacy classes of ℓ-elements of G, respectively G^*. Suppose that ℓ does not divide the determinant of the Cartan matrix of the root system of $\mathbf{G}$ and that centralizers of ℓ-elements in $\mathbf{G}$ and $\mathbf{G}^*$ are connected.*

Then there is a bijection $\mathcal{S}_\ell(G) \to \mathcal{S}_\ell(G^*)$, $t \mapsto t'$ such that the following holds. If $C_{\mathbf{G}}(t)$ is a regular subgroup of $\mathbf{G}$, then $C_{\mathbf{G}^*}(t')$ is a regular subgroup of $\mathbf{G}^*$, and there is an F-equivariant isomorphism $C_{\mathbf{G}}(t) \to C_{\mathbf{G}^*}(t')^*$.

Proof. This is contained in the proof of [39, Proposition 4.2]. □

Assume the hypotheses and the notation of the above proposition. If $C_{\mathbf{G}}(t)$ is not a regular subgroup of $\mathbf{G}$, its dual group is not necessarily the centralizer of a semisimple element. An example is provided by $\mathbf{G} = G_2(\mathbb{F})$, the simple group of type G_2 over $\mathbb{F}$ when $p \neq 3$. Then $\mathbf{G} \cong \mathbf{G}^*$, and the hypotheses of Proposition 3.4 are satisfied. There is an element s of order 3 whose centralizer is isomorphic to $\mathrm{SL}_3(\mathbb{F})$; see [43, Table 4.7.1]. If F is such that $G = \mathbf{G}^F = G_2(q)$ with $3 \mid q-1$, then G contains a representative t of the $\mathbf{G}$-conjugacy class of s, so that $C_G(t) \cong \mathrm{SL}_3(q)$. However, $C_{\mathbf{G}}(t)^* \cong \mathrm{PGL}_3(\mathbb{F})$ has trivial center.

3.5. **Character groups and cocharacter groups of maximal tori.** Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$. Then $X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbb{F}^*)$ and $Y(\mathbf{T}) := \mathrm{Hom}(\mathbb{F}^*, \mathbf{T})$ denote the *character group* and the *cocharacter group* of $\mathbf{T}$, respectively. By $\langle\ ,\ \rangle$ we denote the natural pairing between $X(\mathbf{T})$ and $Y(\mathbf{T})$. We choose an isomorphism

$$\iota : \mathbb{F}^* \to \mathbb{Q}_{p'}/\mathbb{Z} \tag{5}$$

(see [22, Proposition 3.1.3]). This gives rise to an isomorphism of abelian groups $Y(\mathbf{T}) \otimes \mathbb{F}^* \cong Y(\mathbf{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z}$, and, in turn, to an isomorphism

$$Y(\mathbf{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z} \to \mathbf{T} \tag{6}$$

(see [22, Proposition 3.1.2]). Under this isomorphism, T corresponds to the kernel of $F-1$ on $Y(\mathbf{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z}$. This yields a further isomorphism

$$Y(\mathbf{T})/(F-1)Y(\mathbf{T}) \to T \tag{7}$$

(see [22, Proposition 3.2.2] or [25, Proposition 11.1.7(ii)]). Similarly, there is an isomorphism

$$(\text{kernel of } F-1 \text{ on } X(\mathbf{T})) \otimes \mathbb{Q}_{p'}/\mathbb{Z} \to \mathrm{Irr}(T) \tag{8}$$

(see [22, Proposition 3.2.4]). Now let $\chi \in X(\mathbf{T})$ and $a \in \mathbb{Q}_{p'}/\mathbb{Z}$ such that $\chi \otimes a \in X(\mathbf{T}) \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ lies in the kernel of $F-1$, and let $\lambda \in \mathrm{Irr}(T)$ correspond to $\chi \otimes a$ under the isomorphism (8). Next, let $\gamma \in Y(\mathbf{T})$, and let $t \in T$ correspond to $\gamma + (F-1)Y(\mathbf{T}) \in Y(\mathbf{T})/(F-1)Y(\mathbf{T})$ under the isomorphism (7). Then

$$\lambda(t) = \exp(2\pi\sqrt{-1}\langle \chi, \gamma \rangle a), \tag{9}$$

where exp is the exponential function of $\mathbb{C}$, and where the $|G|$th roots of unity of $\mathbb{C}$ are identified with the $|G|$th roots of unity of $\mathcal{K}$ (recall that $\mathcal{K}$ is our large enough field of characteristic 0 containing the character values).

3.6. **Lusztig induction.** If $\mathbf{M}$ is a regular subgroup of $\mathbf{G}$, we write $R_{\mathbf{M}}^{\mathbf{G}}$ for the Lusztig induction map from the class functions of M to the class functions of G. Strictly speaking, this map also depends upon a parabolic subgroup $\mathbf{P}$ containing $\mathbf{M}$ as a Levi complement, so that we should write $R_{\mathbf{M}\leq\mathbf{P}}^{\mathbf{G}}$. We will always implicitly assume that the Mackey formula holds for $\mathbf{G}$, in which case $R_{\mathbf{M}\leq\mathbf{P}}^{\mathbf{G}}$ is independent of such $\mathbf{P}$; see [42, Theorem 3.3.8]. We therefore omit the $\mathbf{P}$ from the notation. By [10], the Mackey formula holds for all connected reductive groups relevant to this work. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(T)$. Suppose that the pair $(\mathbf{T}^*, s)$, where $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in T^*$ corresponds to $(\mathbf{T}, \theta)$ via duality; see [25, Proposition 11.1.16]. We then also write $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, as in [25, p. 167] or [19, Remark 8.22(i)].

Lemma 3.7. *Let $s \in G^*$ be a semisimple ℓ'-element, t an ℓ-element in $C_{G^*}(s)$ and put $\mathbf{C}^* := C_{\mathbf{G}^*}(st)$. Suppose that one of the following conditions is satisfied:*

(a) *The subgroup $\mathbf{C}^* \leq \mathbf{G}^*$ is regular.*

(b) *There is a regular subgroup $\mathbf{M}^* \leq \mathbf{G}^*$ with $\mathbf{C}^* \leq \mathbf{M}^*$ such that ℓ is good for $\mathbf{M}$ and $\ell \nmid |Z(\mathbf{M})/Z(\mathbf{M})^\circ|$, where $\mathbf{M}$ is a regular subgroup of $\mathbf{G}$ dual to $\mathbf{M}^*$.*

(c) *The center of $\mathbf{G}$ is connected and every unipotent character of C^* is uniform.*

Then the class functions $\check{\theta}$ for $\theta \in \mathcal{E}(G, st)$ lie in the $\mathbb{Z}$-span of $\{\check{\chi} \mid \chi \in \mathcal{E}(G, s)\}$ if (a) *or* (b) *holds, and in the $\mathbb{Q}$-span of this set, otherwise.*

Proof. (a) If $\mathbf{C}^*$ is regular, the claim follows with exactly the same proof as that of [39, Theorem 3.1] (with L' replaced by $\mathbf{C}^*$).

(b) Lusztig induction $R_{\mathbf{M}}^{\mathbf{G}}$ establishes a bijection, up to a sign, between $\mathcal{E}(M, st)$ and $\mathcal{E}(G, st)$; see [25, (11.4.3(ii))]. Let $\theta \in \mathcal{E}(G, st)$. Then there is $\mu \in \pm\mathcal{E}(M, st)$ such that $\theta = R_{\mathbf{M}}^{\mathbf{G}}(\mu)$. Denote by γ the characteristic function on the set of ℓ-regular elements of G (or any subgroup of G). Then $\gamma \cdot \theta = \gamma \cdot R_{\mathbf{M}}^{\mathbf{G}}(\mu) = R_{\mathbf{M}}^{\mathbf{G}}(\gamma \cdot \mu)$; see [25, Proposition 10.1.6]. As ℓ is good for $\mathbf{M}$ and does not divide $|Z(\mathbf{M})/Z(\mathbf{M})^\circ|$, we find elements $\nu_1, \ldots, \nu_d$ in $\mathcal{E}(M, s)$ and integers $z_1, \ldots, z_d$ such that $\gamma \cdot \mu = \sum_{i=1}^d z_i(\gamma \cdot \nu_i)$; see [36, Theorem A]. It follows that $\gamma \cdot \theta = \sum_{i=1}^d z_i(\gamma \cdot R_{\mathbf{M}}^{\mathbf{G}}(\nu_i))$. As the irreducible constituents of $R_{\mathbf{M}}^{\mathbf{G}}(\nu_i)$ lie in $\mathcal{E}(G, s)$ for all $1 \leq i \leq d$ (see [19, Proposition 15.7]), our claim follows.

(c) If every unipotent character of C^* is uniform, the same is true for the elements of $\mathcal{E}(G, st)$. Indeed, the assumption implies that the matrix of scalar products between the elements of $\mathcal{E}(C^*, 1)$ and the Deligne–Lusztig characters $R_{\mathbf{T}^*}^{\mathbf{C}^*}(1_{T*})$, where $\mathbf{T}^*$ runs through a set of representatives of the C^*-conjugacy classes of F^*-stable maximal tori of $\mathbf{C}^*$, is square and non-singular. By the compatibility of Lusztig's Jordan decomposition of characters with Deligne–Lusztig induction (see, e.g. [19, Theorem 15.8]), this implies that every element of $\mathcal{E}(G, st)$ is a $\mathbb{Q}$-linear combination of Deligne–Lusztig characters $R^{\mathbf{G}}_{\mathbf{T}^*}(st)$ for F-stable maximal tori $\mathbf{T}$ of $\mathbf{G}$ such that st is contained in a dual $\mathbf{T}^*$ of $\mathbf{T}$. It follows from [25, Proposition 10.1.6] that $R^{\mathbf{G}}_{\mathbf{T}^*}(st)$ and $R^{\mathbf{G}}_{\mathbf{T}^*}(s)$ agree on ℓ-regular elements of G, yielding our assertion. □

Notice that the second condition on ℓ in (b) above is satisfied if $Z(\mathbf{G})$ is connected.

3.8. **Some major results and their consequences.** We will also use the following combination of major results of Bonnafé-Dat-Rouquier, Boltje-Perepelitsky, Cabanes-Enguehard and Puig. The first of these papers generalizes results of Bonnafé-Rouquier [11] and Kessar-Malle [51].

Theorem 3.9. *Let $s \in G^*$ be a semisimple ℓ'-element and assume that $C_{\mathbf{G}^*}(s)$ is connected. Put $\mathbf{M}^* := C_{\mathbf{G}^*}(Z(C_{\mathbf{G}^*}(s))^\circ)$. Then $\mathbf{M}^*$ is a regular subgroup of $\mathbf{G}^*$ minimal with the property that $C_{\mathbf{G}^*}(s) \leq \mathbf{M}^*$. Let $\mathbf{M}$ be a regular subgroup of $\mathbf{G}$ dual to $\mathbf{M}^*$.*

(a) (Bonnafé, Dat, Rouquier [9, Theorems 1.1, 7.7]) *There is a bijection $b \mapsto b'$ between the ℓ-blocks contained in $\mathcal{E}_\ell(G, s)$ and those in $\mathcal{E}_\ell(M, s)$ such that b and b' are Morita equivalent and have a common defect group $D \leq M$.*

(b) (Bonnafé, Dat, Rouquier [9, Theorems 1.1, 7.7]), Puig, [69, Theorem 19.7], Boltje, Perepelitsky [7, Theorems 11.2, 13.4]). *Let b and b' be blocks corresponding as in* (a) *and let $D \leq M$ be a common defect group of b and b'. Then there is a maximal b-Brauer pair (D, b_D) and a maximal b'-Brauer pair (D, b'_D), such that the following statements hold.*

There is a bijection $(R, b_R) \mapsto (R, b'_R)$ between the b-Brauer pairs $(R, b_R) \leq (D, b_D)$ and the b'-Brauer pairs $(R, b'_R) \leq (D, b'_D)$ such that $\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_M(R, b'_R)$. Moreover, (R, b_R) is centric if and only if (R, b'_R) is. In the latter case, the Külshammer-Puig classes α and α' associated to (R, b_R) and (R, b'_R), respectively, correspond to each other under the isomorphism of cohomology groups induced by the isomorphism $\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_M(R, b'_R)$. In particular, $\mathcal{W}_G(R, b_R) = \mathcal{W}_M(R, b'_R)$.

(c) (Cabanes, Enguehard [18]) *If $C_{\mathbf{G}^*}(s) \lneq \mathbf{M}^*$, assume that $\mathbf{M}$ has connected center, is of classical type and that ℓ is odd. Then there is a bijection between the ℓ-blocks contained in $\mathcal{E}_\ell(M, s)$ and those in*

$\mathcal{E}_\ell(C_{G^*}(s),1)$. *Moreover, corresponding blocks have the same number of irreducible ℓ-modular characters and isomorphic defect groups.*

PROOF. Part (a) is directly taken from the cited reference, so it suffices to prove (b) and (c).

(b) The Morita equivalence in (a) is induced by a splendid tilting complex C; see the proof of [9, Theorem 7.7]. Let (D, b_D) be a maximal b-Brauer pair. By [69, Theorem 19.7], there is a maximal b'-Brauer pair and an equivalence between the fusion systems associated to (D, b_D) and (D, b'_D). This yields all but the last statement of (b).

To obtain the last assertion, we follow [7] (which also yields the other statements of (b)). The indecomposable direct summands of the complex C have vertices contained in the diagonal $\Delta(D) \leq G \times M$ by [9, Corollary 3.8]. Hence C yields an ℓ-permutation equivalence between b and b' in the sense of [7, Definition 9.8]. The existence of (D, b_D) and (D, b'_D) follows from [7, Theorem 10.11]. The equivalence between the fusion systems associated to (D, b_D) and (D, b'_D) follows from [7, Theorem 11.2]. The correspondence of the Külshammer-Puig classes α and α' associated to (R, b_R) and (R, b'_R), respectively, follows from [7, Theorem 13.4]. This implies that the twisted group algebras $\mathcal{K}_\alpha \mathrm{Out}_G(R, b_R)$ and $\mathcal{K}_{\alpha'} \mathrm{Out}_M(R, b'_R)$ are isomorphic. As $\mathcal{W}_G(R, b_R) = |\mathrm{Irr}^0(\mathcal{K}_\alpha \mathrm{Out}_G(R, b_R))|$, the last claim follows.

(c) If $C_{\mathbf{G}^*}(s) = \mathbf{M}^*$, the result follows from (a), as the blocks of $\mathcal{E}_\ell(C_{G^*}(s), s)$ and $\mathcal{E}_\ell(C_{G^*}(s), 1)$ are Morita equivalent. The Morita equivalence is given by tensoring with a linear character (see, e.g. [25, Proposition 11.4.8(ii)]) and preserves defect groups.

Now assume that $C_{\mathbf{G}^*}(s) \lneqq \mathbf{M}^*$. Then $\mathbf{M}^*$ is of classical type and ℓ is odd by assumption. As the Mackey formula holds for regular subgroups of $\mathbf{M}$ by [10], the Jordan decomposition of characters for M is compatible with Lusztig induction by [28, Proposition 5.3]; see also [42, Theorem 4.7.2]. The classification of ℓ-blocks of M by Cabanes and Enguehard in [18, Theorem 3.3] then implies that there is a bijection $B \mapsto b$ between the ℓ-blocks B in $\mathcal{E}_\ell(M, s)$ and the ℓ-blocks b in $\mathcal{E}_\ell(C_{M^*}(s), 1) = \mathcal{E}_\ell(C_{G^*}(s), 1)$ such that $|\mathrm{Irr}(B) \cap \mathcal{E}_\ell(M, s)| = |\mathrm{Irr}(b) \cap \mathcal{E}_\ell(C_{M^*}(s), 1)|$. By [39, Theorem 5.1], this implies that B and b have the same number of irreducible ℓ-modular characters, as $\mathbf{M}$ has connected center. Finally, by [18, Remark 3.6], the bijection $B \mapsto b$ preserves isomorphism types of defect groups. □

The following generalizations of a result of Broué and Michel [14, Théorème 3.2] are extremely useful in identifying canonical characters of blocks. It holds under much weaker hypotheses, but its derivation from [9, Theorem 4.14] is simplified under the stronger conditions given.

Lemma 3.10 (Bonnafé, Dat, Rouquier [9]). *Let $s \in G^*$ be a semisimple ℓ'-element such that $C_{\mathbf{G}^*}(s)$ is connected and let $b \subseteq \mathcal{E}_\ell(G,s)$ be an ℓ-block of G. Let (R, b_R) be a b-Brauer pair with R abelian such that $C_{\mathbf{G}}(R)$ and $C_{\mathbf{G}}(y)$ are regular subgroups of $\mathbf{G}$ for all $y \in R$. Let $C_{\mathbf{G}}(R)^*$ denote a regular subgroup of $\mathbf{G}^*$ dual to $C_{\mathbf{G}}(R)$.*

Let $\theta \in \mathrm{Irr}(b_R)$ and suppose that $\theta \in \mathcal{E}(C_G(R), t)$ for some semisimple ℓ'-element $t \in (C_{\mathbf{G}}(R)^*)^{F^*}$. Then t is conjugate to s in G^*. If $C_{\mathbf{G}}(R)^* = C_{\mathbf{G}^*}(R^\diamond)$ for some subgroup $R^\diamond \leq G^*$, there is a G^* conjugate $R^\dagger \leq G^*$ such that $R^\dagger \leq C_{G^*}(s)$.

Proof. This follows from [9, Theorem 4.14]. □

Lemma 3.10 shows the relevance of regular centralizers of ℓ-subgroups. We give a criterion for this property to hold.

Lemma 3.11. *Assume that $\ell \nmid |Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ|$. Let $R \leq G$ be an abelian ℓ-subgroup. Suppose that there is $z \in R$ such that $\mathbf{L} := C_{\mathbf{G}}(z)$ is a regular subgroup of $\mathbf{G}$ and such that ℓ is good for $\mathbf{L}$. Then $C_{\mathbf{G}}(R)$ is regular in $\mathbf{G}$.*

Proof. As R is abelian, we have $R \leq C_{\mathbf{G}}(z)$. Moreover, $C_{\mathbf{G}}(R) \leq \mathbf{L}$, and thus $C_{\mathbf{G}}(R) = C_{\mathbf{L}}(R)$. If $\mathbf{L}^*$ denotes a regular subgroup of $\mathbf{G}^*$ dual to $\mathbf{L}$, we have $\ell \nmid |Z(\mathbf{L}^*)/Z(\mathbf{L}^*)^\circ|$; see, e.g. [8, Proposition 4.2]. By replacing $\mathbf{G}$ with $\mathbf{L}$, we may thus assume that ℓ is good for $\mathbf{G}$. In this case we prove the assertion by induction on $|G|$.

If $R \leq Z(\mathbf{G})$, there is nothing to prove. Thus assume that there is $y \in R \setminus Z(\mathbf{G})$, and put $\mathbf{M} := C_{\mathbf{G}}(y)$. Then $\mathbf{M}$ is a regular subgroup of $\mathbf{G}$, as y is an ℓ-element, $\ell \nmid |Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ|$, and ℓ is good for $\mathbf{G}$; see [39, Corollary 2.6]. It follows that $\ell \nmid |Z(\mathbf{M}^*)/Z(\mathbf{M}^*)^\circ|$, and that ℓ is good for $\mathbf{M}$. Since $y \notin Z(G)$, we have $|M| < |G|$. Hence $C_{\mathbf{M}}(R)$ is a regular subgroup of $\mathbf{M}$ by induction. As $C_{\mathbf{G}}(R) = C_{\mathbf{M}}(R)$, we are done. □

By letting $z = 1$, the conclusion of the above lemma holds in particular if $Z(\mathbf{G}^*)$ is connected and ℓ is good for $\mathbf{G}$.

We record a further useful consequence of Theorem 3.9.

Lemma 3.12. *Let $s \in G^*$ be semisimple such that $\mathbf{L}^* := C_{\mathbf{G}^*}(s)$ is a regular subgroup of $\mathbf{G}^*$. Choose a regular subgroup $\mathbf{L} \leq \mathbf{G}$ dual to $\mathbf{L}^*$.*

Let $b \subseteq \mathcal{E}_\ell(G,s)$ be an ℓ-block of G and $b' \subseteq \mathcal{E}_\ell(L,s)$ the corresponding block according to Theorem 3.9(a). *Assume that $b' = \hat{s} \otimes b_0$, where b_0 is the principal block of L and $\hat{s}$ is a linear character of L corresponding to s via duality; see* [19, (8.19)]. *Let $D \leq L$ be a common defect group of b and b'. Let $(R, b_R) \leq (D, b_D)$ be a centric b-Brauer pair. Then the following statements hold.*

(a) *We have* $\mathrm{Out}_G(R, b_R) = \mathrm{Out}_L(R)$. *If* $\mathcal{W}(R, b_R) \neq 0$, *then* R *is a radical subgroup of* L. *(For the definition of* $\mathcal{W}(R, b_R)$ *see* (2)*).*

(b) *The canonical character* θ_R *of* b_R *extends to* $N_G(R, b_R)$.

(c) *If* $N_L(R)$ *fixes* b_R, *then the homomorphism*

$$N_L(R) \to N_G(R, b_R)/RC_G(R) = \mathrm{Out}_G(R, b_R)$$

is surjective with kernel $RC_L(R)$.

(d) *If* R *is abelian and* $\mathbf{K} := C_{\mathbf{L}}(R)$ *is a regular subgroup of* $\mathbf{L}$, *then* $\mathrm{Out}_G(R, b_R) = W_{\mathbf{L}}(\mathbf{K})^F$.

Proof. By our hypothesis, the group $\mathbf{M}^*$ of Theorem 3.9 is equal to $\mathbf{L}^* = C_{\mathbf{G}^*}(s)$, and $\mathbf{M}$ may be chosen to be equal to $\mathbf{L}$.

(a) By Theorem 3.9(b), there is a centric b'-Brauer pair (R, b'_R) of L such that $\mathrm{Out}_G(R, b_R) = \mathrm{Out}_L(R, b'_R)$. By our assumption on b', the canonical character θ'_R of b'_R equals $\mathrm{Res}^L_{C_L(R)}(\hat{s})$. In particular, $N_L(R, \theta'_R) = N_L(R)$ and θ'_R extends to $N_L(R, \theta'_R)$. We obtain

$$\begin{aligned} \mathrm{Out}_G(R, b_R) &= \mathrm{Out}_L(R, b'_R) \\ &= N_L(R, b'_R)/RC_L(R) \\ &= N_L(R)/RC_L(R) \\ &= \mathrm{Out}_L(R), \end{aligned}$$

giving our first claim.

To prove the second, let $Q := O_\ell(N_L(R))$, so that $R \leq Q$ and $Q \cap RC_L(R) = R$ as b'_R is centric. Now $O_\ell(\mathrm{Out}_L(R)) \cong O_\ell(\mathrm{Out}_G(R, b_R)) \cong \{1\}$, the first isomorphism arising from the first claim, and the second one from (3) and our hypothesis $\mathcal{W}(R, b_R) \neq 0$. This implies that the image of Q in $\mathrm{Out}_L(R)$ is trivial. Hence $Q = R$ and R is a radical ℓ-subgroup of L.

(b) As $\mathrm{Res}^L_{C_L(R)}(\hat{s})$ extends to $N_L(R)$, the Külshammer-Puig class associated to (R, b'_R) is trivial. By Theorem 3.9(b), the Külshammer-Puig class associated to (R, b_R) is trivial as well, hence θ_R extends to $N_G(R, b_R)$.

(c) If $N_L(R)$ fixes b_R, the map $N_L(R) \to N_G(R, b_R)/RC_G(R)$ is well defined and has kernel $RC_L(R)$, hence is surjective by (a).

(d) Now suppose that R is abelian and that $\mathbf{K} = C_{\mathbf{L}}(R)$ is a regular subgroup of $\mathbf{L}$. As $R \leq Z(K)$ is a defect group of b'_R, we have $R = O_\ell(Z(K))$ and thus $N_L(K) = N_L(R) \leq N_L(C_{\mathbf{L}}(R)) = N_L(\mathbf{K}) \leq N_L(K)$. In particular, $N_L(K) = N_L(\mathbf{K})$ and thus

$$\mathrm{Out}_L(R) = N_L(R)/C_L(R) = N_{\mathbf{L}}(\mathbf{K})^F/\mathbf{K}^F = W_{\mathbf{L}}(\mathbf{K})^F.$$

This completes our proof. □

3.13. **Non-regular centralizers.** We now work towards a variant of Lemma 3.12 in the case when $C_{\mathbf{G}^*}(s)$ is not regular, so that $C_{\mathbf{G}^*}(s)^*$ cannot be embedded into $\mathbf{G}$.

Lemma 3.14. *Let $\mathbf{M}$ and $\mathbf{M}^*$ be a pair of dual regular subgroups of $\mathbf{G}$ and $\mathbf{G}^*$, respectively. Then there is an F-F^*-equivariant group isomorphism*

$$W_{\mathbf{G}}(\mathbf{M}) \xrightarrow{\alpha} W_{\mathbf{G}^*}(\mathbf{M}^*), \quad w \mapsto w^*,$$

satisfying the following condition.

Let $\mathbf{S}$ and $\mathbf{S}^*$ be F-stable, respectively F^*-stable, maximal tori of $\mathbf{M}$, respectively $\mathbf{M}^*$, let $\theta \in \mathrm{Irr}(S)$ and $s \in S^*$, such that the M-conjugacy class of the pair $(\mathbf{S}, \theta)$ corresponds to the M^*-conjugacy class of the pair $(\mathbf{S}^*, s)$ under the duality of $\mathbf{M}$ and $\mathbf{M}^*$; see, e.g. [25, Proposition 11.1.16]*. Further, let $w \in W_{\mathbf{G}}(\mathbf{M})^F$ and let $x \in N_G(\mathbf{M})$ and $y \in N_{G^*}(\mathbf{M}^*)$ denote inverse images of w and w^*, respectively. Then*

$$R^{\mathbf{M}}_{\mathbf{S}^*}(s)^x = R^{\mathbf{M}}_{(\mathbf{S}^*)^y}(s^y).$$

Proof. For the first assertion see the concluding remarks of [19, Section 8.2]. The displayed equation is [26, (4.3) Lemma]. □

Lemma 3.15. *Let $\mathbf{M}$ and $\mathbf{M}^*$ be a pair of dual regular subgroups of $\mathbf{G}$ and $\mathbf{G}^*$, respectively, and let $s \in M^*$ be semisimple. Let $x \in N_G(\mathbf{M})$ and $y \in N_{G^*}(\mathbf{M}^*)$ be such that the images of x and y in $W_{\mathbf{G}}(\mathbf{M})^F$ respectively $W_{\mathbf{G}^*}(\mathbf{M}^*)^{F^*}$ correspond under the isomorphism given in* Lemma 3.14*.*

Then $\mathcal{E}(M, s)^x = \mathcal{E}(M, s^y)$. In particular, if $y \in C_{G^*}(s)$, then x stabilizes $\mathcal{E}(M, s)$.

Proof. Let $\mathbf{S}^* \leq \mathbf{M}^*$ be an F^*-stable maximal torus with $s \in S^*$. By Lemma 3.14 we have $R^{\mathbf{M}}_{\mathbf{S}^*}(s)^x = R^{\mathbf{M}}_{(\mathbf{S}^*)^y}(s^y)$. As all irreducible constituents of $R^{\mathbf{M}}_{(\mathbf{S}^*)^y}(s^y)$ lie in $\mathcal{E}(M, s^y)$, the claim follows. □

Part (a) of the next proposition is a corollary to Lemma 3.15.

Proposition 3.16. *Let $R \leq G$ and $R^\dagger \leq G^*$ be abelian radical ℓ-subgroups such that $\mathbf{M} := C_{\mathbf{G}}(R)$ is a regular subgroup of $\mathbf{G}$ and $\mathbf{M}^* := C_{\mathbf{G}^*}(R^\dagger)$ is dual to $\mathbf{M}$. Then $\mathrm{Out}_G(R) = W_{\mathbf{G}}(\mathbf{M})^F$ and $\mathrm{Out}_G(R^\dagger) = W_{\mathbf{G}^*}(\mathbf{M}^*)^{F^*}$.*

Let $s \in M^*$ be a semisimple ℓ'-element and put $\mathbf{L}^* := C_{\mathbf{G}^*}(s) \leq \mathbf{G}^*$. Assume that $\mathbf{L}^*$ is connected.

(a) *Let $b_R \subseteq \mathcal{E}_\ell(C_G(R), s)$ be an ℓ-block of $C_G(R)$ and assume that $N_G(R, b_R) = \{x \in N_G(R) \mid x \text{ stabilizes } \mathcal{E}(C_G(R), s)\}$. Then the isomorphism*

$$W_{\mathbf{G}}(\mathbf{M})^F \xrightarrow{\alpha} W_{\mathbf{G}^*}(\mathbf{M}^*)^{F^*}$$

arising from Lemma 3.14 *maps* $\mathrm{Out}_G(R, b_R)$ *to* $\mathrm{Out}_{L^*}(R^\dagger)$*, i.e. it induces an isomorphism*

$$\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_{L^*}(R^\dagger).$$

(b) *Suppose that* (R, b_R) *is centric and that* $\mathbf{K}^* := C_{\mathbf{L}^*}(R^\dagger)$ *is a regular subgroup of* $\mathbf{L}^*$ *and of* $\mathbf{M}^*$*. Suppose further that*

$$\ell \nmid |Z(\mathbf{M}^*)/Z(\mathbf{M}^*)^\circ||Z(\mathbf{M})/Z(\mathbf{M})^\circ|.$$

Then

$$\mathrm{Out}_{L^*}(R^\dagger) = W_{\mathbf{L}^*}(\mathbf{K}^*)^{F^*}.$$

PROOF. As R is an abelian radical subgroup of G, we have $R = O_\ell(C_G(R))$ and thus $N_G(R) = N_G(C_G(R)) = N_G(M)$ and $N_G(R) = N_G(\mathbf{M})$. Analogously, $N_{G^*}(R^\dagger) = N_{G^*}(M^*) = N_{G^*}(\mathbf{M}^*)$, giving our first assertion.

(a) We show that α maps $N_G(R, b_R)/M \leq N_G(M)/M = W_{\mathbf{G}}(\mathbf{M})^F$ to $\mathrm{Out}_{L^*}(R^\dagger)$, naturally embedded into $\mathrm{Out}_{G^*}(R^\dagger) = W_{\mathbf{G}^*}(\mathbf{M}^*)^{F^*}$.

Let $x \in N_G(\mathbf{M})$ and $y \in N_{G^*}(\mathbf{M}^*)$ be such $\alpha(Mx) = M^*y$. Suppose first that $x \in N_G(R, b_R)$. Then x stabilizes $\mathcal{E}_\ell(C_G(R), s)$ and thus $\mathcal{E}_\ell(C_G(R), s) = \mathcal{E}_\ell(C_G(R), s)^x = \mathcal{E}_\ell(C_G(R), s^y)$ by Lemma 3.15. Hence s^y is conjugate to s in M^*, i.e. $ty \in L^*$ for some $t \in M^*$. Hence α maps $N_G(R, b_R)/M$ into $\mathrm{Out}_{L^*}(R^\dagger)$.

Now assume that $M^*y \in \mathrm{Out}_{L^*}(R^\dagger)$. By multiplying y with a suitable element of M^*, we may assume that $y \in L^*$. With Lemma 3.15 we conclude that x stabilizes $\mathcal{E}(C_G(R), s)$. Our assumption now implies that $x \in N_G(R, b_R)$.

(b) This is very similar to the last part of the proof of Lemma 3.12. We claim that $R^\dagger = O_\ell(Z(C_{L^*}(R^\dagger)))$. Once this claim is proved, we can conclude $N_{L^*}(C_{L^*}(R^\dagger)) = N_{L^*}(R^\dagger) = N_{L^*}(C_{\mathbf{L}^*}(R^\dagger))$, yielding our assertion.

To prove the claim, first notice that $\mathbf{K}^* = \mathbf{L}^* \cap \mathbf{M}^* = C_{\mathbf{M}^*}(s)$. As (R, b_R) is centric, the defect group of b_R equals R, and as R is contained in $Z(M)$, we must have $R = O_\ell(Z(M))$. By [22, Proposition 4.4.5] and our assumption on ℓ, we conclude that $R^\dagger = O_\ell(Z(M^*))$. If $s \in Z(M^*)$, then $M^* = C_{L^*}(R^\dagger)$ and our claim follows. Otherwise, $C_{\mathbf{M}^*}(s)$ is a proper regular subgroup of $\mathbf{M}^*$. Let $\mathbf{M}_s$ denote a regular subgroup of $\mathbf{M}$ dual to $C_{\mathbf{M}^*}(s)$. By Theorem 3.9(a), the block $b_R \subseteq \mathcal{E}_\ell(M, s)$ corresponds to a block of $\mathcal{E}_\ell(M_s, s)$ with a defect group R_s conjugate to R in M. As $Z(M) \leq Z(M_s)$, we get $R = O_\ell(Z(M_s))$. The assumption on ℓ descends to the regular subgroups $\mathbf{M}_s$ and $C_{\mathbf{M}^*}(s)$ of $\mathbf{M}$ and $\mathbf{M}^*$, respectively; see, e.g. [8, Proposition 4.2]. We thus obtain $R^\dagger = O_\ell(Z(C_{M^*}(s)))$. As $C_{M^*}(s) = C_{L^*}(R^\dagger)$, this yields our claim. □

4. The group $F_4(q)$ and some of its Levi subgroups

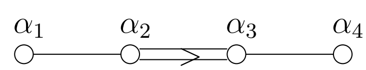
In this section we introduce the group $F_4(q)$ and investigate some of its subgroups. The main new result is contained in Corollary 4.20: Let e be a positive integer. Then every irreducible character of an e-split Levi subgroup of $F_4(q)$ extends to its inertia subgroup. This generalizes the result of Späth in [74, Theorem 1.1] in case of the group $F_4(q)$. For the definition of e-split Levi subgroups see, e.g. [42, 3.5.1].

4.1. **Setup, notation and preliminaries.** Let p be a prime, f a positive integer and $q = p^f$. Further, let $\mathbf{G}$ denote a simple algebraic group of type F_4 over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$, such that $\mathbf{G}$ has a standard Frobenius morphism F_1 with $\mathbf{G}^{F_1} = F_4(p)$. We put $F := F_1^f$, so that $G = \mathbf{G}^F = F_4(q)$. We have

$$|G| = q^{24}\Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_4(q)^2\Phi_6(q)^2\Phi_8(q)\Phi_{12}(q),$$

where Φ_i denotes the ith cyclotomic polynomial.

We choose an F-stable maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, so that $\mathbf{T}_0$ is maximally split and $F(t) = t^q$ for all $t \in \mathbf{T}_0$. Write $W := W_{\mathbf{G}}(\mathbf{T}_0) := N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ for the Weyl group of $\mathbf{G}$ (with respect to $\mathbf{T}_0$). Let $\mathbf{U}$ denote the unipotent radical of $\mathbf{B}$. The root system of $\mathbf{G}$ is denoted by Σ, the root subgroup giving rise to $\alpha \in \Sigma$ by $\mathbf{U}_\alpha$, and $u_\alpha : \mathbb{F} \to \mathbf{U}_\alpha$ the corresponding isomorphism of algebraic groups. The choice of $\mathbf{B}$ determines the set Σ^+ of positive roots and the corresponding base α_i, $i = 1, \ldots, 4$, numbered as in the following Dynkin diagram:



Thus α_1, α_2 are the long simple roots, and α_3, α_4 the short ones.

As $(\mathbf{G}, F)$ is split, F acts trivially on W, so that $W = W^F = N_G(\mathbf{T}_0)/T_0$. Moreover, for each $\alpha \in \Sigma$, the root subgroup $\mathbf{U}_\alpha$ is F-stable, and $F(u_\alpha(t)) = u_\alpha(t^q)$ for all $t \in \mathbb{F}$; in particular, $U_\alpha = \mathbf{U}_\alpha^F = \{u_\alpha(t) \mid t \in \mathbb{F}_q\}$. For $\alpha \in \Sigma$, write $n_\alpha := u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$. Then $n_\alpha \in N_{\mathbf{G}}(\mathbf{T}_0)^{F_1}$, and we write s_α for the image of n_α in W. Let w_0 denote the longest element of W. Then $w_0 \in Z(W)$. If $\dot{w}_0 \in N_{\mathbf{G}}(\mathbf{T}_0)$ is an inverse image of w_0, then $\dot{w}_0$ inverts the elements of $\mathbf{T}_0$ and $\mathbf{U}_\alpha^{\dot{w}_0} = \mathbf{U}_{-\alpha}$ for all $\alpha \in \Sigma$. In the following, we will refer to some computations in Σ using CHEVIE [41]. For easier reference, Table I gives the numbering of the roots of Σ^+ as in CHEVIE. In the column headed α we list the

TABLE I. The positive roots of Σ

α	Root	$\alpha^\dagger$	α	Root	$\alpha^\dagger$	α	Root	$\alpha^\dagger$
1	$[1,0,0,0]$	4	9	$[0,1,2,0]$	6	17	$[1,2,2,1]$	22
2	$[0,1,0,0]$	3	10	$[0,1,1,1]$	11	18	$[1,1,2,2]$	12
3	$[0,0,1,0]$	2	11	$[1,1,2,0]$	10	19	$[1,2,3,1]$	23
4	$[0,0,0,1]$	1	12	$[1,1,1,1]$	18	20	$[1,2,2,2]$	15
5	$[1,1,0,0]$	7	13	$[0,1,2,1]$	14	21	$[1,2,3,2]$	24
6	$[0,1,1,0]$	9	14	$[1,2,2,0]$	13	22	$[1,2,4,2]$	17
7	$[0,0,1,1]$	5	15	$[1,1,2,1]$	20	23	$[1,3,4,2]$	19
8	$[1,1,1,0]$	16	16	$[0,1,2,2]$	8	24	$[2,3,4,2]$	21

CHEVIE number of $\alpha \in \Sigma^+$, and in the column headed "Root" the expansion of α in the base $\{\alpha_1, \ldots, \alpha_4\}$. The significance of the column headed $\alpha^\dagger$ will be explained in Subsection 4.10 below. The negative roots are numbered as $\alpha_{24+i} := -\alpha_i$ for $1 \leq i \leq 24$. If $\alpha = \alpha_i$ for some $1 \leq i \leq 48$, we write $n_i := n_{\alpha_i}$, $s_i := s_{\alpha_i}$ and $u_i := u_{\alpha_i}$.

As already introduced in Subsection 3.5, we write $X := X(\mathbf{T}_0) = \mathrm{Hom}(\mathbf{T}_0, \mathbb{F}^*)$ and $Y := Y(\mathbf{T}_0) = \mathrm{Hom}(\mathbb{F}^*, \mathbf{T}_0)$ for the character group and the cocharacter group of $\mathbf{T}_0$, respectively. Then $X \cong Y \cong \mathbb{Z}^4$ as abelian groups. The homomorphism $h_\alpha : \mathbb{F}^* \to \mathbf{T}_0$ in Y associated to $\alpha \in \Sigma$ as in [21, Theorem 12.1.1] is called the coroot corresponding to α, and will be denoted by $\alpha^\vee$. For every subset $\Gamma \subseteq \Sigma$ write $\Gamma^\vee := \{\alpha^\vee \mid \alpha \in \Gamma\}$. Then $\{\alpha_1, \ldots, \alpha_4\} \subseteq X$ and $\{\alpha_1^\vee, \ldots \alpha_4^\vee\} \subseteq Y$ are $\mathbb{Z}$-bases of X and Y respectively. The root datum of $\mathbf{G}$ is $(X, \Sigma, Y, \Sigma^\vee)$.

Let $\Gamma \subseteq \Sigma$. By $\bar{\Gamma}$ we denote the smallest closed subsystem of Σ containing Γ. We put $\mathbf{L}_\Gamma := \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \bar{\Gamma} \rangle$ and $\mathbf{K}_\Gamma := \langle \mathbf{U}_\alpha \mid \alpha \in \bar{\Gamma} \rangle$. Then $\mathbf{L}_\Gamma$ and $\mathbf{K}_\Gamma$ are connected reductive algebraic groups, $\mathbf{K}_\Gamma$ is semisimple and $\mathbf{K}_\Gamma = [\mathbf{L}_\Gamma, \mathbf{L}_\Gamma]$; see, e.g. [63, Theorem 13.6]. It follows from [63, Proposition 12.14] that $\mathbf{K}_\Gamma$ is simply connected if $\Gamma \subseteq \{\alpha_1, \ldots, \alpha_4\}$, i.e. if $\bar{\Gamma}$ is a parabolic subsystem. Notice that $n_\alpha \in K_\Gamma = \mathbf{K}_\Gamma^F$ if $\alpha \in \bar{\Gamma}$. Moreover, $\mathbf{L}_\Gamma$ and $\mathbf{K}_\Gamma$ are F-stable and $\mathbf{T}_0$, respectively $\mathbf{T}_0 \cap \mathbf{K}_\Gamma$, are maximally split maximal tori of $\mathbf{L}_\Gamma$ and $\mathbf{K}_\Gamma$, respectively; the corresponding root system is $\bar{\Gamma}$ and we have $\bar{\Gamma}^+ = \bar{\Gamma} \cap \Sigma^+$. The root datum of $\mathbf{L}_\Gamma$ with respect to $\mathbf{T}_0$ equals $(X, \bar{\Gamma}, Y, \bar{\Gamma}^\vee)$. Let us also put $W_\Gamma := \langle s_\alpha \mid \alpha \in \Gamma \rangle$. Then the Weyl group of $\mathbf{L}_\Gamma$ with respect to $\mathbf{T}_0$ equals $W_{\bar{\Gamma}}$. If Γ is a base of $\bar{\Gamma}$, then $W_\Gamma = W_{\bar{\Gamma}}$, but, in general, $W_\Gamma \lneq W_{\bar{\Gamma}}$.

If $\Gamma, \Delta \subseteq \Sigma$ are disjoint, closed subsystems of Σ such that $\Delta \cup \Gamma$ is closed, then $[\mathbf{K}_\Delta, \mathbf{K}_\Gamma] = 1$ by the commutator relations for root subgroups.

Lemma 4.2. *Suppose that $\beta_1, \dots, \beta_m$ are m distinct roots of Σ contained in some base (in particular, $m \leq 4$). Then for any $a_1, \dots, a_m \in \mathbb{F}^*$, there is $t \in \mathbf{T}_0$ with $\beta_i(t) = a_i$ for $1 \leq i \leq m$. Moreover, t may be chosen in T_0 if $a_1, \dots, a_m \in \mathbb{F}_q^*$.*

Proof. As $\mathbf{G}$ is adjoint, a base of Σ is a $\mathbb{Z}$-basis of X. The result follows from $\mathbf{T}_0 \cong \mathrm{Hom}(X, \mathbb{F}^*)$; see [22, Proposition 3.1.2(i)]. As the latter isomorphism is F-equivariant, the final statement also follows. □

Of particular importance is the following construction, a special case of twisting; see Subsection 4.6 below. If $g \in \mathbf{G}$ such that $F(g)g^{-1}$ normalizes $\mathbf{T}_0$ and maps to w_0 under the natural epimorphism, then $\mathbf{T}_0^g$, $\mathbf{L}_\Gamma^g$ and $\mathbf{K}_\Gamma^g$ are F-stable for every $\Gamma \subseteq \Sigma$, and F acts trivially on $N_{\mathbf{G}}(\mathbf{T}_0^g)/\mathbf{T}_0^g$. As w_0 acts as $-\mathrm{id}$ on X, we have $F(t) = t^{-q}$ for all $t \in \mathbf{T}_0^g$. A torus which is G-conjugate to $\mathbf{T}_0$, respectively to $\mathbf{T}_0^g$ is called 1-F-split, respectively 2-F-split, where we omit the F if clear from the context.

4.3. **A lift of the longest element.** The existence of a lift of w_0 of order 2 is indicated in [44, Definition (2.23)]. We will make use of a particular such lift with further properties.

Lemma 4.4. *Let w_0 denote the longest element of W. Then there is a lift $\gamma \in N_G(\mathbf{T}_0)$ of w_0 such that*

$$u_\alpha(t)^\gamma = u_{-\alpha}(-t)$$

for all $\alpha \in \{\pm\alpha_1, \dots, \pm\alpha_4\}$ and all $t \in \mathbb{F}$. In particular, $\gamma^2 = 1$.

Moreover, $n_1, \dots, n_4$ commute with γ.

Proof. We start with a particular reduced word for w_0, and let γ denote the corresponding product of the n_α. We then use CHEVIE and [21, Lemma 7.2.1(i)] to verify that $\gamma u_\alpha(t)\gamma^{-1} = u_{-\alpha}(-t)$ for all $\alpha \in \{\alpha_1, \dots, \alpha_4\}$ and all $t \in \mathbb{F}$. The corresponding relation for the negative roots then follows from [21, Proposition 6.4.3]. Now γ^2 fixes $\pm\alpha_1(t), \dots, \pm\alpha_4(t)$ for all $t \in \mathbb{F}$, as well as the elements of $\mathbf{T}_0$, and, as $\mathbf{G} = \langle \mathbf{T}_0, \pm\alpha_1(t), \dots, \pm\alpha_4(t) \mid t \in \mathbb{F} \rangle$, we get $\gamma^2 \in Z(\mathbf{G}) = \{1\}$.

Finally, γ acts as inverse-transpose automorphism on $\mathbf{L}_{\{\alpha_j\}} \cong \mathrm{SL}_2(\mathbb{F})$ in the natural representation of $\mathrm{SL}_2(\mathbb{F})$, for $j = 1, \dots, 4$. Thus γ fixes n_j for $j = 1, \dots, 4$. □

The above proof provides an example of a computation in the *extended Weyl group* of $\mathbf{G}$. This is a group associated to a Coxeter system, introduced and investigated by Tits in [79]. Let $\hat{W} \leq N_{\mathbf{G}}(\mathbf{T}_0)$ denote the subgroup generated by n_j, $1 \leq j \leq 24$; for the definition of these elements see Subsection 4.1. We constructed the n_j as matrices in the adjoint representation of $\mathbf{G}$, as described in [21, Lemma 4.3.1]. One checks that the elements $n_1, \ldots, n_4$ satisfy the relations exhibited in [79, 4.6, Equations (1)–(4)], and that $\hat{W}$ has order $2^4|W|$ if p is odd. Thus in this case, $\hat{W}$ is indeed the extended Weyl group as defined in [79, Définition 2.2]. If $p = 2$, we have $\hat{W} = W$. If p is odd, the map $\hat{W} \to W$ defined by sending n_j to s_j for $j = 1, \ldots, 4$, is surjective with kernel of order 2^4, generated by n_j^2, $j = 1, \ldots, 4$.

4.5. **Automorphisms.** Recall from Subsection 4.1 that F_1 is the Steinberg endomorphism of $\mathbf{G}$ such that $\mathbf{G}^{F_1} = F_4(p)$. If p is odd, let $\sigma_1 := F_1$. If $p = 2$, let σ_1 denote the endomorphism of $\mathbf{G}$ constructed in [78, Theorem 28], such that $\sigma_1^2 = F_1$. Then σ_1 is an automorphism of abstract groups. Following [43, Definition 1.15.1], we write $\mathrm{Aut}_1(\mathbf{G})$ for the set of automorphisms ψ of the abstract group $\mathbf{G}$, such that ψ or ψ^{-1} is an endomorphism of the algebraic group $\mathbf{G}$. By the results summarized in [43, Subsection 1.15], we find $\mathrm{Aut}_1(\mathbf{G}) = \mathrm{Inn}(\mathbf{G}) \rtimes \langle \sigma_1 \rangle$. As $Z(\mathbf{G})$ is trivial we may identify $\mathrm{Inn}(\mathbf{G})$ with $\mathbf{G}$ and $\mathrm{Aut}_1(\mathbf{G})$ with $\mathbf{G} \rtimes \langle \sigma_1 \rangle$.

4.6. **Twisting.** For $g \in \mathbf{G}$ let $\mathrm{ad}_g : \mathbf{G} \to \mathbf{G}$, $x \mapsto g^{-1}xg$ denote conjugation by g. Let $n \in \mathbf{G}$. As usual, we write Fn for the Steinberg morphism of $\mathbf{G}$ defined by $Fn := \mathrm{ad}_n \circ F$, i.e.

$$Fn : \mathbf{G} \to \mathbf{G}, x \mapsto n^{-1}F(x)n.$$

(Notice that, although we compose endomorphisms of $\mathbf{G}$ from right to left, as indicated by the symbol $\circ$, conjugation by g is "conjugation from the right". The reason is that in GAP [34], which we use for numerous computations in the Weyl group of $\mathbf{G}$, this is the default way of conjugating in groups. As a consequence of this convention, $\mathrm{ad}_g \circ \mathrm{ad}_h = \mathrm{ad}_{hg}$ for $g, h \in \mathbf{G}$) .

Let $\mathbf{N}_0 \trianglelefteq \mathbf{M}_0$ denote closed F-stable subgroups of $\mathbf{G}$ normalized by n, so that $\mathbf{N}_0$ and $\mathbf{M}_0$ are Fn-stable. By the Lang-Steinberg theorem, there is $g \in \mathbf{G}$ with $F(g)g^{-1} = n$; choose one such g and put $\mathbf{N} := \mathbf{N}_0^g$ and $\mathbf{M} := \mathbf{M}_0^g$. We say that $\mathbf{M}$ is obtained from $\mathbf{M}_0$ by *(F-)twisting with n.* Notice that $\mathbf{M}$ is F-stable and

$$\mathrm{ad}_g : \mathbf{M}_0^{Fn} \to \mathbf{M}^F$$

is an isomorphism mapping $\mathbf{N}_0^{Fn}$ to $\mathbf{N}^F$. Let $\omega \in \mathrm{Aut}_1(\mathbf{G})$ such that ω stabilizes $\mathbf{N}$ and $\mathbf{M}$. Then $\mathrm{ad}_g^{-1} \circ \omega \circ \mathrm{ad}_g$ stabilizes $\mathbf{N}_0$ and $\mathbf{M}_0$ and we obtain the following commutative diagram of groups and automorphisms

$$\begin{array}{ccc} \mathbf{M}/\mathbf{N} & \xrightarrow{\ \omega\ } & \mathbf{M}/\mathbf{N} \\ {\scriptstyle \mathrm{ad}_g}\uparrow & & \uparrow{\scriptstyle \mathrm{ad}_g} \\ \mathbf{M}_0/\mathbf{N}_0 & \xrightarrow[\mathrm{ad}_g^{-1}\circ\omega\circ\mathrm{ad}_g]{} & \mathbf{M}_0/\mathbf{N}_0 \end{array}$$

where we use ω and ad_g to also denote the induced maps on the factor groups.

Now assume in addition that ω commutes with F. Observe that this is the case, if and only if $\mathrm{ad}_g^{-1} \circ \omega \circ \mathrm{ad}_g$ commutes with Fn. We obtain an analogous diagram for the groups of F-fixed points.

$$\begin{array}{ccc} (\mathbf{M}/\mathbf{N})^F & \xrightarrow{\ \omega\ } & (\mathbf{M}/\mathbf{N})^F \\ {\scriptstyle \mathrm{ad}_g}\uparrow & & \uparrow{\scriptstyle \mathrm{ad}_g} \\ (\mathbf{M}_0/\mathbf{N}_0)^F & \xrightarrow[\mathrm{ad}_g^{-1}\circ\omega\circ\mathrm{ad}_g]{} & (\mathbf{M}_0/\mathbf{N}_0)^F \end{array}$$

4.7. **Subgroups of maximal rank.** Centralizers of semisimple elements in $\mathbf{G}$ are connected, reductive and contain a maximal torus; see, e.g. [22, Theorems 3.5.6, 3.5.4, 3.5.3(i)]. Subgroups with these properties are called *connected reductive subgroups of maximal rank of* $\mathbf{G}$. In particular, the $\mathbf{G}$-conjugates of the groups $\mathbf{L}_\Gamma$, where $\Gamma \subseteq \Sigma$ is a closed subsystem, are subgroups of maximal rank. Once more by [22, Theorem 3.5.3(i)], centralizers of semisimple elements in $\mathbf{G}$ are of this latter form. We will, therefore, restrict the following considerations to subgroups conjugate to $\mathbf{L}_\Gamma$ for closed subsystems Γ.

Let $\Gamma \subseteq \Sigma$ be a closed subsystem and put $\mathbf{L} = \mathbf{L}_\Gamma$. As $N_{\mathbf{G}}(\mathbf{L}) = (N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L}))\mathbf{L}$, we have

$$\begin{aligned} W_{\mathbf{G}}(\mathbf{L}) &= (N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L}))\mathbf{L}/\mathbf{L} \\ &\cong (N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L}))/N_{\mathbf{L}}(\mathbf{T}_0) \\ &\cong [(N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L}))/\mathbf{T}_0] \,/\, [N_{\mathbf{L}}(\mathbf{T}_0)/\mathbf{T}_0] \end{aligned}$$

Let Γ_0 denote a base of Γ. Then $W_{\mathbf{G}}(\mathbf{L}) = \mathrm{Stab}_W(\Gamma_0)$. Moreover, $N_{\mathbf{L}}(\mathbf{T}_0)/\mathbf{T}_0 = W_\Gamma$, and thus the image of $N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L})$ in W is the split extension $W_\Gamma.\mathrm{Stab}_W(\Gamma_0)$. As all bases of Γ are conjugate in W_Γ, we also have $W_\Gamma.\mathrm{Stab}_W(\Gamma_0) = \mathrm{Stab}_W(\Gamma)$.

The G-conjugacy classes of F-stable subgroups of $\mathbf{G}$ which are $\mathbf{G}$-conjugate to $\mathbf{L}$ are in bijection with the conjugacy classes of $W_{\mathbf{G}}(\mathbf{L}) = \mathrm{Stab}_W(\Gamma_0)$. (Recall that F acts trivially on W.) This bijection is determined as follows. Let C be a conjugacy class of $\mathrm{Stab}_W(\Gamma_0)$. Choose an element $v \in W_\Gamma.\mathrm{Stab}_W(\Gamma_0)$, whose image in $\mathrm{Stab}_W(\Gamma_0)$ lies in C. Next, choose an inverse image $\dot{v} \in N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L})$ of v, and an element $g \in \mathbf{G}$ such that $F(g)g^{-1} = \dot{v}$. Then C is mapped to the G-conjugacy class of the group $\mathbf{L}^g$. If $\mathbf{M}$ is an F-stable $\mathbf{G}$-conjugate of $\mathbf{L}$ that corresponds to $v \in W_\Gamma.\mathrm{Stab}_W(\Gamma_0)$ in the above sense, we call $(\Gamma, [v])$ the *F-type* of $\mathbf{M}$, where $[v]$ denotes the conjugacy class in $\mathrm{Stab}_W(\Gamma_0)$ of the coset $W_\Gamma v$. We omit the F from the notation if it is clear from the context. Notice that $\mathbf{M}$ is a regular subgroup of $\mathbf{G}$, if and only if Γ is a parabolic subsystem of Σ, and that $\mathbf{M}$ is a maximal torus, if and only if Γ is the empty set.

Lemma 4.8. *Let $\mathbf{M}$ denote an F-stable connected reductive subgroup of $\mathbf{G}$ of maximal rank which is $\mathbf{G}$-conjugate to $\mathbf{L}_\Gamma$ for some closed subsystem $\Gamma \subseteq \Sigma$. Suppose that the F-type of $\mathbf{M}$ equals $(\Gamma, [v])$.*

Then, for any positive integer m, the F^m-type of $\mathbf{M}$ equals $(\Gamma, [v^m])$.

PROOF. We have $\mathbf{M} = \mathbf{L}_\Gamma^g$ for some $g \in \mathbf{G}$ such that $\dot{v} := F(g)g^{-1} \in N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L}_\Gamma)$ maps to v under the natural homomorphism. Clearly, $\mathbf{T}_0^g$ and $\mathbf{M}$ are F^m-stable and thus $F^m(g)g^{-1} \in N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{L}_\Gamma)$. Using the fact that $F(\dot{v}) \in \mathbf{T}_0\dot{v}$, one shows by induction on m that $F^m(g)g^{-1} = \dot{v}^m t$ for some $t \in \mathbf{T}_0$, which proves our assertion. □

4.9. **Class types.** Our description of the blocks is based upon a classification of the semisimple conjugacy classes of G. These were first parameterized in [71], but we need more precise information on centralizers of semisimple elements, as given in [58]. Let s and s' be two semisimple elements of G. We say that s and s' belong to the same G-class type, respectively $\mathbf{G}$-class type, if and only if $C_{\mathbf{G}}(s)$ and $C_{\mathbf{G}}(s')$ are conjugate in G, respectively $\mathbf{G}$. In the tables of [58] corresponding to $F_4(q)$, the G-class types are labeled by triples (i, j, k), where the first index, i, distinguishes the $\mathbf{G}$-class types, and the second index, j, always takes the value 1, owing to the fact that centralizers of semisimple elements in $\mathbf{G}$ are connected. In the following, we will omit the index j, and talk of the G-class type (i, k) instead of $(i, 1, k)$.

The first index i runs from 1 to 20, and the second index k depends on i. Not every pair (i, k) of indices occurring in the tables in [58] corresponds to a semisimple element of G of class type (i, k). For example, if q is even, there are no elements of class type (i, k) for $i \in \{2, 3, 5, 8, 11, 12, 16\}$. On the other hand, for each (i, k), there is a power q' of a prime p', possibly $p' \neq p$, such that $F_4(q')$ has a semisimple element of class type (i, k).

This indicates that there is a generic description of the class types. Indeed, first notice the root datum $(X, \Sigma, Y, \Sigma^{\perp})$ is generic, i.e. independent of p, up to isomorphism of root data. (In fact, $(X, \Sigma, Y, \Sigma^{\perp})$ is part of the *generic finite reductive group* $(X, \Sigma, Y, \Sigma^{\perp}, W\mathrm{id})$ as defined in [13, Definition in 1.A]; see also [42, Definition 1.6.10]). Up to conjugation in W, there are 19 subsets $\Gamma_i \leq \Sigma$, $2 \leq i \leq 20$ such that $\bar{\Gamma}_i^{\perp} \neq 0$, where $\bar{\Gamma}_i^{\perp} = \{\gamma \in Y \mid \langle \alpha, \gamma \rangle = 0 \text{ for all } \alpha \in \bar{\Gamma}_i\}$, and such that Γ_i is a base of $\bar{\Gamma}_i$. We choose notation such that $\Gamma_{20} = \emptyset$ and put $\Gamma_1 = \{\alpha_1, \ldots, \alpha_4\}$. After the choice of a prime p, i.e. the group $\mathbf{G} = F_4(\bar{\mathbb{F}}_p)$, and a maximal torus $\mathbf{T}_0 \leq \mathbf{G}$ giving rise to the root datum $(X, \Sigma, Y, \Sigma^{\perp})$, one can construct the subgroups $\mathbf{L}_{\Gamma_i} \leq \mathbf{G}$ as in Subsection 4.1. Then $C_{\mathbf{G}}(s)$ is $\mathbf{G}$-conjugate to one of the $\mathbf{L}_{\Gamma_i}$ for every semisimple element $s \in \mathbf{G}$. However, depending on p, the group $\mathbf{L}_{\Gamma_i}$ can have trivial center.

Given $i \in \{1, \ldots, 20\}$, the index k numbers the conjugacy classes of $\mathrm{Stab}_W(\Gamma_i)$. After the choice of a power q of p, there is a Frobenius morphism F of $\mathbf{G}$ such that $\mathbf{T}_0$ is contained in an F-stable Borel subgroup, and $G = \mathbf{G}^F = F_4(q)$. By the results summarized in Subsection 4.7, the G-conjugacy classes of the F-stable $\mathbf{G}$-conjugates of the groups $\mathbf{L}_{\Gamma_i}$ are labeled by the pairs (i, k). Let us write $\mathbf{M}_{i,k}$ for a representative of the corresponding G-conjugacy class of connected reductive subgroups of $\mathbf{G}$ of maximal rank, adopting the convention that $\mathbf{M}_{i,1} := \mathbf{L}_{\Gamma_i}$. Then $C_{\mathbf{G}}(s)$ is G-conjugate to one of the $\mathbf{M}_{i,k}$ for every semisimple element $s \in G$, in which case we say that s has class type (i, k). However, depending on q, the group $Z(\mathbf{M}_{i,k})$ does not necessarily contain elements of G with centralizer $\mathbf{M}_{i,k}$. Notice that, unless $k = 1$, the group $\mathbf{M}_{i,k}$ depends on F, although the index (i, k) does not. Notice also that a semisimple element $s \in G$ of class type (i, k) can have a different class type when viewed as element of $\mathbf{G}^{F^m}$ for a positive integer m; see Lemma 4.8. We therefore sometimes speak of the G-class type of s, respectively the $\mathbf{G}^{F^m}$-class type of s to be precise.

The quasi-isolated semisimple elements of G are exactly those corresponding to $i = 1, \ldots, 5$. The centralizers of the other F-stable semisimple elements are regular, unless $i \in \{8, 11, 12, 16\}$. The trivial element is of class type 1, and the regular semisimple elements are of $\mathbf{G}$-class type 20.

In the first column of Table 23 we list the sets Γ_i for $2 \leq i \leq 19$; these sets are the same as in [58], up to two modifications for $i = 14, 15$. For each pair (i, k), we also give representatives $v \in W_{\bar{\Gamma}_i}.\mathrm{Stab}_W(\Gamma_i)$ for the conjugacy class of $\mathrm{Stab}_W(\Gamma_i) = W_{\bar{\Gamma}_i}.\mathrm{Stab}_W(\Gamma_i)/W_{\bar{\Gamma}_i}$ with label (i, k). This labeling is the same as in [58]. We usually give several values of

$v \in W_{\bar{\Gamma}_i}.\mathrm{Stab}_W(\Gamma_i)$ for a given (i,k), to have more flexibility in the proofs of Section 8. We take one of the groups thus constructed (after a choice of a lift $\dot{v}$ and an element $g \in \mathbf{G}$ with $F(g)g^{-1} = \dot{v}$) as our representative $\mathbf{M}_{i,k}$. The W-conjugacy classes of the elements in the coset $W_{\bar{\Gamma}_i}v$ determine the G-conjugacy classes of the F-stable maximal tori which have some representative in $\mathbf{M}_{i,k}$.

4.10. **Duality.** We will identify the dual group $\mathbf{G}^*$ with $\mathbf{G}$, and thus G^* with G. For later purposes, we will choose a specific identification. The dual root datum $(Y, \Sigma^\vee, X, \Sigma)$ is isomorphic to $(X, \Sigma, Y, \Sigma^\vee)$ via an isomorphism $\delta : X \to Y$ satisfying $\delta(\alpha_i) = \alpha_{5-i}^\vee$ for $i = 1, \ldots, 4$. As δ maps Σ to $\Sigma^\vee$, this yields a permutation $\alpha \mapsto \alpha^\dagger$ of Σ such that $\delta(\alpha^\dagger) = \alpha^\vee$ for all $\alpha \in \Sigma$. This permutation, in fact an involution, is easily determined with CHEVIE. The CHEVIE number of the image of a positive root α under this permutation is given in Table I under the heading $\alpha^\dagger$. The map $s_\alpha \mapsto s_{\alpha^\dagger}$ extends to an automorphism $w \mapsto w^\dagger$ of W. By the isomorphism theorem [76, Theorem 9.6.2], there is an F-equivariant isomorphism $\mathbf{G} \to \mathbf{G}^*$ inducing the isomorphism δ of root data as in [76, 9.6.1], which we use to identify $\mathbf{G}$ with $\mathbf{G}^*$.

In view of (5), (7) and (8), the isomorphism $\delta : X \to Y$ gives rise to a W-equivariant isomorphism $T_0 \to \mathrm{Irr}(T_0), s \mapsto \hat{s}$.

Lemma 4.11. *Let $s \in T_0$, and let $\hat{s} \in \mathrm{Irr}(T_0)$ denote the irreducible character arising from duality. Further, let $\alpha \in \Sigma$ such that $\alpha^\dagger(s) = 1$. Then $\alpha^\vee(t) \in \ker(\hat{s})$ for all $t \in \mathbb{F}_q^*$.*

PROOF. Let $\gamma \in Y$ such that s corresponds to $\gamma + (F-1)Y$ under (7). Then $\alpha^\dagger(s) = \exp(2\pi\sqrt{-1}\iota^{-1}(\langle \alpha^\dagger, \gamma \rangle))$ by (9), and thus $\langle \alpha^\dagger, \gamma \rangle \in \mathbb{Z}$ by assumption.

Let $\chi \in X$ with $\delta(\chi) = \gamma$. Then $\chi \otimes 1$ is in the kernel of $F - 1$ on $X \otimes \mathbb{Q}_{p'}/\mathbb{Z}$ as s is F-stable. By definition, $\hat{s}$ is the character of T_0 which corresponds to $\chi \otimes 1$ under the isomorphism (8). The inverse image of $\alpha^\vee(t)$ under (7) is an element of the form $m\alpha^\vee + (F-1)Y \in Y/(F-1)Y$ for some $m \in \mathbb{Z}$. The claim now follows from Equation (9) as $\langle \chi, m\alpha^\vee \rangle = m\langle \alpha^\dagger, \delta(\chi) \rangle \in \mathbb{Z}$. □

4.12. **Twisting and duality.** We record a basic fact about duality. Let $g, g^* \in \mathbf{G}$ such that $\dot{w} := F(g)g^{-1}$ and $\dot{w}^* := F(g^*)g^{*-1}$ normalize $\mathbf{T}_0$, and write w, w^* for the images of $\dot{w}$ and $\dot{w}^*$ in W. Put $\mathbf{T} := \mathbf{T}_0^g$ and $\mathbf{T}^* := \mathbf{T}_0^{g^*}$. Let $\delta : X(\mathbf{T}_0) \to Y(\mathbf{T}_0)$ denote the duality isomorphism introduced in Subsection 4.10. We then have the following commutative diagram

$$\begin{array}{ccc} X(\mathbf{T}) & \xrightarrow{\delta_{g^*,g}} & Y(\mathbf{T}^*) \\ \mathrm{ad}_g \uparrow & & \uparrow \mathrm{ad}_{g^*} \\ X(\mathbf{T}_0) & \xrightarrow[\delta]{} & Y(\mathbf{T}_0) \end{array} \tag{10}$$

with $\delta_{g^*,g} = \mathrm{ad}_{g^*} \circ \delta \circ \mathrm{ad}_g^{-1}$, where the maps on the vertical arrows are induced by the conjugation maps $\mathrm{ad}_g : \mathbf{T}_0 \to \mathbf{T}$ and $\mathrm{ad}_{g^*} : \mathbf{T}_0 \to \mathbf{T}^*$.

Suppose further that $w^\dagger = w^{*-1}$. Then $\delta_{g^*,g}$ is an isomorphism between $X(\mathbf{T})$ and $Y(\mathbf{T}^*)$, and $F \circ \delta_{g^*,g} = \delta_{g^*,g} \circ F$, i.e. $(\mathbf{T}, F)$ and $(\mathbf{T}^*, F)$ are in duality; see [22, Proposition 4.3.4].

Assume in addition that $\dot{w}$ and $\dot{w}^*$ are F-stable. Then

$$\omega^* := \mathrm{ad}_{g^*} \circ F \circ \mathrm{ad}_{g^*}^{-1}$$

and

$$\omega := \mathrm{ad}_g \circ F \circ \mathrm{ad}_g^{-1}$$

are Steinberg morphisms of $\mathbf{G}$ which commute with F. Moreover, the following diagram commutes

$$\begin{array}{ccc} X(\mathbf{T}) & \xrightarrow{\delta_{g^*,g}} & Y(\mathbf{T}^*) \\ \omega \uparrow & & \uparrow \omega^* \\ X(\mathbf{T}) & \xrightarrow[\delta_{g^*,g}]{} & Y(\mathbf{T}^*) \end{array}$$

where ω and ω^* denote the induced maps on $X(\mathbf{T})$ and $\mathbf{Y}(\mathbf{T}^*)$, respectively. In other words, $(\mathbf{T}, \omega)$ and $(\mathbf{T}^*, \omega^*)$ are in duality.

We now generalize the above considerations to regular subgroups of $\mathbf{G}$. Let Γ denote a parabolic subsystem of Σ. Then $\Gamma^\dagger$ also is a parabolic subsystem, and $\mathbf{L}_\Gamma$ and $\mathbf{L}_{\Gamma^\dagger}$ are dual regular subgroups of $\mathbf{G}$. Let $\mathbf{M}$ denote a regular subgroup of $\mathbf{G}$ of type $(\Gamma, [v])$ for some $v \in \mathrm{Stab}_W(\Gamma_0)$, where Γ_0 is a base of Γ; see Subsection 4.7. If $\mathbf{M}^*$ is a regular subgroup of $\mathbf{G}$ of type $(\Gamma^\dagger, [v^*])$ with $v^\dagger = v^{*-1}$, then the pairs $(\mathbf{M}, F)$ and $(\mathbf{M}^*, F)$ are in duality in the sense of [42, Definition 1.5.17]. Indeed, by replacing v with a suitable element $w \in W_\Gamma v$, putting $w^* := {w^\dagger}^{-1}$, and choosing g and g^* as above, we may assume that $\mathbf{T}$ and $\mathbf{T}^*$ are maximally split tori of $\mathbf{M} := \mathbf{L}_\Gamma^g$, respectively $\mathbf{M}^* := \mathbf{L}_{\Gamma^\dagger}^{g^*}$. The duality of $(\mathbf{M}, F)$ and $(\mathbf{M}^*, F)$ follows from this and the diagram (10). To indicate that $\mathbf{M}^*$ is a subgroup of $\mathbf{G}$, we usually write $\mathbf{M}^\dagger$ instead of $\mathbf{M}^*$ for a group dual to $\mathbf{M}$ constructed in this way. Thus the identification of $\mathbf{G}$ with its dual $\mathbf{G}^*$ induces an involutive bijection $\mathbf{M} \mapsto \mathbf{M}^\dagger$ between G-conjugacy classes of regular subgroups of $\mathbf{G}$, where $\mathbf{M}^\dagger$ is a regular subgroup of $\mathbf{G}$ dual to $\mathbf{M}$. Notice that $\mathbf{T}_0^\dagger = \mathbf{L}_{\emptyset^\dagger} = \mathbf{L}_\emptyset = \mathbf{T}_0$.

4.13. **Some split Levi subgroups.** In this subsection, we describe some split Levi subgroups of G and their normalizers. We begin by introducing two closed subgroups of $\mathbf{G}$ of maximal rank.

Below, γ denotes the lift of w_0 constructed in Lemma 4.4. We choose $h \in \mathbf{G}$ with $F(h)h^{-1} = \gamma$, let $g \in \{1, h\}$ and put $\mathbf{T} := \mathbf{T}_0^g$. We define the parameter ε, by $\varepsilon := 1$ if $g = 1$, and $\varepsilon := -1$, otherwise. We adopt the notation that $\mathrm{SL}_3^\varepsilon(q)$ denotes $\mathrm{SL}_3(q)$ if $\varepsilon = 1$ and $\mathrm{SU}_3(q)$ if $\varepsilon = -1$.

Proposition 4.14. *Let* $J_1 := \{\alpha_1, \alpha_{23}\}$, $J_2 := \{\alpha_3, \alpha_4\}$ *and* $J := J_1 \cup J_2$, *and put* $\mathbf{L}_0 := \mathbf{L}_J$ *and* $\mathbf{L}_0^i := \mathbf{K}_{J_i}$, $i = 1, 2$. *Let* $\mathbf{L} := \mathbf{L}_0^g$ *and* $\mathbf{L}^i := (\mathbf{L}_0^i)^g$ *for* $i = 1, 2$.

(a) *Let* $d = \gcd(3, q^2 - 1)$. *Then* $\mathbf{L} = \mathbf{L}^1 \circ_d \mathbf{L}^2$ *with* $\mathbf{L}^i \cong \mathrm{SL}_3(\mathbb{F})$, $i = 1, 2$. *Moreover,* $N_{\mathbf{G}}(\mathbf{L}) = \langle \mathbf{L}, \gamma \rangle$, *where* γ *normalizes* $\mathbf{L}^i$, *inducing the inverse-transpose automorphism in the isomorphic copy* $\mathrm{SL}_3(\mathbb{F})$, $i = 1, 2$.

(b) *Let* $d = \gcd(3, q - \varepsilon)$. *Then* $L^i \cong \mathrm{SL}_3^\varepsilon(q)$ *for* $i = 1, 2$ *and* $L = \langle L^1 \circ_d L^2, x \rangle$ *for some* $x \in T$ *satisfying the following properties. If* $d = 1$, *then* $x = 1$. *If* $d = 3$, *then* $x^3 \in L^1 \circ_d L^2$ *and* x *normalizes* L^i, *inducing a diagonal automorphism in the isomorphic copy* $\mathrm{SL}_3^\varepsilon(q)$, $i = 1, 2$.

Finally, $N_G(L) = \langle L, \gamma \rangle$. *If* $d = 3$, *the element* γ *inverts* x, *and thus* $N_G(L)/(L^1 \circ_3 L^2)$ *is isomorphic to the symmetric group on three letters.*

Proof. We only give the proof for $g = 1$. The case $g \neq 1$ can be treated by conjugating all relevant structures established for $\mathbf{L}_0$ with g. Alternatively, one can replace the pair $(\mathbf{L}, F)$ by $(\mathbf{L}_0, F\gamma)$.

(a) We employ the notation and results summarized in 4.1. Observe that $\bar{J}_1$ and $\bar{J}_2$ are closed, disjoint and of type A_2, respectively $\tilde{A}_2$. (We use the common notational convention to indicate the irreducible closed subsystems of Σ of type A consisting of short roots by a tilde.) Moreover, $\bar{J}_1 \cup \bar{J}_2 = \bar{J}$ is closed and thus $[\mathbf{L}^1, \mathbf{L}^2] = \{1\}$. As $\bar{J}_1 \cup \bar{J}_2$ has rank 4, we also have $\mathbf{L} = \mathbf{L}^1\mathbf{L}^2$. Finally, $\mathbf{L}^i \cong \mathrm{SL}_3(\mathbb{F})$ for $i = 1, 2$. If $d = 3$, then $\mathbf{L}^1$ and $\mathbf{L}^2$ intersect in a group of order 3, as $|Z(\mathbf{L})| = 3$ (the latter assertion can be verified by a computation with CHEVIE using [76, 8.1.8]). Hence $\mathbf{L} = \mathbf{L}^1 \circ_d \mathbf{L}^2$ as claimed. As $\mathrm{Stab}_W(J) = \langle w_0 \rangle$, we obtain $N_{\mathbf{G}}(\mathbf{L}) = \langle \mathbf{L}, \gamma \rangle$. As α_1 and α_{23} are long, whereas α_3 and α_4 are short, $N_{\mathbf{G}}(\mathbf{L})$ stabilizes $\mathbf{L}^i$ for $i = 1, 2$. It follows from Lemma 4.4, that γ acts on each $\mathbf{L}^i$ as inverse-transpose automorphism (in the natural 3-dimensional representation of $\mathbf{L}^i$), $i = 1, 2$. (The corresponding relation for $\pm\alpha_{23}$ is easily verified with CHEVIE.)

(b) Clearly, $L^i \cong \mathrm{SL}_3(q)$, $i = 1, 2$. Moreover, $L = L^1 \times L^2$ if $d = 1$. Suppose that $d = 3$, and let $z \in Z(\mathbf{L})$ be an element of order 3. Let $x_i \in \mathbf{T} \cap \mathbf{L}^i$, $i = 1, 2$, such that $F(x_1)^{-1}x_1 = z = F(x_2)x_2^{-1}$. Then $x := x_1x_2$ is F-stable and $L = \langle L^1 \circ_3 L^2, x\rangle$. By construction, x normalizes L^1 and L^2. Moreover, x acts as a diagonal automorphism on each of L^i, $i = 1, 2$; see Remark 4.15 below. As $x_i^3 \in L^i$ for $i = 1, 2$, we have $x^3 \in L^1 \circ_3 L^2$. As γ inverts the elements of $\mathbf{T}$, it inverts x. As $x^2 \notin L^1 \circ_3 L^2$, the group $\langle L, \gamma\rangle/(L^1 \circ_3 L^2)$ is not abelian and hence isomorphic to the symmetric group of order 6.

From (a) we get $N_G(\mathbf{L}) = \langle L, \gamma\rangle$. As the latter is a maximal subgroup of G by [55], we obtain $N_G(L) = N_G(\mathbf{L})$. □

We can be more specific in the choice of the element x in Proposition 4.14 in case $3 \mid q^2 - 1$. This shows in particular, that x acts as diagonal automorphism on each of L^1 and L^2.

Remark 4.15. Keep the notation and assumptions of Proposition 4.14 and assume that $3 \nmid q$, i.e. $3 \mid q^2 - 1$. Let e be the minimal positive integer such that $\mathrm{SL}_3^{\varepsilon}(q) \leq \mathrm{GL}_3(q^e)$. Then $e \in \{1, 2\}$. We may think of $\mathbf{L}^1$ in its natural representation and of F acting on $\mathbf{L}^1$ as standard Frobenius endomorphism, if $\varepsilon = 1$, and as standard Frobenius endomorphism followed by the inverse-transpose automorphism if $\varepsilon = -1$. We have $z = \mathrm{diag}(\zeta, \zeta, \zeta)$ for $\zeta \in \mathbb{F}_{q^e}$ a third root of 1. Let $\xi \in \mathbb{F}$ be of 3-power order with $\xi^{1-\varepsilon q} = \zeta$. Then $x_1 := \mathrm{diag}(\xi^{-1}, \xi^{-1}, \xi^2)$ satisfies $F(x_1)^{-1}x_1 = z$. Also, $\xi^3 \in \mathbb{F}_{q^e}$, and x_1 acts on $\mathrm{SL}_3^{\varepsilon}(q)$ as conjugation by $\mathrm{diag}(1, 1, \xi^3) \in \mathrm{GL}_3(q^e)$, i.e. as a diagonal automorphism. If 3^a is the 3-part of $q - \varepsilon$, then $|\xi^3| = 3^a$. An analogous choice can be made for x_2.

Recall that $x_i^3 \in T \cap L^i$ for $i = 1, 2$. Hence $x_2^3 = x_1^{-3}x \in \langle L^1, x\rangle \cap (T \cap L^2)$. Now $L_{\{2,23\}} = \langle L^1, T \cap L^2, x\rangle$ and $Z(L_{\{2,23\}}) = T \cap L^2$. Thus $L_{\{2,23\}} = \langle L^1, x\rangle \circ_{[3^a]} Z(L_{\{2,23\}})$. (See also Proposition 4.17 below.) □

We also note that the group $\mathbf{L}$ defined in Proposition 4.14 is selfdual.

Remark 4.16. Let J and $\mathbf{L} = \mathbf{L}_J$ be as in Proposition 4.14. Then $\mathbf{L}$ is self dual. Indeed, the $\mathbb{Q}$-linear isomorphism $X \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$, determined by mapping the four-tuple $(\alpha_1, \alpha_{23}, \alpha_3, \alpha_4)$ of roots to the four-tuple $(\alpha_{23}^\vee, \alpha_1^\vee, \alpha_3^\vee, \alpha_4^\vee)$ of coroots, restricts to a $\mathbb{Z}$-linear isomorphism $X \to Y$, which defines an isomorphism of the root data of $\mathbf{L}$ and its dual group. □

In the following, if $M \leq G$ and $\chi \in \mathrm{Irr}(M)$, we write $N_G(M, \chi)$ for the stabilizer of χ in $N_G(M)$.

Proposition 4.17. *Let* $J_1 := \{\alpha_1, \alpha_{23}\}$*,* $J_2 := \{\alpha_3, \alpha_4\}$*, and let* $\mathbf{M}_0 := \mathbf{L}_J$ *with* $J \in \{J_1, J_2\}$*. Put* $\mathbf{M} := \mathbf{M}_0^g$ *and* $\mathbf{M}' := [\mathbf{M}, \mathbf{M}]$*.*

(a) *Let* $d = \gcd(3, q^2-1)$*. Then* $\mathbf{M} = Z(\mathbf{M}) \circ_d \mathbf{M}'$*, with* $\mathbf{M}' \cong \mathrm{SL}_3(\mathbb{F})$*. Moreover,*

$$N_{\mathbf{G}}(\mathbf{M}) = \langle \mathbf{N}' \circ_d \mathbf{M}', \gamma \rangle = \mathbf{N}\mathbf{M}',$$

with $Z(\mathbf{M}) \leq \mathbf{N}' \leq \mathbf{N} \leq N_{\mathbf{G}}(\mathbf{T})$*, where* $\mathbf{N} = \langle \mathbf{N}', \gamma \rangle$*. Furthermore,* $\mathbf{N}' \cap \mathbf{M} = \mathbf{N} \cap \mathbf{M} = Z(\mathbf{M})$ *and* $\mathbf{N}'/Z(\mathbf{M}) \cong S_3$ *and* $\mathbf{N}/Z(\mathbf{M}) \cong D_{12}$*. Finally,* $\mathbf{N}' = C_{N_{\mathbf{G}}(\mathbf{M})}(\mathbf{M}')$*. In particular,* $\mathbf{N}'$ *is normal in* $N_{\mathbf{G}}(\mathbf{M})$*.*

(b) *Let* $d = \gcd(3, q - \varepsilon)$*. Then* $M = \langle Z(M) \circ_d M', x \rangle$ *with* $x = 1$*, if* $d = 1$*, and* $x \in T$ *as in* Proposition 4.14*, if* $d = 3$*. Moreover,*

$$N_G(\mathbf{M}) = \langle N' \circ_d M', x, \gamma \rangle = NM$$

with $N'/Z(M) \cong S_3$ *and* $N/Z(M) \cong D_{12}$*.*

(c) *If* $N_G(\mathbf{M}) = N_G(M)$*, every* $\chi \in \mathrm{Irr}(M)$ *extends to* $N_G(M, \chi)$*.*

Proof. We only consider the case $J = J_2$; the other case is proved analogously. We also assume $g = 1$. The case $g \neq 1$ can be treated by conjugating all relevant structures established for $\mathbf{M}_0$ with g. Alternatively, one can replace the pair $(\mathbf{M}, F)$ by $(\mathbf{M}_0, F\gamma)$.

(a) We use the notation of Proposition 4.14. Let $\mathbf{M} = \mathbf{L}_{J_2}$ and put $\mathbf{K} := \mathbf{K}_{J_1}$. Then $Z(\mathbf{M}) = \mathbf{T} \cap \mathbf{K}$. The assertion about the structure of $\mathbf{M}$ follows from Proposition 4.14. To establish the claim about $N_{\mathbf{G}}(\mathbf{M})$, write $\mathbf{N}' := N_{\mathbf{K}}(Z(\mathbf{M}))$. As $\mathbf{N}'$ normalizes $Z(\mathbf{M})$, it also normalizes $\mathbf{M} = C_{\mathbf{G}}(Z(\mathbf{M}))$. Moreover, $\mathbf{N}'$ centralizes $\mathbf{M}'$ as $[\mathbf{M}, \mathbf{M}] = \mathbf{K}_{J_2}$ and $[\mathbf{K}, \mathbf{K}_{J_2}] = 1$. Also, $\mathbf{N}' \cap \mathbf{M} = Z(\mathbf{M})$ and $\mathbf{N}'/Z(\mathbf{M}) \cong W_{J_1} \cong S_3$. Now W_{J_1} is a subgroup of $\mathrm{Stab}_W(J_2)$, which is a dihedral group of order 12; see [48, p. 74]. In particular, $\mathbf{N}' \circ_d \mathbf{M}'$ is a subgroup of $N_{\mathbf{G}}(\mathbf{M})$ of index 2. As $\gamma \in N_{\mathbf{G}}(\mathbf{M}) \setminus \mathbf{N}' \circ_d \mathbf{M}'$, and γ does not centralize $\mathbf{M}'$, we obtain all our claims.

(b) The structure of M follows from Proposition 4.14. Now $x \in T$ normalizes $\mathbf{K}$ and $Z(\mathbf{M})$, hence N'. In particular, $\langle N' \circ_d M', x \rangle = N'M$, and N' and M are invariant under γ, giving the structure of $N_G(\mathbf{M})$. As F acts trivially on W, we get the assertions on $N'/Z(M)$ and $N/Z(M)$.

(c) Let $\chi \in \mathrm{Irr}(M)$. In the considerations to follow, we will make use of the facts summarized in 2.9 for characters of central products. As M is a central product $Z(M) \circ \langle M', x \rangle$, we have $\chi = \lambda\psi$ for $\lambda \in \mathrm{Irr}(Z(M))$ and $\psi \in \mathrm{Irr}(\langle M', x \rangle)$. Let χ' and ψ' denote the restrictions of χ to $Z(M) \circ_d M'$ and of ψ to M', respectively. Then $\chi' = \lambda\psi'$. Put $H := \langle N' \circ_d M', \gamma \rangle = NM'$. Then $H \cap M = Z(M) \circ_d M'$ and $HM = N_G(M)$.

Let $I := N \cap N_G(M, \chi)$ and $I' := N' \cap N_G(M, \chi)$. Then $N_G(M, \chi) = IM$ and $N_G(M, \chi)/M \cong I/Z(M)$. We may assume that $4 \mid |I/Z(M)|$, as otherwise the Schur multiplier of $I/Z(M)$ is trivial. Clearly, I stabilizes λ. To determine the action of N on $Z(M)$, notice that $Z(\mathbf{M})$ is a maximal 1-F-split torus of $\mathbf{K} \cong \mathrm{SL}_3(\mathbb{F})$, that $N'/Z(M) \cong N_{\mathbf{K}}(Z(\mathbf{M}))/Z(\mathbf{M})$, and that γ acts as inverse-transpose automorphism on $\mathbf{K}$. In particular, N only fixes the trivial element of $Z(M)$. By Brauer's permutation lemma, N only fixes the trivial character of $Z(M)$. Thus λ is the trivial character if $I = N$. In this case, we define $\hat{\chi} \in \mathrm{Irr}(N'M)$ by $\hat{\chi}(n'm) = \chi(m)$ for $n' \in N', m \in M$; it follows from (a) that M normalizes N', so that $\hat{\chi}$ is well defined. Then $\hat{\chi}$ is an extension of χ to $N'M$. Moreover, $\hat{\chi}$ is invariant in $N_G(M) = \langle N'M, \gamma\rangle$ as γ stabilizes N' and χ, and so $\hat{\chi}$ extends to $N_G(M)$.

Now suppose that $|I/Z(M)| = 4$. Then $I/Z(M) = \langle s, \gamma\rangle$, with $s \in \{s_1, s_{23}, s_{23}^{s_1}\}$. We only treat the case $s = s_1$; the other cases are handled in an analogous way (or by symmetry). As s_1 stabilizes λ, we conclude that $\alpha_1^\vee(z) \in \ker(\lambda)$ for all $z \in \mathbb{F}_q^*$. In particular, $n_1^2 = \alpha_1^\vee(-1) \in \ker(\lambda)$. Let λ' denote an extension of λ to I'. Now $n_1 \in I'$, as $n_1 \in N'$, and thus $\lambda'(n_1) = \pm 1$. Since γ inverts n_1, it follows that λ' is invariant under γ. As I stabilizes ψ', and either ψ' is irreducible or has exactly three irreducible constituents, we may choose an irreducible constituent ϑ of ψ' which is invariant under I. Hence $\lambda'\vartheta \in \mathrm{Irr}(I' \circ_d M')$ is invariant in IM' and thus extends to IM'.

If $\vartheta = \psi'$, i.e. if χ' is irreducible; then χ extends to $N_G(M, \chi)$ by [74, Lemma 4.1(a)], as $\lambda\psi'$ extends to its inertia group IM' in H. Suppose now that $\vartheta \neq \psi'$. Then $d = 3$ and H has index 3 in $N_G(M)$. Now $\lambda'\vartheta$ is not invariant under x, and thus induces to an irreducible character $\hat{\chi}'$ of $I'M$ which extends χ (notice that x fixes λ and normalizes N', so that x also normalizes I'). As $\lambda'\vartheta$ is invariant under γ, the same is true for $\hat{\chi}'$, which thus extends to IM. □

Notice that $\{\alpha_1, \alpha_2\}$ is conjugate in W to $\{\alpha_1, \alpha_{23}\}$, so that the results established in Proposition 4.17 hold likewise for $\mathbf{L}_{\{\alpha_1,\alpha_2\}}$.

Proposition 4.18. *Let* $\mathbf{M}_0 := \mathbf{L}_J$*, with* $J = \{\alpha_1, \alpha_4\}$ *and put* $\mathbf{M} := \mathbf{M}_0^g$.

(a) *Let* $d := \gcd(2, q-1)$. *Then* $\mathbf{M} = \mathbf{M}_1 \circ_d \mathbf{M}_2$ *with connected, reductive,* F*-stable subgroups* $\mathbf{M}_i$*,* $i = 1, 2$. *Putting* $\mathbf{M}_i' := [\mathbf{M}_i, \mathbf{M}_i]$*, we have* $\mathbf{M}_i' \cong \mathrm{SL}_2(\mathbb{F})$*, and* $Z(\mathbf{M}_i)^\circ$ *is a torus of rank* 1*, for* $i = 1, 2$. *Moreover,* $\mathbf{M}_1 = Z(\mathbf{M}_1)^\circ \times \mathbf{M}_1'$ *and* $\mathbf{M}_2 \cong \mathrm{GL}_2(\mathbb{F})$. *In particular,* $Z(\mathbf{M}_2)$ *is connected and* $\mathbf{M}_2 = Z(\mathbf{M}_2) \circ_d \mathbf{M}_2'$.

Furthermore, there are lifts $m_1, m_2 \in \hat{W}^g$ *of* s_{22}^g *and* s_{17}^g (see Subsection 4.3)*, respectively, such that* m_i *normalizes* $\mathbf{M}_i$ *for* $i = 1, 2$*, and such that* m_1 *centralizes* $\mathbf{M}_1'\mathbf{M}_2$ *and* m_2 *centralizes* $\mathbf{M}_1\mathbf{M}_2'$. *For any* *such pair of elements we have*

$$N_{\mathbf{G}}(\mathbf{M}) = \mathbf{M}_1.2 \circ_d \mathbf{M}_2.2,$$

with $\mathbf{M}_i.2 = \langle \mathbf{M}_i, m_i \rangle$ *for* $i = 1, 2$.

(b) *With the notation of* (a), *we have* $M = \langle M_1 \circ_d M_2, x \rangle$, *where* $x = 1$ *if* $d = 1$, *and* $x^2 \in M_1 \circ_2 M_2$, *otherwise. In the latter case,* x *centralizes* $Z(M_1)$ *and* M_2, *and normalizes* M_1, *inducing a diagonal automorphism on* M_1'. *Moreover,* $M_1 = Z(M_1) \circ_d M_1'$ *and* $M_2 = \langle Z(M_2) \circ_d M_2', y \rangle$ *with* $y = 1$ *if* $d = 1$, *and* $y^2 \in Z(M_2) \circ_d M_2'$, *otherwise. In the latter case,* y *centralizes* $Z(M_2)$ *and induces a diagonal automorphism on* M_2'. *Furthermore,*

$$N_G(\mathbf{M}) = \langle M_1.2 \circ_d M_2.2, x \rangle.$$

with $M_i.2 = \langle M_i, m_i \rangle$, $i = 1, 2$. *Finally,* x *and* y *normalize* $\langle Z(M_i), m_i \rangle$ *and* $\langle M_i, m_i \rangle$ *for* $i = 1, 2$.

(c) *If* $N_G(\mathbf{M}) = N_G(M)$, *every* $\chi \in \mathrm{Irr}(M)$ *extends to* $N_G(M, \chi)$. *Suppose that* $d = 2$ *and that* $N_G(M, \chi) = N_G(M)$. *If* $\mathrm{Res}^M_{M_1 \circ_2 M_2}(\chi)$ *is irreducible, there is an extension* $\hat{\chi} \in \mathrm{Irr}(N_G(M))$ *of* χ *such that* $\mathrm{Res}^{N_G(M)}_{M_1.2 \circ_2 M_2.2}(\hat{\chi})$ *is irreducible. If* $\mathrm{Res}^M_{M_1 \circ_2 M_2}(\chi)$ *is reducible, there is an extension* $\hat{\chi} \in \mathrm{Irr}(N_G(M))$ *of* χ *such that* $\mathrm{Res}^{N_G(M)}_{M_1.2 \circ_2 M_2.2}(\hat{\chi})$ *is reducible.*

Proof. Again, we only prove the assertions for $g = 1$.

(a) Use the notation and results summarized in 4.1. By [48, p. 74], we have $\mathrm{Stab}_W(J) \cong 2^2$. In fact, a computation with CHEVIE [41] shows that $\mathrm{Stab}_W(J) = \langle s_{22}, s_{17} \rangle$. Let $J_1 = \{\alpha_1, \alpha_{22}\}$ and $J_2 = \{\alpha_4, \alpha_{14}\}$. Then $\alpha_{17} \in \bar{J}_2$, and $\bar{J}_1$ and $\bar{J}_2$ are of type A_1A_1 and C_2, respectively. Moreover, $\bar{J}_1 \cap \bar{J}_2 = \emptyset$ and $\Gamma := \bar{J}_1 \cup \bar{J}_2$ is closed. Put $\mathbf{K}_i := \mathbf{K}_{J_i}$ for $i = 1, 2$. Then $[\mathbf{K}_1, \mathbf{K}_2] = 1$ by the remark preceding Lemma 4.2. A calculation with CHEVIE, using [76, 8.1.8, 8.1.9] shows that $\mathbf{K}_1$ has a center of order d^2, and that each of $\mathbf{K}_2$ and $\mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_\Gamma$ has a center of order d; thus $\mathbf{K}_1 \cap \mathbf{K}_2$ is a group of order d. Moreover, $\mathbf{K}_1$ is a direct product of two copies of $\mathrm{SL}_2(\mathbb{F})$, and $\mathbf{K}_2 = \mathrm{Sp}_4(\mathbb{F})$. Let $\mathbf{M}_i$ denote the Levi subgroup of $\mathbf{K}_i$ corresponding to α_1 if $i = 1$, and to α_4 if $i = 2$. Then $\mathbf{M}_1 = Z(\mathbf{M}_1)^\circ \times \mathbf{M}_1'$ and $\mathbf{M}_2 \cong \mathrm{GL}_2(\mathbb{F})$. Moreover, $[\mathbf{M}_1, \mathbf{M}_2] = 1$ and $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$. Also, $\mathbf{M}_1$ and $\mathbf{M}_2$ intersect in a group of order d. This implies the structure of $\mathbf{M}$ asserted in (a).

We now prove the claims on $N_G(\mathbf{M})$. Put $m_1 := n_{22}$ and $m_2 := n_{17}n_{14}^2$. Then $m_i \in \mathbf{K}_i$ and m_i centralizes $\mathbf{M}_1'\mathbf{M}_2'$ for $i = 1, 2$. Moreover, $N_{\mathbf{K}_i}(\mathbf{M}_i) = \langle \mathbf{M}_i, m_i \rangle =: \mathbf{M}_i.2$ for $i = 1, 2$. This yields the claims on the structure of $N_G(\mathbf{M})$.

(b) Notice that $M_1 \circ_d M_2$ has index d in M. In case q is odd, we choose a particular F-stable element x of $M \setminus (M_1 \circ_2 M_2)$ as follows. Let $x_1 \in \mathbf{T} \cap \mathbf{M}_1$ and $x_2 \in Z(\mathbf{M}_2)$ be such that $F(x_1)^{-1}x_1 = F(x_2)x_2^{-1}$ is the unique element of order 2 in $\mathbf{M}_1 \cap \mathbf{M}_2 = M_1 \cap M_2$. Then $x := x_1x_2$ is F-stable and $x \notin M_1 \circ_2 M_2$ so that $M = \langle M_1 \circ_2 M_2, x\rangle$. Notice that x centralizes M_2 and normalizes M_1, inducing a diagonal automorhism on M_1'. Also, x centralizes $Z(M_1)$, as $x = x_1x_2$ with $x_1 \in T \cap M_1$. With an analogous argument we can find $y = y_1y_2 \in M_2$ with $y_1 \in Z(\mathbf{M}_2)$ and $y_2 \in \mathbf{M}_2'$ such that $M_2 = \langle Z(M_2) \circ_d M_2', y\rangle$. In particular, y centralizes $Z(M_2)$ and induces a diagonal automorphism on M_2' if $d = 2$. This gives the structure of M_2 as claimed. The structure of M_1 is clear from (a). Moreover, $z^{-1}m_i^{-1}zm_i \in Z(M_i)$ for $i = 1, 2$ and $z \in \{x, y\}$, so that x and y normalize $\langle Z(M_i), m_i\rangle$ and $\langle M_i, m_i\rangle$ for $i = 1, 2$. As $(N_{\mathbf{G}}(\mathbf{M})/\mathbf{M})^F = N_G(\mathbf{M})/M$, the claims on the structure of $N_G(\mathbf{M})$ in (b) are established.

(c) We use the structure of $N_G(\mathbf{M})$ established in (b) to prove the claim. Let $\chi \in \mathrm{Irr}(M)$. If the inertia subgroup of χ is strictly smaller than $N_G(M)$, the inertia quotient is cyclic, and χ extends. If q is even, then χ clearly extends. We may thus assume that χ is invariant in $N_G(M)$ and that q is odd. In the following, we will frequently use the remarks in 2.9. Let χ' denote the restriction of χ to $M_1 \circ_2 M_2$, and let ψ be an irreducible constituent of χ'. Then $\psi = \psi_1\psi_2$ with $\psi_i \in \mathrm{Irr}(M_i)$, $i = 1, 2$. If χ' is irreducible, i.e., $\chi' = \psi$, then ψ_i is invariant in $M_i.2$ and thus extends to $M_i.2$, $i = 1, 2$. Hence $\chi' = \psi$ extends to $\hat{\psi} \in \mathrm{Irr}(M_1.2 \circ_2 M_2.2)$. By [74, Lemma 4.1(a)], there is an extension $\hat{\chi} \in \mathrm{Irr}(N_G(M))$ such that $\mathrm{Res}^{N_G(M)}_{M_1.2\circ_2 M_2.2} = \hat{\psi}$. If χ' is reducible, we have $\chi' = \psi_1\psi_2 + \psi_1^x\psi_2^x$, as $x^2 \in M_1 \circ_2 M_2$. Moreover, $\psi_2^x = \psi_2$ as x centralizes M_2, and hence $\psi_1^x \neq \psi_1$. Now $N_G(M, \psi)$ has index 2 in $N_G(M)$. Notice that $m_2x \notin N_G(M, \psi)$, as $\psi_1^{m_2x} = \psi_1^x \neq \psi_1$. Suppose that $m_1x \in N_G(M, \psi)$. Then $\psi_1^{m_1} = \psi_1^x$. As $M_1 = Z(M_1) \circ_2 M_1'$, we can write $\psi_1 = \lambda_1\psi_1'$ with $\lambda_1 \in \mathrm{Irr}(Z(M_1))$ and $\psi_1' \in \mathrm{Irr}(M_1')$. However, as x centralizes $Z(M_1)$ and m_1 centralizes M_1', we cannot have $\psi_1^{m_1} = \psi_1^x$ unless $\psi_1^x = \psi_1$. This contradiction shows that $m_1x \notin N_G(M, \psi)$. It follows that $N_G(M, \psi) = M_1.2 \circ_2 M_2.2$. Hence $\psi = \psi_1\psi_2$ extends to $\hat{\psi} \in \mathrm{Irr}(N_G(M, \psi))$, and $\hat{\psi}^x \neq \hat{\psi}$. Then $\hat{\chi} := \mathrm{Ind}^{N_G(M)}_{N_G(M,\psi)}(\hat{\psi})$ is an extension of χ. □

Proposition 4.19. *Let* $\mathbf{M}_0 := \mathbf{L}_J$*, where* J *is one of the sets* $\{\alpha_1\}$*,* $\{\alpha_4\}$ *or* $\{\alpha_2, \alpha_3\}$ *of simple roots of* Σ*. Put* $\mathbf{M} := \mathbf{M}_0^g$ *and* $\mathbf{M}' := [\mathbf{M}, \mathbf{M}]$*. Let* $d := \gcd(2, q-1)$*.*

(a) *We have* $\mathbf{M} = Z(\mathbf{M}) \circ_d \mathbf{M}'$ *with* $\mathbf{M}' \cong \mathrm{SL}_2(\mathbb{F})$ *in the first two cases, and* $\mathbf{M}' \cong \mathrm{Sp}_4(\mathbb{F})$ *in the last case. Furthermore,*

$$N_{\mathbf{G}}(\mathbf{M}) = \mathbf{N}' \circ_d \mathbf{M}'$$

with $Z(\mathbf{M}) \leq \mathbf{N}' \leq N_{\mathbf{G}}(\mathbf{T}_0)$ *and* $\mathbf{N}' \cap \mathbf{M} = Z(\mathbf{M})$. *Furthermore,* $\mathbf{N}'/Z(\mathbf{M}) \cong \mathrm{Stab}_W(J)$, *where* $\mathrm{Stab}_W(J) \cong W(C_3) \cong 2^3.S_3$ *in the first two cases, and* $\mathrm{Stab}_W(J) \cong W(C_2) \cong D_8$ *in the last case. Finally,* $\mathbf{N}' = C_{N_{\mathbf{G}}(\mathbf{M})}(\mathbf{M}')$.

(b) *We have* $M = \langle Z(M) \circ_d M', x\rangle$, *with* $x = 1$ *if* q *is even, and* $x^2 \in Z(M) \circ_2 M'$ *if* q *is odd; in the latter case,* x *induces a diagonal automorphism on* M'. *Moreover,*

$$N_G(\mathbf{M}) = \langle N' \circ_d M', x\rangle.$$

(c) *If* $N_G(\mathbf{M}) = N_G(M)$, *then every* $\chi \in \mathrm{Irr}(M)$ *extends to* $N_G(M, \chi)$.

Proof. Once more, we only prove the case $g = 1$.

(a) We have $\mathbf{M} = Z(\mathbf{M})\mathbf{M}'$ and $\mathbf{M}'$ is a simple, simply connected algebraic group of type A_1 and C_2, respectively. Hence $\mathbf{M}'$ is isomorphic to $\mathrm{SL}_2(\mathbb{F})$, respectively $\mathrm{Sp}_4(\mathbb{F})$. If q is even, $Z(\mathbf{M}')$ is trivial and thus $\mathbf{M} = Z(\mathbf{M}) \times \mathbf{M}'$. If q is odd, $Z(\mathbf{M}')$ has order 2 and lies in $Z(\mathbf{M})$, and thus $\mathbf{M} = Z(\mathbf{M}) \circ_2 \mathbf{M}'$. This proves our first assertion on the structure of $\mathbf{M}$.

To investigate the structure of $N_{\mathbf{G}}(\mathbf{M})$, we will choose a closed subsystem $\Delta \subseteq \Sigma$ with the following properties: $\Delta \cap \Sigma_J = \emptyset$, $\mathrm{rk}(\Delta) + |J| = 4$, $W_\Delta = \mathrm{Stab}_W(J)$ and W_Δ stabilizes each element of J. Moreover, $\tilde{\Delta}$ will denote a closed subsystem of Δ (hence of Σ) with $\mathrm{rk}(\Delta) = \mathrm{rk}(\tilde{\Delta})$ and such that $\tilde{\Delta} \cup \Sigma_J$ is closed in Σ. Put $\mathbf{K} := \mathbf{K}_\Delta$, $\tilde{\mathbf{K}} := \mathbf{K}_{\tilde{\Delta}}$, and $\mathbf{S} := \mathbf{T}_0 \cap \mathbf{K} = \mathbf{T}_0 \cap \tilde{\mathbf{K}}$. Then $\mathbf{S} = Z(\mathbf{M})$. Write $\mathbf{N} := N_{\mathbf{K}}(\mathbf{S})$ and $\tilde{\mathbf{N}} := N_{\tilde{\mathbf{K}}}(\mathbf{S})$. As $\mathbf{N}$ normalizes $\mathbf{S} = Z(\mathbf{M})$, it also normalizes $\mathbf{M} = C_{\mathbf{G}}(Z(\mathbf{M}))$ and $\mathbf{M}'$. Furthermore, $\mathbf{N} \cap \mathbf{M} = Z(\mathbf{M}) = \mathbf{S}$ and $\mathbf{N}/\mathbf{S} = W_\Delta = \mathrm{Stab}_W(J)$. Thus $N_G(\mathbf{M}) = \mathbf{N}\mathbf{M} = \mathbf{N}\mathbf{M}'$. Moreover, $\mathbf{S} \leq \tilde{\mathbf{N}} \leq \mathbf{N}$ and $\tilde{\mathbf{N}}$ centralizes $\mathbf{M}'$ as $\mathbf{M}' = \mathbf{K}_J$ and $[\tilde{\mathbf{K}}, \mathbf{K}_J] = 1$.

We now choose Δ and $\tilde{\Delta}$ in the respective cases. Suppose first that $J = \{\alpha_1\}$. Here, we let $\Delta = \tilde{\Delta}$ to be the closed subsystem of Σ generated by $\{\alpha_3, \alpha_4, \alpha_{14}\}$. Then Δ is of type C_3. If $J = \{\alpha_4\}$, we choose Δ and $\tilde{\Delta}$ to be the closed subsystems of Σ generated by $\{\alpha_1, \alpha_2, \alpha_{13}\}$ and $\{\alpha_1, \alpha_2, \alpha_{22}\}$, respectively. Then Δ is of type B_3 and $\tilde{\Delta}$ of type A_3. If $J = \{\alpha_2, \alpha_3\}$, we choose Δ and $\tilde{\Delta}$ to be the closed subsystems of Σ generated by $\{\alpha_8, \alpha_{16}\}$, and $\{\alpha_{16}, \alpha_{24}\}$, respectively. Then Δ is of type C_2 and $\tilde{\Delta}$ of type A_1A_1.

If $J = \{\alpha_1\}$, we put $\mathbf{N}' := \mathbf{N}$. Notice that in this case, $\mathbf{N}' = \mathbf{N} = \tilde{\mathbf{N}}$ centralizes $\mathbf{M}'$. Suppose that $\tilde{\mathbf{N}} \lneq \mathbf{N}$. Then $[\mathbf{N} : \tilde{\mathbf{N}}] = 2$, as $\tilde{\mathbf{N}}/\mathbf{S} \cong W_{\tilde{\Delta}}$, and thus $\tilde{\mathbf{N}}\mathbf{M}'$ has index 2 in $N_{\mathbf{G}}(\mathbf{M})$. If $J = \{\alpha_2, \alpha_3\}$, let $n := n_8$, and if $J = \{\alpha_4\}$, let $n := n_{13}$. Then $\mathbf{N} = \langle \tilde{\mathbf{N}}, n\rangle$. As the image of n in W lies in W_Δ, which fixes the elements of J, we find that $n^{-1}u_\alpha(s)n = u_\alpha(\pm s)$ for all $\alpha \in J \cup (-J)$ and all $s \in \mathbb{F}$; see [21, Lemma 7.2.1(i)]. In particular, n centralizes $\mathbf{M}'$ if q is even, in which case we set $\mathbf{N}' := \mathbf{N}$. Assume now that q is odd. By Lemma 4.2, we may replace n by $n' = tn$ for a suitable $t \in T_0$ in such a way that n' centralizes $\mathbf{M}'$. Putting $\mathbf{N}' := \langle \tilde{\mathbf{N}}, n' \rangle$, we obtain $N_{\mathbf{G}}(\mathbf{M}) = \mathbf{N}'\mathbf{M}'$ and $[\mathbf{N}', \mathbf{M}'] = 1$, i.e., $\mathbf{N}'\mathbf{M}' = \mathbf{N}' \circ_d \mathbf{M}'$. As $\mathbf{N}'$ normalizes $Z(\mathbf{M})$ and centralizes $\mathbf{M}'$, it normalizes $\mathbf{T}_0 = Z(\mathbf{M})(\mathbf{M}' \cap \mathbf{T}_0)$. Finally, $\mathbf{N}' = C_{\mathbf{G}}(\mathbf{M}') \cap N_{\mathbf{G}}(\mathbf{M}) = C_{N_{\mathbf{G}}(\mathbf{M})}(\mathbf{M}')$.

(b) If q is even, our claims are obvious from (a). Let q be odd. Then there are $x_1 \in Z(\mathbf{M})$ and $x_2 \in \mathbf{T} \cap \mathbf{M}'$ such that $F(x_1)^{-1}x_1 = F(x_2)x_2^{-1}$ is the unique element of order 2 in $Z(\mathbf{M}) \cap \mathbf{M}'$. Hence $x := x_1x_2$ is F-stable and $M = \langle Z(M) \circ_d M', x \rangle$. Notice that x normalizes $\mathbf{M}'$, inducing a diagonal automorphism on M'. This gives our claim for the structure of M. The claim for $N_G(\mathbf{M})$ follows from this and (a).

(c) By (b) we have $M = \langle Z(M) \circ_d M', x \rangle$ and $N_G(\mathbf{M}) = \langle N' \circ_d M', x \rangle$; moreover, $Z(M) = [q-1]^c$, with $c = 3$ or 2. Notice that the groups $\mathbf{S}$, $\mathbf{K}$ and $\tilde{\mathbf{K}}$ introduced in the proof of (a) are F-invariant. Notice also that F acts trivially on $\mathbf{N}/Z(\mathbf{M})$, and we implicitly assume that the inverse images of elements of this group used below are F'-stable.

Let $\chi \in \mathrm{Irr}(M)$. In the considerations to follow, we will make use of the facts summarized in 2.9 for characters of central products. As $M = Z(M) \circ \langle M', x \rangle$ is a central product, we have $\chi = \lambda\psi$ for $\lambda \in \mathrm{Irr}(Z(M))$ and $\psi \in \mathrm{Irr}(\langle M', x \rangle)$. Let χ' and ψ' denote the restrictions of χ to $Z(M) \circ_d M'$ and of ψ to M', respectively. Then $\chi' = \lambda\psi'$.

Recall from the proof of (a) that $N_G(M) = NM$ with $M \cap N = Z(M)$, and that $\tilde{N}$ is a subgroup of N of index 2, containing $Z(M)$ and centralizing M'. Put $I := N_G(M, \chi) \cap N$ and $\tilde{I} := I \cap \tilde{N}$. Then $N_G(M, \chi) = IM$ and thus $N_G(M, \chi)/M \cong I/Z(M)$. Also, $\tilde{I}$ has index at most 2 in I. Clearly, I stabilizes λ and ψ'. It follows from [74, Theorem 1.1], applied to $N_{\mathbf{K}}(\mathbf{S})$, that λ extends to an irreducible character $\hat{\lambda}$ of I. (Notice that Δ is conjugate in W to a parabolic subsystem of Σ, so that $\mathbf{K}$ is simply connected.) Thus there is an extension $\tilde{\lambda}$ of λ to $\tilde{I}$ which is invariant under I.

To continue, suppose first that $\chi' = \lambda\psi'$ is irreducible. Then the inertia group of χ' in NM' equals IM' which contains $\tilde{I}M'$ as a subgroup of index at most 2. By construction, $\tilde{\lambda}\psi' \in \mathrm{Irr}(\tilde{I}M')$ is invariant under IM', as ψ' is invariant under I and $\tilde{\lambda}$ is invariant under I and M', the latter as M' centralizes $\tilde{N}$. Thus $\tilde{\lambda}\psi'$ extends to an irreducible character of IM', which is an extension of $\lambda\psi'$. The claim follows from [74, Lemma 4.1(a)], applied to the subgroup NM' of $NM = N_G(M)$.

Suppose now that $\psi' = \vartheta + \vartheta^x$ for some $\vartheta \in \mathrm{Irr}(M')$ and write $N_G(M, \lambda\vartheta)$ for the inertia subgroups of $\lambda\vartheta \in \mathrm{Irr}(Z(M) \circ_d M')$ in $N_G(M)$. Then $\tilde{I}M' \leq N_G(M, \lambda\vartheta) \leq IM$, where the latter inclusion has index 2. If $N_G(M, \lambda\vartheta) \in \{\tilde{I}M', IM'\}$, there is an extension $\widehat{\lambda\vartheta}$ of $\lambda\vartheta$ to $N_G(M, \lambda\vartheta)$. Now $\widehat{\lambda\vartheta}$ is not invariant under x, as $\widehat{\lambda\vartheta}^x$ is an extension of $\lambda\vartheta^x$. Thus $\mathrm{Ind}_{N_G(M,\lambda\vartheta)}^{N_G(M,\chi)}(\widehat{\lambda\vartheta})$ is an extension of χ to $N_G(M, \chi)$. We may thus assume that $\tilde{I}M' \lneq N_G(M, \lambda\vartheta) \lneq IM$, so that in particular $\tilde{I} \lneq I$. Now $IM/\tilde{I}M'$ is generated by the images of x and n. Moreover, $N_G(M, \lambda\vartheta) \neq \tilde{I}M$ as x does not stabilize $\lambda\vartheta$. Thus there is $\tilde{n} \in \tilde{N}$ such that $N_G(M, \lambda\vartheta) = \langle \tilde{I}M', xn\tilde{n} \rangle$. In particular, $n\tilde{n}$ stabilizes λ, and $xn\tilde{n}$ stabilizes ϑ. As $\tilde{N}$ centralizes M', and n^2 induces an inner automorphism of M', the latter condition can also be written as $\vartheta^x = \vartheta^n$. In particular, this case does not occur if $J = \{\alpha_1\}$, as then N centralizes M'. We claim that there is an $xn\tilde{n}$-stable extension $\tilde{\lambda}$ of λ to $\tilde{I}$. Provided this claim holds, then $\tilde{\lambda}\vartheta \in \mathrm{Irr}(\tilde{I}M')$ is $xn\tilde{n}$-invariant and thus extends to $\widehat{\lambda\vartheta} \in \mathrm{Irr}(\langle \tilde{I}M', xn\tilde{n} \rangle)$, which is not x-invariant. We conclude as in the previous case.

It remains to prove the claim. Suppose first that $J = \{\alpha_2, \alpha_3\}$. In this case $n = n_8$, and s_8 swaps the two roots α_{16} and α_{24}. Thus there are decompositions $Z(M) = [q-1]^2$ and $\tilde{N} = ([q-1].2)^2$ such that n permutes the two direct factors of the latter group. Let $\lambda = \lambda_1 \boxtimes \lambda_2$, where $\lambda_i \in \mathrm{Irr}([q-1])$, for $i = 1, 2$. As $n\tilde{n}$ stabilizes λ, we conclude that $\lambda_2 = \lambda_1$ or $\lambda_2 = \lambda_1^{-1}$. If $\lambda_1 \neq \lambda_1^{-1}$, we also have $\lambda_2 \neq \lambda_2^{-1}$, and then $\tilde{I} = Z(M)$. In this case, the claim is trivially true. Otherwise, $\tilde{I} = \tilde{N}$ and thus $\langle \tilde{I}M', xn\tilde{n} \rangle = \langle \tilde{N}M', xn \rangle$. Let $\tilde{\lambda}_1$ denote an extension of λ_1 to $[q-1].2$, the first factor of the above decomposition of $\tilde{N}$. Define $\tilde{\lambda}_2$ by $(\tilde{\lambda}_1 \boxtimes 1)^{xn} = 1 \boxtimes \tilde{\lambda}_2$. Then $\tilde{\lambda} := \tilde{\lambda}_1 \boxtimes \tilde{\lambda}_2$ is an $xn\tilde{n}$-stable extension of λ to $\tilde{I} = \tilde{N}$, since $(xn)^2$ acts as an inner automorphism on $\tilde{N}$. Suppose finally that $J = \{\alpha_4\}$, in which case $n = n_{13}$. As $\vartheta^x = \vartheta^n$ and as x induces a diagonal automorphism on M', the same must be true for n. Now $n^{-1}u_4(t)n = u_4(-t)$ for all $t \in \mathbb{F}$. As n induces a non-inner automorphism on M', we must have $4 \mid q + 1$. It follows that x_1 may be chosen as an element of order 4 in $Z(\tilde{\mathbf{K}})$. (Recall that $\tilde{\mathbf{K}} \cong \mathrm{SL}_4(\mathbb{F})$.) Hence $x = x_1 x_2$ centralizes $\tilde{N}$, and thus x stabilizes any extension of λ to $\tilde{I}$. As there is an $n\tilde{n}$-invariant such extension, our claim follows. $\square$

Corollary 4.20. *Let* $\mathbf{M}$ *denote an* e*-split Levi subgroup of* $\mathbf{G}$ *for some* $e \in \{1, 2, 3, 4, 6, 8, 12\}$*. Suppose that* $N_G(\mathbf{M}) = N_G(M)$*. Then* M *satisfies the maximal extendibility condition, i.e. every irreducible character of* M *extends to its inertia subgroup in* $N_G(M)$*.*

Proof. If $\mathbf{M}$ is a maximal torus, the result follows from [74, Theorem 1.1]. If the semisimple rank of $\mathbf{M}$ equals 3, then $N_G(\mathbf{M})/M$ is cyclic, and thus the claim also holds. The remaining cases are exactly those treated in Propositions 4.17, 4.18 and 4.19, as every parabolic subsystem of Σ of rank 2 is conjugate in W to one with base $\{\alpha_1, \alpha_2\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_1, \alpha_4\}$ or $\{\alpha_2, \alpha_3\}$. □

We end this section by clarifying the condition in Corollary 4.20.

Lemma 4.21. *Let* $\mathbf{M}$ *denote an* e*-split Levi subgroup of* $\mathbf{G}$ *for some* $e \in \{1, 2, 3, 4, 6, 8, 12\}$*. Then* $N_G(\mathbf{M}) = N_G(M)$ *unless* $(e, q) = (1, 2)$*.*

Proof. First assume that $e = 1$, and let $\mathbf{T} = \mathbf{T}_0$ denote the standard 1-F-split maximal torus of $\mathbf{M}$. Thus T is a complement to the Sylow p-subgroup U_M in a Borel subgroup B_M of M. By the Schur-Zassenhaus theorem, any two such complements are conjugate in B_M. Now M is a finite group with a split BN-pair of characteristic p; see [22, 1.18]. The Bruhat decomposition implies in particular that $N_M(U_M) = B_M$. Hence T^y and T are conjugate in M for all $y \in N_G(M)$. It follows that

$$N_G(M) = (N_G(T) \cap N_G(M))M.$$

Now let $y \in N_G(\mathbf{M})$. Then $\mathbf{T}^y$ is a 1-F-split maximal torus of $\mathbf{M}$, and thus conjugate to $\mathbf{T}$ in M. Hence $N_G(\mathbf{M}) \leq (N_G(\mathbf{T}) \cap N_G(M))M$. In fact,

$$N_G(\mathbf{M}) = (N_G(\mathbf{T}) \cap N_G(M))M.$$

(Although this is well known, we sketch a proof for the readers convenience. We may assume that $\mathbf{M} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Gamma \rangle$ for some closed subsystem $\Gamma \subseteq \Sigma$. Then $M = \langle T, \mathbf{U}_\alpha^F \mid \alpha \in \Gamma \rangle$; see [22, 1.18]. Hence an element $n \in N_G(\mathbf{T}) \cap N_G(M)$ permutes the finite root subgroups $\mathbf{U}_\alpha^F$ for $\alpha \in \Gamma$. But then n also permutes the $\mathbf{U}_\alpha$ for $\alpha \in \Sigma$ and thus $n \in N_G(\mathbf{M})$.)

It thus suffices to show that $N_G(\mathbf{T}) = N_G(T)$. Applying [22, Theorem 3.5.3(i)], we find that $C_{\mathbf{G}}(T) = \mathbf{T}$ unless $q = 2$. Thus, apart from $q = 2$, we have $C_G(T) = T$ which implies that $N_G(T) = T.W = N_G(\mathbf{T})$ and hence our claim.

Now assume that $e \geq 2$. If $\mathbf{M} = C_{\mathbf{G}}(Z(M))$, then $N_G(\mathbf{M}) \leq N_G(M) \leq N_G(Z(M)) \leq N_G(C_{\mathbf{G}}(Z(M))) = N_G(\mathbf{M})$, and hence the claim holds. If there is a semisimple element $s \in G$ with $\mathbf{M} = C_{\mathbf{G}}(s)$, then $\mathbf{M} \leq C_{\mathbf{G}}(Z(M)) \leq C_{\mathbf{G}}(s) = \mathbf{M}$, since $Z(M) = Z(\mathbf{M})^F \leq Z(\mathbf{M})$.

Going through the tables in [58], we find that such an element exists unless $e = 2$ and $q \leq 3$ or $(e, q) \in \{(3, 2), (4, 2), (6, 3)\}$. In the former two cases, we find $C_{\mathbf{G}}(Z(M)) = \mathbf{M}$ by applying [22, Theorem 3.5.3(i)] to the elements of order 3, respectively 4 of $Z(M)$. In the latter three cases, $|Z(M)|$ is divisible by a prime larger than 3, and the claim follows from [62, Proposition 2.3]. □

5. The blocks of $F_4(q)$

We keep the notation introduced in Subsection 4.1. Our aim in this section is to describe the ℓ-blocks of $G = F_4(q)$ for primes $\ell \nmid q$. If b is such a block, we write $d(b)$ for the defect of b and $l(b)$ for the number of its irreducible Brauer characters.

5.1. **The ℓ-blocks for good primes.** Let $\ell > 3$ be a prime dividing $|G|$. Then there is a unique $e \in \{1, 2, 3, 4, 6, 8, 12\}$ such that $\ell \mid \Phi_e(q)$, i.e. Condition (*) of [62] is satisfied. Notice that $e = e_\ell(q)$ is the order of q in the multiplicative group of the field $\mathbb{F}_\ell$.

We recall the description of the ℓ-blocks and their defect groups as summarized in [62, Theorem 3.6]. Let b be an ℓ-block with defect group D. Suppose that $b \subseteq \mathcal{E}_\ell(G, s)$ for some semisimple ℓ'-element $s \in G$. Then there is an e-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ such that $s \in L^\dagger$ (recall the notion $\mathbf{L}^\dagger$ introduced in the last paragraph of Subsection 4.12), and D is the Sylow ℓ-subgroup of $Z(L)$. Moreover, there is $\vartheta \in \mathcal{E}(L, s)$ which is e-cuspidal, has D in its kernel, is of defect 0 when viewed as a character of L/D, and $\mathrm{Irr}(b) \cap \mathcal{E}(G, s)$ is the set of constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\vartheta)$. This *e-cuspidal pair* $(\mathbf{L}, \vartheta)$ is determined by b up to G-conjugacy. If D is non-cyclic, then $e \neq 8, 12$, and $\mathbf{L}$ is a maximal torus of $\mathbf{G}$, unless $e = 1, 2$. If s is not quasi-isolated, then $d(b)$ and $l(b)$ can be determined from the corresponding numbers in $\mathcal{E}_\ell(M, s)$, where $\mathbf{M}^\dagger$ is a regular subgroup of $\mathbf{G}$ minimal with the property that it contains $C_{\mathbf{G}}(s)$; see Theorem 3.9. If $s \neq 1$ is quasi-isolated, the corresponding blocks and their invariants are determined in [51, 47], and for $s = 1$ in [13]. Conversely, every e-cuspidal pair $(\mathbf{L}, \vartheta)$ as above gives rise to a corresponding ℓ-block of G.

The ℓ-blocks and their invariants in those cases, where $C_G(s)$ does not have a cyclic Sylow ℓ-subgroup, are described in Tables 1–19. Moreover, Table 21 contains this information for $\ell \in \{5, 7\}$ for the blocks of the exceptional double cover of $F_4(2)$. The invariants $l(b)$ in these cases can be found in [45].

5.2. **The 2-blocks.** Let us assume that q is odd in this subsection. Let s be a semisimple $2'$-element. Then the $\mathbf{G}$-class type of s is one of

$\{1, 4, 6, 7, 9, 10, 13–15, 17–20\}$. In particular, s is quasi-isolated if and only if $s = 1$ or of $\mathbf{G}$-class type 4 (in which case $3 \nmid q$).

Suppose that s is not quasi-isolated. Then $C_{\mathbf{G}}(s)$ is a proper regular subgroup of $\mathbf{G}$, which is a classical group and thus has a unique unipotent 2-block. It follows from Theorem 3.9(a), that $\mathcal{E}_2(G, s)$ is a single block, whose defect groups are isomorphic to, but in general not conjugate to a Sylow 2-subgroup of $C_G(s)$.

The case $s = 1$ corresponds to the unipotent blocks of G. By [27, Théorème A and Table on p. 349], there are three unipotent 2-blocks of G: the principal block and two blocks of defect 0, containing the characters $F_4[\theta]$ and $F_4[\theta^2]$, respectively. By [37, Table 1], we have $l(b) = 28$ for the principal 2-block b of G.

Now assume that $3 \nmid q$ and let s be of class type $(4, 1)$, respectively $(4, 2)$. The former case occurs if $3 \mid q - 1$, the latter if $3 \mid q + 1$.

Proposition 5.3. *Let s be a quasi-isolated element of order 3 of $\mathbf{G}$-class type 4. Then $\mathcal{E}(G, s)$ is a basic set for $\mathcal{E}_2(G, s)$. In particular, $\mathcal{E}_2(G, s)$ has exactly 9 irreducible Brauer characters. Moreover, if $3 \mid q - 1$, the decomposition matrix of $\mathcal{E}_2(G, s)$ is unitriangular.*

Proof. In this proof, by the *unipotent decomposition matrix* of H, where H is a finite reductive group, we understand the matrix of scalar products of the unipotent characters of H with the projective indecomposable characters in $\mathcal{E}_2(H, 1)$. This is a square matrix whenever $\mathcal{E}(H, 1)$ is a basic set for $\mathcal{E}_2(H, 1)$.

We may assume that $C_{\mathbf{G}}(s) = \mathbf{L}$, the group introduced in Proposition 4.14, so that $L = C_G(s) = (L^1 \circ_3 L^2).3$ with $L^i \cong \mathrm{SL}_3^{\varepsilon}(q)$ for $i = 1, 2$. By [17, Theorem 12], the decomposition matrix of $\mathcal{E}_2(L^i, 1)$ and $\mathcal{E}_2(L, 1)$, $i = 1, 2$, is the same as the decomposition matrix of $\mathcal{E}_2(\mathrm{GL}_3^{\varepsilon}(q), 1)$, respectively $\mathcal{E}_2(\mathrm{GL}_3^{\varepsilon}(q) \times \mathrm{GL}_3^{\varepsilon}(q), 1)$. These facts will be assumed tacitly in the following.

We first prove the assertions if $3 \mid q - 1$. Consider the split Levi subgroup $H := \mathrm{GL}_2(q) \times [q-1]$ of $\mathrm{GL}_3(q)$. The unipotent decomposition matrices of $\mathrm{GL}_3(q)$ and H are given as follows, where we omit entries equal to 0; see [49, Appendix 1]:

(3)	$(2,1)$	(1^3)
1		
	1	
1		1

(2)	(1^2)
1	
1	1

Following [49], we label the columns of these decomposition matrices by partitions of 3, respectively 2. The unipotent part of the projective indecomposable character corresponding to a column labeled by λ is denoted by ρ_λ. Put $\kappa_{(3)} := \rho_{(3)} + 2\rho_{(2,1)}$ and $\kappa_{(2,1)} := \rho_{(2,1)} + \rho_{(1^3)}$. Observe that $\kappa_{(3)}$ and $\kappa_{(2,1)}$ are obtained by Harish-Chandra induction of $\rho_{(2)}$, respectively $\rho_{(1^2)}$ from H to $\mathrm{GL}_3(q)$.

Harish-Chandra inducing the 12 projective indecomposable characters of $\mathcal{E}_2(\mathrm{GL}_3(q) \times H, 1)$ and $\mathcal{E}_2(H \times \mathrm{GL}_3(q), 1)$ to $\mathrm{GL}_3(q) \times \mathrm{GL}_3(q)$, we obtain 12 projective characters in $\mathcal{E}_2(\mathrm{GL}_3(q) \times \mathrm{GL}_3(q), 1)$, whose unipotent parts are $\rho_\lambda \boxtimes \kappa_\mu$ and $\kappa_\mu \boxtimes \rho_\lambda$ with $\lambda \in \{(3), (2,1), (1^3)\}$ and $\mu \in \{(3), (2,1)\}$. We can select 8 of these projective characters of $\mathrm{GL}_3(q) \times \mathrm{GL}_3(q)$ such that the matrix of scalar products of the selected characters with the unipotent characters of $\mathrm{GL}_3(q) \times \mathrm{GL}_3(q)$ is unitriangular. Moreover, if we use this triangular shape to associate a pair of partitions to each of the 8 projective characters thus constructed, then only $((1^3), (1^3))$ is not associated to any of these.

Now $\mathbf{L}$ contains the two split Levi subgroups $\mathbf{M}_1 := \mathbf{L}_{\{1,23,4\}}$ and $\mathbf{M}_2 := \mathbf{L}_{\{1,3,4\}}$ of $\mathbf{G}$, and s is a central element in each of M_1 and M_2. It follows from [25, Proposition 11.4.8(ii)] that $\mathcal{E}_2(M_i^*, s)$ is Morita equivalent to $\mathcal{E}_2(M_i, 1)$, $i = 1, 2$. Moreover, Harish-Chandra induction from $\mathcal{E}_2(M_1, 1)$ to $\mathcal{E}_2(L, 1)$ corresponds to Harish-Chandra induction from $\mathcal{E}_2(\mathrm{GL}_3(q) \times H, 1)$ to $\mathcal{E}_2(\mathrm{GL}_3(q) \times \mathrm{GL}_3(q), 1)$, and likewise for $\mathcal{E}_2(M_2, 1)$.

Every irreducible character of M_i, $i = 1, 2$ is uniform, as $Z(\mathbf{M}_i)$ is connected, each component of $[\mathbf{M}_i, \mathbf{M}_i]$ is of type A, and $\mathbf{M}_1^* \cong \mathbf{M}_2$ by the remarks in Subsection 4.12. Hence Harish-Chandra induction $\mathcal{E}_2(M_i^*, s) \to \mathcal{E}_2(G, s)$ is equal to the composition of Harish-Chandra induction $\mathcal{E}_2(M_i, 1) \to \mathcal{E}_2(L, 1)$ and Lusztig's Jordan decompositions $\mathcal{E}_2(M_i^*, s) \to \mathcal{E}_2(M_i, 1)$ and $\mathcal{E}_2(L, 1) \to \mathcal{E}_2(G, s)$; see [19, Theorem 15.8]. As Harish-Chandra induction preserves projective characters, we obtain 8 projective characters of $\mathcal{E}_2(G, s)$, whose matrix of scalar products with the elements of $\mathcal{E}(G, s)$ is lower unitriangular. If we also consider the restriction of the Gelfand-Graev character of G to $\mathcal{E}_2(G, s)$, we obtain 9 linearly independent projective characters of $\mathcal{E}_2(G, s)$. By Lemma 3.7, there are at most 9 irreducible Brauer characters in $\mathcal{E}_2(G, s)$. The unitriangularity of the projective characters thus constructed implies the unitriangularity of the decomposition matrix. This concludes our proof in case $3 \mid q - 1$.

The proof in case $3 \mid q + 1$ uses an Ennola duality argument. We replace $\mathbf{L}$ and $\mathbf{M}_i$, $i = 1, 2$, by their conjugates with g, where g is as in Proposition 4.14. Then $L = (\mathrm{SU}_3(q) \circ_3 \mathrm{SU}_3(q)).3$, and $\mathbf{M}_i$, $i = 1, 2$, is a 2-split Levi subgroup of $\mathbf{L}$ and of $\mathbf{G}$. The 2-decomposition matrix of the unipotent characters of $\mathrm{SU}_3(q)$ is a lower unitriangular matrix, which has been computed by Erdmann in [29] for the case $4 \mid q - 1$, and by the second author in [46, Appendix] for the case $4 \mid q + 1$. The proof now proceeds as above, except that Harish-Chandra induction $\mathcal{E}(M_i^*, s) \to \mathcal{E}(G, s)$ is replaced by Lusztig induction. As the latter preserves generalized projective characters, we still obtain a set of 9 linearly independent generalized projective characters in $\mathcal{E}_2(G, s)$. □

5.4. **The 3-blocks.** Let us assume that $3 \nmid q$ in this subsection. The 3-blocks of G and their relevant invariants are described in Tables 1–19 and 21 of the appendix, where the latter table contains the information for the exceptional double cover of $F_4(2)$. The derivation of these results is by far the most laborious part of this work. We are now going to prove, in a series of lemmas, that the entries in Columns 5–7 of Tables 1–19 are correct. That is, for a semisimple $3'$-element $s \in G$ and a block $b \subseteq \mathcal{E}_3(G, s)$ of positive defect, we determine a label for b, as well as $d(b)$ and $l(b)$. We define $\varepsilon \in \{1, -1\}$ by the condition that $3 \mid q - \varepsilon$, and write $e := e_3(q)$ for the order of q in $\mathbb{F}_3$. Thus $e = 1$ if $\varepsilon = 1$, and $e = 2$, otherwise.

Lemma 5.5. *Let $s \in G$ be a semisimple $3'$-element. Then every irreducible 3-modular character φ of $\mathcal{E}_3(G, s)$ is a $\mathbb{Z}$-linear combination of elements of $\{\check{\chi} \mid \chi \in \mathcal{E}(G, s)\}$, unless $s = 1$, in which case φ is still a $\mathbb{Q}$-linear combination of this set.*

In particular, the number of irreducible 3-modular characters contained in $\mathcal{E}_3(G, s)$ is equal to the rank of the $\mathbb{Z}$-span of $\{\check{\chi} \mid \chi \in \mathcal{E}(G, s)\}$.

Proof. Let $t \in C_G(s)$ be a non-trivial 3-element. By the list of semisimple class types of G in [58], we find that $C_G(st)$ satisfies one of the hypotheses (a) or (b) of Lemma 3.7, unless $s = 1$ and t is a quasi-isolated element of order 3 and class type 4. In the latter case, $C_G(st) = C_G(t)$ satisfies hypothesis (c) of the lemma, as all components of $[C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)]^F$ are of type A and thus all unipotent characters of $C_G(t)$ are uniform. The result follows. For $s = 1$ our assertion follows from [37, Proposition 7.14] and the fact that decomposition numbers are integers.

The final statement is a consequence of the fact that every $\check{\chi}$ for $\chi \in \mathcal{E}(G, s)$ is a $\mathbb{Z}$-linear combination of irreducible Brauer characters contained in $\mathcal{E}_3(G, s)$. □

We now prove our claims for the unipotent blocks. The unipotent 3-blocks of G have been computed by Enguehard in [27, p. 349–351]. We follow Carter's book [22, p. 478f] for the notation of the unipotent characters of G.

Lemma 5.6. *The unipotent characters of G of 3-defect 0 are $F_4[i]$, $F_4[-i]$, $F_4^I[1]$, $F_4[-1]$ if $\varepsilon = 1$, and $F_4[i]$, $F_4[-i]$, $\phi_{4,8}$, $\phi_{16,5}$, if $\varepsilon = -1$.*

PROOF. See [27, Table on Page 349]. □

Lemma 5.7. *The invariants of the unipotent* 3*-blocks of* G *of positive defect are as given in* Columns 5–7 *of* Table 1*.*

PROOF. It is known that G has exactly 35 unipotent 3-modular characters; see [40, Table in 6.6]. We sketch an alternative proof for this fact. By a result of Shoji [72, 6.2.4(c) and Proposition 6.3] (see also the remarks on [38, p. 42] for the fact that Shoji's results are valid without restrictions on p), the two almost characters corresponding to the unipotent characters $F_4[\theta]$ and $F_4[\theta^2]$ (in the sense of [59, (4.24.1)]) have values 0 except on 3-singular classes. As G has exactly 37 ordinary unipotent characters, and as the almost characters span the same space as the unipotent characters; see [59, Corollary 4.25], it follows that $\{\check{\chi} \mid \chi \in \mathcal{E}(G,1)\}$ spans a space of dimension at most 35. On the other hand, the matrix of values of this set has rank at least 35, as can be checked with the explicit unipotent character table of G computed by Köhler [53], and, independently, by the third author. For this, it suffices to look at the set of unipotent classes and the mixed classes on which the other unipotent almost characters have non-zero values, as well as some classes of elements of the form su, where s is an involution with centralizer of type B_4. Lemma 5.5 now implies the result.

According to [27, THÉORÈME A and the table on Page 349], there is a unique non-principal block b of positive defect corresponding to the e-cuspidal pair $([q-\varepsilon]^2.B_2(q), \zeta_e)$, where ζ_e is the e-cuspidal unipotent character of the semisimple component $B_2(q)$ of $[q-\varepsilon]^2.B_2(q)$. Moreover, b contains exactly the unipotent characters of the e-Harish-Chandra series defined by $([q-\varepsilon]^2.B_2(q), \zeta_e)$. These are the characters $B_{2,1}$, $B_{2,\varepsilon}$, $B_{2,r}$, $B_{2,\varepsilon'}$, $B_{2,\varepsilon''}$ if $\varepsilon = 1$, and $\phi_{4,1}$, $\phi_{4,13}$, $B_{2,r}$, $\phi_{4,7''}$, $\phi_{4,7'}$, otherwise (for the elements in the 2-Harish-Chandra series see [13, Table 2, case 1]). Using the explicit values of the unipotent characters in b, we can check that their restrictions to the 3-regular elements are linearly independent. In view of Lemma 5.5, this shows that $l(b) = 5$. This, together with Lemma 5.6 implies $l(B) = 26$ for the principal 3-block B of G. □

We next show how to determine the number of irreducible 3-modular characters for the non-unipotent blocks.

Proposition 5.8. *Let* $1 \neq s \in G$ *be a semisimple* $3'$*-element. Then* $\mathcal{E}(G,s)$ *is a basic set for* $\mathcal{E}_3(G,s)$*. In particular, if* $b \subseteq \mathcal{E}_3(G,s)$ *is a* 3*-block of* G*, then* $l(b) = |\mathrm{Irr}(b) \cap \mathcal{E}(G,s)|$*.*

Proof. Clearly, the second statement follows from the first. To prove the first statement, we refine the argument of [39, Proposition 4.1]. Let $\mathcal{S}_3(G)$ denote a set of representatives for the G-conjugacy classes of 3-elements of G. For each $t \in \mathcal{S}_3(G)$, let $\mathcal{C}_{3'}(C_G(t))$ be a set of representatives for the $C_G(t)$-conjugacy classes of $3'$-elements of $C_G(t)$, and denote by $\mathcal{S}_{3'}(C_G(t))$ the subset of $\mathcal{C}_{3'}(C_G(t))$ consisting of semisimple elements. Then $\{ts \mid t \in \mathcal{S}_3(G), s \in \mathcal{S}_{3'}(C_G(t))\}$ is a set of representatives for the semisimple conjugacy classes of G.

By t_0 we denote the element of $\mathcal{S}_3(G)$ of class $(4,1)$ if $3 \mid q-1$, respectively $(4,2)$ if $3 \mid q+1$. First, consider $t \in \mathcal{S}_3(G) \setminus \{1, t_0\}$. We claim that in this case $\mathbf{M} := C_{\mathbf{G}}(t)$ is a regular subgroup of $\mathbf{G}$. If not, $|(Z(\mathbf{M})/Z(\mathbf{M})^\circ)^F|$ is a 2-group; see the tables in [58]. As t is a 3-element, Lemma 3.3 implies that $\mathbf{M} \lneq C_{\mathbf{G}}(t)$, a contradiction. Thus $\mathbf{M}$ is regular and there is a regular subgroup $\mathbf{M}^\dagger$ of $\mathbf{G}$ in duality with $\mathbf{M}$. In particular, $Z(\mathbf{M}^\dagger)$ is connected and 3 is a good prime for $\mathbf{M}^\dagger$, so that $\mathbf{M}^\dagger$ satisfies the assumptions of [39, Theorem 5.1]. Putting $M^\dagger := (\mathbf{M}^\dagger)^F$ and identifying $\mathbf{M}$ with $(\mathbf{M}^\dagger)^*$, we find that $|\mathcal{E}(M^\dagger, s)|$ equals the number of irreducible 3-modular characters contained in $\mathcal{E}_3(M^\dagger, s)$ for every $s \in \mathcal{S}_{3'}(M)$, and hence

$$\sum_{s \in \mathcal{S}_{3'}(M)} |\mathcal{E}(M^\dagger, s)| = |\mathcal{C}_{3'}(M^\dagger)|. \tag{11}$$

Now consider the case $t = t_0$ and put $\mathbf{L} := C_{\mathbf{G}}(t_0)$. If $1 \neq s \in \mathcal{S}_{3'}(L)$, then $\mathbf{M} := C_{\mathbf{L}}(s) = C_{\mathbf{G}}(t_0 s)$ is a regular subgroup of $\mathbf{G}$ and hence of $\mathbf{L}$. It follows from the Bonnafé-Rouquier Morita equivalence theorem [11, Théorème 11.8] (see also Theorem 3.9(c)), that $\mathcal{E}_3(L^*, s)$ and $\mathcal{E}_3(M, 1)$ have the same number of irreducible 3-modular characters. As $\mathbf{M}$ has connected center and 3 is a good prime for $\mathbf{M}$, the latter number equals $|\mathcal{E}(M, 1)|$ by [39, Theorem 5.1]. By the Jordan decomposition of characters, we have $|\mathcal{E}(M, 1)| = |\mathcal{E}(L^*, s)|$. Thus the number of irreducible 3-modular characters in $\mathcal{E}_3(L^*, s)$ equals $|\mathcal{E}(L^*, s)|$. Let us now compute the number of irreducible 3-modular characters in $\mathcal{E}_3(L^*, 1)$. We have $L = (\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)).3$, where the outer automorphism of order 3 acts as a simultaneous diagonal automorphism on each of the factors $\mathrm{SL}_3^\varepsilon(q)$ of L; see Subsection 6.1 below and Remark 4.15. Now $\mathrm{SL}_3^\varepsilon(q)$ has exactly 5 unipotent 3-modular characters, three of which lie in an orbit under the outer automorphism of order 3. (A reference for the latter two statements in case of $\mathrm{SU}_3(q)$ is [35, Theorem 4.5]; the corresponding results for $\mathrm{SL}_3(q)$ are proved in the same way.) Thus $\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)$ has 25 unipotent 3-modular characters, four of which are fixed under the outer automorphism of order 3, and the other 21 lie in 7 orbits of length 3. It follows that L has exactly 11 unipotent

3-modular characters. On the other hand, L has exactly 9 ordinary unipotent characters. We conclude that

$$\sum_{s\in\mathcal{S}_{3'}(L)} |\mathcal{E}(L^*,s)| = -2 + |\mathcal{C}_{3'}(L^*)|. \tag{12}$$

Let $c \in \mathbb{Z}$ be such that

$$\sum_{s\in\mathcal{S}_{3'}(G)} |\mathcal{E}(G,s)| = c + |\mathcal{C}_{3'}(G)|. \tag{13}$$

Now we use the fact that $|\mathcal{E}(G,ts)| = |\mathcal{E}(C_G(t)^*,s)|$ for all $t \in \mathcal{S}_3(G)$ and all $s \in \mathcal{S}_{3'}(C_G(t))$. This follows from the identification $G = G^*$ and the Jordan decomposition of characters:

$$\begin{aligned} |\mathcal{E}(C_G(t)^*,s)| &= |\mathcal{E}(C_{C_G(t)}(s),1)| \\ &= |\mathcal{E}(C_G(ts),1)| \\ &= |\mathcal{E}(C_{G^*}(ts),1)| \\ &= |\mathcal{E}(G,ts)|. \end{aligned}$$

We find, using (13), (12) and (11), that

$$\begin{aligned} \text{number of conjugacy classes of } G &= \sum_{t\in\mathcal{S}_3(G)} \sum_{s\in\mathcal{S}_{3'}(C_G(t))} |\mathcal{E}(G,ts)| \\ &= \sum_{t\in\mathcal{S}_3(G)} \sum_{s\in\mathcal{S}_{3'}(C_G(t))} |\mathcal{E}(C_G(t)^*,s)| \\ &= c - 2 + \sum_{t\in\mathcal{S}_3(G)} |\mathcal{C}_{3'}(C_G(t)^*)|. \end{aligned}$$

We claim that $\sum_{t\in\mathcal{S}_3(G)} |\mathcal{C}_{3'}(C_G(t)^*)|$ equals the number of conjugacy classes of G. First, if $t = 1$ or $t = t_0$, then $C_{\mathbf{G}}(t)^* \cong C_{\mathbf{G}}(t)$ and $C_G(t)^* \cong C_G(t)$ by Remark 4.16. It follows from Proposition 3.4 that there is a permutation $t \mapsto t'$ of $\mathcal{S}_3(G) \setminus \{1, t_0\}$ such that $C_G(t) \cong C_G(t')^*$. Hence $\sum_{t\in\mathcal{S}_3(G)} |\mathcal{C}_{3'}(C_G(t)^*)| = \sum_{t\in\mathcal{S}_3(G)} |\mathcal{C}_{3'}(C_G(t))|$, giving our claim.

We conclude that $c = 2$. By Table 1, the number of unipotent 3-modular characters is two less than $|\mathcal{E}(G,1)|$. By Lemma 5.5, for all $s \in \mathcal{S}_{3'}(G)$, the number of irreducible 3-modular characters in $\mathcal{E}_3(G,s)$ is at most equal to $|\mathcal{E}(G,s)|$. We conclude from (13) that equality holds for all non-trivial such s. This implies our claim. □

Lemma 5.9. *The invariants contained in* Columns 5–7 *of* Tables 2, 3 *and* 5 *are correct.*

Proof. The blocks and their labels are described in [51, Table 2]. The defects of these blocks can be derived from [51, Proposition 3.2].

The numbers of the irreducible 3-modular characters in these blocks can be determined with Proposition 5.8. □

Lemma 5.10. *The invariants contained in* Columns 5–7 *of* Tables 6–19 *and* 21 *are correct.*

Proof. Let s be a semisimple $3'$-element of G such that $C_{\mathbf{G}}(s) = C_{\mathbf{G}^*}(s)$ is contained in a proper regular subgroup $\mathbf{M}^*$ of $\mathbf{G}^* = \mathbf{G}$. We choose $\mathbf{M}^*$ as in Theorem 3.9. As every component of such an $\mathbf{M}^*$ is of classical type, we may apply Theorem 3.9. By appealing to [31] and [32], we easily obtain all the entries of Tables 6–19.

For the invariants in Table 21 see [45]. □

5.11. **The action of outer automorphisms.** Recall that p is a prime and that $q = p^f$ for some positive integer f. Recall also the definition of σ_1 and $\mathrm{Aut}_1(\mathbf{G}) = \mathbf{G} \rtimes \langle \sigma_1 \rangle$ from Subsection 4.5. Let us put $f' := f$ if p is odd, and $f' := 2f$ if $p = 2$. Then $F = \sigma_1^{f'}$. In particular, σ_1 commutes with F and thus G is σ_1-invariant. We tacitly use the symbol σ_1 to also denote the restriction of σ_1 to G. With this notation, $\mathrm{Aut}(G) = G \rtimes \langle \sigma_1 \rangle$ and $\mathrm{Out}(G) = \langle G\sigma_1 \rangle$ is cyclic of order f'; see [43, Theorem 2.5.12(a),(e)]. In particular, every subgroup of $\langle \sigma_1 \rangle \leq \mathrm{Aut}(G)$ is of the form $\langle \sigma_1^{m'} \rangle$ for some integer m' with $m' \mid f'$. Notice also that every automorphism of G extends to an element of $\mathrm{Aut}_1(\mathbf{G})$ which commutes with F.

Let $\sigma := \sigma_1^{m'}$ for some positive integer m' with $m' \mid f'$. The restriction of σ to G, also denoted by σ, has order f'/m'. We define m by $m := m'/2$ if m' and p are even and by $m := m'$, otherwise. Then $m \mid f$ and $F = (F_1^m)^{f/m}$ in all cases; moreover $\sigma = F_1^m$ if either m' and p are even or if p is odd, and $\sigma^2 = F_1^m$, otherwise. The field automorphism of G of order f/m' is induced by σ if p is odd, and by σ^2 if $p = 2$.

As always, we identify $\mathbf{G}$ with its dual $\mathbf{G}^*$ and G with G^*. We let ℓ be an odd prime different from p (we do not assume $\ell = 3$ here) and write $e = e_\ell(q)$ for the order of q in the multiplicative group of the field $\mathbb{F}_\ell$. We will need the following consequence of the Lang-Steinberg theorem.

Lemma 5.12. *Let $s \in G$ be semisimple such that $\sigma(s)$ is G-conjugate to s. Then some G-conjugate t of s satisfies $\sigma(t) = t$.*

Proof. Let C denote the $\mathbf{G}$-conjugacy classes of s. Our hypothesis implies that C is σ-stable. An application of the Lang-Steinberg theorem shows that $C \cap \mathbf{G}^\sigma$ is non-empty and that $C \cap G$ is a conjugacy class of G; see, e.g. [42, Example 1.4.10]. If $t \in C \cap \mathbf{G}^\sigma$, then $t, s \in C \cap G$ yielding our claim. □

We can now prove the main result of this subsection.

Proposition 5.13. *Let $s \in G$ be a semisimple ℓ'-element. Then*

(14) $\sigma(\mathcal{E}(G,s)) = \mathcal{E}(G,\sigma^{-1}(s))$ and $\sigma(\mathcal{E}_\ell(G,s)) = \mathcal{E}_\ell(G,\sigma^{-1}(s))$.

Suppose that $\sigma(s)$ is conjugate to s in G. Then σ stabilizes $\mathcal{E}(G,s)$ and $\mathcal{E}_\ell(G,s)$.

(a) *Suppose that $p = 2$ and that m' is odd. Then either $C_{\mathbf{G}}(s)$ is a maximal torus, or the G-class type of s is one of $(1,1)$, $(4,1)$, $(4,2)$, $(14,1)$, $(14,4)$, $(15,1)$, $(15,3)$ or $(15,5)$. In these cases, the non-trivial σ-orbits on $\mathcal{E}(G,s)$ have length 2, and the number of such orbits is as given in the last column of the following table.*

(15)

i	k	$C_G(s)$	no.
1	1	G	8
4	1, 2	$(\mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)).3$	3
14	1, 4	$[q-\varepsilon]^2 \times \mathrm{SL}_2(q)^2$	1
15	1, 3	$[q-\varepsilon]^2 \times \mathrm{Sp}_4(q)$	1
15	5	$[q^2+1] \times \mathrm{Sp}_4(q)$	1

Here, $\varepsilon \in \{-1,1\}$, and the two values for k in the second column, if present, correspond to the cases $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

(b) *Suppose that p is odd or that $p = 2$ and m' is even, so that $\sigma = F_1^m$. Then either σ fixes every element of $\mathcal{E}(G,s)$, or the G-class type of s is one of $(12,1)$, $(16,1)$ or $(16,9)$. In these cases, the non-trivial σ-orbits on $\mathcal{E}(G,s)$ have length 2, and the number of such orbits is as given in the last column of the following table.*

(16)

i	k	$C_G(s)$	no.
12	1	$([q-1] \circ_2 (\mathrm{SL}_2(q)^2 \circ_2 \mathrm{SL}_2(q)).2).2$	2
16	1, 9	$([q-\varepsilon]^2 \circ_2 \mathrm{SL}_2(q)^2).2$	1

Here, the same conventions regarding ε are used as in the table displayed in (15). Moreover, the following conditions hold. Let t be a σ-stable G-conjugate of s. If the G-class type of t is $(12,1)$, then the $\mathbf{G}^\sigma$-class type of t is $(12,2)$ or $(12,4)$ and f/m is even. If the G-class type of t is $(16,1)$, the $\mathbf{G}^\sigma$-class type of t is $(16,k)$ for $k \in \{3,4,7,10\}$ and f/m is even, or the $\mathbf{G}^\sigma$-class type of t is $(16,8)$ and $4 \mid f/m$. If the G-class type of t is $(16,9)$, the $\mathbf{G}^\sigma$-class type of t is $(16,8)$ and $f/m \equiv 2 \pmod 4$.

(c) *Let $b \subseteq \mathcal{E}_\ell(G,s)$ be a σ-stable ℓ-block. Then σ stabilizes $\mathcal{E}(G,s)$. Moreover, the permutation actions of $\langle\sigma\rangle$ on $\mathrm{IBr}(b)$ and on $\mathrm{Irr}(b) \cap \mathcal{E}(G,s)$ are equivalent, unless $s = 1$ and $\ell = 3$.*

(d) *Suppose that* $s = 1$ *and* $\ell = 3$. *If* p *and* m' *are as in Case (i), then* σ *has exactly* 7 *orbits of length* 2 *on* $\mathrm{IBr}(b)$ *if* b *is the principal block, and exactly one orbit of length* 2 *on* $\mathrm{IBr}(b)$ *if* b *is the non-principal unipotent block of positive defect. The other unipotent* 3*-modular characters are fixed by* σ. *If* p *and* m' *are as in Case (ii), then* σ *fixes every element of* $\mathrm{IBr}(b)$.

Proof. Notice that σ is a Steinberg morphism of $\mathbf{G}$, and thus in particular an isogeny. The equalities in (14) follow from [24, Corollary 9.3(ii)]. Now suppose that $\sigma(s)$ is conjugate in G to s. By Lemma 5.12 we may and will assume that $\sigma(s) = s$. Then $\mathcal{E}(G, s)$ and hence also $\mathcal{E}_\ell(G, s)$ are invariant under σ by (14).

(a) Suppose that p is even and that m' is odd. Since $\sigma(s) = s$, we have $C_{\mathbf{G}}(s) = \sigma(C_{\mathbf{G}}(s))$. We can also assume that $C_{\mathbf{G}}(s)$ is not a torus. As σ_1 interchanges the long root subgroups of $\mathbf{G}$ with its short root subgroups, the tables in [58] only leave the possibilities $(1, 1)$, $(4, k)$, $(14, k)$ or $(15, k)$ for the G-class types of s. As σ interchanges the maximal tori of type $(20, 12)$ and $(20, 17)$, it also swaps the classes of type $(14, 2)$ and $(14, 3)$ and the classes of type $(15, 2)$ and $(15, 4)$. Let us now prove the remaining statements.

If s is of class type $(4, k)$, $k = 1, 2$, then $\ell > 3$, as s is an element of order 3. Moreover, σ interchanges the two components of $C_G(s)$ of type A_2, as one is a long root subgroup, and the other one a short root subgroup. Thus σ has three orbits of length 2 on the set of unipotent characters of $C_G(s)$ and fixes its other unipotent characters. The unipotent characters of $C_G(s)$ are uniform functions, hence there is a bijection between $\mathcal{E}(C_G(s), 1)$ and $\mathcal{E}(G, s)$ which commutes with σ by [24, Corollary 9.2]. It follows that the number of orbits of σ of length two on $\mathcal{E}(G, s)$ equals 3 and that σ fixes the other elements of $\mathcal{E}(G, s)$. If s is of class type $(14, k)$, $k = 1, 4$, then σ swaps the two components of $C_G(s)$ of type A_1, as one is a short root subgroup and the other one a long root subgroup. Thus σ interchanges the two unipotent characters of $C_G(s)$ of degree q and fixes the other unipotent characters. If s is of class type $(15, k)$, $k = 1, 3, 5$, the semisimple component of $C_G(s)$ is of type C_2, and σ induces the exceptional graph automorphism of $\mathrm{Sp}_4(q)$ on $C_G(s)$. It follows from [61, Theorem 2.5(c)], that σ interchanges the two unipotent principal series characters of $C_G(s)$ of the same degree and fixes the other unipotent characters. In theses cases, $C_{\mathbf{G}}(s)$ is a regular subgroup of $\mathbf{G}$, and the Lusztig induction map corresponding to $C_{\mathbf{G}}(s)$ yields a bijection between $\mathcal{E}(C_G(s), 1)$ and $\mathcal{E}(G, s)$. As this map commutes with the action of σ (see [24, Corollary 9.2]), this gives the entries of the table displayed in (15) in case $s \neq 1$. In case $s = 1$, the entry is determined in [61, Theorem 2.5(e)].

(b) Suppose that p is odd or that $p = 2$ and m' is even. Then $\sigma = F_1^m$. Consider the statement:

$$\sigma \text{ fixes every element of } \mathcal{E}(G, s). \tag{17}$$

We are going to determine the cases for which (17) holds. First of all, (17) is true if $s = 1$ by [24, Proposition 6.6]. In the following, we will need to compute the degrees of the unipotent characters of the groups $C_G(s)$. These are easily determined using Jean Michel's extension of CHEVIE; see [64]. Suppose the unipotent characters of $C_G(s)$ have pairwise distinct degrees. Then the same is true for the elements of $\mathcal{E}(G, s)$ by the Jordan decomposition of characters. In this case (17) trivially holds. Considering the tables in [58], it remains to investigate the cases where the G-class type of s is one of $(2, 1)$, $(11, 1)$, $(11, 2)$, $(12, 1)$, $(12, 3)$, $(14, k)$, $1 \leq k \leq 4$, $(15, k)$, $1 \leq k \leq 5$ or $(16, k)$ with $k \in \{1, 2, 5, 6, 9\}$. If s is of G-class type $(2, 1)$, the elements of $\mathcal{E}(G, s)$ are distinguished by their degrees, except for two characters $\chi, \chi' \in \mathcal{E}(G, s)$ of equal degree, which correspond, via the Jordan decomposition, to the unipotent characters of $C_G(s)$ labeled by the bipartitions $(21^2, -)$ and $(-, 31)$, respectively. There is a σ-stable split Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ such that s is contained in a σ-stable dual $\mathbf{M}^* \leq \mathbf{G}$ (in fact we may take $\mathbf{M} =_{\mathbf{G}^\sigma} \mathbf{L}_{\{2,3,4\}}$ and $\mathbf{M}^* =_{\mathbf{G}^\sigma} \mathbf{L}_{\{1,2,3\}}$ in the notation of Subsection 4.1), and a σ-stable element $\psi \in \mathcal{E}(M, s)$ such that the Harish-Chandra induced character $R_{\mathbf{M}}^{\mathbf{G}}(\psi)$ contains χ with multiplicity 1 and χ' with multiplicity 0. This follows from the compatibility of Harish-Chandra induction and the Jordan decomposition (see [24, Corollary 9.2]) with a computation of unipotent characters in $C_G(s) = \mathrm{Spin}_9(q)$. Hence σ fixes χ and so also χ'.

In the remaining cases, put $\mathbf{M}^* := C_{\mathbf{G}}(Z(C_{\mathbf{G}}(s))^\circ)$. Then $\mathbf{M}^*$ is a σ-stable regular subgroup of $\mathbf{G}$ and we choose a σ-stable regular subgroup $\mathbf{M} \leq \mathbf{G}$ dual to $\mathbf{M}^*$. In these cases, $\mathbf{M}$ and $C_{\mathbf{G}}(s)$ are of classical type. Hence the elements of $\mathcal{E}(C_G(s), 1)$ and of $\mathcal{E}(M, s)$ are uniquely determined by their multiplicities in the Deligne–Lusztig characters; see, e.g. [19, Theorem 15.8]. The latter theorem, together with [24, Corollary 9.2] then implies that there is a σ-equivariant bijection $\mathcal{E}(C_G(s), 1) \to \mathcal{E}(M, s)$. Also, Lusztig induction with respect to $\mathbf{M}$ induces a bijection $\mathcal{E}(M, s) \to \mathcal{E}(G, s)$, which is σ-equivariant by [24, Corollary 9.2]. To show that (17) is satisfied, it suffices therefore to show that σ fixes every element of $\mathcal{E}(C_G(s), 1)$. If s is of G-class type $(11, 1)$, $(11, 2)$, or $(15, k)$, $1 \leq k \leq 5$, then σ fixes the unipotent characters of $C_G(s)$ by [61, Theorem 2.5(c)]. If s is of G-class type

$(14, k)$, $1 \leq k \leq 4$, then σ fixes every unipotent character of $C_G(s)$, as σ stabilizes the two semisimple factors of type A_1 of $C_G(s)$, one being a short root subgroup, the other one a long root subgroup.

Suppose that the G-class type of s is $(12, 1)$ or $(12, 3)$ and that σ does not fix $\mathcal{E}(G, s)$ element-wise. Then σ does not fix every element of $C_G(s)$, again by [24, Corollary 9.2]. This implies that σ permutes the two long root components of $C_G(s)$ of type A_1, which happens if the $\mathbf{G}^\sigma$-class type of s is one of $(12, 2)$ or $(12, 4)$. But then the G-class type of s equals $(12, 1)$ by Lemma 4.8. We obtain two σ-orbits of length 2 on $\mathcal{E}(G, s)$ in this case. The closed subsystem Γ of Σ which gives rise to the centralizer of an element of $\mathbf{G}$-class type 16 has stabilizer in W isomorphic to $2 \times D_8$, as is easily computed with CHEVIE. This group has ten conjugacy classes giving rise to the ten G-class types $(16, k)$, $1 \leq k \leq 10$. Using Lemma 4.8 and Table 24, one finds that the class types $(16, 5)$ and $(16, 8)$ correspond to the conjugacy classes of elements of order 4, whereas the class type $(16, 9)$ corresponds to the squares of these elements. This information is enough to prove the statements for s of G-class type $(16, k)$.

(c) Suppose that $\ell \neq 3$ or that $s \neq 1$. Then Proposition 5.8 and [39, Theorem 5.1] imply that $\mathrm{Irr}(b) \cap \mathcal{E}(G, s)$ is a basic set for $\mathrm{IBr}(b)$. By Lemma 2.8, the numbers of fixed points of σ on $\mathrm{IBr}(b)$ and on $\mathrm{Irr}(b) \cap \mathcal{E}(G, s)$ are the same. The analogous statement holds for every power of σ, which gives our claim.

(d) Suppose that $\ell = 3$ and $s = 1$. If we are in case (ii), σ satisfies (17) by (a), and Lemma 5.5 implies our claim. Hence assume that we are in case (i). Let M denote the (37×37)-block diagonal Fourier transform matrix for the unipotent characters of G; see [22, Section 13.6]. Furthermore, let P denote the permutation matrix arising from the permutation of σ on the set of unipotent characters of G as given by [61, Theorem 2.5(e)]. One then checks that $MP = PM$, i.e. σ permutes the set of almost characters of G in the same way as it permutes the unipotent characters. In particular, the number of fixed points of σ on the set of unipotent characters is the same as on the set of almost characters. Consequently, σ has exactly 8 orbits of length 2 on the set of almost characters. The two almost characters corresponding to $F_4[\theta]$ and $F_4[\theta^2]$ are fixed by σ and vanish on the 3-regular classes of G; see [72, 6.2.4(c) and Proposition 6.3] and [38, p. 42]. If we denote by $\mathcal{U}$ the set of almost characters with the latter two characters removed, then Lemma 5.5 implies that $\mathcal{U}$ satisfies the hypotheses of Lemma 2.8 for $\ell = 3$ and the union of unipotent blocks. Thus σ has 8 orbits of length 2 on the set of unipotent Brauer characters. We may apply the same lemma to the unique non-principal unipotent 3-block of positive defect, to see that σ has exactly one orbit of length 2 on this block. As σ fixes the unipotent defect 0 characters, it follows that σ has exactly 7 orbits of length 2 on each of the set of irreducible ordinary and irreducible Brauer characters of the principal block. This completes our proof. □

Corollary 5.14. *Suppose that $p = 2$, that $\ell > 3$ and that the Sylow ℓ-subgroups of G are non-cyclic. Let b denote the principal ℓ-block of G. Then the number of non-trivial orbits of σ_1 on* $\mathrm{IBr}(b)$ *is as given in the following table.*

(18)

e	1	2	3	4	6
no.	7	7	6	0	6

(See Subsection 5.1 *for the significance of e.)*

Proof. This follows from Proposition 5.13(a), in connection with the action of σ_1 on the set of unipotent characters of G; see [61, Theorem 2.5(e)]. □

We finally consider the case of the exceptional covering group of $F_4(2)$. Observe that the automorphism σ_1 of $F_4(2)$ lifts to an automorphism of its double cover, also denoted by σ_1.

Proposition 5.15. *Let $\hat{G} = 2.F_4(2)$ denote the exceptional double cover of $F_4(2)$ and let $\ell \in \{3, 5, 7\}$. Let b be an ℓ-block of $\hat{G}$ of non-cyclic defect. Then σ_1 fixes b unless $\ell = 3$ and b is one of the two blocks of defect 2, which are swapped by σ_1.*

If σ_1 fixes b, the non-trivial orbits of σ_1 on $\mathrm{IBr}(b)$ *have length* 2*, and the number of such orbits is as given in the table below.*

ℓ	3	5	7
no.	4	6	6

Proof. This follows by inspecting the character table of $\hat{G}$ in the Atlas [23] and the decomposition matrices in [45], in connection with Lemma 2.8. □

6. Radical 3-subgroups and defect groups of $F_4(q)$

In this section, let $G = F_4(q)$ with $q = p^f$ and $p \neq 3$. We keep the notation introduced in Subsection 5.4. We let $\varepsilon \in \{\pm 1\}$ be defined by $3 \mid q - \varepsilon$, and put $e = e_3(q)$, i.e. $e = 1$ if $\varepsilon = 1$, and $e = 2$, otherwise. Also, the positive integer a is defined by $(q - \varepsilon)_3 = 3^a$. In addition, we put $d := \gcd(2, q - 1)$. The radical 3-subgroups of G are classified in [4, 6] and we state the main results here. We also derive some consequences needed later on. For our notation for groups and their extensions used below, see Subsection 2.1.

6.1. **Elements of order 3 in G.** By [43, Table 4.7.3], the group G contains exactly three conjugacy classes of elements of order 3, called 3A, 3B, and 3C, with representatives z_{A}, z_{B}, and z_{C}, such that for each $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ the elements z_X and z_X^{-1} are conjugate. Moreover, the local structure is as follows:

$$\begin{aligned}
C_G(z_{\mathrm{A}}) &= \langle [q-\varepsilon] \circ_d \mathrm{Sp}_6(q), x_{\mathrm{A}} \rangle, & N_G(\langle z_{\mathrm{A}} \rangle) &= \langle C_G(z_{\mathrm{A}}), \gamma_{\mathrm{A}} \rangle,\\
C_G(z_{\mathrm{B}}) &= \langle [q-\varepsilon] \circ_d \mathrm{Spin}_7(q), x_{\mathrm{B}} \rangle, & N_G(\langle z_{\mathrm{B}} \rangle) &= \langle C_G(z_{\mathrm{B}}), \gamma_{\mathrm{B}} \rangle,\\
C_G(z_{\mathrm{C}}) &= \langle \mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q), x_{\mathrm{C}} \rangle, & N_G(\langle z_{\mathrm{C}} \rangle) &= \langle C_G(z_{\mathrm{C}}), \gamma_{\mathrm{C}} \rangle,
\end{aligned}$$

where the actions of γ_X and x_X are determined by:

- $\gamma_X = (\iota : 1)$: for $X \in \{\mathrm{A}, \mathrm{B}\}$, the action of γ_X on $[q-\varepsilon]$ is by $x \mapsto x^{-1}$, and trivial on $O^{p'}(C_G(z_X))$;
- $\gamma_{\mathrm{C}} = (\gamma : \gamma)$: γ_{C} acts as the order 2 graph automorphism, i.e. inverse-transpose, on each factor $\mathrm{SL}_3^\varepsilon(q)$ of $O^{p'}(C_G(z_{\mathrm{C}}))$;
- $x_X = (1 : d)$: for $X \in \{\mathrm{A}, \mathrm{B}\}$, the action of x_X on $[q-\varepsilon]$ is trivial; on $O^{p'}(C_G(z_X))$ it is trivial, if $d = 1$, and by the diagonal automorphism of order 2 if $d = 2$;
- $x_{\mathrm{C}} = (x_1 : x_2)$: each x_i acts on $\mathrm{SL}_3^\varepsilon(q)$ by the diagonal automorphism of order 3.

Notice that $C_G(z_{\mathrm{C}})$ is conjugate in G to the group L introduced in Proposition 4.14. Note also that x_{A} does not necessarily have order 2; we only know that $x_{\mathrm{A}}^2 \in [q-\varepsilon] \circ_d \mathrm{Sp}_6(q)$; similarly for $x_{\mathrm{B}}, x_{\mathrm{C}}$ and γ_X. For example, as discussed in Remark 4.15, we have $x_{\mathrm{C}}^3 \in \mathrm{SL}_3^\varepsilon(q) \circ_3 \mathrm{SL}_3^\varepsilon(q)$, and there is a choice of x_{C} such that each x_i acts on $\mathrm{SL}_3^\varepsilon(q)$ in the same way as $\mathrm{diag}(1, 1, \xi^3) \in \mathrm{GL}_3^\varepsilon(q)$ acts on $\mathrm{SL}_3^\varepsilon(q)$ by conjugation; here, $\xi \in \mathbb{F}^*$ is an element of order 3^{a+1}. Then x_i^3 acts as conjugation with $\mathrm{diag}(1, 1, \xi^9)$ and thus as the inner automorphism $\mathrm{diag}(\xi^{-3}, \xi^{-3}, \xi^6)$, $i = 1, 2$.

6.2. **Radical 3-subgroups of $\mathrm{SL}_3^\varepsilon(q)$.** We first recall the classification of the radical 3-subgroups of $L_\varepsilon := \mathrm{SL}_3^\varepsilon(q)$ and fix notation. We view L_ε as a subgroup of $\mathrm{GL}_3^\varepsilon(q)$, where, in case of $\varepsilon = -1$, the latter group is defined with respect to the standard diagonal hermitian form. In particular, $\mathrm{GL}_3^\varepsilon(q)$ and L_ε are invariant under the inverse-transpose automorphism.

In order to simplify notation, we suppress the subscript ε from the objects associated to L_ε, although these may depend on ε. In particular, we put $L := L_\varepsilon = \mathrm{SL}_3^\varepsilon(q)$. We begin by describing various elements and subgroups of L. Let $\xi \in \mathbb{F}^*$ have order 3^{a+1}, and put $\zeta := \xi^{3^a}$. Next, let $y := \mathrm{diag}(1, \zeta, \zeta^{-1}) \in L$. Further, let $c, t \in L$ be defined by

$$c := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad t := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Finally, let x denote the diagonal automorphism of L which acts on L in the same way as $\mathrm{diag}(1, 1, \xi^3)$. Notice that x centralizes t and that x^3 is an inner automorphism of L.

Put $H := C_L(\mathrm{diag}(1, 1, \zeta)) \cong \mathrm{GL}_2^\varepsilon(q)$, let Z denote the center of L, and T the subgroup of diagonal matrices in L, i.e. $T \cong [q-\varepsilon]^2$. We also let $D := \langle O_3(T), c\rangle$; then $D \in \mathrm{Syl}_3(L)$. Finally, we put $K := \langle y, c\rangle$. Notice that if $a = 1$, then $K = D$ and $K^x = K$.

The next lemma merges [4, Lemma 4.4] and [6, Lemma 3.1(2)].

Lemma 6.3. *Let the notation be as above. Then representatives for $\mathcal{R}_3(L)/L$, the set of L-conjugacy classes of radical 3-subgroups of L is given as follows.*

Conditions	Representatives for $\mathcal{R}_3(L)/L$
$q = 2$	D
$q = 4$	$Z,\ D$
$q = 8$	$Z,\ O_3(H),\ K,\ K^x,\ K^{x^2},\ D$
$q \geq 5, a = 1$	$Z,\ O_3(T),\ D$
$q \geq 17, a \geq 2$	$Z,\ O_3(H),\ K,\ K^x,\ K^{x^2},\ O_3(T),\ D$

The structure of $C_L(R)$, $N_L(R)$ and $\mathrm{Out}_L(R)$ for the radical 3-subgroups R of L as above is given in the following table.

R	Struc.	$C_L(R)$	$N_L(R)$	$\mathrm{Out}_L(R)$
Z	3	L	L	1
$O_3(H)$	3^a	H	H	1
$K, a \geq 2$	3_+^{1+2}	Z	$K.\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$
$D, a = 1$	3_+^{1+2}	Z	$D.Q_8$	Q_8
$D, a \geq 2$	$[3^a]^2.3$	Z	$D.2 = \langle D, t\rangle$	2
$O_3(T)$	$[3^a]^2$	T	$T.S_3 = \langle T, c, t\rangle$	S_3

6.4. **Radical 3-subgroups of $F_4(q)$.** Let $L^1, L^2 \leq C_G(z_{\mathrm{C}})$ denote the subgroups with $L^1 \cong \mathrm{SL}_3^\varepsilon(q) \cong L^2$, such that $C_G(z_{\mathrm{C}}) = \langle L^1 \circ_3 L^2, x_{\mathrm{C}}\rangle$

and $N_G(\langle z_{\rm C}\rangle) = \langle C_G(z_{\rm C}), \gamma_{\rm C}\rangle$, where $x_{\rm C} = (x_1{:}x_2)$ and $\gamma_{\rm C} = (\gamma_1{:}\gamma_2)$ act as described in Subsection 6.1. Notice that L^1 and L^2 have the same meaning as in Proposition 4.14. In particular, L^1 is of Dynkin type $A_2^\varepsilon(q)$, and L^2 of Dynkin type $\tilde{A}_2^\varepsilon(q)$ so that L^1 is a long root subgroup and L^2 is a short root subgroup. Notice that $L^1 \cap L^2 = Z(C_G(z_{\rm C}))$.

The radical 3-subgroups of G are classified in [6] and [4].

Theorem 6.5. *The non-trivial radical* 3*-subgroups of* G*, up to conjugacy, are as given in* Table 26. □

It turns out that the Sylow 3-subgroups of the centralizers of semisimple elements of G are radical 3-subgroups.

Proposition 6.6. *Let* $\mathbf{M} = \mathrm{C}_{\mathbf{G}}(s)$ *denote the centralizer of the semisimple element* $s \in G$*, and let* P *be a Sylow* 3*-subgroup of* M*. Then* P *is a radical* 3*-subgroup of* G*, and the conjugacy class of* P *in* $\mathcal{R}_3(G)$ *is as given in the fourth column of* Table 25*.*

Proof. We enumerate the possibilities for (the G-conjugacy classes) of $\mathbf{M}$ by the pairs (i,k) according to the tables in [58]. Assume first that $i = 20$, i.e. that $\mathbf{M}$ is a maximal torus of $\mathbf{G}$. Using Table 24 for $i \in \{4, 6, 10\}$, we find that one of the following cases occurs:

(a) $P = 1$,
(b) $\mathbf{M} \leq C_{\mathbf{G}}(z)$ for some element z in the conjugacy class 3C,
(c) $P = [3^a]$ and $P \leq Z(C_{\mathbf{G}}(z))$ for some z in class 3A or 3B, or
(d) $P = [3^a]^2$ and $P \leq Z(C)$, where $\mathbf{C} \leq \mathbf{G}$ is the centralizer of a semisimple element of G of class type $(15,1)$ if $\varepsilon = 1$, and of class type $(15,3)$ if $\varepsilon = -1$.

By the construction of the radical 3-subgroups of $\mathrm{SL}_3^\varepsilon(q)$ in Lemma 6.3, we find that P is a radical 3-subgroup of G in Case (b). Unless $k \in \{14, 15, 19, 20\}$, the conjugacy class of P in this case is determined from $|P|$ and the types of the characteristics (see Subsection 9.7 for this notion) of the abelian radical 3-subgroups of G, in connection with the information given in Table 24 on the number of elements in $Z(M)$ belonging to other class types. For example, if M is of class type $(20,12)$ and $\varepsilon = 1$, then $|P| = 3^{3a}$ and $Z(M)$ contains 8 elements of class type $(4,1)$ and 12 elements of class type $(6,1)$, as well as 6 elements of class type $(10,1)$. Thus $P =_G R_{16}$. To identify the conjugacy class of P in $\mathcal{R}_3(G)$ in the remaining cases, suppose first that $\varepsilon = 1$. We may then choose notation such that $P =_G R_6$ if $\mathbf{M} =_G \mathbf{M}_{20,14}$, and $P =_G R_7$ if $\mathbf{M} =_G \mathbf{M}_{20,19}$. By looking at the two classes of maximal tori in each of $\mathbf{M}_{18,3}$ and $\mathbf{M}_{19,8}$, which occur as the centralizers of P in the respective cases, we find that $R_6 \leq_G R_{11}$ and $R_7 \leq_G R_{12}$. Now in case $\varepsilon = -1$, the two classes of maximal tori in each of $\mathbf{M}_{18,7}$ and $\mathbf{M}_{19,9}$ show that $P =_G R_6$ if $\mathbf{M} =_G \mathbf{M}_{20,15}$, and $P =_G R_7$ if $\mathbf{M} =_G \mathbf{M}_{20,20}$. In Case (c) we have $C_G(P) = C_G(z)$ is conjugate to the centralizer of R_2, respectively R_3, showing that P is conjugate to R_2 respectively R_3. In Case (d), we have $C = C_G(R)$ for some radical subgroup $R =_G R_{10}$, and thus $P = O_3(Z(C)) =_G R_{10}$.

Now assume that $i \neq 20$. Here, we use the following general argument. Suppose that $z \in G$ is semisimple and $s \in Z(C_G(z))$. Then $C_G(z) \leq C_G(s) = M$. If, in addition, $|C_G(s)|_3 = |C_G(z)|_3$, the two centralizers have conjugate Sylow 3-subgroups. Applying this argument in case $C_{\mathbf{G}}(z)$ is a maximal torus, yields all the entries of Table 25 in the cases where P is abelian. We give a sample argument for one of the other cases. Suppose that z is of class type $(13, k)$, where $1 \leq k \leq 6$. We may then assume that $C_{\mathbf{G}}(z)$ equals the regular subgroup $\mathbf{M}_{13,k}$ of $\mathbf{G}$ of type A_2 corresponding to the long roots; see Subsections 4.9 and 9.3. Suppose further that ε is such that the Sylow 3-subgroups of $M_{13,k}$ are non-abelian. Then the center of $M_{13,k}$ contains elements of class type $(4, 1)$, if $\varepsilon = 1$, and of class type $(4, 2)$, if $\varepsilon = -1$, i.e. 3C-elements. Thus $M_{13,k} \leq C_G(z_{\mathrm{C}}) = L$. If $k = 1$ or $k = 6$, according as $\varepsilon = 1$ or $\varepsilon = -1$, respectively, we have $C_G(s) = (L^1 \circ_3 [q - \varepsilon]^2).3$. Thus a Sylow 3-subgroup of $C_G(z_{\mathrm{C}})$ equals $(D_1 \circ_3 [3^a]^2).3 =_G R_{34}$. Now in the respective cases for ε, we find that $Z(M_{13,k})$ contains elements s of class types $(8, 1), (8, 4)$ and (i, k) for $i = 3, 6, 7, 8$ and $k = 1, 2$, as well as $(2, 1)$. Since $|M_{13,k}|_3 = |C_G(s)|_3$ in all these cases, we obtain the corresponding entries in Table 25. (This argument also works in case $q = 4$, where there is no element $z \in G$ such that $C_{\mathbf{G}}(z) =_G \mathbf{M}_{13,1}$.) The remaining cases are treated similarly. □

We collect a few further results on the radical subgroups of G. The following lemma describes the regular subgroups of $\mathbf{G}$ that occur as centralizers of abelian radical 3-subgroups (not conjugate to one of R_1, R_8 or R_{15}).

Lemma 6.7. *Let R be a non-trivial, abelian radical 3-subgroup of G, but $R \notin_G \{R_1, R_8, R_{15}\}$. Then $C_{\mathbf{G}}(R) =_G \mathbf{M}_{i,k}$, with (i, k) as in the following table. The two cases for k correspond to $\varepsilon = 1$ and $\varepsilon = -1$,* *respectively.*

R	i	k	Rem
R_2	10	$1,2$	
R_3	6	$1,2$	
R_4	9	$1,2$	$a \geq 2$
R_5	7	$1,2$	$a \geq 2$
R_6	18	$3,7$	
R_7	19	$8,9$	
R_9	14	$1,4$	
R_{10}	15	$1,3$	

R	i	k
R_{11}	20	$7,8$
R_{12}	20	$4,5$
R_{13}	17	$1,6$
R_{14}	13	$1,6$
R_{16}	18	$1,10$
R_{17}	19	$1,10$
R_{18}	20	$1,2$

Notice that $C_{\mathbf{G}}(R) \leq \mathbf{G}$ is regular, and, unless $R \in_G \{R_6, R_7, R_{11}, R_{12}\}$, also e-split. Moreover, R is the Sylow 3-subgroup of $Z(C_G(R))$.

PROOF. Let (i,k) be as in the table corresponding to the row containing R. Then, by Proposition 6.6, unless $R \notin_G \{R_3, R_4, R_{13}, R_{14}\}$, there is a maximal torus $\mathbf{S} \leq \mathbf{M}_{i,k}$ such that $R \in \mathrm{Syl}_3(S)$, and $|S|_3 = |Z(M_{i,k})|_3$. Comparing with $|C_G(R)|$, as given in Table 26, we find $C_G(R) = M_{i,k}$ as claimed.

Now, suppose that $\varepsilon = 1$ and $(i,k) = (17,1)$. Then $\mathbf{M}_{17,1}$ embeds into $\mathbf{L} = C_{\mathbf{G}}(z_{\mathrm{C}})$ such that $[\mathbf{M}_{17,1}, \mathbf{M}_{17,1}] = \mathbf{L}^2$. Moreover, $Z(\mathbf{M}_{17,1})$ is the intersection of maximal tori G-conjugate to $\mathbf{M}_{20,7}$ and $\mathbf{M}_{20,17}$, whose Sylow 3-subgroups are conjugate to R_{11} and R_{17}, respectively; see Tables 24 and 25. On the other hand, by the construction of the radical 3-subgroups (see Subsection 6.4), the intersection of a suitable conjugate of R_{11} and R_{17} is a radical 3-subgroup of G conjugate to R_{13}, as can be seen by the type of its characteristic; see Subsection 9.7. The analogous argument works for $R =_G R_{14}$ and also for $\varepsilon = -1$. Finally suppose that $\varepsilon = 1$, $R =_G R_4$ and $(i,k) = (9,1)$. Again, $\mathbf{M}_{17,1}$ embeds into $\mathbf{L}$. Also, $Z(\mathbf{M}_{17,1})$ is contained in the torus of type $(20,14)$, whose Sylow 3-subgroup is conjugate to R_6. If $a \geq 2$, then the Sylow 3-subgroup of $Z(M_{17,1})$ is a radical 3-subgroup of G, which we denote by R_4. By construction $R_4 \leq_G R_6$. The remaining cases are proved similarly.

As $\mathbf{M}_{i,k}$ is a regular subgroup of $\mathbf{G}$ in all cases, and $\mathbf{M}_{i,k}$ is e-split except for $R =_G R_6, R_7$, the penultimate assertion follows. Inspection gives the last claim. □

The centralizers of most of the non-abelian radical 3-subgroups of G are connected.

Lemma 6.8. *Let R be a non-abelian radical 3-subgroup of G, but*

$$R \notin_G \{R_{21}, R_{22}, R_{35}–R_{38}\}.$$

Then $C_{\mathbf{G}}(R)$ is a closed, connected reductive subgroup of $\mathbf{G}$.

PROOF. By the construction of R as a subgroup of $L = (L^1 \circ_3 L^2).3$, we find that $C_{\mathbf{G}}(R) \leq \mathbf{L}$, i.e. $C_{\mathbf{G}}(R) = C_{\mathbf{L}}(R)$. As indicated in Table 26, the latter is equal to a regular subgroup of $\mathbf{L}^i$, for $i \in \{1, 2\}$. □

It turns out that the abelian, characteristic subgroups of certain non-abelian radical 3-subgroups exhibited in the next lemma play a crucial role in the following.

Lemma 6.9. *Let $R \in_G \{R_{29}–R_{34}\}$. Then R has a unique maximal abelian normal subgroup Q which is again a radical subgroup of G. The conjugacy class of Q is as given in the following table.*

R	R_{29}	R_{30}	R_{31}	R_{32}	R_{33}	R_{34}
Q	R_{12}	R_{11}	R_{16}	R_{17}	R_{18}	R_{18}

In particular, the conjugacy class of R in $\mathcal{R}_3(G)/G$ is determined by the conjugacy class of Q, unless $R \in_G \{R_{33}, R_{34}\}$.

PROOF. Realize R as a Sylow 3-subgroup of a suitable $M_{i,k}$ according to Table 25. Using Table 25, choose a maximal torus of $M_{i,k}$, whose Sylow 3-subgroup is a radical 3-subgroups Q of the class indicated in the table of the lemma. Now $|R\colon Q| = 3$, and thus $[R, R] \leq Q \trianglelefteq R$. From the type of the characteristic of R (recall that the characteristic of R equals $\Omega_1([R, R])$ in these cases) given in Table 26, we find that $[R, R] \not\leq Z(R)$. Hence $Q = C_R([R, R])$. Thus Q is characteristic in R.

Let A be any maximal abelian normal subgroup of R and suppose that $A \neq Q$. Then $R = AQ$ and thus $R/A \cong Q/(A \cap Q)$ is abelian. Thus $[R, R] \leq A$ and hence $A \leq C_R([R, R]) = Q$, a contradiction. □

6.10. **Duality of radical 3-subgroups.** There is an involutive, inclusion preserving bijection $R \mapsto R^\dagger$ on the set $\mathcal{R}_3(G)/G$ such that $C_{\mathbf{G}}(R^\dagger) \cong C_{\mathbf{G}}(R)^*$, unless $R \in_G \{R_{19}–R_{22}, R_{27}, R_{28}, R_{35}–R_{38}\}$. In the latter cases, we put $R^\dagger := R$. The groups $R^\dagger$ are given in the last column of Table 26.

6.11. **Radical subgroups and Brauer pairs.** In this subsection, we record a couple of preliminary results. Let (R, b_R) denote a centric Brauer pair. Recall the notation $\mathcal{W}(R, b_R)$ introduced in (2). If

$\mathcal{W}(R, b_R) \neq 0$, then R is a radical 3-subgroup of G and the canonical character θ_R of b_R is of 3-defect zero, viewed as a character of $C_G(R)/Z(R)$; see the results summarized in Subsection 2.10.

We begin with the following result on irreducible characters of $\mathrm{GL}_2^\varepsilon(q)$ and $\mathrm{SL}_3^\varepsilon(q)$.

Lemma 6.12. *Let H be one of $\mathrm{GL}_2^\varepsilon(q)$ or $\mathrm{SL}_3^\varepsilon(q)$. Let $\psi \in \mathrm{Irr}(H)$ be such that $O_3(Z(H))$ is in the kernel of ψ and that ψ is of* 3*-defect zero as character of $H/O_3(Z(H))$. Then one of the following occurs.*

(a) *We have $H = \mathrm{GL}_2^\varepsilon(q)$ and ψ is Lusztig induced from a character in general position of the Coxeter torus $[q^2 - 1]$ of H.*

(b) *We have $H = \mathrm{SL}_3^\varepsilon(q)$ and ψ is Lusztig induced from a character in general position of the Coxeter torus $[q^2 + \varepsilon q + 1]$ of H.*

(c) *We have $a = 1$ and $H = \mathrm{SL}_3^\varepsilon(q)$ and ψ is Lusztig induced from a character in general position of the torus $[q^2 - 1]$ of H.*

Moreover, in cases (b) *and* (c)*, the character ψ, viewed as a character of $H/O_3(Z(H)) = \mathrm{PSL}_3^\varepsilon(q)$, extends to a character of $\mathrm{PGL}_3^\varepsilon(q)$.*

PROOF. This follows from the known character tables of H and Deligne–Lusztig theory. For the character degrees of H one may consult the tables in [57]. The last statement follows from Clifford theory. □

We now investigate the existence of centric Brauer pairs (R, b_R) for $R \in_G \{R_1, R_8, R_{15}\}$.

Lemma 6.13. *Let $R \in_G \{R_1, R_8, R_{15}\}$ and let (R, b_R) be a centric Brauer pair. Then $R \neq_G R_1$ and one of the following holds.*

(a) *We have $R =_G R_8$ and $\mathrm{Out}_G(R, b_R) = 1$. In this case (R, b_R) is a maximal b-Brauer pair for a block b of G with $b = \mathcal{E}_3(G, s)$ for a semisimple $3'$-element s of class type* $(20, 9)$ *or* $(20, 10)$*, according as $\varepsilon = 1$ or $\varepsilon = -1$, respectively. In particular, $R^\dagger =_G R \leq_G C_G(s)$.*

(b) *We have $R =_G R_8$ and $|\mathrm{Out}_G(R, b_R)| = 3$, so that $\mathcal{W}(R, b_R) = 0$.*

(c) *We have $R =_G R_{15}$, and b_R is the principal block of $C_G(R)$. In particular, $\mathrm{Out}_G(R, b_R) = \mathrm{SL}_2(3)$ and $\mathcal{W}(R, b_R) = 1$. Moreover, R is not a defect group of any* 3*-block of G.*

PROOF. Assume that $R =_G R_1$. Then $C_G(R) = (L^1 \circ_3 L^2).3$ with $L^i \cong \mathrm{SL}_3^\varepsilon(q)$, $i = 1, 2$. Put $L' := L^1 \circ_3 L^2 \leq C_G(R)$. Then $R \leq L'$. The canonical character θ_R of b_R is a 3-defect zero character of $C_G(R)/R$. Hence the restriction of θ_R to L'/R splits into a sum of three 3-defect zero characters of $L'/R \cong \mathrm{PSL}_3^\varepsilon(q) \times \mathrm{PSL}_3^\varepsilon(q)$. On the other hand, a 3-defect zero character of $\mathrm{PSL}_3^\varepsilon(q)$ extends to an irreducible character of $\mathrm{PGL}_3^\varepsilon(q)$; see the last statement of Lemma 6.12. As an outer element of order 3 of $(L^1 \circ_3 L^2).3$ induces a diagonal automorphism on each factor L^i, $i = 1, 2$ (see Subsection 6.1), every 3-defect zero character of L'/R extends to $C_G(R)/R$, a contradiction.

Next, assume that $R =_G R_8$. Then, by Tables 26 and 25, we find that $C_G(R) = S.3$, where $S = \mathbf{S}^F$ is a maximal torus of G of type $(20, 9)$ or $(20, 10)$, according as $\varepsilon = 1$ or -1, respectively. We also have $N_G(R) = N_G(S) = S.W_{\mathbf{G}}(\mathbf{S})^F$ with $W_{\mathbf{G}}(\mathbf{S})^F = 3.\mathrm{SL}_2(3)$, and there is an order preserving $W_{\mathbf{G}}(\mathbf{S})^F$-equivariant bijection between S and $\mathrm{Irr}(S)$, as $\mathbf{S} \cong \mathbf{S}^*$; see Subsections 4.6, 4.12 and 4.10. Let θ_R denote the canonical character of b_R. Then $\theta_R(1) = 3$, and thus $\mathrm{Res}^{S.3}_S(\theta_R)$ is the sum of three irreducible characters of $3'$-order. Let ξ be one of these constituents. If ξ is in general position, i.e. ξ lies in a regular orbit of $W_{\mathbf{G}}(\mathbf{S})^F$ on $\mathrm{Irr}(S)$, then $\mathrm{Out}_G(R, b_R) = 1$, and thus (R, b_R) is a maximal Brauer pair for some 3-block of G. On the other hand, each regular orbit of $W_{\mathbf{G}}(\mathbf{S})^F$ on S yields a G-conjugacy class of $3'$-elements of class type $(20, 9)$ or $(20, 10)$, respectively. Let $s \in S$ be a $3'$-element which lies in a regular $W_{\mathbf{G}}(\mathbf{S})^F$-orbit, and let D be a defect group of the block $b := \mathcal{E}_3(G, s)$. We claim that $D =_G R$. If not, $a = 1$ and $D \cong 3^2$ is conjugate to one of R_9, R_{10}, R_{13} or R_{14}. In all these cases, $C_{\mathbf{G}}(D)$ is a regular subgroup of $\mathbf{G}$ by Lemma 6.7. Then, up to conjugation in G, we have $s \in C_G(D)^* = C_G(D^\dagger)$ by Lemma 3.10. Hence $D^\dagger \leq C_G(s) \cong [q + \varepsilon q + 1]^2$, and thus $D^\dagger \in \mathrm{Syl}_3(C_G(s))$. This contradicts Table 25.

The number of centric Brauer pairs (R, b_R) with $\mathrm{Out}_G(R, b_R) = 1$ equals the number of regular orbits of $W_{\mathbf{G}}(\mathbf{S})^F$ on the set of $3'$-elements of $\mathrm{Irr}(S)$. The latter number is also equal to the number of blocks of the form $\mathcal{E}_3(G, s)$, where $s \in S$ is a $3'$-element of class type $(20, 9)$, respectively $(20, 10)$. As all these blocks have R as defect group, we have proved (a).

Now assume that ξ does not lie in a regular $W_{\mathbf{G}}(\mathbf{S})^F$-orbit. Then $\mathrm{Out}_G(R, b_R) \neq 1$ and, under the above bijection, ξ corresponds to a semisimple $3'$-element $t \in S$ such that t lies in a class type different from $(20, 9)$, respectively $(20, 10)$. The tables in [58] show that t is of class type $(13, 5)$ or $(17, 5)$, or $(13, 4)$ or $(17, 4)$, respectively. Notice that $N_G(R)$ stabilizes $\mathbf{S} = C_{\mathbf{G}}(R)^\circ$ and hence S. Every element of $N_G(R, b_R)$ fixes θ_R and thus the orbit of t under $C_G(R) = S.3$. In particular, an element of $N_G(R, b_R)$ either permutes all three elements of this orbit or else centralizes t. For each possible class type of t, the relative Weyl group in $C_G(t)$ of S is cyclic of order 3. Thus $3 \mid |\mathrm{Out}_G(R, b_R)|$ and $2 \nmid |\mathrm{Out}_G(R, b_R)|$. It follows that $|\mathrm{Out}_G(R, b_R)| = 3$, as $N_G(R)/C_G(R) \cong \mathrm{SL}_2(3)$. This yields (b).

Finally, assume that $R =_G R_{15}$. Then $C_G(R) = R$, and thus b_R is the principal block of $C_G(R)$. Hence $\mathrm{Out}_G(R, b_R) = N_G(R)/R \cong \mathrm{SL}_3(3)$ by Table 26. Since this is not a $3'$-group, R is not a defect group. Now θ_R extends to $N_G(R)$, and as $\mathrm{SL}_3(3)$ has a unique 3-defect zero character, we obtain $\mathcal{W}(R, b_R) = 1$ by Equation (3). □

Now let (Q, b_Q) denote a centric Brauer pair with Q abelian. Assume that (Q, b_Q) is a b-Brauer pair for some 3-block b of G with $b \subseteq \mathcal{E}_3(G, s)$, where s is a semisimple $3'$-element of G. If we assume that $Q \notin_G \{R_1, R_8, R_{15}\}$, then $\mathbf{M} := C_{\mathbf{G}}(Q)$ is a regular subgroup of $\mathbf{G}$ by Lemma 6.7. In these cases, if we let $\mathbf{M}^*$ denote a regular subgroup of $\mathbf{G}$ dual to $\mathbf{M}$, and let $t \in \mathbf{M}$ be a semisimple $3'$-element such that $b_Q \subseteq \mathcal{E}_3(M^*, t)$, then t is conjugate to s in G; see Lemma 3.10. In the following lemma we list the candidates for the pairs (Q, θ_Q).

Lemma 6.14. *Let Q be a non-trivial, abelian radical* 3*-subgroup of G, but $Q \notin_G \{R_1, R_2, R_3, R_8, R_{15}\}$. Then $\mathbf{M} := C_{\mathbf{G}}(Q)$ is a regular subgroup of $\mathbf{G}$ by* Lemma 6.7. *Choose a regular subgroup $\mathbf{M}^*$ of $\mathbf{G}$ dual to $\mathbf{M}$. Let $\theta \in \mathcal{E}(M, t)$ for some semisimple $3'$-element $t \in M^*$. Then $\theta(1)_3 = |M/Q|_3$ in exactly one of the following cases.*

(a) *We have $Q =_G R_{10}$, $t \in Z(M^*)$ and $\theta = \hat{t}\psi$, where $\hat{t}$ is the linear character of M that corresponds to t by duality, and ψ is the unipotent character of M of degree $q(q-\varepsilon)^2/2$.*

(b) *We have $\theta = \pm R^{\mathbf{M}}_{\mathbf{S}^*}(t)$, where $\mathbf{S}^*$ is an F-stable maximal torus of $\mathbf{M}^*$, and $t \in S^*$ is in general position (with respect to $\mathbf{M}^*$). The cases for Q and $\mathbf{S}^*$ are as given the following table, where the conjugacy class of $\mathbf{S}^*$ in $\mathbf{G}$ is given by the integer k, if $\mathbf{S}^* =_G \mathbf{M}_{20,k}$ as in* Table 25.

Q	R_6	R_7	R_9	R_{10}	R_{11}	R_{12}	R_{16}	R_{17}	R_{18}
$\mathbf{S}^*$	$14, 15$	$19, 20$	$22, 22$	$3, 3; 23, 24$	$7, 8$	$4, 5$	$17, 18$	$12, 13$	$1, 2$

Except for $Q =_G R_{10}$, there is a unique such torus $\mathbf{S}^*$, whereas there are two for $Q =_G R_{10}$. The two values for k separated by a comma correspond to the cases $\varepsilon = 1$, and $\varepsilon = -1$, respectively. We have $t \in Z(M^*)$ if and only if $Q \in_G \{R_{11}, R_{12}, R_{18}\}$. In these cases, $\mathbf{S}^* = \mathbf{M}^*$.

Proof. Suppose first that $t \in Z(M^*)$. Then $\mathcal{E}(M, t) = \hat{t}\mathcal{E}(M, 1)$, so that $\theta = \hat{t}\psi$ for some unipotent character $\psi \in \mathcal{E}(M, 1)$. Now $\theta(1)_3 = |M/Q|_3$, if and only if ψ is a 3-defect zero character of $M/Z(M)$. The possibilities for M can be determined from Lemma 6.7 and Table 25. Looking at the degrees of the unipotent characters of these groups, we get exactly the cases in (a) and those in (b) for $Q \in_G \{R_{11}, R_{12}, R_{18}\}$.

Now assume that $t \notin Z(M^*)$ and that $Q =_G R_{10}$. As $\mathbf{M} = \mathbf{M}^*$ in this case and $Z(\mathbf{M})$ is connected, the character degrees of M and those of $M/Z(M)$ agree. Now $M/Z(M) = (\mathbf{M}/Z(\mathbf{M}))^F$, and $\mathbf{M}/Z(\mathbf{M})$ is a simple group of adjoint type C_2. The character degrees of $M/Z(M)$ can be found in the tables [57], yielding our claim.

Finally, assume that $Q \notin_G \{R_{10}\}$. Then $\mathbf{M}$ is of type A, and $\theta = \pm R_{\mathbf{L}}^{\mathbf{M}}(\hat{t}\nu)$, where $\mathbf{L}$ is a regular subgroup of $\mathbf{M}$ with $\mathbf{L}^* = C_{\mathbf{M}^*}(t)$, and $\nu \in \mathcal{E}(L, 1)$. As $Q \leq Z(M) \leq L$, we have $\theta(1)_3 = |M/Q|_3$ if and only if $\nu(1)_3 = |L/Q|_3$. Also, $\mathbf{L}$ is a proper subgroup of $\mathbf{M}$, since $t \notin Z(M^*)$. Now L has a unipotent character ν with $\nu(1)_3 = |L/Q|_3$, if and only if L is a maximal torus of M and $|L/Q|_3 = 1$. This rules out the possibilities $Q \in_G \{R_4, R_5, R_{13}, R_{14}\}$, and gives exactly the remaining cases in (b). □

Corollary 6.15. *Let the notation be as in* Lemma 6.14*, and suppose that (Q, b_Q) is a centric Brauer pair with $b_Q \subseteq \mathcal{E}_3(M, t)$. Then (Q, b_Q) satisfies the hypotheses of* Proposition 3.16(a)*. In particular,* $\operatorname{Out}_G(Q, b_Q) \cong \operatorname{Out}_{C_G(t)}(Q^\dagger)$ *(recall that we have identified* $\mathbf{G}$ *with* $\mathbf{G}^*$*).*

Proof. This is clear in case (a) of Lemma 6.14, as ψ is invariant under all automorphisms of $\mathrm{Sp}_4(q)$ (for odd q this follows from [60, Remarks on p. 159] and for even q from [61, Theorem 2.5]). In case (b), $b_Q = \mathcal{E}_3(M, t)$, as t is in general position in $\mathbf{S}^*$. □

6.16. **Defect groups of $F_4(q)$.** Let b be a 3-block of G with defect group D. Then $b \subseteq \mathcal{E}_3(G, s)$ for a semisimple $3'$-element $s \in G^*$. As usual, we identify $\mathbf{G}$ with $\mathbf{G}^*$ and G with G^*. By Tables 1–19, we know that b is uniquely determined by the conjugacy class of s in G and its defect group D (even by the conjugacy class of s and $|D|$). We label the block b by (s, D). We begin with a lemma.

Lemma 6.17. *Let* $\mathbf{L}, \mathbf{L}^1, \mathbf{L}^2$ *and* $\mathbf{T}$ *be defined as in* Proposition 4.14*, and put $T_i := T \cap L^i$ for $i = 1, 2$. The identification of* $\mathbf{G}$ *with* $\mathbf{G}^*$ *induces a bijection $T \to \mathrm{Irr}(T), s \mapsto \hat{s}$ such that the following holds for all $3'$-elements $s \in T$.*

We have $s \in T_1$ if and only if T_1 is contained in the kernel of $\hat{s}$.

Proof. We only prove the claim for $\varepsilon = 1$. The other case is proved by twisting with w_0. Recall that $\mathbf{L}^1 = [\mathbf{L}_{\{1,23\}}, \mathbf{L}_{\{1,23\}}]$ and $\mathbf{L}^2 = [\mathbf{L}_{\{3,4\}}, \mathbf{L}_{\{3,4\}}]$. Also, $\mathbf{T}_i = \mathbf{T} \cap \mathbf{L}^i$, for $i = 1, 2$.

Let $s \in T$ be a $3'$-element. Suppose that $s \in T_1 = \langle \alpha_1^\vee(t), \alpha_{23}^\vee(t) \mid t \in \mathbb{F}_q^* \rangle$. Then s is in the kernel of $\hat{s}$ by Lemma 4.11. Similarly, s is in the kernel of $\hat{s}$ if $s \in T_2$. Suppose that $T_1 \leq \ker(\hat{s})$. As T_1T_2 has index 3 in T, we have $s \in T_1T_2$. Write $s = s_1s_2$ with $s_i \in T_i$ for $i = 1, 2$. By the above, $T_1 \leq \ker(\hat{s}_1^{-1})$. In turn, $T_1 \leq \ker(\hat{s}_1^{-1}\hat{s})$. As $\hat{s}_1^{-1}\hat{s} = \hat{s}_2$, it follows that $\hat{s}_2 = 1_T$, and thus $s_2 = 1$. □

Proposition 6.18. *Let b be a 3-block of G labeled by (s, D). Then the following statements hold.*

(a) *We have $D^\dagger \leq C_G(s)$ for some $D^\dagger$ dual to D (in the sense of* Subsection 6.10*).*

(b) *The conjugacy class of D in $\mathcal{R}_3(G)$ is as given in* Column 9 *of the row corresponding to b in* Table i*, where i is the* **G**-*class type of s.*

Proof. To prove (a), we begin with the case that D is abelian and let (D, b_D) denote a maximal b-Brauer pair. If $D =_G R_8$, the claim follows from Lemma 6.13(a). Hence assume that $D \neq_G R_8$ in the following. Then $C_{\mathbf{G}}(D)$ is a regular subgroup of $\mathbf{G}$ by Lemmas 6.14, 6.13 and 6.7, and (a) follows from Lemma 3.10.

Assume next that s is not quasi-isolated and let $\mathbf{M}^\dagger$ denote the regular subgroup of $\mathbf{G}$ defined in Theorem 3.9, which is minimal with the property that $C_{\mathbf{G}}(s) \leq \mathbf{M}^\dagger$. Let $\mathbf{M}$ denote a regular subgroup of $\mathbf{G}$ dual to $\mathbf{M}^\dagger$. By Theorem 3.9(a), we may assume that $D \leq M$, and hence $D^\dagger \leq M^\dagger$. Now if $|D| = |M|_3$, we may choose $D^\dagger \leq C_G(s)$, which proves (a) in these cases. If $|D| < |M|_3$, which is the case if s has $\mathbf{G}$-class type 11 or 12, then the Sylow 3-subgroups of $C_G(s)$ are abelian; see Tables 25 and 26. Then D is abelian by Theorem 3.9(c), a case we have already settled.

If $s = 1$, the claim is trivially satisfied. We are left with the case that D is non-abelian and that $s \neq 1$ is isolated, i.e. that the $\mathbf{G}$-class type of s is one of 2, 3 or 5. Then $|D| = 3^{4a+1}$ by Tables 2–5, and D has a normal subgroup conjugate to R_{18} by [51, Proposition 3.5]. Let P denote a Sylow 3-subgroup of $C_G(s)$. Table 26 implies that $D \in_G \{P, P^\dagger\}$, and thus $D \in_G \{R_{33}, R_{34}\}$. Choose $Q^\dagger = Q =_G R_{18}$ such that $Q^\dagger \leq P$ is the maximal abelian normal subgroup of P described in Lemma 6.9. Then $s \in C_G(Q^\dagger) \cong C_G(Q)^*$. As $C_G(Q) = T$ is a maximal torus, $b_Q := \mathcal{E}_3(C_G(Q), s)$ is a block of $C_G(Q)$. Let b' be the unique block of G such that that $(1, b') \leq (Q, b_Q)$; see [50, Theorem 2.9]. Choose a maximal Brauer pair $(D', b_{D'})$ containing (Q, b_Q). Then D' is a defect group of b'. By Lemma 3.10, applied to the b'-Brauer pair (Q, b_Q), we may assume that $b' \subseteq \mathcal{E}_3(G, s)$. From Columns 6 of Tables 2–5 we conclude that $Q \lneq D'$ and thus $|D| = |D'|$. As b is uniquely determined by $(s, |D|)$, we conclude that $b' = b$ and $D =_G D' = R$. Thus assume that $D = D'$ and so $Q \leq D$ in the following. Then $(Q, b_Q) \leq (D, b_D)$. We have $C_G(Q) = T$, and $C_G(D) \in \{T_1, T_2\}$; see Table 26. In particular, $\theta_Q = \hat{s}$, where $\hat{s} \in \operatorname{Irr}(T)$ corresponds to $s \in T$ by duality, and θ_D is the restriction of $\hat{s}$ to $C_G(D)$. As s is a $3'$-element, $s \in C_G(D) \cup C_G(D^\dagger) = T_1 \cup T_2$ and $\theta_D \neq 1$, we obtain $s \in C_G(D^\dagger)$ by Lemma 6.17. Hence $D^\dagger =_G P$ as claimed. The proof of (a) is complete.

To prove (b), assume first that b corresponds to the principal block of $C_G(s)$ via Theorem 3.9 (i.e. the label in Columns 5 of Tables 2–19 equals 1). Then $|D| = |C_G(s)|_3$, so that D is conjugate in G to a Sylow 3-subgroup of $C_G(s)$ by (a). In this case, the claim follows from Table 25. Assume that b does not correspond to the principal block of $C_G(s)$. Then D is abelian by [51, Theorem 1.2] for quasi-isolated s, and by Theorem 3.9 in the other cases. Using the knowledge of $d(b)$ and Table 26, we find that $D \in_G \{[3^a]^2, [3^a], 1\}$. Suppose first that $D = [3^a]^2$. If $D =_G R_8$, the claim holds by Lemma 6.13. We are left with the cases $D \in_G \{R_9, R_{10}\}$ by Lemma 6.14. In particular, $D =_G D^\dagger$. The possible class types for s can be found in Tables 1–19. It follows from this and Table 25, that there is a conjugate Q of R_9 and a conjugate R of R_{10} such that $C_G(Q) \leq C_G(s)$ and $C_G(R) \leq C_G(s)$, unless s is of class type $(15, 1)$ or $(15, 3)$. Thus in the former cases, we may assume that $s \in Z(C_G(D^\dagger)) = Z(C_G(D)^*)$, and hence $D =_G R_{10}$ by Lemma 6.14. In the latter cases, assume that $D =_G R_9$. Lemmas 3.10 and 6.14 imply that s lies in a maximal torus of $C_{\mathbf{G}}(D)$ of class type $(20, 22)$. However, $C_{\mathbf{G}}(s)$ does not have a maximal torus of this type, as Table 24 shows. This contradiction shows that $D =_G R_{10}$ in these cases as well.

Finally, suppose that $D = [3^a]$. Then $D \in_G \{R_2, R_3\}$ by Lemmas 6.13 and 6.14. Recall that $R_2^\dagger =_G R_3$ and $R_3^\dagger =_G R_2$. The possible class types of s can be found in Tables 1–19. In particular, s is not quasi-isolated. By (a), we may assume that $D^\dagger \leq C_G(s)$. Put $\mathbf{K}^\dagger := C_{\mathbf{G}}(D^\dagger) \cap \mathbf{M}^\dagger$, where $\mathbf{M}^\dagger$ is as in the proof of (a). Let $t \in G$ with $D^\dagger = \langle t \rangle$. Then $C_{\mathbf{G}}(st) = C_{\mathbf{G}}(D^\dagger) \cap C_{\mathbf{G}}(s) \leq \mathbf{K}^\dagger$, with equality if $\mathbf{M}^\dagger = C_{\mathbf{G}}(s)$. The latter holds unless s is of class type $(11, k)$. If $D^\dagger =_G R_2$, respectively R_3, then $C_{\mathbf{G}}(D^\dagger)$ is G-conjugate to $\mathbf{M}_{10,k}$, respectively $\mathbf{M}_{6,k}$ with $k = 1, 2$; see Lemma 6.7. By running through the possible class types for s, we can determine $\mathbf{K}^\dagger$ in each case by comparing the maximal tori fusing into $C_{\mathbf{G}}(D^\dagger)$ respectively $\mathbf{M}^\dagger$; see Table 24. We give more details in case $\varepsilon = 1$. Then $C_{\mathbf{G}}(D^\dagger) \in_G \{\mathbf{M}_{6,1}, \mathbf{M}_{10,1}\}$. It turns out that, unless the class type of s equals $(11, 2)$, there is a unique $i \in \{6, 10\}$ such that the intersection of a G-conjugate of $\mathbf{M}_{i,1}$ with $\mathbf{M}^\dagger$ can be the centralizer of st. A variant of this argument also works in the remaining case, where $\mathbf{M}^\dagger =_G \mathbf{M}_{10,2}$. We have thus determined $D^\dagger$. The same proof works for $\varepsilon = -1$. □

The classification of the defect groups for the abelian 3-blocks of G reveals an analogous behavior as in the case of ℓ-blocks for $\ell > 3$.

Corollary 6.19. *Let $s \in G$ be a semisimple $3'$-element and let $b \subseteq \mathcal{E}_3(G, s)$ be a 3-block of G with a non-cyclic abelian defect group D.*

Then there is an e-split Levi subgroup $\mathbf{M}$ *of* $\mathbf{G}$ *such that* D *is the Sylow* 3*-subgroup of* $Z(M)$*, unless the* G*-class type of* s *is one of* $(18, 3)$ *or* $(19, 8)$ *if* $e = 1$*, respectively* $(18, 7)$ *or* $(19, 9)$ *if* $e = 2$.

PROOF. This follows from Proposition 6.18 in connection with Lemma 6.7. □

6.20. **Candidates for weight subgroups.** Let b be a 3-block of G with defect group D. If D is non-abelian and not a Sylow 3-subgroup of G, it is conjugate to one of R_{29}–R_{34} by Tables 1–19. Recall from the summary given in 2.10, that if (Q, b_Q) is a b-Brauer pair giving rise to a b-weight, then (Q, b_Q) is centric and Q is a radical 3-subgroup of G. By Equation (1), we have $Z(D) \leq Z(R) \leq R \leq D$ for some conjugate R of Q. Using these conditions, the following lemma restricts the set of candidates for Q in case D is non-abelian.

Lemma 6.21. *Let* b *be a* 3*-block of* $G = F_4(q)$ *and let* (D, b_D) *denote a maximal* b*-Brauer pair, where* $D \in_G \{R_{29}$–$R_{34}\}$*. Let* $Q \lneq D$ *be such that the Brauer pair* $(Q, b_Q) \leq (D, b_D)$ *is centric and* $\mathcal{W}(Q, b_Q) \neq 0$*. Then* Q *or* $Q^\dagger$ *is one of the groups contained in the following table.*

D	$Q/Q^\dagger$
R_{33}	R_{16}, R_{18}, R_{25}
R_{31}	$R_{16}, R_{19}(a = 1), R_{23}(a \geq 2)$
R_{29}	R_{19}

PROOF. As (Q, b_Q) is a centric b-Brauer pair but not maximal, $|\mathrm{Out}_G(Q, b_Q)|$ is divisible by 3 (this is a consequence of Brauer's First Main Theorem; see [56, Theorem 6.7.6(v)]). Moreover, $\mathrm{Out}_G(Q, b_Q)$ has an irreducible projective character of degree divisible by 3, since $\mathcal{W}(Q, b_Q) \neq 0$. These conditions exclude the cases $Q \in_G \{R_1$–$R_7\}$, $Q \in_G \{R_9$–$R_{14}\}$ and $Q \in_G \{R_{27}$–$R_{32}\}$. Indeed, by Table 26, in these cases $\mathrm{Out}_G(Q, b_Q)$ is a 2-group or a subgroup of $[6] \times S_3$. No such group of order divisible by 3 has a projective character of degree divisible by 3. By Lemma 6.13, the radical subgroup R_8 is also excluded.

Now suppose that Q is one of the other radical 3-subgroups of G. From $Q \leq D$ we obtain $C_G(D) \leq C_G(Q)$. Together with $Z(D) \leq Z(Q)$ our claim follows from Table 26. For example, if $D =_G R_{33}$, we have $Z(D) = [3^a]^2 =_G R_{13}$ and the type of the characteristic of $Z(D)$ equals $3\mathrm{A}_6\mathrm{C}_2$. This yields R_{25} as only non-abelian candidate for Q, and R_{16}, R_{18} as the only abelian candidates. □

7. Alperin's blockwise weight conjecture for $F_4(q)$

Let $G = F_4(q)$ with q a power of the prime p. We are now ready to prove the first main result of this article.

Theorem 7.1. *Let ℓ be an odd prime not dividing q. Let b be an ℓ-block of $G = F_4(q)$ and let (D, b_D) denote a maximal b-Brauer pair. Then the following two statements hold.*

(a) *The number of G-conjugacy classes of b-weights equals the number of irreducible Brauer characters of b, i.e. the Alperin weight conjecture is satisfied for b.*

(b) *Let (R, b_R) denote a centric b-Brauer pair contained in (D, b_D), i.e. $(1, b) \leq (R, b_R) \leq (D, b_D)$ and b_R has defect group $Z(R)$. If $\mathcal{W}(R, b_R) \neq 0$, the canonical character θ_R of b_R extends to $N_G(R, b_R)$. In particular, $\mathcal{W}(R, b_R) = |\mathrm{Irr}^0(\mathrm{Out}_G(R, b_R))|$ by* Equation (3).

Proof. Suppose first that $\ell > 3$. Then (a) follows from (b) by [62, Corollary 3.7]. As (b) holds by [62, Proposition 2.3(a)] and Corollary 4.20, we are done in this case.

For the remainder of the proof we assume $\ell = 3$. Let $s \in G$ be a semisimple $3'$-element with $b \subseteq \mathcal{E}_3(G, s)$. If $C_{\mathbf{G}}(s)$ is a maximal torus of $\mathbf{G}$, then D is abelian, $l(b) = 1$ and $|\mathrm{Out}_G(D, b_D)| = 1$, so that both claims hold. To prove statement (a), it thus suffices to show that the entries in Columns 9–11 of each of the Tables 1–19 are correct. Let (R, b_R) denote a centric b-Brauer pair contained in (D, b_D) as in (b). The entries in Column 9 corresponding to b list those R, for which $\mathcal{W} := \mathcal{W}(R, b_R)$ is non-zero, where the first entry in this column is the defect group D. The latter has already been determined in Proposition 6.18. For each R in Column 9, the values of $\mathrm{Out}_G(R, b_R)$ and $\mathcal{W}$ are given in Columns 10 and 11, respectively. In view of the last statement of (b), $\mathcal{W}$ can be determined from the entries in Column 10. Notice that for R to appear in Column 9, we must have $Z(D) \leq Z(R)$ by Equation (1). In Lemma 6.21 we have listed the candidates for these R in case D is non-abelian. If D is abelian, then $R = D$ and $\mathcal{W}(R, b_R) = \mathcal{W}(D, b_D)$. If D is cyclic, statement (a) is known to hold by [54, Theorem 1.1]. If R is cyclic, so is $\mathrm{Out}_G(R, b_R)$ by Table 26, and thus θ_R extends to $N_G(R, b_R)$. In what follows we will therefore always assume that D and R are non-cyclic.

We first prove our assertions for the principal block. Let b be the principal block of G, so that $D \in \mathrm{Syl}_3(G)$ and b_D is the principal block of $C_G(D)$. Moreover, b_R is the principal block of $C_G(R)$ and θ_R is the trivial character. Thus $|C_G(R)/Z(R)|_3 = 1 = \theta_R(1)_3$ and $C_G(R)$ is abelian by Table 26. Also, $N_G(R, b_R) = N_G(R)$ and $\mathrm{Out}_G(R, b_R) =$

$\mathrm{Out}_G(R)$. In particular, $\mathcal{W}(R, b_R) = |\mathrm{Irr}^0(\mathrm{Out}_G(R))|$. The set of irreducible characters of $\mathrm{Out}_G(R)$ is easy to compute with the help of a computer algebra system such as MAGMA [12] or GAP [34]. If $C_G(R) = Z(R)$, we may suppose that

$$R \in_G \{R_{15}, R_{21}, R_{22}, R_{35}, R_{36}, R_{37}, R_{38}\}$$

by Table 26. If $C_G(R) \neq Z(R)$, then $\mathrm{Irr}^0(\mathrm{Out}_G(R)) = \emptyset$ by Table 26, except when $R =_G R_{18}$, in which case $\mathrm{Out}_G(R) = W(F_4)$. This completes the proof for the principal block.

From now on we assume that b is not the principal block, and continue with a preliminary consideration. Suppose that R is abelian. Then $R \notin_G \{R_1, R_8, R_{15}\}$ by Tables 1–19, and thus $C_{\mathbf{G}}(R)$ is a regular subgroup of $\mathbf{G}$ by Lemma 6.7. By Lemma 3.10, we may assume that $s \in C_G(R^\dagger) \cong C_G(R)^*$ and that $b_R \subseteq \mathcal{E}_3(C_G(R), s)$. By Lemma 6.14 and Corollary 6.15, we have $\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_{C_G(s)}(R^\dagger)$. Put

$$\mathbf{K}^\dagger := C_{\mathbf{G}}(R^\dagger) \cap C_{\mathbf{G}}(s).$$

Lemma 3.11 implies that $\mathbf{K}^\dagger = C_{C_{\mathbf{G}}(s)}(R^\dagger)$ is a regular subgroup of $C_{\mathbf{G}}(s)$, as $3 \nmid |Z(C_{\mathbf{G}}(s)/Z(C_{\mathbf{G}}(s)^\circ|$ and 3 is a good prime for $C_{\mathbf{G}}(s)$. Also, $\mathbf{K}^\dagger = C_{C_{\mathbf{G}}(R^\dagger)}(s)$ is a regular subgroup of $C_{\mathbf{G}}(R^\dagger)$ by [39, Corollary 2.6]. Proposition 3.16(b) implies that $\mathrm{Out}_{C_G(s)}(R^\dagger) = W_{C_{\mathbf{G}}(s)}(\mathbf{K}^\dagger)^F$.

We proceed to prove our assertions in case of abelian defect groups. Assume that D is abelian. By Tables 1–19, we have

$$D \in_G \{R_9, R_{10}, R_{11}, R_{12}, R_{16}, R_{17}, R_{18}\}.$$

In each case we find $C_{\mathbf{G}}(D^\dagger)$ from Lemma 6.7. We now run through all the possibilities for s and D, using the information contained in Table 24 on the inclusion of maximal tori. Except if $D =_G R_{10}$, it turns out that $C_{\mathbf{G}}(s)$ and $C_{\mathbf{G}}(D^\dagger)$ have a unique common F-stable maximal torus $\mathbf{S}^\dagger$, hence $\mathbf{K}^\dagger = C_{\mathbf{G}}(D^\dagger) \cap C_{\mathbf{G}}(s) = \mathbf{S}^\dagger$. In the other cases, we have $s \in Z(C_{\mathbf{G}}(D^\dagger))$ so that $\mathbf{K}^\dagger = C_{\mathbf{G}}(D^\dagger)$. In any case, the relative Weyl group $W_{C_{\mathbf{G}}(s)}(\mathbf{K}^\dagger)^F$ and hence the isomorphism type of $\mathrm{Out}_G(R, b_R)$ is easy to determine using standard methods.

We now show that θ_D extends to $N_G(D, b_D)$. If $D \in_G \{R_{11}, R_{12}\}$, this is clear as $\mathrm{Out}_G(D, b_D)$ is cyclic. If $D =_G R_{18}$, the assertion follows from [74, Theorem 1.1] and, in the remaining cases, from Propositions 4.18 and 4.19. This completes the proof for abelian defect groups.

We now proceed to prove our claims for non-abelian defect groups. Suppose that b is not the principal block and that D is non-abelian. Then $D \in_G \{R_{29}\text{–}R_{34}\}$ by Tables 1–19. It suffices to consider the cases $D \in_G \{R_{29}, R_{31}, R_{33}\}$, as the other cases are proved in an analogous way. For each such D, we discuss the possibilities for R determined in

Lemma 6.21 case by case. If $C_{\mathbf{G}}(s)$ is a regular subgroup of $\mathbf{G}$, then, by Tables 1–19, the block b' of $M := C_G(s)^\dagger$ corresponding to b by Theorem 3.9(a) is of the form $\hat{s} \otimes b_0$, where b_0 is the principal block of M, so that we may apply Lemma 3.12, which gives the extendibility of θ_R to $N_G(R, b_R)$.

Suppose first that $D =_G R_{29}$. Then s is of class type $(13, 4)$ or $(13, 5)$, according as $\varepsilon = -1$ or $\varepsilon = 1$, respectively. By Lemma 6.21 we have $\mathcal{W}(R, b_R) = 0$ or $R \in_G \{R_{19}, D\}$. We apply Lemma 3.12 with the subgroup $\mathbf{M} =_G \mathbf{M}_{17,4}$ respectively $\mathbf{M} =_G \mathbf{M}_{17,5}$, so that $\mathbf{M}^\dagger$ is G-conjugate to $C_{\mathbf{G}}(s)$. If $R =_G R_{19}$, Lemma 3.12 yields $\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_M(R) \cong \mathrm{SL}_2(3)$. Similarly, the case $R = D$ yields $\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_M(R) = 2$.

Next, assume that $D =_G R_{31}$. Then the class type of s is one of

$$\{(6, 1), (6, 2), (8, 2), (8, 3), (13, 2), (13, 3)\}.$$

By Lemma 6.21, we are left with the possibilities

$$R \in_G \{R_{16}, R_{19}\, (a = 1), R_{23}\, (a \geq 2), D\}. \tag{19}$$

Assume that s is of class type $(13, 2)$ or $(13, 3)$, according as $\varepsilon = 1$ or -1, respectively. To apply Lemma 3.12, we take $\mathbf{M} =_G \mathbf{M}_{17,3}$, respectively $\mathbf{M} =_G \mathbf{M}_{17,2}$. In case $R =_G R_{16}$ we get $\mathrm{Out}_G(R, b_R) \cong \mathrm{Out}_M(R) \cong S_3$, and thus $\mathcal{W}(R, b_R) = 0$. In the other cases, Lemmas 3.12 and 6.3, and the considerations on central products in Subsection 2.9 yield the corresponding entries in Table 13. Now suppose that s is of class type $(6, 1)$ or $(6, 2)$, according as $\varepsilon = -1$ or $\varepsilon = 1$, respectively. We then choose $\mathbf{M}$ as $\mathbf{M} =_G \mathbf{M}_{10,1}$, respectively $\mathbf{M} =_G \mathbf{M}_{10,2}$. Let $Q \leq D$ denote the abelian normal subgroup of D of index 3 considered in Lemma 6.9. Then $Q =_G R_{16}$, and θ_Q is an extension of θ_D to $C_M(Q) = S$, where $\mathbf{S}$ is a maximal e-split torus of $\mathbf{M}$. We obtain $\mathrm{Out}_G(Q, b_Q) = \mathrm{Out}_M(Q) \cong W(C_3)$. If R is non-abelian, we proceed as follows. Notice that D is conjugate to a Sylow 3-subgroup of K, where $\mathbf{K} \leq \mathbf{M}$ is a regular subgroup conjugate to $\mathbf{M}_{17,3}$ if $\varepsilon = 1$, respectively $\mathbf{M}_{17,2}$, if $\varepsilon = -1$; see Table 24. Then $RC_G(R) \leq K \leq M$ by Table 26. As M contains an element which induces the graph automorphism of K and normalizes R, we also have $N_G(R) \leq M$, hence $\mathrm{Out}_M(R) = \mathrm{Out}_G(R)$, and thus the required entries in Table 6.

Now suppose that s is of class type $(8, 2)$ if $\varepsilon = 1$, respectively $(8, 3)$, if $\varepsilon = -1$. In these cases we choose $\mathbf{M} =_G \mathbf{M}_{10,2}$, respectively $\mathbf{M} =_G \mathbf{M}_{10,1}$, as the regular subgroup of $\mathbf{G}$ minimal with the property that $\mathbf{M}^\dagger$ contains $C_{\mathbf{G}}(s)$. Let $b' \subseteq \mathcal{E}_3(M, s)$ denote the block of M corresponding to b via Theorem 3.9(a). By Theorem 3.9(b) we have $\mathrm{Out}_G(R, b_R) = \mathrm{Out}_M(R, b'_R) \leq \mathrm{Out}_M(R)$. The latter has already been determined in the previous paragraph. We let $\mathbf{K} \leq \mathbf{M}$ be as in that paragraph and assume that $D \leq K$. Suppose that $R_{16} =_G Q \leq D$. Using Table 24, we find that $C_{\mathbf{K}}(Q) = C_{\mathbf{M}}(Q) = \mathbf{S}$, a maximal torus of type $(20, 12)$ if $\varepsilon = 1$, and $(20, 13)$, if $\varepsilon = -1$. By the description of the centralizers in Table 26, this implies that $C_{\mathbf{K}}(R) \leq \mathbf{S}$ and hence $C_M(R) \leq S$ for all $R \leq D$ with R occurring in (19). In turn, θ'_R is the restriction of $\hat{s}$ to $C_M(R)$, where $\hat{s} \in \mathrm{Irr}(S)$ corresponds to $s \in S^*$ via duality. Once more by Table 24, we may assume that s is a central element of $\mathbf{K}^\dagger$, where $\mathbf{K}^\dagger$ is a suitable regular subgroup dual to $\mathbf{K}$. Now $\hat{s}$ extends to a linear character of K, as $s \in Z(K^\dagger)$; see [42, Proposition 2.5.20]. In particular, $\hat{s}$ is invariant in $\mathrm{Out}_K(R)$. Observe that $\mathrm{Out}_K(R)$ has index 2 in $\mathrm{Out}_M(R)$ for all R occurring in (19). We may assume that R is invariant under the graph automorphism of K induced by M, and we claim that $\hat{s}$ is not invariant under this automorphism. This will then give the desired values for $\mathrm{Out}_M(R, \hat{s})$. By Corollary 6.15, we have $\mathrm{Out}_M(Q, \hat{s}) = W(A_3)$ for $Q =_G R_{16}$, by Corollary 6.15. This implies the claim.

If $R =_G R_{16}$, the extendibility of θ_R follows from Proposition 4.19. In the other cases, the Schur multiplier of $\mathrm{Out}_G(R, b_R)$ is trivial, which gives the extendibility.

Finally, assume that $D =_G R_{33}$. Then the class type of s is one of

$$\{(2,1), (3,1), (3,2), (6,1), (6,2), (7,1), (7,2), (8,1), (8,4), (13,1), (13,6)\}.$$

By Lemma 6.13, we are left with the possibilities

$$R \in_G \{R_{16}, R_{18}, R_{25}, D\}.$$

Aiming at a contradiction, assume that $R =_G R_{16}$. Considering the various possibilities for s and using Table 24, we find that $s \in Z(C_G(R^\dagger))$, unless s is of type $(8, 1)$ or $(13, 1)$ if $\varepsilon = 1$, or of type $(8, 4)$ or $(13, 6)$ if $\varepsilon = -1$. Now $\theta_R \in \mathcal{E}(C_G(R), s)$ by Lemma 3.10 and satisfies $\theta_R(1)_3 = |C_G(R)/R|_3$. By Lemma 6.14, this excludes the cases with $s \in Z(C_G(R^\dagger))$, and shows that s lies in a torus of type $(20, 17)$ if $\varepsilon = 1$, and of type $(20, 18)$ if $\varepsilon = -1$. But then s cannot be of class type $(8, 1)$, $(13, 1)$, $(8, 4)$ nor $(13, 6)$, once more by Table 24. This contradiction shows that $R \neq_G R_{16}$.

Let $Q \leq D$ denote the abelian normal subgroup of D of index 3 considered in Lemma 6.9. Then $Q =_G R_{18}$, and θ_Q is an extension of θ_D to $C_G(Q) = T$, the maximal e-split torus of G. If $R = Q$, we obtain $\mathrm{Out}_G(R, b_R)$ from Corollary 6.15, and the extendibility of θ_R from [74, Theorem 1.1]. We are thus left with the cases $R \in_G \{R_{25}, D\}$.

If the class type of s is one of $(6, 1)$, $(6, 2)$, $(8, 1)$, $(8, 4)$, $(13, 1)$ or $(13, 6)$, we proceed exactly as in the corresponding cases for $R =_G R_{31}$.

The case that s is of class type $(7,1)$ or $(7,2)$ can also be dealt with analogously.

Next, assume that s is of class type $(2,1)$. Then q is odd. We choose s in the center of the standard e-split Levi subgroup $\mathbf{K}^\dagger$ of type $\mathbf{M}_{13,1}$ respectively $\mathbf{M}_{13,6}$ according as $\varepsilon = 1$ or -1, and we let $\mathbf{K}$ denote the standard e-split Levi subgroup of type $\mathbf{M}_{17,1}$, respectively $\mathbf{M}_{17,6}$, dual to $\mathbf{K}^\dagger$. Notice that $\mathbf{K}$ and $\mathbf{K}^\dagger$ are subgroups of $\mathbf{L} = C_{\mathbf{G}}(z_{\mathrm{C}}) = \mathbf{L}^1 \circ_3 \mathbf{L}^2$; see Proposition 4.14. In fact, $\mathbf{L}^1 = [\mathbf{K}^\dagger, \mathbf{K}^\dagger]$ and $\mathbf{L}^2 = [\mathbf{K}, \mathbf{K}]$. Let $\mathbf{T}$ denote the standard e-split maximal torus of $\mathbf{L}$, and put $\mathbf{T}_i = \mathbf{T} \cap \mathbf{L}^i$, for $i = 1, 2$. Then $T_1 = Z(K)$ and $T_2 = Z(K^\dagger)$. As $s \in Z(K^\dagger)$, we have $T_2 \in \ker(\hat{s})$ by Lemma 6.17, and as T_1T_2 has index 3 in T we may view $\hat{s}$ as an element of $\mathrm{Irr}(T_1)$. As $\hat{s}$ has order 2, it extends to a linear character of $K = T_1 \circ \langle L^2, x_{\mathrm{C}} \rangle$, with $\langle L^2, x_{\mathrm{C}} \rangle$ in its kernel; see Subsection 2.9. Also $s \in T_2 = Z(K^\dagger) \cong [q-\varepsilon]^2$ may be taken to be the element corresponding to $(-1,-1)$ under the latter isomorphism. By duality, $\hat{s} \in \mathrm{Irr}(T_1) = [q-\varepsilon]^2$ corresponds to the linear character sending a generator of each cyclic factor of $[q-\varepsilon]^2$ to -1.

We adopt the notation of Subsections 6.1 and 6.2, as well as that of Lemma 6.3; the subgroups and elements of $\mathrm{SL}_3^\varepsilon(q)$ introduced in Subsection 6.2 are provided with an index i here, indicating their association to L^i, $i = 1, 2$; see also Subsection 9.7. By [6, Tables 6–8], we have $N_G(R) = N_{L.2}(R)$ with $L.2 = \langle L, \gamma_{\mathrm{C}} \rangle$. (Recall that $R = D$ or $R =_G R_{25}$ at this stage of the proof.) If $R = D$, then

$$N_{L.2}(D) = (\langle T_1, t_1 \rangle \circ_3 \langle D_2, t_2 \rangle).\langle x_C, \gamma_C \rangle, \tag{20}$$

and thus $N_G(D)$ stabilizes $\hat{s}$ (recall that D_2 denotes a particular Sylow 3-subgroup of L^2). Moreover, $\hat{s}$ extends to $N_G(D) \leq N_G(Q)$, as $\hat{s}$ extends to $N_G(Q)$. Now assume that $R =_G R_{25}$ and that $a \geq 2$. Then

$$N_{L.2}(R) = (T_1.\langle t_1, c_1 \rangle \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)).\langle \gamma_C \rangle, \tag{21}$$

and thus

$$N_G(R, b_R) = (T_1.\langle t_1 \rangle \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)).\langle \gamma_C \rangle,$$

as c_1 does not stabilize $\hat{s}$. It follows that $\mathrm{Out}_G(R, b_R) = (2 \times \mathrm{SL}_2(3)).2$. Clearly, $\hat{s}$ extends to $T_1.\langle t_1 \rangle \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)$ such that $3_+^{1+2}.\mathrm{SL}_2(3)$ is in the kernel of this extension and $\langle \gamma_C \rangle$ stabilizes this extension. It follows that $\theta_R = \hat{s}$ extends to $N_G(R, b_R)$. If $R =_G R_{25}$ and $a = 1$, we have

$$N_{L.2}(R) = (T_1.\langle t_1 \rangle \circ_3 3_+^{1+2}.Q_8).\langle x_C, \gamma_C \rangle, \tag{22}$$

and thus $N_G(R, b_R) = N_G(R)$, and $\mathrm{Out}_G(R, b_R) = (2 \times \mathrm{SL}_2(3)).2$. As $\langle x_C, \gamma_C \rangle \cong S_3$ has trivial Schur multiplier, $\theta_R = \hat{s}$ extends to $N_G(R, b_R)$. Finally assume that s is of type $(3,1)$ if $\varepsilon = 1$ or of type $(3,2)$, if $\varepsilon = -1$. In the former case, $4 \mid q-1$ and s is contained in the center of a group of type $\mathbf{M}_{7,1}$. In the latter case, $4 \mid q+1$ and s is contained in the center of a group of type $\mathbf{M}_{7,2}$. As above, we may view $\hat{s}$ as a linear character of $M_{9,1}$, respectively $M_{9,2}$, i.e. of a central product $\mathrm{GL}_2^\varepsilon(q) \circ \langle L^2, x_{\mathrm{C}}\rangle$, such that $\langle L^2, x_{\mathrm{C}}\rangle$ is in the kernel of $\hat{s}$. As $|s| = 4$, the linear character $\hat{s}$ of $\mathrm{GL}_2^\varepsilon(q) \leq L^1$ has order 4, and is fixed by t_1. On the other hand, $\hat{s}$ is not fixed by γ_C, which induces the inverse-transpose automorphism on $\mathrm{GL}_2^\varepsilon(q)$. The claims for $R = D$ and $R =_G R_{25}$ now follow from (20), (21) and (22). □

Corollary 7.2. *The non-cyclic weight subgroups of G are exactly the following (up to conjugation):*

$$\{R_8\text{–}R_{12}, R_{15}\text{–}R_{26}, R_{29}\text{–}R_{38}\}.$$

Of these, R_{15}, R_{21}, R_{22} and R_{35}–R_{38}, occur only as weight subgroups of the principal block.

We finally deal with the exceptional double cover of $F_4(2)$.

Proposition 7.3. *Let $G := F_4(2)$ and $\hat{G} := 2.G$, the exceptional double cover of G. Let $\ell \in \{3, 5, 7\}$, and let b be an ℓ-block of $\hat{G}$ with a non-cyclic defect group containing faithful characters. Then the invariants of b given in* Columns 8–11 *of* Table 21 *are correct.*

Proof. Assume first that $\ell = 3$. We identify the 3-subgroups of $\hat{G}$ and G via the canonical map $\hat{G} \to G$. A 3-subgroup R of $\hat{G}$ is radical, if and only if its image in G is radical. The radical 3-subgroups of G are given in Table 26.

We use the permutation representation of $\hat{G}$ on 139 776 points from Wilson's Atlas of Finite Group Representations [80] to compute in $\hat{G}$ with GAP, and thus determine $N_{\hat{G}}(R)/R$ for each radical 3-subgroup R of $\hat{G}$. We find $N_{\hat{G}}(R)/R = 2^4$, $2 \times \mathrm{GL}_2(3)$, $2 \times \mathrm{SL}_3(3)$, $2 \times W(F_4)$, $2^{\cdot}[(Q_8 \times Q_8){:}\,S_3]$ and $2 \times D_8$ for $R =_G R_{38}$, R_{37}, R_{15}, R_{18}, R_{21} and R_{10}, respectively. As $2^{\cdot}[(Q_8 \times Q_8){:}\,S_3]$ has exactly two faithful defect zero character, we obtain the numbers in Column 11.

In case $\ell > 3$, the defect group of b is a Sylow ℓ-subgroup of $\hat{G}$, and the invariants are easily computed with GAP. □

8. The action of the automorphism group on the set of weights

Let p, $q = p^f$, $\mathbf{G}$ and F be as in Subsection 4.1, so that $\mathbf{G}^F = G = F_4(q)$ is the simple Chevalley group of type F_4. Let ℓ be an odd prime with $\ell \neq p$. We aim to prove the inductive blockwise Alperin weight condition for G at ℓ; see Subsection 2.12.

Recall that $F = F_1^f$ for a Steinberg morphism of $\mathbf{G}$ with $\mathbf{G}^{F_1} = F_4(p)$. Let σ_1 be the Steinberg morphism of $\mathbf{G}$ defined in Subsection 4.5 (i.e. $\sigma_1 = F_1$ if p is odd, and $\sigma_1^2 = F_1$ if $p = 2$). In this section, our reference torus $\mathbf{T}_0$ is assumed to be σ_1-stable and contained in a σ_1-stable Borel subgroup of $\mathbf{G}$.

8.1. **Preliminaries.** As in Subsection 5.11, we put $f' = f$ if p is odd, and $f' = 2f$ if $p = 2$. Then $F = \sigma_1^{f'}$ and $\mathrm{Out}(G) = \langle G\sigma_1 \rangle$ is cyclic of order f'. Unless $q = 2$, the Schur multiplier of G is trivial, and hence any ℓ'-covering group of G is equal to G. Thus, by Remark 2.14, it suffices to verify the conditions of Hypothesis 2.13 for every ℓ-block of G. An equivalent set of conditions is formulated in Hypothesis 2.15. The case of the exceptional covering group of $F_4(2)$ will be dealt with separately.

Let b denote an ℓ-block of G. We may and will assume that b has non-cyclic defect groups, as the inductive blockwise Alperin weight condition is known to hold for blocks with cyclic defect groups by the work of Koshitani and Späth [54]. In Theorem 7.1, we have established the blockwise Alperin weight condition, i.e. we have already verified Hypothesis 2.15(1) for b. It remains to settle the equivariance condition Hypothesis 2.15(2). As this is trivial if b has a unique irreducible Brauer character, we will also assume $|l(b)| > 1$.

Put $A := N_{\mathrm{Aut}(G)}(b)$. Then $A = G \rtimes \langle \sigma \rangle$ for some $\sigma = \sigma_1^{m'}$, where m' is a positive integer with $m' \mid f'$. In fact $\langle \sigma \rangle = \mathrm{Stab}_{\langle \sigma_1 \rangle}(b)$. Even more can be said. Suppose that $s \in G$ is a semisimple ℓ'-element such that $b \subseteq \mathcal{E}_\ell(G, s)$. Then, except in case $\ell \mid q^2 + 1$ the block b is uniquely determined in $\mathcal{E}_\ell(G, s)$ by its defect group. Thus, unless $\ell \mid q^2 + 1$, as a consequence of (14) in Proposition 5.13, we have $\mathrm{Stab}_{\langle \sigma_1 \rangle}(b) = \mathrm{Stab}_{\langle \sigma_1 \rangle}([s])$, where $[s]$ denotes the G-conjugacy class of s.

Notice that every element of $A \setminus G$ naturally extends to a Steinberg morphism of $\mathbf{G}$ which commutes with F. Indeed, an element of $A \setminus G$ is of the form $\mathrm{ad}_g \circ \sigma^j$ for some $g \in G$ and some positive integer j, where σ is tacitly assumed to be restricted to G. We may now view $\mathrm{ad}_g \circ \sigma^j$ as a Steinberg morphism of $\mathbf{G}$. In particular, we may view A as a subgroup of $\mathrm{Aut}_1(\mathbf{G})$.

For easy reference, we list a set of recurring assumptions.

Hypothesis 8.2. Let b be an ℓ-block of G with non-cyclic defect groups and with $l(b) > 1$.

Put $A = N_{\mathrm{Aut}(G)}(b)$ and define $\sigma \in \langle \sigma_1 \rangle$ by $A = G \rtimes \langle \sigma \rangle$. Let m' be the positive integer with $m' \mid f'$ and $\sigma = \sigma_1^{m'}$. Following the conventions of Subsection 5.11, we put $m := m'/2$ if m' and p are even and $m := m'$, otherwise. Then $\sigma = F_1^m$ if either m' and p are even or if p is odd. If $p = 2$ and m' is odd, we have $m = m'$ and $m \mid f$. In this case $\sigma^2 = F_1^m$; see Subsection 5.11.

Let s be a σ-stable semisimple ℓ'-element such that $b \subseteq \mathcal{E}_\ell(G, s)$. (Such an s exists by Proposition 5.13 and Lemma 5.12.) □

Let D be a defect group of b. According to Hypothesis 2.15, we have to consider $N_A(D)$. If D is σ-invariant then $N_A(D) = N_G(D) \rtimes \langle\sigma\rangle$, a case which is somewhat easier to handle. To deal with the general case, observe that $N_A(D)/N_G(D)$ embeds into $\langle\sigma\rangle$, and hence $N_A(D) = \langle N_G(D), \omega\rangle$ for some $\omega \in N_A(D)$. In particular, $C_{\mathbf{G}}(D)$ is ω-invariant. The next lemma explores this situation.

Lemma 8.3. *Assume* Hypothesis 8.2 *and let D be a defect group of b. Let $\sigma' \in A$ be such that $A = G \rtimes \langle\sigma'\rangle$. Suppose that $\mathbf{M}$ is a regular subgroup of $\mathbf{G}$ such that D is the Sylow ℓ-subgroup of $Z(M)$. In particular, D is abelian. Then $N_G(D) = N_G(\mathbf{M}) = N_G(M)$ and $N_A(D) = N_A(\mathbf{M}) = N_A(M)$.*

Let $\mathbf{M}_0$ denote a σ_1-stable regular subgroup of $\mathbf{G}$ containing $\mathbf{T}_0$. Suppose that $\mathbf{M} = \mathbf{M}_0^g$ for some $g \in \mathbf{G}$ such that $\dot{w} := F(g)g^{-1}$ is σ'-stable and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0) \cap N_{\mathbf{G}}(\mathbf{M}_0)$.

Then ad_g induces an isomorphism between $N_A(M)$ and $N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}} \rtimes \langle\sigma'\rangle$. In particular

$$N_A(M) = N_G(M) \rtimes \langle\omega\rangle$$

with $\omega = \mathrm{ad}_g \circ \sigma' \circ \mathrm{ad}_g^{-1}$. Hence $N_A(D) = \langle N_G(D), \omega\rangle$.

Proof. As $C_{\mathbf{G}}(D)$ is a regular subgroup of $\mathbf{G}$ by Lemma 6.7 and [62, Proposition 2.3(a)], we have $C_{\mathbf{G}}(D) = \mathbf{M}$ and thus obtain $N_G(D) = N_G(\mathbf{M}) = N_G(M)$. As $\mathbf{G}$ is normalized by A and the elements of A commute with F, we also get $N_A(M) \leq N_A(D) \leq N_A(C_{\mathbf{G}}(D)) = N_A(\mathbf{M}) \leq N_A(M)$.

Put $\mathbf{A} := \mathbf{G} \rtimes \langle\sigma\rangle = \mathbf{G} \rtimes \langle\sigma'\rangle \leq \mathrm{Aut}_1(\mathbf{G})$. Observe that $\mathbf{A}$ is normal in $\mathrm{Aut}_1(\mathbf{G})$ and that A consists exactly of those elements of $\mathbf{A}$ which commute with F. Hence

$$\begin{aligned} N_A(\mathbf{M}) &= \{\omega \in N_{\mathbf{A}}(\mathbf{M}) \mid \omega \text{ commutes with } F\} \\ &= \{\mathrm{ad}_g^{-1} \circ \omega_0 \circ \mathrm{ad}_g \mid \omega_0 \in N_{\mathbf{A}}(\mathbf{M}_0), \omega_0 \text{ commutes with } F\dot{w}\}. \end{aligned}$$

As σ' stabilizes $\mathbf{M}_0$, we have $N_{\mathbf{A}}(\mathbf{M}_0) = N_{\mathbf{G}}(\mathbf{M}_0) \rtimes \langle\sigma'\rangle$. As σ' commutes with $F\dot{w}$, an element $\mathrm{ad}_h \circ (\sigma')^j \in N_{\mathbf{G}}(\mathbf{M}_0) \rtimes \langle\sigma'\rangle$ with $h \in N_{\mathbf{G}}(\mathbf{M}_0)$ and $j \in \mathbb{Z}$ commutes with $F\dot{w}$, if and only if $h \in N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}}$.
□

In case A acts trivially on $\mathrm{IBr}(b)$, we have to show that $N_A(D)$ fixes all the conjugacy classes of weights associated to b. In the next subsection, we prepare for the necessary arguments to establish these results.

8.4. **Preparations.** Assume Hypothesis 8.2. Below, we will make use of the notation introduced in Subsection 2.10. A centric b-Brauer pair (R, b_R) is called *relevant*, if $\mathcal{W}(R, b_R) \neq 0$. In this case we say that the radical ℓ-subgroup R of G is b-*relevant*. This notion is also used for G-conjugacy classes of b-Brauer pairs and radical ℓ-subgroups of G. We first have the following easy observation, which is the basis of our proof.

Lemma 8.5. *Let (R, b_R) be a relevant b-Brauer pair and suppose that $\omega \in A$ stabilizes (R, b_R). If, in addition, ω fixes every element in $\mathrm{Irr}^0(N_G(R, \theta_R) \mid \theta_R)$, then ω stabilizes each b-weight (R, φ) with φ lying above θ_R.*

Recall that by Theorem 7.1(b) *the canonical character θ_R of b_R extends to $N_G(R, b_R)$. If ω stabilizes one such extension, as well as each $\xi \in \mathrm{Irr}^0(\mathrm{Out}_G(R, b_R))$, then ω fixes every element in* $\mathrm{Irr}^0(N_G(R, \theta_R) \mid \theta_R)$.

Proof. This is straightforward. Since ω stabilizes (R, b_R), it also stabilizes $C_G(R)$, $N_G(R)$, θ_R and $N_G(R, \theta_R) = N_G(R, b_R)$. If (R, φ) is a b-weight with φ lying above θ_R, then $\varphi = \mathrm{Ind}_{N_G(R,\theta_R)}^{N_G(R)}(\zeta)$ for some $\zeta \in \mathrm{Irr}^0(N_G(R, \theta_R) \mid \theta_R)$. By assumption, ζ, and hence φ is fixed by ω.

Let $\hat{\theta}_R$ denote an extension of θ_R to $N_G(R, \theta_R)$. The elements of $\mathrm{Irr}^0(N_G(R, \theta_R) \mid \theta_R)$ are of the form $\hat{\theta}_R\xi$ for $\xi \in \mathrm{Irr}^0(\mathrm{Out}(R, b_R))$, and thus fixed by ω, if $\hat{\theta}_R$ and ξ are ω-stable. □

In the following lemma we use the classification of the defect groups and their centralizers; see Tables 1–19 and 26.

Lemma 8.6. *Let D be a defect group of b. Assume that $|D|$ equals the ℓ-part of $|C_G(s)|$. Let $\omega \in A$ be such that $N_A(D) = \langle N_G(D), \omega\rangle$. If D is abelian, put $Q := D$, otherwise let Q denote the maximal abelian normal subgroup of D exhibited in* Lemma 6.9.

Let $\mathbf{M} := C_{\mathbf{G}}(Q)$. Then $\mathbf{M}$ is an ω-stable regular subgroup of $\mathbf{G}$. Suppose that $\mathbf{M}^\dagger \leq \mathbf{G}$ is a regular subgroup with $s \in \mathbf{M}^\dagger$ and such that $(\mathbf{M}, F)$ and $(\mathbf{M}^\dagger, F)$ are in duality.

View ω as a Steinberg morphism of $\mathbf{G}$ in the natural way; then ω commutes with F. Suppose further that $\omega^\dagger$ is a Steinberg morphism of $\mathbf{G}$ which commutes with F, fixes $\mathbf{M}^\dagger$ and s, and that $(\mathbf{M}, \omega)$ and $(\mathbf{M}^\dagger, \omega^\dagger)$ are in duality.

Then there is a maximal b-Brauer pair (D, b_D) with $\omega(b_D) = b_D$; moreover, $b_D \subseteq \mathcal{E}_\ell(C_G(D), s)$ if D is abelian.

PROOF. As $(\mathbf{M}, \omega)$ and $(\mathbf{M}^\dagger, \omega^\dagger)$ are in duality, $\omega|_{\mathbf{M}}$ and $\omega^\dagger|_{\mathbf{M}^\dagger}$ are dual isogenies in the sense of [25, Proposition 11.1.11]. As s is $\omega^\dagger$-stable and $\mathbf{M}$ is ω-stable, [24, Corollary 9.3(ii)] implies that $\mathcal{E}(M, s)$ is ω-stable, too.

By the description of the defect groups in Tables 1–19, either $\mathbf{M}$ is a maximal torus, or else $e \in \{1, 2\}$ and $\mathbf{M}$ is an e-split Levi subgroup of $\mathbf{G}$ which is G-conjugate to one of $\mathbf{M}_{i,k}$ with $k \in \{13, 14, 15, 17, 18, 19\}$ (and k is such that $\mathbf{M}_{i,k}$ is e-split). Notice that the cases $i = 13, 17$ only occur for $\ell > 3$.

Let (D, b'_D) denote a maximal b-Brauer pair, and let (Q, b'_Q) be the b-Brauer pair with $(Q, b'_Q) \leq (D, b'_D)$. If $t \in M^\dagger$ such that $b'_Q \subseteq \mathcal{E}_\ell(M, t)$, then s and t are conjugate in G by Lemma 3.10. Now consider $\mathbf{K}^\dagger := C_{\mathbf{M}^\dagger}(t)$.

By going through the various cases for s and $\mathbf{M}^\dagger$, and using Table 24, we find that any two F-stable maximal tori of $\mathbf{M}^\dagger$ which have some G-conjugate in $C_{\mathbf{G}}(t)$ are conjugate in $M^\dagger$. In particular, $\mathbf{K}^\dagger$ is a maximal torus of $\mathbf{M}^\dagger$, and $t \in K^\dagger$ is in general position with respect to $M^\dagger$. Notice that the class type of t determines the type of $\mathbf{K}^\dagger$. As s and t are conjugate in G, it follows that $\mathbf{S}^\dagger := C_{\mathbf{M}^\dagger}(s)$ is an F-stable maximal torus of $\mathbf{M}^\dagger$, and $\mathbf{K}^\dagger$ is conjugate in $M^\dagger$ to $\mathbf{S}^\dagger$. Conjugation inside $M^\dagger$ does not affect $\mathcal{E}_\ell(M, t)$, so that we may assume that $t \in \mathbf{S}^\dagger$. Choose an F-stable maximal torus $\mathbf{S}$ of $\mathbf{M}$ dual to $\mathbf{S}^\dagger$. Now $\theta_t := \pm R_{\mathbf{S}}^{\mathbf{M}}(\hat{t}) \in \mathcal{E}(M, t)$ is a character of M of central defect, and $\operatorname{Res}^M_{C_G(D)}(\theta_t)$ is the canonical character of b'_D. This follows from Lemma 6.14 in case $\ell = 3$, but the case $\ell > 3$ is proved analogously. Put $\theta_s := \pm R_{\mathbf{S}}^{\mathbf{M}}(\hat{s}) \in \mathcal{E}(M, s)$, let $b_Q \subseteq \mathcal{E}_\ell(M, s)$ denote the ℓ-block of M containing θ_s and let b_D denote the ℓ-block of $C_G(D)$ containing $\operatorname{Res}^M_{C_G(D)}(\theta_s)$. Then (D, b_D) is a centric Brauer pair with $(Q, b_Q) \leq (D, b_D)$. As θ_s is the unique character in $\mathcal{E}(M, s)$, it follows that (Q, b_Q) is ω-invariant. In turn (D, b_D) is ω-invariant.

Notice that $\operatorname{Out}_G(D, b'_D) = \operatorname{Out}_G(Q, b'_Q)$ and $\ell \nmid |\operatorname{Out}_G(D, b'_D)|$, as (D, b'_D) is a maximal b-Brauer pair. We also have $\operatorname{Out}_G(D, b_D) = \operatorname{Out}_G(Q, b_Q) \leq \operatorname{Out}_G(M)$. To prove that (D, b_D) is a b-Brauer pair, we claim that $\ell \nmid |\operatorname{Out}_G(D, b_D)|$. This is clear if $\mathbf{M}$ is not a torus or if $\ell > 3$, as $\ell \nmid |\operatorname{Out}_G(M)|$ in these cases. Suppose that $\ell = 3$ and that $\mathbf{M} = \mathbf{S}$ is a torus. The argument in the proof of Proposition 3.16(a) gives $\operatorname{Out}_G(Q, b_Q) \cong W_{C_{\mathbf{G}}(s)}(\mathbf{S}^\dagger)$ and $\operatorname{Out}_G(Q, b'_Q) \cong W_{C_{\mathbf{G}}(t)}(\mathbf{S}^\dagger)$. Now s and t are conjugate in $W_{\mathbf{G}}(\mathbf{S}^\dagger)$; see [22, Proposition 3.7.1]. If $\mathbf{S}^\dagger$ is the maximally e-split torus of $\mathbf{G}$, then $W_{\mathbf{G}}(\mathbf{S}^\dagger) = W_{\mathbf{G}}(\mathbf{S}^\dagger)^F$, and thus

$W_{C_{\mathbf{G}}(s)}(\mathbf{S}^\dagger) = \{w \in W_{\mathbf{G}}(\mathbf{S}^\dagger) \mid s^w = s\}$ and $W_{C_{\mathbf{G}}(t)}(\mathbf{S}^\dagger)$ are conjugate in G. The only case remaining is when $Q \in_G \{R_{11}, R_{12}\}$, and hence s is of class type $(13,4)$, $(13,5)$, $(17,4)$, $(17,5)$, $(18,3)$, $(18,7)$, $(19,8)$ or $(19,9)$. In these cases, $W_{C_{\mathbf{G}}(s)}(\mathbf{S}^\dagger)$ has order 2, as $S^\dagger = M^\dagger \cong [q-\varepsilon] \times [q^3-\varepsilon]$ in these cases. This proves our claim.

Now let b' denote the ℓ-block of G such that $(1, b') \leq (Q, b_Q)$. By Lemma 3.10, we have $b' \subseteq \mathcal{E}_\ell(G, s)$. As b' is the Brauer correspondent of b_D, a defect group of b' equals D. As b is the unique block in $\mathcal{E}_\ell(G,s)$ with defect group D, it follows that $b = b'$. □

8.7. **Proofs for the non-unipotent blocks.** Let b, σ and s be as in Hypothesis 8.2. In addition, suppose that b is non-unipotent, and that $\sigma = F_1^m$. In this subsection we also put $G_m := \mathbf{G}^\sigma$.

Let us outline the strategy of proof in the generic case, namely when the order of a defect group of b equals the ℓ-part of $|C_G(s)|$. Recall that this condition holds exactly when the entry corresponding to b in Column 5 of Tables 1–19 equals 1. Starting from s and σ, we construct a pair of regular subgroups $\mathbf{M}$ and $\mathbf{M}^\dagger$ of $\mathbf{G}$ such that the Sylow ℓ-subgroup D of $Z(M)$ is a defect group of b. Simultaneously, we construct a pair ω and $\omega^\dagger$ of Steinberg morphisms of $\mathbf{G}$ such that $N_A(D) = \langle N_G(D), \omega \rangle$, and such that the hypotheses of Lemma 8.6 are satisfied. This yields an ω-stable maximal b-Brauer pair (D, b_D). We then have to investigate the action of ω on $N_G(D, b_D)$ and $\mathrm{Out}_G(D, b_D)$ in order to apply Lemma 8.5. This is done by describing $\mathrm{Out}_G(D, b_D)$ in terms of suitable structures inside the Weyl group W. It turns out that, in most of the cases, ω acts trivially on $\mathrm{Out}_G(D, b_D)$.

To construct $\mathbf{M}$ and $\mathbf{M}^\dagger$, we exhibit a pair $y, y^* \in \mathbf{G}$ such that $\dot{u} := F(y)y^{-1}$ and $\dot{u}^* := F(y^*){y^*}^{-1}$ normalize $\mathbf{T}_0$, and that the images u and u^* of $\dot{u}$, respectively $\dot{u}^*$ in W are related by $u^* = {u^\dagger}^{-1}$. We also choose a suitable element $v \in W$ which determines the G_m-class type of s. Putting $\mathbf{S} := \mathbf{T}_0^y$ and $\mathbf{S}^\dagger := \mathbf{T}_0^{y^*}$, and defining ω and $\omega^\dagger$ accordingly, we obtain a pair of commutative diagrams

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\ \omega\ } & \mathbf{S} \\ {\scriptstyle \mathrm{ad}_y}\uparrow & & \uparrow{\scriptstyle \mathrm{ad}_y} \\ \mathbf{T}_0 & \xrightarrow[\ \sigma\dot{v}\ ]{} & \mathbf{T}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{S}^\dagger & \xrightarrow{\ \omega^\dagger\ } & \mathbf{S}^\dagger \\ {\scriptstyle \mathrm{ad}_{y^*}}\uparrow & & \uparrow{\scriptstyle \mathrm{ad}_{y^*}} \\ \mathbf{T}_0 & \xrightarrow[\ \sigma\dot{v}\ ]{} & \mathbf{T}_0 \end{array}$$

where $\dot{v} \in N_{\mathbf{G}}(\mathbf{T}_0)$ is a suitable lift of v (whose choice only becomes relevant in the diagrams below). Notice that the F-stable maximal tori $\mathbf{S}$ and $\mathbf{S}^\dagger$ are in duality by the facts summarized in Subsection 4.12. We choose our data such that the Sylow ℓ-subgroup D of S is a defect group of b, and that $s \in S^\dagger$ is $\omega^\dagger$-stable. Let $D^\dagger$ denote the Sylow ℓ-subgroup of $S^\dagger$ and put $\mathbf{M} := C_{\mathbf{G}}(D)$ and $\mathbf{M}^\dagger := C_{\mathbf{G}}(D^\dagger)$. We obtain the following pair of commutative diagrams

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{\ \omega\ } & \mathbf{M} \\ {\scriptstyle \mathrm{ad}_y}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{ad}_y} \\ \mathbf{M}_0 & \xrightarrow[\sigma\dot{v}]{} & \mathbf{M}_0 \end{array} \qquad \begin{array}{ccc} \mathbf{M}^\dagger & \xrightarrow{\ \omega^\dagger\ } & \mathbf{M}^\dagger \\ {\scriptstyle \mathrm{ad}_{y^*}}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{ad}_{y^*}} \\ \mathbf{M}_0 & \xrightarrow[\sigma\dot{v}]{} & \mathbf{M}_0 \end{array}$$

for suitable standard Levi subgroups $\mathbf{M}_0$ and $\mathbf{M}_0^\dagger$ such that $(\mathbf{M}_0, F)$ and $(\mathbf{M}_0^\dagger, F)$ are in duality. We may now apply Lemmas 8.3 and 8.6 to find $N_A(D) = \langle N_G(D), \omega\rangle$ and to get an ω-stable b-Brauer pair (D, b_D).

To describe $N_G(D, b_D)$ and the action of ω on $\mathrm{Out}_G(D, b_D)$, let $\mathbf{L}_0$ denote a standard Levi subgroup such that $\mathbf{L}^* := \mathbf{L}_0^{y^*} = C_{\mathbf{G}}(s)$. Then $\mathbf{M}^\dagger \cap \mathbf{L}^* = \mathbf{S}^\dagger$ in our situation, and we have the following commutative diagram

$$\begin{array}{ccccccccc} N_{\mathbf{G}}(\mathbf{M})^F & \xrightarrow{\ \omega\ } & N_{\mathbf{G}}(\mathbf{M})^F & \xrightarrow{\ \kappa\ } & W_{\mathbf{G}}(\mathbf{M})^F & \xrightarrow{\ \alpha\ } & W_{\mathbf{G}}(\mathbf{M}^\dagger)^F & \xleftarrow{\ \iota\ } & W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F \\ {\scriptstyle \mathrm{ad}_y}\big\uparrow & & {\scriptstyle \mathrm{ad}_y}\big\uparrow & & {\scriptstyle \mathrm{ad}_y}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{ad}_{y^*}} & & \big\uparrow{\scriptstyle \mathrm{ad}_{y^*}} \\ N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{u}} & \xrightarrow[\sigma\dot{v}]{} & N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{u}} & \xrightarrow[\kappa]{} & W_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{u}} & \xrightarrow[\dagger]{} & W_{\mathbf{G}}(\mathbf{M}_0^\dagger)^{F\dot{u}} & \xleftarrow[\iota]{} & W_{\mathbf{L}_0}(\mathbf{T}_0)^{F\dot{u}} \end{array}$$

where α is an isomorphism, κ denotes a canonical epimorphism and ι an embedding. By Lemma 8.6, we have $N_G(D) = N_G(\mathbf{M})$. Moreover, by Proposition 3.16, the image in $W_{\mathbf{G}}(\mathbf{M}^\dagger)^F$ of $N_G(D, b_D)$ under $\alpha \circ \kappa$ equals the image of $W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F$ under ι. Thus $N_G(D, b_D) = \kappa^{-1}(\alpha^{-1}(\iota(W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F)))$. We may now transfer the computations to the bottom line of the above diagram. The inverse image of $W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F$ under ad_{y^*} equals $C_W(u^*) \cap W_{\mathbf{L}_0}(\mathbf{T}_0) \leq W$. The inverse image under $\dagger$ of the latter group equals $C_W(u) \cap W_{\mathbf{L}_0}(\mathbf{T}_0)^\dagger$. Thus $\mathrm{Out}_G(D, b_D) \cong C_W(u) \cap W_{\mathbf{L}_0}(\mathbf{T}_0)^\dagger \leq W_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{u}}$. We now have to investigate the action of $\sigma\dot{v}$ on $C_W(u) \cap W_{\mathbf{L}_0}(\mathbf{T}_0)^\dagger$ and on $\kappa^{-1}(C_W(u) \cap W_{\mathbf{L}_0}(\mathbf{T}_0)^\dagger)$.

We basically follow this outline in Proposition 8.10 below, where many more details are worked out. The proof of Proposition 8.12 also proceeds along these lines, but with fewer details. In Proposition 8.10, the element u of the above considerations has the form $v^{f/m}w$, where v and w are suitable elements from Table 23. Our approach requires the derivation of further properties of these elements, which is done in Lemma 8.9 below. The crucial group $C_W(u) \cap W_{\mathbf{L}_0}(\mathbf{T}_0)^\dagger$ corresponds to the group $\bar{C}_0$ of Definition 8.8 below. This finishes the outline of our proofs.

Recall that e denotes the order of q modulo ℓ. In accordance with our previous usage, we put $\varepsilon := 1$ if $e = 1$, and $\varepsilon := -1$ if $e = 2$. Recall that † stands for the "duality" isomorphism of W; see Subsection 4.10.

Definition 8.8. Let $\Gamma \subseteq \Sigma$, $i, k \in \mathbb{Z}$, $e \in \{1, 2\}$ and $v, w \in W$ such that some row of Table 23 has the values $(\Gamma^\dagger, (i, k), e, -, v^\dagger, -, w^\dagger, -, -)$. Put

$$\bar{C}_0 := C_{W_\Gamma}(v^{f/m}w).$$

If $e > 2$, put $\mathbf{M}_0^\dagger := \mathbf{T}_0$. If $e \leq 2$, let $\mathbf{T}_0'$ denote the subtorus of $\mathbf{T}_0$ on which $Fv^{\dagger f/m}w^\dagger$ acts as $t \mapsto t^{\varepsilon q}$ and and put $\mathbf{M}_0^\dagger := C_{\mathbf{G}}(\mathbf{T}_0')$.

Then $\mathbf{M}_0^\dagger = \mathbf{L}_{\Delta^\dagger}$ for a subset $\Delta \subseteq \Sigma^+$ which is W-conjugate to a subset of $\{\alpha_1, \ldots, \alpha_4\}$. Assume that $|\Delta| \leq 2$ and put $\mathbf{M}_0 := \mathbf{L}_\Delta$. □

We have excluded the case $|\Delta| = 3$ in the above definition, as this corresponds to cyclic defect groups. Recall the definition of W_Γ from Subsection 4.1, namely $W_\Gamma = \langle s_\alpha \mid \alpha \in \Gamma \rangle$. In particular, $W_\Gamma^\dagger = W_{\Gamma^\dagger}$. Notice that $\Gamma^\dagger$ is a base of the closed subsystem $\overline{\Gamma^\dagger}$ of Σ by the conventions adopted in Table 23 and thus $W_{\Gamma^\dagger} = W_{\overline{\Gamma^\dagger}}$. However, Γ need not be a base of $\bar{\Gamma}$, in which cases we have $W_\Gamma \lneq W_{\bar{\Gamma}}$.

Lemma 8.9. *Assume the notation of* Definition 8.8. *Recall that* $\hat{W}$ *denotes the subgroup of* $N_{\mathbf{G}}(\mathbf{T}_0)$ *generated by* $n_1, \ldots, n_4$*; see* Subsection 4.4. *Then there is a subgroup* $C_0 \leq \hat{W}$ *which maps to* $\bar{C}_0$ *under the natural epimorphism* $\hat{W} \to W$*, and the following statements hold.*

(a) *The group* C_0 *normalizes* $\mathbf{M}_0$.

(b) *Suppose that* f/m *is even or that the commutator factor group of* $\bar{C}_0$ *does not have order* 2 *or* 4. *Then* C_0 *centralizes* $[\mathbf{M}_0, \mathbf{M}_0]$ *and there are inverse images* $\dot{v}, \dot{w} \in \hat{W}$ *of* v *and* w *respectively, such that the following conditions are satisfied.*

- (i) *If* $\Delta \neq \emptyset$*, then* $\dot{v} \in [\mathbf{M}_0, \mathbf{M}_0]$ *unless* $w \neq 1$ *and* (i, k) *is one of* $(18, 6), (18, 4), (19, 6), (19, 7)$.
- (ii) *Unless* $(i, k) \in \{(12, 2), (12, 4), (16, k) \mid k \in \{3, 4, 7, 8, 10\}\}$*, the element* $\dot{v}$ *centralizes* C_0.
- (iii) *If* $w \neq 1$ *then* $\dot{w} \in Z(C_0)$.

Suppose that $w \neq 1$ *and that* $\Delta \neq \emptyset$. *Then* $e > 1$. *In these cases, the first and second column of* Table III *give the parameters* i *and* Δ*, the third and fourth column display generators for* $\bar{C}_0$ *and* C_0*, respectively.*

Suppose that $w = 1$ *and that* $\Delta \neq \emptyset$. *Then* $e = 1$. *In these cases, the first three columns of* Table IV *give the parameters* i, k, Δ*, the fourth and fifth column display generators for* $\bar{C}_0$ *and* C_0*, respectively.*

(c) *The elements of* C_0 *as well as the elements* $\dot{v}$ *and* $\dot{w}$ *constructed in* (b) *are* σ*-stable.*

TABLE III. Lifts of some centralizers

i	Δ	$\bar{C}_0$	C_0
3	22	$s_1, s_4, s_4s_{17}, (s_4s_{17})^{s_3}$	$n_1, n_4n_3^2, n_4n_{17}, (n_4n_{17})^{n_3}n_{22}n_3^2$
4	18, 10	s_5, s_{19}	$n_5, n_{19}n_3^2$
6	24	s_2, s_3, s_4	n_2, n_3, n_4
7	20, 19	s_1, s_7	$n_1, n_7n_3^2$
8	1, 13	$s_4, s_4s_{17}, (s_4s_{17})^{s_3}$	$n_4n_3^2, n_4n_{17}, (n_4n_{17})^{n_3}n_{22}n_3^2$
9	23, 15	s_5, s_4	$n_5, n_4n_3^2$
10	21	s_1, s_2, s_3	$n_1, n_2, n_3n_4^2$
11	8	s_2, s_3, s_{21}	$n_2, n_3n_4^2, n_{21}n_4^2$
12	14	s_1, s_3, s_{21}	$n_1, n_3n_4^2, n_{21}$
14	22, 17	s_1, s_4	$n_1, n_4n_3^2$
15	8, 16	s_2, s_3	$n_2, n_3n_4^2$
16	1, 6	s_3, s_{21}	$n_3n_4^2, n_{21}n_1^2$
18	17	s_4	$n_4n_3^2$
19	22	s_1	n_1

TABLE IV. Lifts of some centralizers (cont.)

i	k	Δ	$\bar{C}_0$	C_0
3	2	22	$s_1, s_4, s_4s_{17}, (s_4s_{17})^{s_3}$	$n_1, n_4n_3^2, n_4n_{17}, (n_4n_{17})^{n_3}n_{22}n_3^2$
11	2	8	s_2, s_3, s_{21}	$n_2, n_3n_4^2, n_{21}n_4^2$
12	$\neq 1$	14	s_1, s_3, s_{21}	$n_1, n_3n_4^2, n_{21}$
13	4, 5	1, 23	s_3, s_4	n_3, n_4
16	$\neq 1$	1, 6	s_3, s_{21}	$n_3n_4^2, n_{21}n_1^2$
17	4, 5	4, 21	s_1, s_2	n_1, n_2
18	4, 6	1, 13	s_4	$n_4n_3^2$
18	3, 7	1, 2	s_4	n_4
19	6, 7	4, 14	s_1	n_1
19	8, 9	3, 4	s_1	n_1

PROOF. The explicit computations described below are performed with CHEVIE.

By construction, $C_W(v^{\dagger f/m}w^\dagger)$ normalizes $W_{\mathbf{M}_0^\dagger}(\mathbf{T}_0)$. Hence $\bar{C}_0 \leq C_W(v^{\dagger f/m}w^\dagger)^\dagger$ normalizes $W_{\mathbf{M}_0^\dagger}(\mathbf{T}_0)^\dagger = W_{\mathbf{M}_0}(\mathbf{T}_0)$. Thus, in any case, there is $C_0 \leq \hat{W}$ normalizing $\mathbf{M}_0$ and mapping to $\bar{C}_0$. However, to meet the conditions in (b), we need specific lifts, in particular those indicated in Tables III and IV, so that we have to check (a) for these cases. CHEVIE can be used to show that the elements given in the respective columns of Tables III and IV do indeed generate $\bar{C}_0$. The corresponding entries in the columns headed by C_0 are lifts of these generators in $\hat{W}$ and normalize $\mathbf{M}_0 = \mathbf{L}_\Delta$ (a trivial fact if $\Delta = \emptyset$, in which case $\mathbf{M}_0 = \mathbf{T}_0$).

(b) If $\Delta \neq \emptyset$, we choose C_0 as in Tables III and IV. If $w = 1$, put $\dot{w} := 1$. Suppose that $\Delta = \emptyset$ and that $w \neq 1$. Then $e \geq 2$ by Table 23. If $e = 2$, either $w^\dagger = w_0$ or $v^{\dagger f/m} \neq 1$. In the former case, which occurs exactly for $i \in \{2, 5\}$, we also have $w_0 \in \bar{C}_0$, and we choose C_0 such that it contains the lift γ of w_0. In the latter case, which only occurs for $i \in \{15, 16\}$, we take C_0 as in the row corresponding to i of Table III. In case $e \in \{3, 6\}$, the group $\bar{C}_0$ has odd order, so that we may lift it to an isomorphic group C_0 in the extended Weyl group. In case $e = 4$, we put $C_0 := \langle n_3n_2, n_4n_3n_8n_4 \rangle$. If $\Delta = \emptyset$ and $w = 1$, we define the groups C_0 in our proof of (ii). In any case, $C_0 \leq \hat{W}$.

Recall that $\bar{C}_0 = C_{W_\Gamma}(v^{f/m}w) = C_{\Gamma^\dagger}(v^{\dagger f/m}w^\dagger)^\dagger$ in the notation of Remark 9.4. Notice also that if $e \in \{1, 2\}$ then $\Delta = \emptyset$ if and only if the corresponding entry in the "cl"-column of Table 23 equals 1 or 2. Indeed, these are exactly the cases in which $\mathbf{T}_0' = \mathbf{T}_0$ in the notation of Definition 8.8. In particular, the conditions in (b) on the commutator factor group of $\bar{C}_0$ and the non-emptiness of Δ are easily checked from the entries of Table 23. Assume that $\Delta \neq \emptyset$. To find $\Delta^\dagger$ and hence Δ, we determine the ε-eigenspace Y' of $v^{\dagger f/m}w^\dagger$ on $Y = Y(\mathbf{T}_0)$. We then choose $\Delta^\dagger$ as a base of the subsystem of Σ of roots perpendicular to Y'; then $\mathbf{M}_0^\dagger = \mathbf{L}_{\Delta^\dagger}$. This yields the corresponding entries for Δ in Tables III and IV.

To show that C_0 commutes with $[\mathbf{M}_0, \mathbf{M}_0]$, we may assume $\Delta \neq \emptyset$. Using the formulas in [21, Lemmas 7.2.1(i), 6.4.4(i)], one checks that the generators of C_0 given in Tables III and IV indeed centralize the root subgroups corresponding to the roots in $\pm\Delta$.

Observe that v is given in Table 23 as a product of s_j's and w_0. We lift each s_j to n_j and w_0 to γ, and we lift v to the corresponding product $\dot{v}$ of the n_js and γ. Then $\dot{v} \in \hat{W}$.

(i) With the above choice of $\dot{v}$, we check $\dot{v} \in [\mathbf{M}_0, \mathbf{M}_0]$ from Tables 23, III and IV, except in the cases listed in the statement.

(ii) Suppose first that $\Delta \neq \emptyset$. Then the claim follows from (i) and the fact that C_0 centralizes $[\mathbf{M}_0, \mathbf{M}_0]$ which we have already proved, except in the cases excluded in (i). If $(i, k) = (18, 6)$, we have $v = s_{13}s_1$, and one checks that $n_{13}n_1$ commutes with $n_4n_3^2$. The same argument applies for $(i,k) = (18,4)$ if we lift w_0 to γ. If $(i,k) = (19,6)$, we have $v = s_{14}s_4$, and one checks that $n_{14}n_4$ commutes with n_1. The same argument applies for $(i,k) = (19,7)$ if we lift w_0 to γ.

Suppose that $\Delta = \emptyset$ and $w \neq 1$. Then $v \in \{1, w_0\}$ unless $i \in \{15, 16\}$. In the former case, our claim clearly holds. In the latter case, it follows from our choice of C_0 and from what we have already proved above. Suppose finally that $\Delta = \emptyset$ and $w = 1$. Then $e = 1$. If $v \in \{1, w_0\}$, there is nothing to prove. Suppose that there is an element $w \neq 1$ such that $(\Gamma^\dagger, (i,k), -, -, v^\dagger, -, w^\dagger, -, -)$ also is a line in Table 23 with the same value of f/m and with corresponding Δ satisfying $|\Delta| \leq 2$ and such that $C_{\Gamma^\dagger}({v^\dagger}^{f/m} w^\dagger)^\dagger = W_\Gamma$. Then the claim follows from what we have already proved, as $C_{\Gamma^\dagger}({v^\dagger}^{f/m})^\dagger \leq W_\Gamma$. We now discuss the remaining cases. If $(i,k) = (8,2)$ and $v = s_1$, we lift $\bar{C}_0$ to the group generated by n_3 and the group given in the row corresponding to $i = 8$ in Table III. As n_1 commutes with n_3, we obtain our claim. The analogous argument works for $(i,k) = (8,3)$ and $v = w_0s_4$. If $(i,k) \in \{(13,2), (13,3), (13,4), (13,5)\}$, we have $\bar{C}_0 = \langle s_3, s_4 \rangle$, which we lift to $C_0 := \langle n_3, n_4 \rangle$. As n_1 and n_{23} commute with each of n_3 and n_4, our claim follows. The analogous argument works in case $i = 17$. In the cases occurring for $i = 18$, we lift $\bar{C}_0$ to $\langle n_4n_3^2 \rangle$. Then C_0 centralizes n_1, n_2 and n_{13}, giving our claim. For $i = 19$, we lift $\bar{C}_0$ to $\langle n_1 \rangle$. As this commutes with n_3, n_4 and n_{14}, we are done.

(iii) If $i \in \{2, 5\}$, we have $w_0 \in \bar{C}_0$, and our claim follows from Lemma 4.4. In the other cases, we determine the order of $Z(C_0)$ as well as of the intersection of $Z(C_0)$ with the normal subgroup of the extended Weyl group spanned by n_j^2, $j = 1, \ldots, 4$. This shows that in each case $Z(C_0)$ projects onto $Z(\bar{C}_0)$; as the latter group contains w, this proves our claim.

(c) This is clear as $\hat{W}$ consists of σ-stable elements. □

We are now ready to obtain the main structural results about centralizers of abelian defect groups and the corresponding inertia groups.

Proposition 8.10. *Let b, σ and s be as described at the beginning of this subsection and assume in addition that b has abelian defect groups. Then there is a maximal b-Brauer pair (D, b_D) and an element $\omega \in N_A(D)$ with $N_A(D) = \langle N_G(D), \omega \rangle$ such that ω fixes b_D. Moreover, ω centralizes* $\mathrm{Out}_G(D, b_D)$ *except in the cases listed in* (a) *and* (b) *below.*

Suppose from now on that f/m is even or that the commutator quotient of $\mathrm{Out}_G(D, b_D)$ *does not have order* 2 *or* 4. *Put $\mathbf{M} := C_{\mathbf{G}}(D)$. Then there is $N'' \leq N_G(D, b_D)$ with $Z(M) \leq N'' \leq C_G([\mathbf{M}, \mathbf{M}])$*

and $N'' \cap M = Z(M)$ such that $N_G(D, b_D) = N''M$. In particular, $\mathrm{Out}_G(D, b_D) \cong N''/Z(M)$. Moreover, every coset of $N''/Z(M)$ contains an ω-stable element, except in cases (a) *and* (b) *below. Finally, if $\mathbf{M}$ is not a torus, ω acts as a field automorphism on $M' = [\mathbf{M}, \mathbf{M}]^F$.*

(a) *The G_m-class type of s is one of $(12, 2)$ or $(12, 4)$ and f/m is even. In this case, $\mathrm{Out}_G(D, b_D) \cong 2^3$, and the group of ω-fixed points on $\mathrm{Out}_G(D, b_D)$ has order 4.*

(b) *The G_m-class type of s is $(16, k)$ for $k \in \{3, 4, 7, 8, 10\}$ and f/m is even. In all these cases, $\mathrm{Out}_G(D, b_D) \cong 2^2$, and the group of ω-fixed points on $\mathrm{Out}_G(D, b_D)$ has order 2.*

In each of the cases listed in (a) *and* (b)*, let $\lambda \in \mathrm{Irr}(Z(G))$ be defined by $\mathrm{Res}^M_{Z(M)}(\theta_D) = \lambda\theta_D(1)$, where θ_D denotes the canonical character of b_D. Then there is an ω-stable extension of λ to N''.*

Proof. First, we deal with the case that the order of a defect group of b is smaller than the ℓ-part of $|C_G(s)|$. In this case, $e \in \{1, 2\}$ and the defect groups of b are G-conjugate to $R_{10,\ell}$; see Table 1–19. In turn, the centralizers in $\mathbf{G}$ of the defect groups are G-conjugate to $\mathbf{M}_{15,1}$ if $e = 1$ and to $\mathbf{M}_{15,3}$ if $e = 2$. Moreover, the $\mathbf{G}$-class type of s equals i with $i \in \{2, 5, 6, 10, 11\}$. Let $g \in \mathbf{G}$ be such that $\sigma(g)g^{-1} = \gamma$, with $\gamma \in N_{\mathbf{G}}(\mathbf{T}_0)$ as in Lemma 4.4. Put $\mathbf{L}^* := \mathbf{M}_{i,1}$ if the G_m-class type of s equals $(i, 1)$ and $e = 1$, and $\mathbf{L}^* := \mathbf{M}_{i,1}^g$, otherwise. Then $\mathbf{M}_{i,1}^g =_{G_m} \mathbf{M}_{i,1}$ for $i \in \{2, 5\}$. If $i \in \{6, 10, 11\}$ and $e = 2$, the G-class type of s equals $(i, 2)$, as otherwise b would have cyclic defect. But then the G_m-class type of s must be $(i, 2)$ as well. We may thus assume that $\mathbf{L}^* = C_{\mathbf{G}}(s)$. Now $\mathbf{M}_{i,1}$ contains $\mathbf{M}_{15,1}$ as a subgroup of maximal rank. Put $\mathbf{M} := \mathbf{M}_{15,1}$ if the G_m-class type of s equals $(i, 1)$ and $e = 1$, and $\mathbf{M} := \mathbf{M}_{15,1}^g$, otherwise. Then $\mathbf{M}$ is σ-stable, $s \in Z(M)$ and $D := O_\ell(Z(M))$ is a defect group of b. Notice that we can identify $\mathbf{M}$ with $\mathbf{M}^\dagger$ and D with $D^\dagger$ in this case. As D is σ-stable, we obtain $N_A(D) = \langle N_G(D), \sigma\rangle$. Let $b_D \subseteq \mathcal{E}_\ell(M, s)$ denote the ℓ-block of M which covers the ℓ-block of $M' = \mathrm{Sp}_4(q)$ containing the unipotent ℓ-defect zero character ψ of M'. Now $\mathcal{E}_\ell(M, s)$ is σ-stable by (14), and thus b_D is σ-stable, as b_D is uniquely determined by s and ψ. Thus (D, b_D) is a centric, σ-stable Brauer pair with $\ell \nmid |\mathrm{Out}_G(D, b_D)|$. Let b' denote the ℓ-block of G with $(1, b') \leq (D, b_D)$. Then b' has defect group D, and lies in $\mathcal{E}_\ell(G, s)$ by Lemma 3.10, and thus $b' = b$ by uniqueness. Let $N' \leq N_G(\mathbf{M})$ be as in Proposition 4.19. As $\mathbf{T}_0$ is 1-σ-split, this proposition, applied with σ instead of F, shows that every coset of $N'/Z(M)$ contains a σ-stable element. Putting $N'' := N' \cap N_G(D, b_D)$, we obtain the claims about the structure of $N_G(D, b_D)$. Clearly, $\omega = \sigma$ acts as a field automorphism on M'. We have now proved all assertions in case the order of a defect group of b is smaller than the ℓ-part of $|C_G(s)|$.

Suppose now that the order of a defect group of b is equal to the ℓ-part of $|C_G(s)|$. We begin by choosing a pair $\mathbf{S}$, $\mathbf{S}^\dagger$ of F-stable maximal tori of $\mathbf{G}$ and a pair of elements $h^*, h \in \mathbf{G}$ with the following properties.

(i) The pairs $(\mathbf{S}, F)$ and $(\mathbf{S}^\dagger, F)$ are in duality.

(ii) The Steinberg morphisms $\omega := \mathrm{ad}_h^{-1} \circ \sigma \circ \mathrm{ad}_h$ and $\omega^\dagger := \mathrm{ad}_{h^*}^{-1} \circ \sigma \circ \mathrm{ad}_{h^*}$ commute with F and stabilize $\mathbf{S}$ and $\mathbf{S}^\dagger$, respectively. Moreover, the pairs $(\mathbf{S}, \omega)$ and $(\mathbf{S}^\dagger, \omega^\dagger)$ are in duality.

(iii) The element s lies in $S^\dagger := {\mathbf{S}^\dagger}^F$ and is fixed by $\omega^\dagger$.

(iv) Let D and $D^\dagger$ denote the Sylow ℓ-subgroup of S, respectively $S^\dagger$. Then D is a defect group of b.

(v) Put $\mathbf{M} := C_{\mathbf{G}}(D)$ and $\mathbf{M}^\dagger := C_{\mathbf{G}}(D^\dagger)$. Then $\mathbf{M}$ and $\mathbf{M}^\dagger$ are ω-stable, respectively $\omega^\dagger$-stable regular subgroups of $\mathbf{G}$, such that $(\mathbf{M}, F)$ and $(\mathbf{M}^\dagger, F)$ as well as $(\mathbf{M}, \omega)$ and $(\mathbf{M}^\dagger, \omega^\dagger)$ are in duality. The corresponding duality isomorphism $W_{\mathbf{G}}(\mathbf{M}) \xrightarrow{\alpha} W_{\mathbf{G}}(\mathbf{M}^\dagger)$ satisfies $\alpha \circ \omega = \omega^\dagger \circ \alpha$ and $\alpha \circ F = F \circ \alpha$.

To choose the pairs $\mathbf{S}$, $\mathbf{S}^\dagger$ and h^*, h, we proceed as follows. First, we write $u^* := {u^\dagger}^{-1}$ for $u \in W$. Suppose that the G_m-class type of s equals (i, k) and let $\mathbf{L}_0 := \mathbf{M}_{i,1}$, with $\mathbf{M}_{i,1} = \mathbf{L}_{\Gamma_i}$ as in Subsection 4.9. For each e, choose a pair $v, w \in W$ such that (v^*, w^*) is as given in the row of Table 23 corresponding to the tuple $(i, k, e, f/m)$. In particular, v^* and w^* satisfy the conditions described in Remark 9.4, so that $w^* \in W_{\Gamma_i} = W_{\mathbf{L}_0}(\mathbf{T}_0)$. Choose inverse images $\dot{v}, \dot{v}^*, \dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ and $\dot{w}^* \in N_{\mathbf{L}_0}(\mathbf{T}_0)$ of v, v^*, w, w^*, respectively, with the following properties. The elements $\dot{v}$ and $\dot{v}^*$ are σ-stable, $\dot{w}$ is $\sigma\dot{v}$-stable and $\dot{w}^*$ is $\sigma\dot{v}^*$-stable. The latter two conditions hold automatically if $\dot{w}$ and $\dot{w}^*$ are σ-stable and commute with $\dot{v}$, respectively $\dot{v}^*$. In any case, these conditions can be met as v^* and w^* commute. We tacitly assume that the lifts $\dot{v}$ and $\dot{w}$ satisfy, whenever applicable, the stronger properties listed in Lemma 8.9. Choose elements $g, g^* \in \mathbf{G}$ with $\sigma(g)g^{-1} = \dot{v}$ and $\sigma(g^*){g^*}^{-1} = \dot{v}^*$. Next, choose elements $h \in \mathbf{G}$ and $h^* \in \mathbf{L}_0^{g^*}$ with $F(h)h^{-1} = g^{-1}\dot{w}g$ and $F(h^*){h^*}^{-1} = {g^*}^{-1}\dot{w}^*g^*$. Take each of the elements g, h, g^*, h^* as 1, if the corresponding elements in W equal 1.

Putting $\mathbf{L}^* := \mathbf{L}_0^{g^*}$, we may and will now assume that $C_{\mathbf{G}}(s) = \mathbf{L}^*$ (recall that σ-twisting $\mathbf{L}_0 = \mathbf{M}_{i,1}$ with v^* yields a group in the G_m-conjugacy class of $\mathbf{M}_{i,k}$). Let $\mathbf{S}^\dagger := \mathbf{T}_0^{g^*h^*}$ and $\mathbf{S} := \mathbf{T}_0^{gh}$. Put $u := v^{f/m}w$, $u^* := {v^*}^{f/m}w^*$, $\dot{u} := \dot{v}^{f/m}\dot{w}$ and $\dot{u}^* := (\dot{v}^*)^{f/m}\dot{w}^*$. Notice that $u^* = {u^\dagger}^{-1}$ as v and w commute. Now $F(gh)(gh)^{-1} = F(g)g^{-1}\dot{w} = \dot{v}^{f/m}\dot{w} = \dot{u}$, and thus $\mathbf{S}$ is an F-stable maximal torus of F-type $(\emptyset, [u])$. Similarly, $\mathbf{S}^\dagger$ is an F-stable maximal torus of F-type $(\emptyset, [u^*])$. This implies (i) by [22, Proposition 4.3.4]. By construction, ω stabilizes $\mathbf{S}$, and ω commutes with F since σ fixes $F(h)h^{-1} = g^{-1}\dot{w}g$. The analogous argument applies for $\omega^\dagger$. An elementary computation shows the duality of $(\mathbf{S}, \omega)$ and $(\mathbf{S}^\dagger, \omega^\dagger)$; see Subsection 4.12. Thus (ii) holds. Property (iii) is clear by the choice of $h^* \in C_{\mathbf{G}}(s)$. The fact that D is a defect group of b follows from Tables 1–19, in conjunction with Table 25. To prove (v), first notice that $\mathbf{M}$ is ω-stable, since D is. The same argument applies to $\mathbf{M}^\dagger$. Next, observe that $\mathbf{M} = \mathbf{S}$, unless $e \in \{1, 2\}$. In these cases, $\mathbf{M}$ is the centralizer of the subtorus of $\mathbf{S}$, on which F acts as $t \mapsto t^{\varepsilon q}$. Thus $\mathbf{M} = \mathbf{M}_0^{gh}$, where $\mathbf{M}_0$ is the centralizer of the subtorus of $\mathbf{T}_0$, on which Fu acts as $t \mapsto t^{\varepsilon q}$ (recall that $\varepsilon = 1$ if $e = 1$, and $\varepsilon = -1$ if $e = 2$). The analogous construction applies to $\mathbf{M}^\dagger$, where Fu is replaced by Fu^*. We obtain the following commutative diagram of relative Weyl groups,

$$\begin{array}{ccccccc}
W_{\mathbf{G}}(\mathbf{M}) & \xrightarrow{\ \omega\ } & W_{\mathbf{G}}(\mathbf{M}) & \xrightarrow{\ \alpha\ } & W_{\mathbf{G}}(\mathbf{M}^\dagger) & \xrightarrow{\ \omega^\dagger\ } & W_{\mathbf{G}}(\mathbf{M}^\dagger) \\
\uparrow {\scriptstyle \mathrm{ad}_{gh}} & & \uparrow {\scriptstyle \mathrm{ad}_{gh}} & & \uparrow {\scriptstyle \mathrm{ad}_{g^*h^*}} & & \uparrow {\scriptstyle \mathrm{ad}_{g^*h^*}} \\
W_{\mathbf{G}}(\mathbf{M}_0) & \xrightarrow[\ \sigma v\ ]{} & W_{\mathbf{G}}(\mathbf{M}_0) & \xrightarrow[\ \dagger\ ]{} & W_{\mathbf{G}}(\mathbf{M}_0^\dagger) & \xrightarrow[\ \sigma v^*\ ]{} & W_{\mathbf{G}}(\mathbf{M}_0^\dagger)
\end{array}$$

where α is defined such that the middle square commutes. The duality of $(\mathbf{M}_0, F)$ and $(\mathbf{M}_0^\dagger, F)$ gives rise to the isomorphism $\dagger$ and implies the duality of $(\mathbf{M}, F)$ and $(\mathbf{M}^\dagger, F)$; see Subsection 4.12. Similarly, the duality of $(\mathbf{M}_0, \sigma)$ and $(\mathbf{M}_0^\dagger, \sigma)$ implies the duality of $(\mathbf{M}_0, \sigma\dot{v})$ and $(\mathbf{M}_0^\dagger, \sigma\dot{v}^*)$ and thus of $(\mathbf{M}, \omega)$ and $(\mathbf{M}^\dagger, \omega^\dagger)$. This gives all the claims of (v).

By Lemma 8.9(b), the element $\dot{w}$ is σ-stable if f/m is even. If f/m is odd, and $v = \dot{v} = 1$, then $\dot{w}$ is σ-stable as it is $\sigma\dot{v}$-stable by our choice. As b has non-cyclic defect, no other cases occur. Thus $F(gh)(gh)^{-1} = \dot{v}^{f/m}\dot{w}$ is σ-stable and hence $N_A(D) = \langle N_G(D), \omega \rangle$ by Lemma 8.3. Lemma 8.6 implies the existence of an ω-stable maximal b-Brauer pair (D, b_D) with $b_D \subseteq \mathcal{E}_\ell(M, s)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
N_{\mathbf{L}^*}(\mathbf{S}^\dagger) & \xrightarrow{\ \omega^\dagger\ } & N_{\mathbf{L}^*}(\mathbf{S}^\dagger) \\
\uparrow {\scriptstyle \mathrm{ad}_{g^*h^*}} & & \uparrow {\scriptstyle \mathrm{ad}_{g^*h^*}} \\
N_{\mathbf{L}_0}(\mathbf{T}_0) & \xrightarrow[\ \sigma\dot{v}^*\ ]{} & N_{\mathbf{L}_0}(\mathbf{T}_0)
\end{array}$$

We have

$$\mathrm{ad}_{g^*h^*}^{-1}(N_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F) = N_{\mathbf{L}_0}(\mathbf{T}_0)^{F\dot{u}^*}.$$

In particular, $(g^*h^*)^{-1}$ conjugates $W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F$ to

$$N_{\mathbf{L}_0}(\mathbf{T}_0)^{F\dot{u}^*}/\mathbf{T}_0^{F\dot{u}^*} = C_W(u^*) \cap W_{\mathbf{L}_0}(\mathbf{T}_0).$$

By the definition of $\mathbf{M}^\dagger$ we have $W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F \leq W_{\mathbf{G}}(\mathbf{M}^\dagger)^F$, so that $\alpha^{-1}(W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F) \leq W_{\mathbf{G}}(\mathbf{M})^F$, as α is F-equivariant. The first commutative diagram above implies that $(gh)^{-1}$ conjugates $\alpha^{-1}(W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F)$ to

$$\bar{C}_0 := C_W(u) \cap W_{\mathbf{L}_0}(\mathbf{T}_0)^\dagger.$$

Proposition 3.16(a) implies that $\mathrm{Out}_G(D, b_D) = \alpha^{-1}(W_{\mathbf{L}^*}(\mathbf{S}^\dagger)^F)$.

Notice that $\bar{C}_0$ is exactly the group associated in Lemma 8.9 to the tuple $(i, k, e, f/m)$, as $W_{\mathbf{L}_0}(\mathbf{T}_0) = W_{\Gamma^\dagger}$ if $\mathbf{L}_0 = \mathbf{L}_{\Gamma^\dagger}$. The action of ω on $\mathrm{Out}_G(D, b_D)$ corresponds to the action of σv on $\bar{C}_0$. By Lemma 8.9(a), the latter action is trivial unless we are in one of the cases described in (a) or (b). This gives the second statement of our proposition, namely that ω centralizes $\mathrm{Out}_G(D, b_D)$. In the situations of (a) and (b) we have to compute the number of orbits of v on $\bar{C}_0$. This is done with a CHEVIE computation confirming the assertions.

Now assume that f/m is even or that the quotient of $\mathrm{Out}_G(D, b_D)$ by its commutator subgroup does not have order 2 or 4 (this includes the cases of (a) and (b)). Let C_0 denote the inverse image of $\bar{C}_0$ exhibited in Lemma 8.9. In particular, C_0 consists of σ-stable elements and centralizes $\dot{w}$ and $[\mathbf{M}_0, \mathbf{M}_0]$. Moreover, C_0 centralizes $\dot{v}$ except in the cases listed in (a) and (b). In the latter cases, C_0 still centralizes $\dot{u}$. Thus, in all cases, $C_0 \leq N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{u}}$. By the considerations above, $N_G(D, b_D)$ is conjugate, under $(gh)^{-1}$, to $C_0\mathbf{M}_0^{F\dot{u}}$. Now define $N_0'' := Z(M_0)C_0$ and $N'' := (N_0'')^{gh}$. Then $N_G(D, b_D) = N''M$, and $Z(M) \leq N'' \leq C_{\mathbf{G}}([\mathbf{M}, \mathbf{M}])$. As N'' centralizes $[\mathbf{M}, \mathbf{M}]$, we also have $N'' \cap M = Z(M)$ by Propositions 4.17, 4.18 and 4.19. Unless we are in either the situation (a) or (b), the elements of C_0 are fixed by $\sigma\dot{v}$, and thus every coset of $N''/Z(M)$ contains an ω-stable element. Suppose that $\mathbf{M}_0$ is not a torus. Then $\dot{v} \in [\mathbf{M}_0, \mathbf{M}_0]^{F\dot{u}}$ by Lemma 8.9(b), unless the G_m-class type of s is one of $(18, 6)$, $(18, 4)$, $(19, 6)$ or $(19, 7)$. In the latter cases, $[\mathbf{M}_0, \mathbf{M}_0]$ has semisimple rank 1, and if $\alpha \in \Sigma^+$ is the positive root of $[\mathbf{M}_0, \mathbf{M}_0]$, we have $u_{\pm\alpha}(t)^{\dot{v}} = u_{\pm\alpha}(-t)$ for all $t \in \mathbb{F}$; this follows from the corresponding entries in Table 23 and [21, Lemma 7.2.1]. As f/m is even, either q is even or $4 \mid q - 1$, and thus $\dot{v}$ acts as an inner automorphism on $[\mathbf{M}_0, \mathbf{M}_0]^{F\dot{u}}$. In each case, $\sigma\dot{v}$ acts as a field automorphism on $[\mathbf{M}_0, \mathbf{M}_0]^{F\dot{u}}$, and hence ω acts as a field automorphism on M'.

We finally prove the last assertion. Suppose first that s is as described in (a). We have $N'' = \langle Z(M), n_1^{gh}, (n_3n_4^2)^{gh}, n_{21}^{gh}\rangle$ and that the elements n_1^{gh}, $(n_3n_4^2)^{gh}$, n_{21}^{gh} are ω-stable. One checks that $(n_3n_4^2)^2 = 1$, and that the kernel of the natural epimorphism $\langle n_1, n_3n_4^2, n_{21}\rangle \to \langle s_1, s_3, s_{21}\rangle$ is generated by n_1^2 and n_{21}^2. By [21, Lemma 6.4.4] we have $n_i^2 = \alpha_i^\vee(-1)$ for all $1 \leq i \leq 24$. By Lemma 4.11, the elements of S of the form $\alpha_i^\vee(t)^{gh}$ for $t \in \mathbb{F}_q^*$ and $i \in \{1, 3, 21\}$, are contained in the kernel of $\hat{s}$. As λ is the restriction of $\hat{s}$ to $Z(M)$, and as $N''/Z(M)$ is elementary abelian, $\langle (n_1^2)^{gh}, (n_{21}^2)^{gh}\rangle \leq \ker(\lambda)$. Thus $Z(M)/\ker(\lambda)$ has an ω-stable complement in $N''/\ker(\lambda)$. View λ as a character of $Z(M)/\ker(\lambda)$. As such, it has a trivial extension to $N''/\ker(\lambda)$, and this is ω-invariant. Exactly the same argument applies for elements s as in (b). □

We also need the following result on the action of field automorphisms on certain characters.

Lemma 8.11. *Let M' be one of the groups $\mathrm{SL}_2(q)$ or $\mathrm{Sp}_4(q)$ with q odd, or $\mathrm{SL}_3(q)$ with $3 \mid q-1$. Let $d = 2$ in the first two cases, and $d = 3$ in the last case.*

Let $\nu \in \mathrm{Irr}(M')$ which is not invariant under a diagonal automorphism κ of M', and let $\nu_1, \ldots, \nu_d$ denote the distinct conjugates of ν under κ. In case of $M' = \mathrm{SL}_3(q)$, assume that $\nu(1) = (q-1)^2(q+1)/3$, and in case of $M' = \mathrm{Sp}_4(q)$ assume that $\nu(1) = (q \pm 1)^2(q^2+1)/2$.

Let ω be a field automorphism of M' which permutes $\nu_1, \ldots, \nu_d$. Then ω fixes at least one of $\nu_1, \ldots, \nu_d$.

Proof. Suppose first that $M' = \mathrm{SL}_2(q)$. Then $d = 2$ and ν_1 and ν_2 differ on a unipotent element which is fixed by ω; see the character table of M' given in [33]. If $M' = \mathrm{SL}_3(q)$, the three characters ν_1, ν_2, ν_3 take two distinct values on a unipotent conjugacy class of M' which is fixed by ω; see the character table given in [73]. Thus ω fixes at least one of these characters. If $M' = \mathrm{Sp}_4(q)$, we use the character table computed by Srinivasan in [77]. If $\nu(1) = (q-1)^2(q^2+1)/2$, then $\{\nu_1, \nu_2\} = \{\xi_{21}'(k), \xi_{22}'(k)\}$ for some value of k. This follows from the description of the conjugacy classes of $\mathrm{Sp}_4(q)$ in [77, p. 489-491] and its character table in [77, p. 516]. Next, ν_1 and ν_2 differ on the class A_{41}, which is fixed by ω. This gives the result. The same argument works for $\nu_1(1) = (q+1)^2(q^2+1)/2$. (The article [68] of Przygocki contains a few corrections to Srinivasan's character table of $\mathrm{Sp}_4(q)$, but these changes do not affect our argument.) □

We next consider the blocks with non-abelian defect groups. These only occur for $\ell = 3$.

Proposition 8.12. *Let* $\ell = 3$. *Assume* Hypothesis 8.2*, and in addition that* b *is a non-unipotent block with a non-abelian defect group. Assume furthermore that* $\sigma = F_1^m$.

Then there is a maximal b*-Brauer pair* (D, b_D) *fixed by* σ. *Moreover,* D*, and the* b*-Brauer pairs* $(R, b_R) \leq (D, b_D)$ *with* $\mathcal{W}(R, b_R) > 1$ *can be chosen such that* $\sigma(R) = R$ *and* $N_G(R, b_R) \leq RC_G(R)N_{\mathbf{G}}(R)^\sigma$*, except that the latter condition does not always hold in the following situations:*

The G_m*-class type of* s *is one of* $(7, 2)$ *or* $(9, 2)$ *and* f/m *is even, and* R *is non-abelian and properly contained in a defect group of* b*. In this case,* σ *fixes the two central factors of* $N_G(R, b_R)$ *isomorphic to* $[q-1]^2.2$ *and* $3_+^{1+2}.\mathrm{SL}_2(3)$*, respectively, and centralizes an element in* $[q-1]^2.2 \setminus [q-1]^2$.

Proof. The possible class types of s and the b-relevant radical 3-subgroups can be read off from Tables 1–19. In particular, the defect groups of b are G-conjugate to one of R_{29}–R_{34} and thus their centralizers in $\mathbf{G}$ are connected reductive groups by Lemma 6.8. Let (i, k) and (i, k') denote the G-class type, respectively the G_m-class type of s. Define $\varepsilon' \in \{-1, 1\}$ by the condition that $3 \mid p^m - \varepsilon'$. Notice that $\varepsilon = -1$ implies $\varepsilon' = -1$. Also, if f/m is even, $\varepsilon = 1$.

Suppose first that $(\varepsilon, k) = (\varepsilon', k')$ or that the G-class type of s is one of $(3, 1)$ or $(3, 2)$. Let $\mathbf{L}$ denote the centralizer of a 3C-element $z_{\mathrm{C}} \in G_m$ as described in Proposition 4.14. We let D and $D^\dagger$ denote a pair of σ-stable radical subgroups of L such that D is a defect group of b, such that $(C_{\mathbf{L}}(D), F)$ and $(C_{\mathbf{L}}(D^\dagger), F)$ are in duality, and such that $N_G(D) = DC_G(D)N_{G_m}(D)$. Such a pair exists by the construction of R_{29}–R_{34}; see Subsections 6.2, 6.4 and Lemma 6.3. Moreover, in every G-conjugacy class of b-relevant radical subgroups we can choose a representative R with $R \leq D$ and $N_G(R) = RC_G(R)N_{G_m}(R)$. Observe that $D_m := D \cap G_m$ and $D_m^\dagger := D^\dagger \cap G_m$ are radical 3-subgroups of G_m of the same type as D, respectively $D^\dagger$. It suffices to show that $s \in C_G(D^\dagger)$, as this implies the existence of a σ-stable b-Brauer pair (D, b_D) by Lemma 8.6. Now D_m is a defect group of a 3-block of $\mathcal{E}_3(G_m, s)$ by our assumption on s and (ε, k). In particular, $D_m^\dagger$ is conjugate in G_m to a subgroup of $C_{G_m}(s)$ by Proposition 6.18(a). By replacing s by a G_m-conjugate, we may assume that $s \in C_{G_m}(D_m^\dagger)$. As $C_{\mathbf{G}}(D_m^\dagger) = C_{\mathbf{G}}(D^\dagger)$ by Lemma 6.8, we have $s \in C_G(D^\dagger)$, and we are done.

We are left with the cases $i \in \{6, 7, 8, 9, 10, 13, 17\}$ and either $k = k'$ and $\varepsilon \neq \varepsilon'$ or $k \neq k'$. If $\varepsilon \neq \varepsilon'$, then $(\varepsilon, \varepsilon') = (1, -1)$. In this case, f/m is even. If $k \neq k'$, then f/m is even by Table 23, unless $i \in \{13, 17\}$

and $k' \in \{4, 5\}$. If $(i, k') \in \{(13, 5), (17, 5)\}$ we have $(\varepsilon, \varepsilon') = (1, -1)$, and f/m is even and not divisible by 3. In particular, $k = k' = 5$. If $(i, k') \in \{(13, 4), (17, 4)\}$ we have $(\varepsilon, \varepsilon') = (-1, -1)$, and f/m is odd and divisible by 3. In this case, $k = 6$.

We put $\mathbf{L}_0^* := \mathbf{M}_{i,1}$, as constructed in Table 23. If $k \neq k'$, choose $v \in W$ according to the following table.

i	k	k'	ε	ε'	v
6	1	2	1	1	s_{24}
6	1	2	1	-1	w_0
7	1	2	1	± 1	w_0
9	1	2	1	± 1	w_0
8	1	4	1	± 1	w_0
8	1	2	1	± 1	s_1
8	1	3	1	± 1	$w_0 s_1$
10	1	2	1	1	s_{21}
10	1	2	1	-1	w_0

i	k	k'	ε	ε'	v
13	1	6	1	± 1	w_0
13	1	2	1	± 1	s_1
13	1	3	1	± 1	$w_0 s_1$
13	6	4	-1	-1	$w_0 s_1 s_{23}$
13	5	5	1	± 1	$s_1 s_{23}$
17	1	6	1	± 1	w_0
17	1	2	1	± 1	$w_0 s_4$
17	1	3	1	± 1	s_4
17	6	4	-1	-1	$w_0 s_4 s_{19}$
17	5	5	1	± 1	$s_4 s_{19}$

We first deal with the cases $i \neq 8$. Then $\mathbf{L}_0^*$ is a regular subgroup of $\mathbf{G}$ and the set of subgroups thus defined contains a dual of each of its members; we let $\mathbf{L}_0$ denote this dual. If $k = k'$, let $\mathbf{L} := \mathbf{L}_0$. Otherwise, choose a σ-stable lift $\dot{v} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of v with $\dot{v} = \gamma$ if $v = w_0$. Let $g \in \mathbf{G}$ with $\sigma(g)g^{-1} = \dot{v}$ and put $\mathbf{L} := \mathbf{L}_0^g$ and $\mathbf{T} := \mathbf{T}_0^g$. By Table 23, the G_m-conjugacy class of $\mathbf{L}$ contains a group dual to $\mathbf{M}_{i,k'}$, and we may assume that $\mathbf{L} = C_{\mathbf{G}}(s)^*$. As $\dot{v}$ is σ-stable, the F-type of $\mathbf{L}$ equals $(\bar{\Gamma}^{\dagger}, [v^{f/m}])$, where $\bar{\Gamma}$ is the closed subsystem of Σ defining the G-class type i, and the F-type of $\mathbf{T}$ equals $(\emptyset, [v^{f/m}])$. The action of σ on $\mathbf{L}$ corresponds to the action of $\sigma\dot{v}$ on $\mathbf{L}_0$. In particular, an element of $N_{\mathbf{L}}(\mathbf{T})$ is σ-stable, if and only if it is the g-conjugate of a $\sigma\dot{v}$-stable element of $N_{\mathbf{L}_0}(\mathbf{T}_0)$. One checks that v centralizes $W_{\mathbf{L}_0}(\mathbf{T}_0)$, so that σ fixes every element of $W_{\mathbf{L}}(\mathbf{T})$. Thus the elements of $W_{\mathbf{L}}(\mathbf{T})$ can be lifted to σ-stable elements of $N_{\mathbf{L}}(\mathbf{T})$.

Let Q denote the Sylow 3-subgroup of T. Then Q is σ-stable and Q is G-conjugate to one of R_{11}, R_{12} or R_{18}. As $\mathrm{Out}_L(Q) = W_{\mathbf{L}}(\mathbf{T})$, we find that $N_L(Q) = C_L(Q)N_{\mathbf{L}}(Q)^{\sigma}$. There is an element u of order 3 in $W_{\mathbf{L}}(\mathbf{T})$, which lifts to a σ-stable element $\dot{u} \in N_{\mathbf{L}}(\mathbf{T})$ such that $D := \langle Q, \dot{u} \rangle$ is a Sylow 3-subgroup of L. Moreover, $\mathrm{Out}_L(D)$ is generated by elements of order 2 in $W_{\mathbf{L}}(\mathbf{T})$. It follows that $N_L(D) = DC_L(D)N_{\mathbf{L}}(D)^{\sigma}$.

As D is a σ-stable defect group of b, and $\sigma(s) = s$, we obtain a σ-stable b-Brauer pair (D, b_D) by Lemma 8.6, applied with $\omega = \sigma$

and a radical 3-subgroup $Q^\dagger$ such that $C_{\mathbf{G}}(Q^\dagger)$ is dual to $C_{\mathbf{G}}(Q)$ and $s \in C_G(Q^\dagger)$; see Proposition 6.18. Let $(R, b_R) \leq (D, b_D)$ be a b-Brauer pair with $\mathcal{W}(R, b_R) > 0$. If $R \lneq D$ and R is abelian, we get $R = Q$, so that R is σ-stable. In turn, (R, b_R) is σ-stable. Now suppose that $R \lneq D$ is non-abelian. Then $i \in \{6, 7, 9, 10\}$ and we let $\mathbf{M}_0$ denote the standard Levi subgroup of $\mathbf{L}_0$ of type A_2, respectively $\tilde{A}_2$, and put $\mathbf{M} := \mathbf{M}_0^g$. Then $\mathbf{M} =_G \mathbf{M}_{13,1}$ if $i \in \{6, 7\}$, and $\mathbf{M}_{17,1}$ if $i \in \{9, 10\}$. Thus $M = \langle Z(M) \circ_3 M', x\rangle$ with x as in Proposition 4.17, $Z(M) \cong [q-1]^2$ and $M' = \mathrm{SL}_3(q)$. Now $R = Q_0 \circ_3 Q_1$, where Q_0 is the Sylow 3-subgroup of $Z(M)$, and $Q_1 \leq M'$ is isomorphic to 3_+^{1+2}. By the construction described in Subsection 6.2, we may choose Q_1 to be σ-stable. In particular, R and thus (R, b_R) are σ-stable. If $i \in \{7, 9\}$, there is a σ-stable element $n \in L$ of order 4 normalizing $Z(M)$ and centralizing $[\mathbf{M}, \mathbf{M}]$; see Proposition 4.17(b). Thus $N_L(R) = \langle Z(M), n\rangle \circ_3 N_{M'}(Q_1)$ if $a \geq 2$. In case $a = 1$, there is a σ-stable element $x' \in M \setminus M'$ which induces a diagonal automorphism on M', normalizes Q_1 and with $x'^3 \in Z(M')$; see Proposition 4.17 with F repaced by $\sigma\dot{v}$. As $N_{M'}(Q_1) \cong 3_+^{1+2}.Q_8$ in this case, we obtain $N_L(R) = \langle Z(M), n\rangle \circ_3 \langle N_{M'}(Q_1), x'\rangle$. In either case, $N_L(R) = RC_L(R)N_{\mathbf{L}}(R)^\sigma \cong [q-1]^2.2 \circ_3 3_+^{1+2}.\mathrm{SL}_2(3)$. If $i \in \{6, 10\}$, we may assume that $N_{M'}(Q_1)$ respectively $\langle N_{M'}(Q_1), x'\rangle$ consists of σ-stable elements. Indeed, $[\mathbf{M}, \mathbf{M}]^\sigma = \mathrm{SL}_3^{\varepsilon'}(p^m)$, and thus Q_1 can be chosen such that $N_{M'}(Q_1) \leq \mathbf{M}^\sigma$. As there is a σ-stable element in $[\mathbf{L}, \mathbf{L}]$ which induces the inverse transpose automorphism on $[\mathbf{M}, \mathbf{M}]$, we obtain $N_L(R) = RC_L(R)N_{\mathbf{L}}(R)^\sigma$.

For arbitrary $R \leq D$, as $RC_G(R) \leq L$, we get $N_G(R, b_R) = N_L(R)$ by Lemma 3.12(c); to apply this lemma notice that $N_L(R)$ fixes b_R, as the canonical character of b_R equals $\hat{s} \in \mathrm{Irr}(C_G(R))$. From $N_L(R) = RC_L(R)N_{\mathbf{L}}(R)^\sigma$ we conclude $N_G(R, b_R) \leq RC_G(R)N_{\mathbf{G}}(R)^\sigma$.

It remains to consider the case $i = 8$. If $k' = 1$, put $\mathbf{L}^* := \mathbf{L}_0^*$. If $k' \neq 1$, let $\mathbf{L}^*$ and $\mathbf{T}^\dagger$ denote the groups obtained by σ-twisting $\mathbf{L}_0^*$ respectively $\mathbf{T}_0$ with $v^\dagger$. We may then assume that $\mathbf{L}^* = C_{\mathbf{G}}(s)$ by Table 23. Notice that $W_{\mathbf{L}_0^*}(\mathbf{T}_0) = \langle s_1, s_2, s_{24}\rangle$. Let $\mathbf{M}_0^\dagger$ denote the parabolic subgroup of $\mathbf{L}_0^*$ of type A_2 corresponding to the roots α_1 and α_2. Let $\mathbf{M}$ and $\mathbf{T}$ denote the σ-stable subgroups of $\mathbf{G}$ obtained from $\mathbf{M}_0^\dagger$ respectively $\mathbf{T}_0$ by σ-twisting with v. Then $\mathbf{M}$ is G_m-conjugate to $\mathbf{M}_{17,k'}$ if $k' \in \{1, 2, 3\}$, and to $\mathbf{M}_{17,6}$ if $k' = 4$; see Table 23. Let Q denote the Sylow 3-subgroup of T contained in a σ-stable Sylow 3-subgroup D of M. By Lemma 8.6, there is a σ-stable b-Brauer pair (D, b_D). This satisfies our assertion by what we have already proved for $i = 17$. Now let $(R, b_R) \leq (D, b_D)$ with $R := Q$. Then (R, b_R) is σ-stable. By Proposition 3.16, $\mathrm{Out}_G(R, b_R)$ is sent to

$W_{\mathbf{L}^*}(\mathbf{T}^\dagger)$ under duality. Thus $N_G(R, b_R)$ is $\mathbf{G}$-conjugate to an extension of T_0 by $\langle s_4, s_3, s_{17}\rangle \cong W(A_3)$. One checks that n_1 centralizes $\langle n_4, n_3, n_{17}\rangle$, and thus $N_G(R, b_R) = C_G(R)N_{\mathbf{G}}(R)^\sigma$. □

We remark that the exceptional cases do occur. For example, let $p = 13$, $f = 2$ and $m = 1$. Then $\varepsilon = \varepsilon' = 1$. Now assume that s is of G_m-class type $(7, 2)$. Adopt the notation of Proposition 8.12. By Theorem 3.9(a), there is a b-Brauer pair (R, b_R) with $R =_G R_{25}$ such that $R \leq L$. Now $\mathbf{L}^\sigma \cong \mathrm{SL}_2(13) \times 7 \times \mathrm{SU}_3(13)$. As this group does not have any subgroup isomorphic to the quaternion group Q_8, we cannot have $N_L(R) = RC_G(R)N_{\mathbf{L}}(R)^\sigma$.

The following proposition contains the essential arguments for the invariance of the b-weights in case of non-unipotent blocks.

Proposition 8.13. *Let b, σ and s be as in* Hypothesis 8.2*, and assume in addition that b is a non-unipotent block and that $\sigma = F_1^m$. Then there is a maximal b-Brauer pair (D, b_D) and an element $\omega \in N_A(D)$ with $N_A(D) = \langle N_G(D), \omega\rangle$ such that ω fixes b_D. Moreover, the following holds.*

Let $(R, b_R) \leq (D, b_D)$ be a relevant b-Brauer pair fixed by ω and let θ_R denote the canonical character of b_R. Then some extension of θ_R to $N_G(R, b_R)$ is ω-stable. In particular, ω stabilizes every element of $\mathrm{Irr}^0(N_G(R, \theta_R) \mid \theta_R)$*, unless b is one of the blocks described in* Proposition 8.10(a),(b).

Proof. If the defect groups of b are abelian, choose (D, b_D) and ω as in Proposition 8.10. Otherwise, let $\omega := \sigma$, and choose (D, b_D) and the subpairs $(R, b_R) \leq (D, b_D)$ as in Proposition 8.12. Then (D, b_D) and (R, b_R) are ω-stable, so that our first statement holds. Moreover, ω centralizes $\mathrm{Out}_G(R, b_R)$, up to the exceptional cases listed in Propositions 8.10(a),(b) and 8.12. Thus, in view of Lemma 8.5, the last statement follows from the penultimate one in the non-exceptional cases. Write $\theta := \theta_R$. Then $N_G(R, b_R) = N_G(R, \theta)$. In particular, ω fixes θ.

Suppose first that the defect groups of b are non-abelian. Then $\ell = 3$ and $\omega = \sigma$. If $\theta(1) = 1$, the assertion follows from Proposition 8.12 and Lemma 2.4. If (R, b_R) is as in one of the exceptional cases listed in Proposition 8.12, the two elements of $\mathrm{Irr}^0(N_G(R, \theta) \mid \theta)$ are σ-stable by Lemma 2.4 and the considerations on central products in Subsection 2.9. We have now proved our claim in case the G-class type of s is one of $(13, k)$ or $(17, k)$ with $k \in \{4, 5\}$, as then either $|\mathrm{Irr}^0(N_G(R, \theta) \mid \theta)| = 1$ or $\theta(1) = 1$. Thus we exclude these cases in the discussion to follow. If f/m is even, then $C_G(s)$ is G-conjugate to one of $M_{i,1}$ with $i \in \{2, 3, 5, \ldots, 10, 13, 17\}$; see Table 23. In these instances, $C_G(R)$ is abelian, hence $\theta(1) = 1$, a case we have already considered. We may thus assume that f/m is odd and that $\theta(1) \neq 1$. Then R is G-conjugate to one of R_{19}, R_{20}, R_{23}, R_{24}, R_{31} or R_{32}. Here, $|\mathrm{Irr}^0(N_G(R, \theta) \mid \theta)|$ is even and at most equal to 4, which implies that σ fixes at least one element of $\mathrm{Irr}^0(N_G(R, \theta) \mid \theta)$.

Suppose now that b has abelian defect groups. If f/m is odd and the commutator quotient of $\mathrm{Out}_G(D, \theta)$ has order 2 or 4, there is an invariant extension of θ, and we are done. Suppose from now on that f/m is even or that the commutator quotient of $\mathrm{Out}_G(D, \theta)$ does not have order 2 or 4. Put $\mathbf{M} := C_{\mathbf{G}}(D)$. By Proposition 8.10, we have $N_G(D, \theta) = N''C_G(D)$ with $Z(M) \leq N'' \leq C_G([\mathbf{M}, \mathbf{M}])$, and every coset of $N''/Z(M)$ contains an ω-stable element. Thus, if $\theta(1) = 1$, every extension of θ to $N_G(D, \theta)$ is ω-invariant by Lemma 2.4. Suppose then that $\theta(1) \neq 1$. Then $e \in \{1, 2\}$ and D is G-conjugate to one of $R_{9,\ell}$, $R_{10,\ell}$, $R_{11,\ell}$, $R_{12,\ell}$, $R_{16,\ell}$ or $R_{17,\ell}$, where $R_{11,\ell}$ and $R_{12,\ell}$ only occur for $\ell > 3$. Also, $\mathbf{M}$ is an e-split Levi subgroup of $\mathbf{G}$ which is G-conjugate to one of the standard regular subgroups described in Propositions 4.17, 4.18 and 4.19. As $C_G(D) = M$ and D is the Sylow ℓ-subgroup of $Z(M)$, we also have $N_G(D) = N_G(M)$.

Recall that $d = \gcd(3, q-1)$ if $D \in_G \{R_{11,\ell}, R_{12,\ell}\}$ and that $d = \gcd(2, q-1)$ in all other cases. Suppose first that $R \neq_G R_{9,\ell}$. Then M is of the form $M = \langle Z(M) \circ_d M', x' \rangle$ and $M' = [\mathbf{M}, \mathbf{M}]^F$. Moreover, $x' = 1$ if $d = 1$, and x' is an arbitrary element of $M \setminus (Z(M) \circ_d M')$ if $d \neq 1$. By the above description of $N_G(D, \theta)$, we have $N_G(D, \theta) = N_G(M, \theta) = \langle N'' \circ_d M', x' \rangle$. Now ω stabilizes $\mathbf{M}$ and commutes with F, and thus $Z(M)$, $M' = [\mathbf{M}, \mathbf{M}]^F$ and $N'' = C_{N_G(D,\theta)}(M')$ are also stabilized by ω. As M is a central product of $Z(M)$ with $\langle M', x' \rangle$, we can write $\theta = \lambda\psi$ for irreducible characters λ of $Z(M)$ and ψ of $\langle M', x' \rangle$, respectively. Hence the restriction of θ to $Z(M) \circ_d M'$ is of the form $\theta' = \lambda\psi'$, where ψ' denotes the restriction of ψ to M'. It follows that ω fixes λ and ψ', as ω fixes θ and stabilizes $Z(M)$ and M'. If $d = 1$, then $M' = M$ and $\psi' = \psi$. Otherwise, ω acts as a field automorphism on M' by Proposition 8.10, and thus ω fixes at least one irreducible constituent of ψ' by Lemma 8.11.

Let $\hat{\theta}$ denote an extension of θ to $N_G(R, \theta)$, write $\hat{\theta}'$ for its restriction to $N''M' = N'' \circ_d M'$, and let $\hat{\lambda}\psi''$ be an irreducible constituent of $\hat{\theta}'$. Then $\hat{\lambda}$ is an extension of λ to N'' and ψ'' is an irreducible constituent of ψ'. By what we have said above, we may assume that ψ'' is ω-invariant. The distinct extensions of θ to $N''M$ and the distinct extensions of λ to N'' are of the form $\hat{\theta}\xi$, respectively $\hat{\lambda}\xi$, where ξ runs through the irreducible characters of $N''/Z(M) = N_G(D, \theta)/M$. If we are in one of the cases (a) or (b) of Proposition 8.10, the last statement of this proposition shows that we may choose $\hat{\theta}$ such that $\hat{\lambda}$ is ω-invariant. In the other cases, Lemma 2.4 implies that ω stabilizes $\hat{\lambda}$. It follows that ω fixes $\hat{\lambda}\psi''$. If $\hat{\theta}'$ is reducible, i.e. if $\psi'' \neq \psi'$, then $\hat{\theta}$ is ω-invariant, as $\hat{\theta}$ is induced from $\hat{\lambda}\psi''$. We may thus assume that $\hat{\theta}' = \hat{\lambda}\psi'$ is irreducible and fixed by ω. Then $\theta' = \lambda\psi'$ is irreducible, and Corollary 2.6, applied with $N = N''M'$, implies that ω fixes $\hat{\theta}$.

In case of $R_{9,\ell}$ we use a variant of this argument. Let s denote an ω-stable semisimple ℓ'-element such that $b \subseteq \mathcal{E}_\ell(G, s)$. An inspection of Tables 1–19 gives the possible G-class types of s and also shows that $\mathrm{Out}_G(D, \theta)$ has order 2 or 4. We may thus assume that f/m is even. The case that s has G-class type $(16, 7)$ is one of the exceptional cases of Proposition 8.10(b) and thus excluded. Table 23 then shows that the G-class type of s is one of $\{(4, 1), (7, 1), (9, 1), (14, 1)\}$; for example, every σ-stable semisimple element of $\mathbf{G}$-class type 14 has G-class type $(14, 1)$ and no σ-stable element of $\mathbf{G}$-class type 13 can have G-class type $(13, 2)$ or $(13, 3)$. From Tables 4, 7, 9 and 14 we then get that $\ell \mid q+1$, that $\mathrm{Out}_G(D, \theta) \cong 2^2$ and thus that $N_G(D, \theta) = N_G(M)$.

In Proposition 8.10, the group $\mathbf{M} = C_{\mathbf{G}}(D)$ was constructed by twisting $\mathbf{L}_\Delta$ for $\Delta = \{\alpha_{22}, \alpha_{17}\}$ with $s_1 s_4$; see also Definition 8.8 and Lemma 8.9. In Proposition 4.18, the standard 2-split Levi subgroup of $\mathbf{G}$ in the class containing $\mathbf{M}_{14,4}$ was constructed by twisting $\mathbf{L}_\Gamma$ for $\Gamma = \{\alpha_1, \alpha_4\}$ with w_0, the longest element of W. Now there is $z \in W$ that maps the four-tuple $(\alpha_{22}, \alpha_{17}, \alpha_1, \alpha_4)$ of roots to $(\alpha_1, \alpha_4, \alpha_{22}, \alpha_{17})$. Choose a σ-stable lift $\dot{z} \in \hat{W}$. Then $\mathbf{L}_\Delta^{\dot{z}} = \mathbf{L}_\Gamma$. Conjugation by $\dot{z}$ sends the elements n_1, $n_4 n_3^2$ of Lemma 8.9 to lifts m_1 and m_2 of s_{22} and s_{17}, respectively, satisfying the properties of Proposition 4.18(a). The Steinberg morphism $\sigma\dot{v}$ with $v = s_{22}s_{17}$ on $\mathbf{L}_\Delta$ is transformed to the Steinberg morphism $\sigma\dot{v}'$ with $v' = s_1 s_4$. Also, $\mathbf{M}^{\dot{z}}$ is obtained by twisting $\mathbf{L}_\Gamma$ with $s_{22}s_{17}$. As $w_0 = s_{22}s_{17}s_1s_4$, the group $\mathbf{M}^{\dot{z}}$ is a representative of the G-conjugacy class $\mathbf{M}_{14,4}$ of 2-split Levi subgroups corresponding to the closed subsystem with base $\{\alpha_1, \alpha_4\}$; see the discussion in Subsection 4.7. We may therefore replace $\mathbf{M}$ by this standard copy and use the notation of Proposition 4.18. By Lemma 8.9, the automorphism ω then acts on $\mathbf{M}$ as $\sigma\dot{v}'$ for some $\dot{v}' \in [\mathbf{M}, \mathbf{M}]^F = M'$. In particular, ω acts as a field automorphism on M', normalizes M_1 and M_2 and fixes m_1 and m_2.

We have $N_G(M, \theta) = N_G(M) = \langle M_1.2 \circ_d M_2.2, x\rangle$, where $x \in M$ centralizes $Z(M_1)$ and M_2 and acts as a diagonal automorphism on M_1 if $d = 2$. Moreover, $M_i.2$ is ω-invariant for $i = 1, 2$, as $\dot{v}' \in M' = M_1' \circ_d M_2'$. Put $N := M_1.2 \circ_d M_2.2$. Then $[N_G(M) : N] = d$ and $M \cap N =$

$M_1 \circ_d M_2$. Let $\hat{\theta}$ denote an extension of θ to $N_G(M)$, satisfying the additional properties exhibited in Proposition 4.18(c) in case $d = 2$. Let $\hat{\theta}_1\hat{\theta}_2$ with $\hat{\theta}_i \in \mathrm{Irr}(M_i.2)$ for $i = 1, 2$, denote an irreducible constituent of $\mathrm{Res}_N^{N_G(M)}(\hat{\theta})$, and put $\theta_i = \mathrm{Res}_{M_i}^{M_i.2}(\hat{\theta}_i)$, $i = 1, 2$. If $\mathrm{Res}_N^{N_G(M)}(\hat{\theta})$ is reducible, then $d = 2$ and $\hat{\theta}$ is induced from $\hat{\theta}_1\hat{\theta}_2$ by Proposition 4.18(c). If $\mathrm{Res}_N^{N_G(M)}(\hat{\theta})$ is irreducible, so is $\mathrm{Res}_{M\cap N}^{N_G(M)}(\hat{\theta}) = \mathrm{Res}_{M\cap N}^{M}(\theta)$, once more by Proposition 4.18(c). In either case, $\theta_i \in \mathrm{Irr}(M_i)$ for $i = 1, 2$ and $\theta_1\theta_2$ is an irreducible constituent of $\mathrm{Res}_{M\cap N}^{M}(\theta)$. Thus it suffices to show that $\hat{\theta}_1$ and $\hat{\theta}_2$ are ω-invariant, using Corollary 2.6 in case $\mathrm{Res}_N^{N_G(M)}(\hat{\theta})$ is irreducible and $d = 2$.

Recall that $M_1 = Z(M_1) \circ_d M_1'$, so that we can write $\theta_1 = \lambda_1\theta_1'$ for $\lambda_1 \in \mathrm{Irr}(Z(M_1))$ and $\theta_1' \in \mathrm{Irr}(M_1')$. If $\mathrm{Res}_{M\cap N}^{M}(\theta)$ is irreducible, θ_1 is ω-invariant. Otherwise, $\mathrm{Res}_{M\cap N}^{M}(\theta) = \theta_1\theta_2 + \theta_1^x\theta_2$ with $\theta_1^x \neq \theta_1$. Now $\theta_1^x = \lambda_1^x(\theta_1')^x = \lambda_1(\theta_1')^x$, as x centralizes $Z(M_1)$. Thus $(\theta_1')^x \neq \theta_1'$. As ω acts as a field automorphism on M_1' by Proposition 8.10, it fixes θ_1' by Lemma 8.11. In particular, ω does not interchange θ_1 and θ_1^x. As ω fixes $\theta_1\theta_2 + \theta_1^x\theta_2$, it must fix θ_1. It follows that in either case, ω fixes λ_1 and θ_1'. Now any extension of θ_1 to $M_1.2 = \langle Z(M_1), m_1\rangle \circ_d M_1'$ is of the form $\hat{\lambda}_1\theta_1'$ for some extension $\hat{\lambda}_1$ of λ to $\langle Z(M_1), m_1\rangle$. As ω fixes m_1, Lemma 2.4 implies that $\hat{\lambda}_1$ and thus $\hat{\lambda}_1\theta_1'$ are ω-invariant.

Write $\hat{\theta}_2'$ for the restriction of $\hat{\theta}_2$ to $\langle Z(M_2), m_2\rangle \circ_d M_2'$, and θ_2' for the restriction of θ to $Z(M_2) \circ_d M_2'$. If $\hat{\theta}_2'$ is irreducible, so is θ_2', as otherwise the two constituents of θ_2' would be m_2-conjugate as well as y-conjugate, which is impossible; see Proposition 4.18(b). An argument as in the previous paragraph, with x replaced by y, shows that $\hat{\theta}_2'$ is ω-invariant. As θ_2' is ω-invariant, Corollary 2.6, applied to the normal subgroups $\langle Z(M_2), m_2\rangle \circ_d M_2'$ and M_2 of $M_2.2$ implies that $\hat{\theta}_2$ is ω-invariant. This completes our proof. □

8.14. **Proof for the unipotent blocks.** Here, we assume that b is a unipotent ℓ-block of G of positive defect, and we write $G_1 := \mathbf{G}^{F_1} = F_4(p)$. If b is not the principal block, then $e \in \{1, 2, 4\}$. If $e = 4$, there are two such blocks, which are swapped by σ_1 if $p = 2$. In the other cases, there is a unique such block; see the references given in Subsection 5.1 and Table 1.

Under our hypothesis, $A = N_{\mathrm{Aut}(G)}(b) = \mathrm{Aut}(G) = G \rtimes \langle\sigma_1\rangle$, unless $p = 2$, $e = 4$ and b is a non-principal block; in the latter case, $A = N_{\mathrm{Aut}(G)}(b) = G \rtimes \langle F_1\rangle$. For the notation refer to Subsection 8.1.

We begin with the proof for the principal blocks in case $\ell > 3$.

Proposition 8.15. *Let $\ell > 3$ and let D denote a Sylow ℓ-subgroup of G. Assume that D is non-cyclic and let b be the principal ℓ-block of G. Then $N_A(D)$ fixes every element of $\mathrm{Irr}(N_G(D)/C_G(D))$, unless q is even and $e \in \{1, 2, 3, 6\}$. In the latter cases, the non-trivial orbits of $N_A(D)$ on $\mathrm{Irr}(N_G(D)/C_G(D))$ have length two, and the number of non-trivial orbits is as given in the following table.*

e	1	2	3	6
no.	7	7	6	6

PROOF. Let $\mathbf{S} := C_{\mathbf{G}}(D)$. Then $\mathbf{S}$ is an F-stable maximal torus of $\mathbf{G}$ as $\ell > 3$. In particular, $N_G(D) = N_G(\mathbf{S})$, and thus $N_G(D)/C_G(D) = (N_{\mathbf{G}}(\mathbf{S})/\mathbf{S})^F$.

Suppose that $\mathbf{S}$ is obtained from $\mathbf{T}_0$ by twisting with $w \in W$. We will find an element $\sigma_1' \in A$ with $A = G \rtimes \langle \sigma_1' \rangle$, and a σ_1'-stable inverse image $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of w, and choose $g \in \mathbf{G}$ with $F(g)g^{-1} = \dot{w}$. Then σ_1' commutes with $F\dot{w}$, and we obtain $N_A(D) = N_G(D) \rtimes \langle \omega \rangle$ by Lemma 8.3. From the considerations of Subsection 4.6 we obtain the following commutative diagram:

$$\begin{array}{ccc}
(N_{\mathbf{G}}(\mathbf{S})/\mathbf{S})^F & \xrightarrow{\quad\omega\quad} & (N_{\mathbf{G}}(\mathbf{S})/\mathbf{S})^F \\
\uparrow{\scriptstyle \mathrm{ad}_g} & & \uparrow{\scriptstyle \mathrm{ad}_g} \\
(N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0)^{F\dot{w}} & \xrightarrow[\quad\sigma_1'\quad]{} & (N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0)^{F\dot{w}}
\end{array}$$

We may thus replace the action of ω on $(N_{\mathbf{G}}(\mathbf{S})/\mathbf{S})^F$ by the action of σ_1' on $(N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0)^{F\dot{w}} = C_W(w)$.

Suppose first that $w \in W$ can be chosen to be σ_1-stable. This is the case if q is odd, as then $\sigma_1 = F_1$ acts trivially on W. If $e \in \{1, 2\}$, then $w \in Z(W)$ and thus is fixed by σ_1. If $e = 4$, a computation with CHEVIE [41] shows that there is a σ_1-stable element in the conjugacy class $D_4(a_1)$ of W, which gives rise to the maximal torus $\mathbf{S}$ of $\mathbf{G}$ with $S \cong [q^2 + 1] \times [q^2 + 1]$ (cf. Table 22). As σ_1 is a Steinberg morphism of $\mathbf{G}$, we may choose $\dot{w}$ to be σ_1-stable, and let $\sigma_1' := \sigma_1$ in these cases. The action of σ_1 on the set of conjugacy classes of $C_W(w)$ is trivial if q is odd, and if q is even it can be computed with CHEVIE. If $e = 4$, every conjugacy class of $C_W(w)$ is fixed by σ_1, and if $e \in \{1, 2\}$, there are exactly 7 pairs of σ_1-conjugate conjugacy classes of $C_W(w) = W$. This yields our claim.

It remains to consider the case that q is even and $e \in \{3, 6\}$. We give the proof in case $e = 3$; the other case is analogous. A CHEVIE computation shows that the W-conjugacy class $A_2 + \tilde{A}_2$, which gives

TABLE V. Action on weights for the principal block

R	$\mathrm{Out}_G(R, b_R)$	$\mathcal{W}$	Rem	#
R_{38}	2^3	8		2
R_{37}	$\mathrm{GL}_2(3)$	2		
R_{15}	$\mathrm{SL}_3(3)$	1		
R_{18}	$W(F_4)$	4		1
R_{21}	$(Q_8 \times Q_8).S_3$	11	$a = 1$	4
R_{21}	$(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)).2$	2	$a \geq 2$	
R_{22}	$\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)$	1	$a \geq 2$	
R_{35}, R_{36}	$(\mathrm{SL}_2(3) \times 2).2, (2 \times \mathrm{SL}_2(3)).2$	$4, 4$	$a \geq 2$	4

rise to the maximal torus $\mathbf{S}$ of $\mathbf{G}$ with $S = [q^2 + q + 1] \times [q^2 + q + 1]$ contains an element w such that $C_W(w)$ is σ_1-stable, but that this class does not contain any σ_1-stable element. Now let w be an element in the conjugacy class $A_2 + \tilde{A}_2$ such that $n^{-1}\sigma_1(w)n = w$ for some $n \in N_W(C_W(w))$. Choose an inverse image $\dot{n} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of n and put $\sigma_1' := \sigma_1\dot{n}$. Then σ_1' centralizes w, and we may assume that it also centralizes $\dot{w}$, as σ_1' is a Steinberg morphism of $\mathbf{G}$. A computation with CHEVIE shows that σ_1' has exactly 6 orbits of length two on the set of conjugacy classes of $C_W(w)$ and fixes the remaining conjugacy classes. This completes the proof. □

The following prepares the proof for the principal block in case $\ell = 3$.

Lemma 8.16. *Let b be the principal 3-block of G. Then every G-conjugacy class of b-relevant radical 3-subgroups of G contains an F_1-stable representative R such that $N_G(R) = RC_G(R)N_{G_1}(R)$. If $p = 2$, such a representative can be chosen to be σ_1-invariant, unless $R \in_G \{R_{35}, R_{36}\}$, in which case σ_1 maps R_{35} to a conjugate of R_{36}.*

PROOF. Fix a σ_1-stable 3C-element z_{C}. The existence of z_{C} is clear if p is odd. For $p = 2$, we can use GAP [34] to compute the class fusion of ${}^2F_4(2) = \mathbf{G}^{\sigma_1}$ into $F_4(2) = G_1 = \mathbf{G}^{F_1}$; this shows that the elements of order 3 of ${}^2F_4(2)$ fuse into the 3C-class of $F_4(2)$. Put $\mathbf{L} := C_{\mathbf{G}}(z_{\mathrm{C}})$. Then $\mathbf{L} = \mathbf{L}^1 \circ_3 \mathbf{L}^2$, and σ_1 swaps the two simple components $\mathbf{L}^i \cong \mathrm{SL}_3(\mathbb{F})$, $i = 1, 2$, if $p = 2$, and fixes them if p is odd; for the notation see Proposition 4.14(a). In particular, $\mathbf{L}^i$ is F_1-stable for $i = 1, 2$. Let $e' \in \{1, 2\}$ denote the order of 3 modulo p. Then there is a σ_1-stable e'-F_1-split maximal torus $\mathbf{T}$ of $\mathbf{L}$. Again, this is clear if p is odd. If $p = 2$, choose a maximal F_1-stable torus $\mathbf{T}_1 \leq \mathbf{L}^1$ with $\mathbf{T}_1^{F_1} \cong 3^2$, put $\mathbf{T}_2 := \sigma_1(\mathbf{T}_1)$ and $\mathbf{T} := \mathbf{T}_1\mathbf{T}_2$. Clearly, $N_G(\mathbf{T}) = TN_{G_1}(\mathbf{T})$. By Proposition 4.14(b) and in the notation of Subsection 6.1, we have $L = \langle L^1 \circ_3 L^2, x_{\mathrm{C}}\rangle$ with $x_{\mathrm{C}} \in T$.

Let Q denote the Sylow 3-subgroup of T. Then $N_L(Q)$ is σ_1-stable, and as $N_L(Q) = T.(S_3 \times S_3)$, we obtain a σ_1-stable Sylow 3-subgroup P of G with $Q \leq P$.

Suppose for the moment that $p = 2$, and consider $Q_0 := \mathbf{T}^{\sigma_1}$. This is an elementary abelian subgroup of $\mathbf{G}^{\sigma_1} \leq G$ of type $3\mathrm{C}^2$. By [4, Lemma 2.7(b)], the G-conjugacy class of Q_0 equals $(3\mathrm{C}^2)_1$. Put $C := C_{G_1}(Q_0)$. Then C is σ_1-stable, $C = \mathbf{T}^{F_1}.3 = 3^4.3$ by [4, Table 2], and $C =_{G_1} R_{37}$ by [4, Case (M_9), p. 567]. One can check that C contains a σ_1-stable elementary abelian subgroup of type $(3\mathrm{C}^3)_1$. Indeed, using the permutation representation of G_1 on 69 888 points given in [80] and GAP, we construct a copy of $\mathbf{T}^{F_1} \leq C$ in G_1. An explicit computation shows that $Z(C) = Q_0 = [C, C]$, and that the exponent of C equals 3. In particular, C/Q_0 is elementary abelian of order 27. Thus C has exactly 13 subgroups of order 27 containing Q_0, and 4 of these lie in $\mathbf{T}^{F_1}$. As $F_1 = \sigma_1^2$ fixes the elements of C, the orbits of σ_1 on the subgroups of C have lengths at most 2. In particular, σ_1 fixes an elementary abelian subgroup R of C of order 27 with $Q_0 \leq R \not\leq \mathbf{T}^{F_1}$. Let $y \in C \setminus \mathbf{T}^{F_1}$ such that $R = \langle Q_0, y\rangle$. Then $C_{G_1}(R) = C_C(y)$. A GAP computation shows that $|C_C(y')| = 3^3$ for every $y' \in C \setminus \mathbf{T}^{F_1}$, and thus $C_{G_1}(R) = R$. This implies that $R =_{G_1} (3\mathrm{C}^3)_1 =_{G_1} R_{15}$ by [4, Lemma 2.7, and proof of Proposition 3.11].

Let us return to the general case. Let R be a b-relevant radical 3-subgroup of G. We now consider the possibilities for R as listed in Table 1. If $R \in_G \{R_{15}, R_{21}\}$, we may assume that $R \leq G_1$. As $|\mathrm{Out}_{G_1}(R)| = |\mathrm{Out}_G(R)|$ we obtain our claim. Suppose that $p = 2$. Then we may even choose R to be σ_1-stable. Namely, if $R =_G R_{21}$, choose R as $R := D \circ_3 \sigma_1(D)$, where D is a Sylow 3-subgroup of $\mathrm{SU}_3(2) \leq (L^1 \cap F_4(2))$; see Lemma 6.3. If $R =_G R_{15}$, then R is of type $(3\mathrm{C}^3)_1$, and the claim follows from what we have noted above. Now suppose that $R =_G R_{37}$. It follows from [4, 6] that there is an elementary abelian subgroup $E \leq \mathbf{T}^{\sigma_1}$ of type $(3\mathrm{C}^2)_1$ such that $R := C_P(E) = Q.3$ is a radical subgroup of G of type R_{37}. To be more specific, if p is odd, refer to [6, Proofs of Cases (3.1), (3.2) and Table 9]. If $p = 2$, use [4, Case (M_9), p. 567], together with the considerations in the previous paragraph. Now R is σ_1-stable, as E is. Thus if R is conjugate to one of R_{18}, R_{37} or R_{38}, we may choose a σ_1-stable representative R in $N_G(\mathbf{T})$. In each case, $R \cap T = Q = C_R([R, R])$ is characteristic in R and in T. In particular, R is F_1-stable and $N_G(R) \leq N_G(\mathbf{T})$. This gives our result in these cases.

We are left with the case that $R \in_G \{R_{35}, R_{36}\}$. It suffices to prove one of these cases, say $R =_G R_{35}$. Here, we may choose a representative R such that $Z(R) = \langle z_{\mathrm{C}} \rangle$ with z_{C} as above. Hence $R \leq C_G(z_{\mathrm{C}}) = \langle L^1 \circ_3 L^1, x_{\mathrm{C}} \rangle$ and $N_G(\langle z_{\mathrm{C}} \rangle) = \langle C_G(z_{\mathrm{C}}), \gamma_C \rangle$, where $\gamma_{\mathrm{C}} \in G_1$; see Subsection 6.1. By the construction of R indicated in Subsections 6.2 and 6.4, we may assume $R = K \circ_3 D \leq L^1 \circ_3 L^2$, where $K \leq L^1$ and $D \leq L^2$ are F_1-stable and as in Lemma 6.3. In particular, R is F_1-stable and $\gamma_{\mathrm{C}} \in N_G(R)$. As $N_G(R) = \langle N_{L^1}(K) \circ_3 N_{L^2}(D), \gamma_{\mathrm{C}} \rangle$, the claim $N_G(R) = RC_G(R)N_{G_1}(R)$ is reduced to an analogous assertion in L^1 and L^2, which clearly holds. The final statement in case $p = 2$ is also clear, as σ_1 swaps L^1 and L^2. □

We can now give the proof for the principal block in case $\ell = 3$.

Proposition 8.17. *Let $\ell = 3$, and let b be the principal 3-block of G. If q is odd, σ_1 fixes every conjugacy class of b-weights. If q is even, σ_1 has exactly seven orbits of length two on the conjugacy classes of b-weights, and fixes the remaining G-conjugacy classes of b-weights.*

Proof. Let R be a b-relevant radical 3-subgroup of G.

Suppose first that q is odd. By Lemma 8.16 we may assume that σ_1 fixes R and hence $N_A(R) = N_G(R) \rtimes \langle \sigma_1 \rangle$. Let (R, b_R) be a b-Brauer pair. Then b_R is the principal ℓ-block of R and thus $N_G(R, b_R) = N_G(R)$. In particular, θ_R is the trivial character of $RC_G(R)$ and extends to the trivial character of $N_G(R)$. As we may also assume that $N_G(R) = RC_G(R)N_{G_1}(R)$ by Lemma 8.16, every element of $N_A(R)$ fixes the irreducible characters of $N_G(R)/RC_G(R) = \mathrm{Out}_G(R, b_R)$. Our assertion now follows from Lemma 8.5.

Suppose now that q is even. By Lemma 8.16, we may assume that R is σ_1-stable, unless $R =_G R_{35}$ or $R =_G R_{36}$. In the latter case, we choose σ_1-conjugate representatives in the classes containing R_{35}, respectively R_{36}. If $a \geq 2$, this gives rise to 4 non-trivial σ_1-orbits on the set of conjugacy classes of b-weights.

If R is σ_1-stable, we have to count the number of non-trivial σ_1-orbits on $\mathrm{Out}(R, b_R)$. These numbers are given in the last column of Table V. This table proves the proposition. □

We conclude this subsection with the proof for the non-principal unipotent blocks.

Proposition 8.18. *Let $e \in \{1, 2\}$ and let b be the unique non-principal unipotent ℓ-block of G of positive defect and let D be a defect group of b. If q is odd, $N_A(D)$ fixes every G-conjugacy class of b-weights. If q is even, $N_A(D)$ has a unique orbit of length two on the G-conjugacy* *classes of b-weights and fixes the remaining G-conjugacy classes of b-weights.*

Proof. Put $\mathbf{M}_0 := \mathbf{L}_{\{2,3\}}$ in the notation of Subsection 4.1. By twisting $\mathbf{M}$ suitably, we obtain an F-stable regular subgroup $\mathbf{M}$ of $\mathbf{G}$ such that $D := Z(M) =_G R_{10,\ell}$ is a defect group of b.

Let us now specify our choices. In case $e = 1$, let $g = 1 = \dot{w}$. In case $e = 2$, choose a σ_1-stable inverse image $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of w_0, and an element $g \in \mathbf{G}$ with $F(g)g^{-1} = \dot{w}$, put $\mathbf{M} := \mathbf{M}_0^g$, and $\omega := \mathrm{ad}_g \circ \sigma_1 \circ \mathrm{ad}_g^{-1} \in \mathrm{Aut}_1(\mathbf{G})$. As σ_1 stabilizes $\mathbf{M}_0$ and commutes with $F\dot{w}$, Lemma 8.3 yields $N_A(D) = \langle N_G(D), \omega\rangle$ with $\omega = \mathrm{ad}_g \circ \sigma_1 \circ \mathrm{ad}_g^{-1}$. We obtain the commutative diagram

$$\begin{array}{ccc} N_{\mathbf{G}}(\mathbf{M})^F & \xrightarrow{\ \omega\ } & N_{\mathbf{G}}(\mathbf{M})^F \\ {\scriptstyle \mathrm{ad}_g}\uparrow & & \uparrow{\scriptstyle \mathrm{ad}_g} \\ N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}} & \xrightarrow[\ \sigma_1\ ]{} & N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}} \end{array}$$

so that we can replace $N_G(D) = N_{\mathbf{G}}(\mathbf{M})^F$ and the action of $\langle\omega\rangle$ by $N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}}$ and the action of $\langle\sigma_1\rangle$. Now $\mathbf{M}$ is the standard e-split Levi subgroup of $\mathbf{G}$ corresponding to $\{\alpha_2, \alpha_3\}$ considered in Proposition 4.19 and $N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}} \cong N_{\mathbf{G}}(\mathbf{M})^F$.

The canonical character of b is unipotent, so has $Z(M)$ in its kernel. It thus suffices to look at the central quotient $N_{\mathbf{G}}(\mathbf{M}_0)^{F\dot{w}}/Z(\mathbf{M}_0^{F\dot{w}})$, whose structure can be determined with Proposition 4.19. This central quotient is of the form $\bar{N}' \times \mathrm{PSp}_4(q).d$ with $d = \gcd(2, q-1)$, where $\bar{N}' \cong D_8$ arises from the stabilizer $\langle s_8, s_{16}\rangle$ of $\{\alpha_2, \alpha_3\}$ in W. If q is odd, σ_1 acts trivially on $\mathrm{Irr}(\bar{N}')$, and if q is even, σ_1 has a unique orbit of length two on $\mathrm{Irr}(\bar{N}')$, as σ_1 swaps the two roots α_8 and α_{16}; see Table I. Since the b-weights are in bijection with $\mathrm{Irr}(\bar{N}')$, our claim follows. □

8.19. **The proof for $2.F_4(2)$.** Here, we let $G := F_4(2)$ and $\hat{G} := 2.G$, the exceptional double cover of G. The automorphism σ_1 of G lifts to an automorphism of $\hat{G}$, also denoted by σ_1.

Proposition 8.20. *Let $\ell \in \{3, 5, 7\}$ and let b be the non-principal ℓ-block of $\hat{G}$ of maximal defect.*

Then the non-trivial σ_1-orbits on the set of $\hat{G}$-conjugacy classes of b-weights have length 2, and the number of non-trivial orbits is as given in the table below. For $\ell = 3$, the four orbits of length 2 are distributed among the weight subgroups as indicated in the last column of

TABLE VI. Action on weights for the faithful 3-block of $2.F_4(2)$

R	$N_{\hat{G}}(R)/R$	$\mathcal{W}$	#
R_{38}	2^4	8	2
R_{37}	$2 \times \mathrm{GL}_2(3)$	2	
R_{15}	$2 \times \mathrm{SL}_3(3)$	1	
R_{18}	$2 \times W(F_4)$	4	1
R_{21}	$2^{\cdot}[(Q_8 \times Q_8) : S_3]$	2	1

Table VI.

ℓ	3	5	7
no.	4	6	6

PROOF. Wilson's Atlas [80] contains a representation of the group $G.2$. Thomas Breuer noticed that its construction also works for the group $\hat{G}.2$, providing a permutation representation on 279 552 points. Using this representation and GAP [34], it is straightforward to verify the claimed multiplicities. □

8.21. **Proof in case $p = 2$ and m' odd.** Let b, σ and s be as in Hypothesis 8.2. Assume further that $p = 2$, that m' is odd and that b is a non-unipotent block. By Proposition 5.13(a), this can only occur in the following cases: the G-class type of s is one of $(14, 1)$, $(14, 4)$, $(15, 1)$, $(15, 3)$ or $(15, 5)$, or $\ell > 3$ and the G-class type of s is one of $(4, 1)$ or $(4, 2)$. In the latter two cases, $e \in \{1, 2, 3\}$ and $e \in \{1, 2, 6\}$, respectively.

Proposition 8.22. *Assume the hypothesis and notation introduced at the beginning of this subsection.*

Then the non-trivial orbits of $N_A(D)$ on the set of conjugacy classes of b-weights have length two, and the number of such orbits equals 3 *in case s has class type* $(4, 1)$ *and* $e \in \{1, 3\}$ *or* $(4, 2)$ *and* $e \in \{2, 6\}$, *and* 1 *in all other cases.*

PROOF. Suppose first that the G-class type of s is $(4, 1)$ or $(4, 2)$. We may assume that $\sigma_1(s) = s$, so that $\sigma = \sigma_1$. Moreover, σ fixes $C_{\mathbf{G}}(s)$ swapping its two simple components $\mathbf{L}^1$ and $\mathbf{L}^2$. By the description of the defect groups in Table 4, this implies that there is a σ_1-stable maximal torus $\mathbf{S}$ of $C_{\mathbf{G}}(s)$, such that the Sylow ℓ-subgroup D of S is a defect group of b. In fact $\mathbf{S} = \mathbf{S}_1\mathbf{S}_2$, where $\mathbf{S}_i$ is a maximal torus of $\mathbf{L}^i$ and σ swaps $\mathbf{S}_1$ with $\mathbf{S}_2$. Moreover, $\mathbf{S}$ is its own dual with respect to σ. Hence there is a σ-stable b-Brauer pair (D, b_D) by Lemma 8.6. Now $\mathrm{Out}_G(D, b_D)$ may be identified with $W_{C_{\mathbf{G}}(s)}(\mathbf{S})^F$ by Proposition 3.16, and $W_{C_{\mathbf{G}}(s)}(\mathbf{S})^F = W_{\mathbf{L}^1}(\mathbf{S}_1)^F \times W_{\mathbf{L}^2}(\mathbf{S}_2)^F$, where σ swaps the two factors $W_{\mathbf{L}^1}(\mathbf{S}_1)^F$ and $W_{\mathbf{L}^2}(\mathbf{S}_2)^F$. This gives our claim.

The proof in case of class types $(14, 1)$ and $(14, 4)$ is analogous.

It remains to consider the case that the G-class type of s is one of $(15, 1)$, $(15, 3)$ or $(15, 5)$. By the facts summarized in Hypothesis 8.2, we have that $m = m'$, that m divides f and that $\sigma^2 = F_1^m$. We proceed as in Proposition 8.10, but only sketch the arguments. As $p = 2$, the elements $n_j \in N_G(\mathbf{T}_0)$, $j = 1, \ldots, 4$, have order 2. We may and will thus identify $W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ with the subgroup of $N_{\mathbf{G}}(\mathbf{T}_0)$ generated by n_j, $j = 1, \ldots, 4$. Let $\mathbf{L}_0 := \mathbf{L}_{\{\alpha_2,\alpha_3\}} =_G \mathbf{M}_{15,1}$. If the G-class type of s is $(15, 1)$, take $v = 1$. If the G-class type of s is $(15, 3)$ or $(15, 5)$, let $v = s_8$. Then v centralizes s_2 and s_3. respectively. Notice that $\sigma^2 = F_1^m$ fixes $\sigma(v)v$, and if $g \in \mathbf{G}$ with $\sigma(g)g^{-1} = v$, we have $F(g)g^{-1} = (\sigma(v)v)^{f/m}$. By σ-twisting $\mathbf{L}_0$ with $v^\dagger$, we obtain σ-stable regular subgroups $\mathbf{L}^\dagger$ of F-type $(\Gamma^\dagger, [(\sigma(v^\dagger)v^\dagger)^{f/m}])$, where Γ is the parabolic subsystem of Σ spanned by $\{\alpha_2, \alpha_3\}$. If $v = s_8$, then $v^\dagger = s_{16}$, and $(\sigma(v^\dagger)v^\dagger) = s_8 s_{16}$ lies in conjugacy class 23 of W. Using Lemma 4.8, the 2-power map on W given in Table 22, and the fusion of the maximal tori of Table 24, we conclude that $\mathbf{L}^\dagger =_G \mathbf{M}_{15,5}$ if f/m is odd, and that $\mathbf{L}^\dagger =_G \mathbf{M}_{15,3}$ if $f/m \equiv 0(4)$. By the results summarized in Subsection 4.7, the centralizer of every semisimple σ-stable element of these G-class types arises in this way. We may thus identify $\mathbf{L}^\dagger$ with $C_{\mathbf{G}}(s)$.

By σ-twisting $\mathbf{L}_0$ with v, we obtain a σ-stable regular subgroup $\mathbf{L}$ such that $(\mathbf{L}^\dagger, \sigma)$ and $(\mathbf{L}, \sigma)$ are in duality. If $\ell \mid q - 1$, let $w = 1$. Otherwise, let w be the longest element of $W_{\mathbf{L}_0}(\mathbf{T}_0) = W_{\{\alpha_2,\alpha_3\}}$. By first σ-twisting $\mathbf{T}_0$ with $v^\dagger$, respectively v, and then F-twisting with $w^\dagger$, respectively w, as in the proof of Proposition 8.10, we obtain a pair of F-stable maximal tori $\mathbf{S}^\dagger \leq \mathbf{L}^\dagger$ and $\mathbf{S} \leq \mathbf{L}$, and a pair of Steinberg morphisms $\omega^\dagger$ and ω such that $\omega^\dagger$ fixes $\mathbf{S}^\dagger$ and ω fixes $\mathbf{S}$ and such that $(\mathbf{S}^\dagger, \omega^\dagger)$ and $(\mathbf{S}, \omega)$ as well as $(\mathbf{S}^\dagger, F)$ and $(\mathbf{S}, F)$ are in duality. Let $D^\dagger$ and D denote the Sylow ℓ-subgroups of $S^\dagger$ and S, respectively, and put $\mathbf{M}^\dagger := C_{\mathbf{G}}(D^\dagger)$ and $\mathbf{M} := C_{\mathbf{G}}(D)$. Then D is a defect group of b, and we obtain an ω-stable b-Brauer pair (D, b_D) by Lemma 8.6. Moreover, $N_A(D) = \langle N_G(D), \omega \rangle$ by Lemma 8.3. Finally, ω acts as the exceptional isogeny on $M' = [\mathbf{M}, \mathbf{M}]^F \cong \mathrm{Sp}_4(q)$.

By Proposition 4.19, we have $M = Z(M) \times M'$. Notice that $S \leq M$ and that $N_G(D, b_D) = N_M(S) = S.W_M$, a split extension with an

ω-stable subgroup $W_M \cong \mathrm{Out}_M(S) \cong D_8$. In particular, the canonical character of b_D has an ω-invariant extension to $N_G(D, b_D)$. As ω interchanges two generators of W_M, our claim follows. □

8.23. **The equivariance condition.** We can now prove the second main result of this article.

Theorem 8.24. *Let $G = F_4(q)$ and let ℓ be an odd prime with $\ell \nmid q$. Then G satisfies the inductive blockwise Alperin weight condition at the prime ℓ.*

Proof. We have to verify the conditions of Hypothesis 2.15 for all ℓ-blocks of G, and an analogous set of conditions for the ℓ-blocks of the double cover $2.F_4(2)$ of $F_4(2)$ containing faithful characters. For blocks with cyclic defect groups the inductive Alperin weight condition is known to hold by [54], so that it holds for all blocks of $2.F_4(2)$ containing faithful characters by Propositions 5.15 and 8.20.

Let b denote an ℓ-block of G with a non-cyclic defect group. In Theorem 7.1 we have already verified Condition (1) of Hypothesis 2.15. Thus it remains to verify the equivariance Condition (2) of this hypothesis. If $l(b) = 1$, equivariance trivially holds. We will thus assume that $l(b) > 1$ in the following. The orbits of $N_{\mathrm{Aut}(G)}(b)$ are described in Propositions 5.13, and we will assume these results in what follows.

Suppose that b is the principal ℓ-block. For $\ell = 3$, equivariance follows from Proposition 8.17, and for $\ell > 3$ from Corollary 5.14 and Proposition 8.15. Now let b be a unipotent non-principal block. Then $e \in \{1, 2\}$ and equivariance follows from Proposition 8.18 in view of the action of σ_1 on the set of unipotent characters of G described in [61, Theorem 2.5]).

From now on we assume that b is a non-unipotent block.

In case $p = 2$ and b is stabilized by some odd power of σ_1, Proposition 8.22 establishes our assertion. We may thus assume $\sigma = F_1^m$. In this case, Proposition 8.13 reduces to the instances listed in Proposition 8.10(a),(b). In the latter cases, there is at least one ω-invariant extension of θ to $N_G(D, b_D)$. Thus the ω-orbits on the set of b-weights correspond to the ω-orbits on $\mathrm{Irr}(\mathrm{Out}_G(D, b_D))$. By Brauer's permutation lemma, the number of such orbits equals the number of orbits of ω on $\mathrm{Out}_G(D, b_D)$. As there are only two orbit lengths, the total number of orbits determines the number of orbits of length two. □

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9. Appendix

9.1. **The ℓ-blocks of $F_4(q)$ and their invariants.** This first subsection of the appendix gives information on the ℓ-blocks of the group $G = F_4(q)$ for odd primes ℓ not dividing q. There is one table, which may consist of several parts, for 19 out of the 20 geometric class types of semisimple ℓ'-elements, where the table number agrees with the number of the geometric class type in [58]. There is no table for the geometric class type 20, as here the centralizers are maximal tori, and thus the corresponding blocks have only one irreducible Brauer character. As the $\mathbf{G}$-class type 4 corresponds to elements of order 3, Table 4 is only valid for primes $\ell > 3$. There is an additional Table 21, which gives the relevant information for those ℓ-blocks of $2.F_4(2)$ with non-cyclic defect groups which contain faithful characters.

Recall that G and its dual group G^* are identified. Let $s \in G$ be a semisimple ℓ'-element of G-class type (i, k) and let b be an ℓ-block inside $\mathcal{E}_\ell(G, s)$ of positive defect. Then Table i describes b and some of its invariants, unless the Sylow ℓ-subgroups of $C_G(s)$ are cyclic.

We use the following notation. First of all, $e = e_\ell(q)$ denotes the order of q in the multiplicative group of $\mathbb{F}_\ell$. Thus, only the cases $e \in \{1, 2, 3, 4, 6\}$ are relevant, as otherwise the Sylow ℓ-subgroups of G are cyclic. Moreover, $e > 2$ only occurs in Tables 1, 2 and 4, as otherwise the Sylow ℓ-subgroups of $C_G(s)$ are cyclic. In all other cases, $e \in \{1, 2\}$, and then we let the parameter $\varepsilon \in \{-1, 1\}$ be such that $\ell \mid q - \varepsilon$. Also, if ℓ is understood from the context, a denotes the positive integer such that ℓ^a is the highest power of ℓ dividing $\Phi_e(q)$ (recall that Φ_e denotes the eth cyclotomic polynomial). For example, if $\ell = 3$, the 3-part $|G|_3$ of $|G|$ equals 3^{4a+2}.

Let us now explain the structure and the contents of the tables. These have up to 13 columns, numbered $1, \ldots, 13$ in the first row of the table. The column number determines the content type of the column; if this content is not relevant for a table, the corresponding column is omitted. The second row of the tables contains the column headings, and the remaining rows between the second and last set of double rules constitute the body of the tables, containing the desired information on the blocks.

The contents of the table is separated by horizontal rules according to the following scheme: Rules beginning in Column 6 separate the information for different ranges of ℓ, if necessary. Rules beginning in Column 5 separate the information for different blocks; it turns out that $\mathcal{E}_\ell(G, s)$ contains at most two blocks of positive defect, except in the case $e = 4$. Rules beginning in Column 3 separate the information for the two values of ε in case $e \in \{1, 2\}$. If such a rule is present, Column 2 usually contains two values for k, say k_1, k_2, separated by a comma. The first set of rows between this horizontal rule beginning in Column 3 corresponds to k_1, if $\varepsilon = 1$, and to k_2, otherwise. Likewise, the second set of rows corresponds to k_2, if $\varepsilon = 1$, and to k_1, otherwise. An analogous convention is used if there are two entries in Column 9. If the entries in Column 2 are separated by a slash, there is a corresponding and matching pair of entries in Column 9. If their are four entries in Column 2, they are grouped in pairs, separated by a slash, and the conventions above apply to each of these pairs. Finally, a rule beginning in Column 2 separates the information for the different values of k. The values of k are contained in the second column. An entry in a row of the body of the table is effective for all rows below it up to the next entry or to a separating horizontal rule.

Columns 3 and 4 describe the structure of the centralizer $C := C_G(s)$ by printing the order of $Z := Z(C)$, as well as the components of $[C_{\mathbf{G}}(s), C_{\mathbf{G}}(s)]^F$, under the heading $[C, C]$. These components are described by their Dynkin type. We use the convention that $\tilde{A}_j$ designates a component of type A_j corresponding to short root elements. Finally, we write $A_j^{\varepsilon}(q)$ and $\tilde{A}_j^{\varepsilon}(q)$ to denote a linear group if $\varepsilon = 1$, and a unitary group if $\varepsilon = -1$.

Columns 5, 6 and 7 give a label for b, its defect $d(b)$ and the number $l(b)$ of its irreducible Brauer characters, respectively. The label of b describes an e-cuspidal pair associated to b in the following way. Suppose first that s is not quasi-isolated, i.e. the centralizer $C_{\mathbf{G}}(s)$ is contained in a proper Levi subgroup of $\mathbf{G}$. This is the case if and only if s belongs to a class type $i \geq 6$. As all proper regular subgroups of $\mathbf{G}$ are of classical type, we may apply Theorem 3.9. Thus b corresponds to a unipotent block b' of $\mathcal{E}_\ell(C_G(s), 1)$, and the label of b is taken to be the e-cuspidal pair (L, ζ) associated to b'; see [16, Theorem]. This label is given as L, ζ in Column 5 of Tables 1–19, unless L is a torus (and thus ζ the trivial character), where we just write 1 and suppress L. If $L = B_2(q)$, we write ζ_1 for the cuspidal unipotent character, ζ_2 for the 2-cuspidal unipotent character labelled by the bipartition $(1, 1)$, and ζ_4, ζ_4' for the two 4-cuspial unipotent characters labelled by the bipartitions $(1^2, -)$ and $(-, 2)$, respectively. In case of $L = A_2^{\varepsilon}(q)$, we write $(2, 1)$ for the e-cuspidal unipotent character labelled by the partition $(2, 1)$. If s is quasi-isolated, i.e. $s \in \{1, 2, 3, 5\}$, the label of b describes the e-cuspidal pair of G defining b, following [27, p. 349] (for $i = 1$) and [51, Table 2] (for $i \in \{2, 3, 5\}$), respectively.

Column 8 describes the isomorphism type of a defect group $D(b)$ of b, in case $D(b)$ is abelian. If $D(b)$ is non-abelian, then $\ell = 3$ and $D(b)$ is G-conjugate to a Sylow 3-subgroup of a group $M_{i',k'}$ dual to $C_G(s)$; in this case $D(b)$ is identified by the pair (i', k'). (In case $i = 13$ or $i = 17$ and $k \in \{2, 3\}$, we use the convention that (i', k') is such that $\{i, i'\} = \{13, 17\}$ and $\{k, k'\} = \{2, 3\}$ but $i \neq i'$ and $k \neq k'$.)

Column 9 contains the weight subgroups of b, described by their conjugacy class in the set of radical ℓ-subgroups. The first row corresponding to b gives its defect group. The radical 3-subgroups occurring as weight subgroups in the tables below are named according to Table 26. In order to be consistent with this labelling, the defect groups in case $\ell > 3$ and $e \in \{1, 2\}$ are named as $R_{j,\ell}$, with $j \in \{2, 3, 9, 10, 11, 12, 16, 17, 18\}$. These groups are Sylow ℓ-subgroups of suitable maximal tori $M_{20,k}$, with j and k as in the following table, where the two cases for k in some of the columns correspond to the cases $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

j	2	3	9	10	11	12	16	17	18
k	21	16	22	3	7, 8	4, 5	12, 13	17, 18	1, 2

If the description of a block is restricted to the case $\ell = 3$, the second index in $R_{j,\ell}$ is omitted. In the tables for $e > 2$, we give the defect groups as Sylow ℓ-subgroups of a maximal torus $M_{20,k}$, denoted by $S_{k,\ell}$.

Column 10 gives the structure of $\mathrm{Out}_G(R, b_R) = N_G(R, b_R)/RC_G(R)$ for the weight subgroups R of b, and Column 11, headed by $\mathcal{W}$ contains the numbers $\mathcal{W}(R, b_R)$; see (2) in Subsection 2.10 for this notation. In Columns 12 and 13, if present, we remark a case distinction between $a = 1$ and $a \geq 2$, respectively between $\ell = 3$, $\ell > 3$ and $\ell \geq 3$.

Our naming of the groups follows standard conventions. For example, S_3, D_8 and Q_8 designate the symmetric group on three letter, the dihedral and the quaternion group, respectively, of order 8. Further, $W(X_j)$ denotes the Weyl group of the root system X_j.

TABLE 1. The unipotent ℓ-blocks of $F_4(q)$ of positive defect

$\ell = 3$

1	2	3	4	5	6	7	8	9	10	11	12
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	Rem
1	1	1	$F_4(q)$	1	$4a+2$	26	$(1,1)$	R_{38}	2^3	8	
								R_{37}	$\mathrm{SL}_2(3).2$	2	
								R_{15}	$\mathrm{SL}_3(3)$	1	
								R_{18}	$W(F_4)$	4	
								R_{21}	$(Q_8 \times Q_8).S_3$	11	$a = 1$
								R_{21}	$(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)).2$	2	$a \geq 2$
								R_{22}	$\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)$	1	$a \geq 2$
								R_{35}	$(\mathrm{SL}_2(3) \times 2).2$	4	$a \geq 2$
								R_{36}	$(2 \times \mathrm{SL}_2(3)).2$	4	$a \geq 2$
				$B_2(q), \zeta_e$	$2a$	5	$[3^a]^2$	R_{10}	D_8	5	

TABLE 1. The unipotent ℓ-blocks of $F_4(q)$ of positive defect (cont.)

$e = 1, 2, \quad \ell > 3$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
1	1	1	$F_4(q)$	1	$4a$	26	$[\ell^a]^4$	$R_{18,\ell}$	$W(F_4)$	26
				$B_2(q), \zeta_e$	$2a$	5	$[\ell^a]^2$	$R_{10,\ell}$	D_8	5

$e = 3, 6$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
1	1	1	$F_4(q)$	1	$2a$	21	$[\ell^a]^2$	$S_{9,\ell}, S_{10,\ell}$	$\mathrm{SL}_2(3) \times 3$	21

$e = 4$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
1	1	1	$F_4(q)$	1	$2a$	16	$[\ell^a]^2$	$S_{6,\ell}$	$\mathrm{SL}_2(3).[4]$	16
				$B_2(q), \zeta_4$	a	4	$[\ell^a]$	$R_{10,\ell}$	$[4]$	4
				$B_2(q), \zeta_4'$	a	4	$[\ell^a]$	$R_{10,\ell}$	$[4]$	4

TABLE 2. The ℓ-blocks of $F_4(q)$ of geometric type 2 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
2	1	2	$B_4(q)$	1	$4a+1$	20	$(5,1)$	R_{33}	2^3	8	3
								R_{25}	$(2 \times \mathrm{SL}_2(3)).2$	4	
								R_{18}	$W(B_4)$	8	
					$4a$	20	$[\ell^a]^4$	$R_{18,\ell}$	$W(B_4)$	20	> 3
				$B_2(q), \zeta_e$	$2a$	5	$[\ell^a]^2$	$R_{10,\ell}$	D_8	5	≥ 3

$e = 4$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
2	1	2	$B_4(q)$	1	$2a$	14	$[\ell^a]^2$	$S_{6,\ell}$	$[4]^2.2$	14
				$B_2(q), \zeta_4$	a	4	$[\ell^a]$	$R_{10,\ell}$	$[4]$	4
				$B_2(q), \zeta_4'$	a	4	$[\ell^a]$	$R_{10,\ell}$	$[4]$	4

TABLE 3. The ℓ-blocks of $F_4(q)$ of geometric type 3 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
3	$1, 2$	4	$A_3^{\varepsilon}(q)\tilde{A}_1(q)$	1	$4a+1$	10	$(5, 1)$	R_{33}	2^2	4	3
								R_{25}	$2 \times \mathrm{SL}_2(3)$	2	
								R_{18}	$W(A_3) \times 2$	4	
					$4a$	10	$[\ell^a]^4$	$R_{18,\ell}$	$W(A_3) \times 2$	10	> 3
			$A_3^{-\varepsilon}(q)\tilde{A}_1(q)$	1	$3a$	10	$[\ell^a]^3$	$R_{16,\ell}$	$D_8 \times 2$	10	≥ 3

TABLE 4. The ℓ-blocks of $F_4(q)$ of geometric type 4 (exist only when $3 \nmid q$ and $\ell > 3$)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
4	1, 2	3	$A_2^{\varepsilon}(q)\tilde{A}_2^{\varepsilon}(q)$	1	$4a$	9	$[\ell^a]^4$	$R_{18,\ell}$	$S_3 \times S_3$	9
			$A_2^{-\varepsilon}(q)\tilde{A}_2^{-\varepsilon}(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{9,\ell}$	2^2	4

$e = 3$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
4	1	3	$A_2(q)\tilde{A}_2(q)$	1	$2a$	9	$[\ell^a]^2$	$S_{9,\ell}$	3^2	9

$e = 6$

1	2	3	4	5	6	7	8	9	10	11
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$
4	2	3	$A_2^{-1}(q)\tilde{A}_2^{-1}(q)$	1	$2a$	9	$[\ell^a]^2$	$S_{10,\ell}$	3^2	9

TABLE 5. The ℓ-blocks of $F_4(q)$ of geometric type 5 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
5	1	2	$C_3(q)A_1(q)$	1	$4a+1$	20	$(2,1)$	R_{34}	2^3	8	3
								R_{26}	$(\mathrm{SL}_2(3) \times 2).2$	4	
								R_{18}	$W(C_3) \times 2$	8	
					$4a$	20	$[\ell^a]^4$	$R_{18,\ell}$	$W(C_3) \times 2$	20	> 3
				$B_2(q), \zeta_e$	$2a$	4	$[\ell^a]^2$	$R_{10,\ell}$	2^2	4	≥ 3

TABLE 6. The ℓ-blocks of $F_4(q)$ of geometric type 6

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	12	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	Rem	ℓ
6	$1, 2$	$q - \varepsilon$	$B_3(q)$	1	$4a + 1$	10	$(10, k)$	R_{33}	2^2	4		3
								R_{25}	$\mathrm{SL}_2(3).2$	2		
								R_{18}	$W(C_3)$	4		
					$4a$	10	$[\ell^a]^4$	$R_{18,\ell}$	$W(C_3)$	10		> 3
				$B_2(q), \zeta_e$	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}$	2	2		≥ 3
		$q + \varepsilon$	$B_3(q)$	1	$3a + 1$	10	$(10, k)$	R_{31}	2^2	4		3
								R_{23}	$\mathrm{SL}_2(3).2$	2	$a \geq 2$	
								R_{19}	$\mathrm{SL}_2(3).2$	2	$a = 1$	
								R_{16}	$W(C_3)$	4		
					$3a$	10	$[\ell^a]^3$	$R_{16,\ell}$	$W(C_3)$	10		> 3
				$B_2(q), \zeta_e$	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2		≥ 3

TABLE 7. The ℓ-blocks of $F_4(q)$ of geometric type 7

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\lvert Z\rvert$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
7	$1, 2$	$q - \varepsilon$	$A_2^{\varepsilon}(q)\tilde{A}_1(q)$	1	$4a + 1$	6	$(9, k)$	R_{33}	2^2	4	3
								R_{25}	$2 \times \mathrm{SL}_2(3)$	2	
					$4a$	6	$[\ell^a]^4$	$R_{18,\ell}$	$2 \times S_3$	6	> 3
		$q + \varepsilon$	$A_2^{-\varepsilon}(q)\tilde{A}_1(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{9,\ell}$	2^2	4	≥ 3
				$A_2^{-\varepsilon}(q), (2, 1)$	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2	≥ 3

TABLE 8. The ℓ-blocks of $F_4(q)$ of geometric type 8 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	12	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	Rem	ℓ
8	$1, 4$	$2(q-\varepsilon)$	$A_3^{\varepsilon}(q)$	1	$4a+1$	5	$(5, 1)$	R_{33}	2	2		3
								R_{25}	$\mathrm{SL}_2(3)$	1		
								R_{18}	$W(A_3)$	2		
					$4a$	5	$[\ell^a]^4$	$R_{18,\ell}$	$W(A_3)$	2		> 3
		$2(q+\varepsilon)$	$A_3^{-\varepsilon}(q)$	1	$2a$	5	$[\ell^a]^2$	$R_{10,\ell}$	D_8	5		≥ 3
	$2, 3$	$2(q+\varepsilon)$	$A_3^{\varepsilon}(q)$	1	$3a+1$	5	$(10, 2), (10, 1)$	R_{31}	2	2		3
								R_{23}	$\mathrm{SL}_2(3)$	1	$a \geq 2$	
								R_{19}	$\mathrm{SL}_2(3)$	1	$a = 1$	
								R_{16}	$W(A_3)$	2		
					$3a$	5	$[\ell^a]^3$	$R_{16,\ell}$	$W(A_3)$	2		> 3
		$2(q-\varepsilon)$	$A_3^{-\varepsilon}(q)$	1	$3a$	5	$[\ell^a]^3$	$R_{16,\ell}$	D_8	5		≥ 3

TABLE 9. The ℓ-blocks of $F_4(q)$ of geometric type 9

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\lvert Z\rvert$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
9	$1, 2$	$q-\varepsilon$	$\tilde{A}_2^{\varepsilon}(q)A_1(q)$	1	$4a+1$	6	$(7, k)$	R_{34}	2^2	4	3
								R_{26}	$\mathrm{SL}_2(3)\times 2$	2	
					$4a$	6	$[\ell^a]^4$	$R_{18,\ell}$	$S_3\times 2$	6	>3
		$q+\varepsilon$	$\tilde{A}_2^{-\varepsilon}(q)A_1(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{9,\ell}$	2^2	4	≥ 3
				$\tilde{A}_2^{-\varepsilon}(q), (2, 1)$	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2	≥ 3

TABLE 10. The ℓ-blocks of $F_4(q)$ of geometric type 10

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	12	13
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	Rem	ℓ
10	$1,2$	$q-\varepsilon$	$C_3(q)$	1	$4a+1$	10	$(6,k)$	R_{34}	2^2	4		3
								R_{26}	$\mathrm{SL}_2(3).2$	2		
								R_{18}	$W(C_3)$	4		
					$4a$	10	$[\ell^a]^4$	$R_{18,\ell}$	$W(C_3)$	10		> 3
				$B_2(q), \zeta_e$	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}$	2	2		≥ 3
		$q+\varepsilon$	$C_3(q)$	1	$3a+1$	10	$(6,k)$	R_{32}	2^2	4		3
								R_{24}	$\mathrm{SL}_2(3).2$	2	$a \geq 2$	
								R_{20}	$\mathrm{SL}_2(3).2$	2	$a = 1$	
								R_{17}	$W(C_3)$	4		
					$3a$	10	$[\ell^a]^3$	$R_{17,\ell}$	$W(C_3)$	10		> 3
				$B_2(q), \zeta_e$	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2		≥ 3

TABLE 11. The ℓ-blocks of $F_4(q)$ of geometric type 11 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
11	1, 2	$2(q-\varepsilon)$	$C_2(q)A_1(q)$	1	$4a$	10	$[\ell^a]^4$	$R_{18,\ell}$	$D_8 \times 2$	10	≥ 3
				$B_2(q), \zeta_e$	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}$	2	2	≥ 3
		$2(q+\varepsilon)$	$C_2(q)A_1(q)$	1	$3a$	10	$[\ell^a]^3$	$R_{17,\ell}$	$D_8 \times 2$	10	≥ 3
				$B_2(q), \zeta_e$	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2	≥ 3

TABLE 12. The ℓ-blocks of $F_4(q)$ of geometric type 12 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\lvert Z\rvert$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
12	$1, 3$	$2(q-\varepsilon)$	$A_1(q)A_1(q)\tilde{A}_1(q)$	1	$4a$	8	$[\ell^a]^4$	$R_{18,\ell}$	2^3	8	≥ 3
		$2(q+\varepsilon)$	$A_1(q)A_1(q)\tilde{A}_1(q)$	1	$3a$	8	$[\ell^a]^3$	$R_{16,\ell}$	2^3	8	≥ 3
	$2, 4$	$2(q-\varepsilon)$	$A_1(q^2)\tilde{A}_1(q)$	1	$3a$	4	$[\ell^a]^3$	$R_{16,\ell}$	2^2	4	≥ 3
		$2(q+\varepsilon)$	$A_1(q^2)\tilde{A}_1(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{10,\ell}$	2^2	4	≥ 3

TABLE 13. The ℓ-blocks of $F_4(q)$ of geometric type 13

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	12	13
i	k	$\|Z\|$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	Rem	ℓ
13	$1,6$	$(q-\varepsilon)^2$	$A_2^{\varepsilon}(q)$	1	$4a+1$	3	$(17,k)$	R_{33}	2	2		3
								R_{25}	$\mathrm{SL}_2(3)$	1		
					$4a$	3	$[\ell^a]^4$	$R_{18,\ell}$	S_3	3		>3
		$(q+\varepsilon)^2$	$A_2^{-\varepsilon}(q)$	1	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2		≥ 3
	$2,3$	q^2-1	$A_2^{\varepsilon}(q)$	1	$3a+1$	3	$(17,k')$	R_{31}	2	2		3
								R_{23}	$\mathrm{SL}_2(3)$	1	$a \geq 2$	
								R_{19}	$\mathrm{SL}_2(3)$	1	$a = 1$	
					$3a$	3	$[\ell^a]^3$	$R_{16,\ell}$	S_3	3		>3
		q^2-1	$A_2^{-\varepsilon}(q)$	1	$2a$	2	$[\ell^a]^2$	$R_{9,\ell}$	2	2		≥ 3
				$A_2^{-\varepsilon}(q), (2,1)$	a	1	$[\ell^a]$	$R_{2,\ell}$	1	1		≥ 3
	$5,4$	$q^2+\varepsilon q+1$	$A_2^{\varepsilon}(q)$	1	$2a+2$	3	$(17,k)$	R_{29}	2	2		3
								R_{19}	$\mathrm{SL}_2(3)$	1		
					$2a$	3	$[\ell^a]^2$	$R_{12,\ell}$	S_3	3		>3
		$q^2-\varepsilon q+1$	$A_2^{-\varepsilon}(q)$	1	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2		≥ 3

TABLE 14. The ℓ-blocks of $F_4(q)$ of geometric type 14

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\|Z\|$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
14	1, 4	$(q-\varepsilon)^2$	$A_1(q)\tilde{A}_1(q)$	1	$4a$	4	$[\ell^a]^4$	$R_{18,\ell}$	2^2	4	≥ 3
		$(q+\varepsilon)^2$	$A_1(q)\tilde{A}_1(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{9,\ell}$	2^2	4	≥ 3
	2	q^2-1	$A_1(q)\tilde{A}_1(q)$	1	$3a$	4	$[\ell^a]^3$	$R_{16,\ell}, R_{17,\ell}$	2^2	4	≥ 3
	3	q^2-1	$A_1(q)\tilde{A}_1(q)$	1	$3a$	4	$[\ell^a]^3$	$R_{17,\ell}, R_{16,\ell}$	2^2	4	≥ 3

TABLE 15. The ℓ-blocks of $F_4(q)$ of geometric type 15

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\lvert Z\rvert$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
15	$1, 3$	$(q-\varepsilon)^2$	$C_2(q)$	1	$4a$	5	$[\ell^a]^4$	$R_{18,\ell}$	D_8	5	≥ 3
				$B_2(q), \zeta_e$	$2a$	1	$[\ell^a]^2$	$R_{10,\ell}$	1	1	≥ 3
		$(q+\varepsilon)^2$	$C_2(q)$	1	$2a$	5	$[\ell^a]^2$	$R_{10,\ell}$	D_8	5	≥ 3
	$2/4$	q^2-1	$C_2(q)$	1	$3a$	5	$[\ell^a]^3$	$R_{17,\ell}/R_{16,\ell}$	D_8	5	≥ 3
				$B_2(q), \zeta_2$	a	1	$[\ell^a]$	$R_{3,\ell}/R_{2,\ell}$	1	1	≥ 3
	5	q^2+1	$C_2(q)$	1	$2a$	5	$[\ell^a]^2$	$R_{10,\ell}$	D_8	5	≥ 3

TABLE 16. The ℓ-blocks of $F_4(q)$ of geometric type 16 (exist only when q is odd)

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\lvert Z\rvert$	$[C, C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
16	$1, 9$	$2(q-\varepsilon)^2$	$A_1(q)A_1(q)$	1	$4a$	4	$[\ell^a]^4$	$R_{18,\ell}$	2^2	4	≥ 3
		$2(q+\varepsilon)^2$	$A_1(q)A_1(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{10,\ell}$	2^2	4	≥ 3
	$2/6$	$2(q^2-1)$	$A_1(q)A_1(q)$	1	$3a$	4	$[\ell^a]^3$	$R_{16,\ell}/R_{17,\ell}$	2^2	4	≥ 3
	5	$2(q^2+1)$	$A_1(q)A_1(q)$	1	$2a$	4	$[\ell^a]^2$	$R_{10,\ell}$	2^2	4	≥ 3
	$3, 10$	$2(q-\varepsilon)^2$	$A_1(q^2)$	1	$3a$	2	$[\ell^a]^3$	$R_{16,\ell}$	2	2	≥ 3
		$2(q+\varepsilon)^2$	$A_1(q^2)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2	≥ 3
	$4/7$	$2(q^2-1)$	$A_1(q^2)$	1	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}/R_{9,\ell}$	2	2	≥ 3
	8	$2(q^2+1)$	$A_1(q^2)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2	≥ 3

TABLE 17. The ℓ-blocks of $F_4(q)$ of geometric type 17

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	12	13
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	Rem	ℓ
17	$1,6$	$(q-\varepsilon)^2$	$\tilde{A}_2^{\varepsilon}(q)$	1	$4a+1$	3	$(13,k)$	R_{34}	2	2		3
								R_{26}	$\mathrm{SL}_2(3)$	1		
					$4a$	3	$[\ell^a]^4$	$R_{18,\ell}$	S_3	3		>3
		$(q+\varepsilon)^2$	$\tilde{A}_2^{-\varepsilon}(q)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2		≥ 3
	$3,2$	q^2-1	$\tilde{A}_2^{\varepsilon}(q)$	1	$3a+1$	3	$(13,k')$	R_{32}	2	2		3
								R_{24}	$\mathrm{SL}_2(3)$	1	$a \geq 2$	
								R_{20}	$\mathrm{SL}_2(3)$	1	$a = 1$	
					$3a$	3	$[\ell^a]^3$	$R_{17,\ell}$	S_3	3		>3
		q^2-1	$\tilde{A}_2^{-\varepsilon}(q)$	1	$2a$	2	$[\ell^a]^2$	$R_{9,\ell}$	2	2		≥ 3
				$\tilde{A}_2^{-\varepsilon}(q),(2,1)$	a	1	$[\ell^a]$	$R_{3,\ell}$	1	1		≥ 3
	$5,4$	$q^2+\varepsilon q+1$	$\tilde{A}_2^{\varepsilon}(q)$	1	$2a+2$	3	$(13,k)$	R_{30}	2	2		3
								R_{20}	$\mathrm{SL}_2(3)$	1		
					$2a$	3	$[\ell^a]^2$	$R_{11,\ell}$	S_3	3		>3
		$q^2-\varepsilon q+1$	$\tilde{A}_2^{-\varepsilon}(q)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2		≥ 3

TABLE 18. The ℓ-blocks of $F_4(q)$ of geometric type 18

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\|Z\|$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
18	$1, 10$	$(q-\varepsilon)^3$	$A_1(q)$	1	$4a$	2	$[\ell^a]^4$	$R_{18,\ell}$	2	2	≥ 3
		$(q+\varepsilon)^3$	$A_1(q)$	1	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2	≥ 3
	$2, 8/9, 5$	$(q^2-1)(q-\varepsilon)$	$A_1(q)$	1	$3a$	2	$[\ell^a]^3$	$R_{16,\ell}/R_{17,\ell}$	2	2	≥ 3
		$(q^2-1)(q+\varepsilon)$	$A_1(q)$	1	$2a$	2	$[\ell^a]^2$	$R_{9,\ell}/R_{10,\ell}$	2	2	≥ 3
	$3, 7$	$q^3-\varepsilon$	$A_1(q)$	1	$2a+1$	2	$[3^{a+1}] \times [3^a]$	R_{12}	2	2	3
					$2a$	2	$[\ell^a]^2$	$R_{12,\ell}$	2	2	> 3
		$q^3+\varepsilon$	$A_1(q)$	1	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2	≥ 3
	$6, 4$	$(q^2+1)(q-\varepsilon)$	$A_1(q)$	1	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}$	2	2	≥ 3
		$(q^2+1)(q+\varepsilon)$	$A_1(q)$	1	a	2	$[\ell^a]$	$R_{3,\ell}$	2	2	≥ 3

TABLE 19. The ℓ-blocks of $F_4(q)$ of geometric type 19

$e = 1, 2$

1	2	3	4	5	6	7	8	9	10	11	13
i	k	$\lvert Z\rvert$	$[C,C]$	b	$d(b)$	$l(b)$	$D(b)$	R	$\mathrm{Out}(b_R)$	$\mathcal{W}$	ℓ
19	$1, 10$	$(q-\varepsilon)^3$	$\tilde{A}_1(q)$	1	$4a$	2	$[\ell^a]^4$	$R_{18,\ell}$	2	2	≥ 3
		$(q+\varepsilon)^3$	$\tilde{A}_1(q)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2	≥ 3
	$2, 5/3, 4$	$(q^2-1)(q-\varepsilon)$	$\tilde{A}_1(q)$	1	$3a$	2	$[\ell^a]^3$	$R_{16,\ell}/R_{17,\ell}$	2	2	≥ 3
		$(q^2-1)(q+\varepsilon)$	$\tilde{A}_1(q)$	1	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}/R_{9,\ell}$	2	2	≥ 3
	$8, 9$	$q^3-\varepsilon$	$\tilde{A}_1(q)$	1	$2a+1$	2	$[3^{a+1}] \times [3^a]$	R_{11}	2	2	3
					$2a$	2	$[\ell^a]^2$	$R_{11,\ell}$	2	2	> 3
		$q^3+\varepsilon$	$\tilde{A}_1(q)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2	≥ 3
	$6, 7$	$(q^2+1)(q-\varepsilon)$	$\tilde{A}_1(q)$	1	$2a$	2	$[\ell^a]^2$	$R_{10,\ell}$	2	2	≥ 3
		$(q^2+1)(q+\varepsilon)$	$\tilde{A}_1(q)$	1	a	2	$[\ell^a]$	$R_{2,\ell}$	2	2	≥ 3

TABLE 21. The faithful ℓ-blocks of $\hat{G} = 2.F_4(2)$ of non-cyclic defect

$\ell = 3$

6	7	8	9	10′	11
$d(b)$	$l(b)$	$D(b)$	R	$N_{\hat{G}}(R)/R$	$\mathcal{W}$
6	17	$(1,1)$	R_{38}	2^4	8
			R_{37}	$2 \times \mathrm{GL}_2(3)$	2
			R_{15}	$2 \times \mathrm{SL}_3(3)$	1
			R_{18}	$2 \times W(F_4)$	4
			R_{21}	$2^{\cdot}[(Q_8 \times Q_8) : S_3]$	2
2	5	3^2	R_{10}	$2 \times D_8 \times \mathrm{Sp}_4(2)$	5
2	5	3^2	R_{10}	$2 \times D_8 \times \mathrm{Sp}_4(2)$	5

$\ell = 5$

6	7	8	9	10′	11
$d(b)$	$l(b)$	$D(b)$	R	$N_{\hat{G}}(R)/R$	$\mathcal{W}$
2	16	$[5]^2$	$[5]^2$	$2 \times \mathrm{SL}_2(3) : [4]$	16

$\ell = 7$

6	7	8	9	10′	11
$d(b)$	$l(b)$	$D(b)$	R	$N_{\hat{G}}(R)/R$	$\mathcal{W}$
2	21	$[7]^2$	R_8	$2 \times \mathrm{SL}_2(3) \times 3$	21

9.2. **Maximal tori.** This subsection contains information on the maximal tori of $F_4(q)$. The G-conjugacy classes of the F-stable maximal tori of $\mathbf{G}$ are in bijection with the conjugacy classes of W. For each such conjugacy class, Table 22 gives the order of a representative w, the order of the centralizer $|C_W(w)|$, the 2- and 3-power maps on the conjugacy classes, the action of the automorphism † of W (which swaps s_1 with s_4 and s_2 with s_3), as well as the names of the conjugacy classes following [20] and the structure of the corresponding maximal tori. The first column of Table 22 numbers the conjugacy classes. The reflections corresponding to the long respectively short roots lie in conjugacy class 12, respectively 17.

TABLE 22. Maximal tori of G

No.	$\|w\|$	$\|C_W(w)\|$	2p	3p	†	Name	Structure
1	1	1152	1	1	1	A_0	$[\Phi_1(q)]^4$
2	2	1152	1	2	2	$4A_1$	$[\Phi_2(q)]^4$
3	2	64	1	3	3	$2A_1$	$[d] \times [\Phi_1(q)\Phi_2(q)/d] \times [\Phi_1(q)\Phi_2(q)]$
4	3	36	4	1	7	A_2	$[\Phi_1(q)] \times [\Phi_1(q)\Phi_3(q)]$
5	6	36	4	2	8	D_4	$[\Phi_2(q)] \times [\Phi_2(q)\Phi_6(q)]$
6	4	96	2	6	6	$D_4(a_1)$	$[\Phi_4(q)]^2$
7	3	36	7	1	4	$\tilde{A}_2$	$[\Phi_1(q)] \times [\Phi_1(q)\Phi_3(q)]$
8	6	36	7	2	5	$C_3 + A_1$	$[\Phi_2(q)] \times [\Phi_2(q)\Phi_6(q)]$
9	3	72	9	1	9	$A_2 + \tilde{A}_2$	$[\Phi_3(q)]^2$
10	6	72	9	2	10	$F_4(a_1)$	$[\Phi_6(q)]^2$
11	12	12	10	6	11	F_4	$[\Phi_{12}(q)]$
12	2	96	1	12	17	A_1	$[\Phi_1(q)]^2 \times [\Phi_1(q)\Phi_2(q)]$
13	2	96	1	13	18	$3A_1$	$[\Phi_2(q)]^2 \times [\Phi_1(q)\Phi_2(q)]$
14	6	12	7	12	19	$\tilde{A}_2 + A_1$	$[\Phi_1(q)\Phi_2(q)\Phi_3(q)]$
15	6	12	7	13	20	C_3	$[\Phi_1(q)\Phi_2(q)\Phi_6(q)]$
16	4	16	3	16	21	A_3	$[d] \times [\Phi_1(q)\Phi_2(q)\Phi_4(q)/d]$
17	2	96	1	17	12	$\tilde{A}_1$	$[\Phi_1(q)]^2 \times [\Phi_1(q)\Phi_2(q)]$
18	2	96	1	18	13	$2A_1 + \tilde{A}_1$	$[\Phi_2(q)]^2 \times [\Phi_1(q)\Phi_2(q)]$
19	6	12	4	17	14	$A_2 + \tilde{A}_1$	$[\Phi_1(q)\Phi_2(q)\Phi_3(q)]$
20	6	12	4	18	15	B_3	$[\Phi_1(q)\Phi_2(q)\Phi_6(q)]$
21	4	16	3	21	16	$B_2 + A_1$	$[d] \times [\Phi_1(q)\Phi_2(q)\Phi_4(q)]/d]$
22	2	16	1	22	22	$A_1 + \tilde{A}_1$	$[\Phi_1(q)\Phi_2(q)]^2$
23	4	32	3	23	23	B_2	$[\Phi_1(q)] \times [\Phi_1(q)\Phi_4(q)]$
24	4	32	3	24	24	$A_3 + \tilde{A}_1$	$[\Phi_2(q)] \times [\Phi_2(q)\Phi_4(q)]$
25	8	8	6	25	25	B_4	$[\Phi_8(q)]$

9.3. **The construction of the centralizers.** Recall that $F = F_1^f$ with F_1 as in Subsection 4.1. In this subsection we assume that m is a positive integer dividing f. Let $\mathbf{T}_0$ denote a 1-F_1-split maximal torus of $\mathbf{G}$, and let $W := N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$.

Recall from Subsection 4.9 that the G-class types of semisimple elements of G are numbered by pairs of integers (i, k), with $1 \leq i \leq 20$ and k depending on i. The integer i labels certain subsets $\Gamma_i \subseteq \Sigma$ with $\Gamma_1 = \{\alpha_1, \dots, \alpha_4\}$ and $\Gamma_{20} = \emptyset$, and such that Γ_i is a base of $\bar{\Gamma}_i$ for $1 \leq i \leq 20$. Moreover, k labels the conjugacy classes of $\mathrm{Stab}_W(\Gamma_i)$.

Let us now explain the contents of Table 23, referring to Subsections 4.1 and 4.9 for the notation. For each pair (i, k) as above, with $2 \leq i \leq 19$, this table lists several pairs of elements $v, w \in W$; the choice of these pairs depends on two parameters, namely on certain congruence classes of f/m and on integers $e \in \{1, 2, 3, 4, 6\}$. The first column of Table 23 contains the sets Γ_i as list of integers $j_1, \dots, j_c$ if $\Gamma_i = \{\alpha_{j_1}, \dots, \alpha_{j_c}\}$, and the second column contains (i, k). Column 3 and 4 give the values of e and of the congruence class of f/m, respectively, where a hyphen indicates that there is no condition on f/m. Columns 5–7 of Table 23 contain the elements v, $v^{f/m}$ and w, respectively, where we use the following conventions. Let Γ be one of the Γ_i. As Γ is a base of $\bar{\Gamma}$, we have $W_\Gamma = W_{\bar{\Gamma}}$, and thus W_Γ is a Weyl group. The longest element of W_Γ is denoted by w_Γ and w_0 is the longest element of W. Also, $s_j \in W$ denotes the reflection corresponding to the root α_j and r_j denotes a representative of the W-conjugacy class with number j, according to Table 22. For $w \in W$, we denote by w' any primitive power of w, i.e. any power of w of the same order as w.

The content of the last two columns of Table 23 and further properties of the data listed are collected in the following remark, whose assertions are easily checked with CHEVIE.

Remark 9.4. Let $\Gamma \subseteq \Sigma$, $i, k \in \mathbb{Z}$, $e \in \{1, 2, 3, 4, 6\}$ and $v, w \in W$ such that some row of Table 23 has the values $(\Gamma, (i, k), -, -, v, -, w, -, -)$. Put $C_\Gamma(v^{f/m}w) := C_W(v^{f/m}w) \cap W_\Gamma = \mathrm{C}_{W_\Gamma}(v^{f/m}w)$. Then the following holds.

(i) We have $v \in W_\Gamma.\mathrm{Stab}_W(\Gamma)$, and the centralizers of semisimple elements of G of G-class type (i, k) are of F-type $(\bar{\Gamma}, [v])$.

(ii) We have $w \in W_\Gamma \leq W$ and the number, according to Table 22, of the conjugacy class of $v^{f/m}w$ in W is as given in the column of Table 23 headed with cl.

(iii) The elements v and w commute.

(iv) The element v centralizes $C_\Gamma(v^{f/m}w)$, unless f/m is even and $(i, k) \in \{(12, 2), (12, 4), (16, k), k \in \{3, 4, 7, 8, 10\}\}$.

(v) The structure of $C_\Gamma(v^{f/m}w)$ is as given in the last column of Table 23. $\square$

TABLE 23. Construction of Centralizers

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$1,2,3,48$	$(2,1)$	1	–	1	1	1	1	$W(B_4)$
		2	–	1	1	w_0	2	$W(B_4)$
		4	–	1	1	r_6	6	$[4]^2.2$
$1,2,4,48$	$(3,1)$	1	–	1	1	1	1	$W(A_3)\times 2$
		2	–	1	1	$s_1s_4s_{22}$	18	$D_8\times 2$
	$(3,2)$	1	$\not\equiv 0(2)$	s_{17}	s_{17}	1	17	$D_8\times 2$
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	$W(A_3)\times 2$
$1,3,4,48$	$(4,1)$	1	–	1	1	1	1	$S_3\times S_3$
		2	–	1	1	s_7s_{23}	22	2^2
		3	–	1	1	r_9	9	3^2
	$(4,2)$	1	$\not\equiv 0(2)$	$s_{12}s_{11}$	$s_{12}s_{11}$	1	22	2^2
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	$S_3\times S_3$
		6	$\not\equiv 0(2)$	w_0	w_0	r_9	10	3^2
$2,3,4,48$	$(5,1)$	1	–	1	1	1	1	$W(C_3)\times 2$
		2	–	1	1	w_0	2	$W(C_3)\times 2$
$1,2,3$	$(6,1)$	1	–	1	1	1	1	$W(B_3)$
		2	–	1	1	w_Γ	18	
	$(6,2)$	1	$\equiv 0(2)$	w_0	1	1	1	
		2	$\equiv 0(2)$	s_{21}	1	w_Γ	18	
		1	$\not\equiv 0(2)$	s_{21}	s_{21}	1	17	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
$1,2,4$	$(7,1)$	1	–	1	1	1	1	$2\times S_3$
		2	–	1	1	w_Γ	22	2^2
	$(7,2)$	1	$\equiv 0(2)$	w_0	1	1	1	$2\times S_3$
		2	$\equiv 0(2)$	$s_{15}s_{23}$	1	w_Γ	22	2^2
		1	$\not\equiv 0(2)$	$s_{15}s_{23}$	$s_{15}s_{23}$	1	22	2^2
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	$2\times S_3$

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$1,2,48$	$(8,1)$	1	$-$	1	1	1	1	$W(A_3)$
		2	$-$	1	1	s_1s_{22}	3	D_8
	$(8,4)$	1	$\equiv 0(2)$	w_0	1	1	1	$W(A_3)$
		2	$\equiv 0(2)$	s_4s_{17}	1	s_1s_{22}	3	D_8
		1	$\not\equiv 0(2)$	s_4s_{17}	s_4s_{17}	1	3	D_8
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	$W(A_3)$
	$(8,2)$	1	$\equiv 0(2)$	s_4	1	1	1	$W(A_3)$
		2	$\equiv 0(2)$	s_4	1	s_1s_{22}	3	D_8
		1	$\not\equiv 0(2)$	s_4	s_4	1	17	$W(A_3)$
		2	$\not\equiv 0(2)$	w_0s_{17}	w_0s_{17}	1	18	D_8
	$(8,3)$	1	$\equiv 0(2)$	w_0s_4	1	1	1	$W(A_3)$
		2	$\equiv 0(2)$	s_{17}	1	s_1s_{22}	3	D_8
		1	$\not\equiv 0(2)$	s_{17}	s_{17}	1	17	D_8
		2	$\not\equiv 0(2)$	w_0s_4	w_0s_4	1	18	$W(A_3)$
$1,3,4$	$(9,1)$	1	$-$	1	1	1	1	$S_3\times 2$
		2	$-$	1	1	w_Γ	22	2^2
	$(9,2)$	1	$\equiv 0(2)$	w_0	1	1	1	$S_3\times 2$
		2	$\equiv 0(2)$	$s_{20}s_{19}$	1	w_Γ	22	2^2
		1	$\not\equiv 0(2)$	$s_{20}s_{19}$	$s_{20}s_{19}$	1	22	2^2
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	$S_3\times 2$
$2,3,4$	$(10,1)$	1	$-$	1	1	1	1	$W(C_3)$
		2	$-$	1	1	w_Γ	13	
	$(10,2)$	1	$\equiv 0(2)$	w_0	1	1	1	
		2	$\equiv 0(2)$	s_{24}	1	w_Γ	13	
		1	$\not\equiv 0(2)$	s_{24}	s_{24}	1	12	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$2,3,48$	$(11,1)$	1	$-$	1	1	1	1	$D_8 \times 2$
		2	$-$	1	1	w_Γ	13	
	$(11,2)$	1	$\equiv 0(2)$	w_0	1	1	1	
		2	$\equiv 0(2)$	s_{16}	1	w_Γ	13	
		1	$\not\equiv 0(2)$	s_{16}	s_{16}	1	12	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
$2,4,48$	$(12,1)$	1	$-$	1	1	1	1	2^3
		2	$-$	1	1	w_Γ	18	
	$(12,3)$	1	$\equiv 0(2)$	w_0	1	1	1	
		2	$\equiv 0(2)$	s_{13}	1	w_Γ	18	
		1	$\not\equiv 0(2)$	s_{13}	s_{13}	1	17	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
	$(12,2)$	1	$\equiv 0(2)$	s_{17}	1	1	1	2^2
		2	$\equiv 0(2)$	s_{17}	1	w_Γ	18	
		1	$\not\equiv 0(2)$	s_{17}	s_{17}	1	17	
		2	$\not\equiv 0(2)$	$w_0s_{13}s_{17}$	$w_0s_{13}s_{17}$	1	3	
	$(12,4)$	1	$\equiv 0(2)$	$s_{13}s_{17}$	1	1	1	
		2	$\equiv 0(2)$	$s_{13}s_{17}$	1	w_Γ	18	
		1	$\not\equiv 0(2)$	$s_{13}s_{17}$	$s_{13}s_{17}$	1	3	
		2	$\not\equiv 0(2)$	w_0s_{17}	w_0s_{17}	1	18	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$1,2$	$(13,1)$	1	$-$	1	1	1	1	S_3
		2	$-$	1	1	w_Γ	12	2
	$(13,6)$	1	$\equiv 0(2)$	w_0	1	1	1	S_3
		2	$\equiv 0(2)$	w_0w_Γ	1	w_Γ	12	2
		1	$\not\equiv 0(2)$	w_0w_Γ	w_0w_Γ	1	13	2
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	S_3
	$(13,2)$	1	$\equiv 0(2)$	s_4	1	1	1	S_3
		2	$\equiv 0(2)$	s_4	1	w_Γ	12	2
		1	$\not\equiv 0(2)$	s_4	s_4	1	17	S_3
		2	$\not\equiv 0(2)$	s_4w_Γ	s_4w_Γ	1	22	2
	$(13,3)$	1	$\equiv 0(2)$	w_0s_4	1	1	1	S_3
		2	$\equiv 0(2)$	w_0s_4	1	w_Γ	12	2
		1	$\not\equiv 0(2)$	$w_0s_4w_\Gamma$	$w_0s_4w_\Gamma$	1	22	2
		2	$\not\equiv 0(2)$	w_0s_4	w_0s_4	1	18	S_3
	$(13,5)$	1	$\equiv 0(3)$	s_4s_{19}	1	1	1	S_3
		2	$\equiv 0(3)$	s_4s_{19}	1	w_Γ	12	2
		1	$\not\equiv 0(3)$	s_4s_{19}	$(s_4s_{19})'$	1	7	S_3
		2	$\not\equiv 0(3)$	$s_4s_{19}w_\Gamma$	$(s_4s_{19}w_\Gamma)'$	1	14	2
	$(13,4)$	1	$\equiv 0(6)$	$w_0s_4s_{19}$	1	1	1	S_3
		2	$\equiv 0(6)$	$w_0s_4s_{19}$	1	w_Γ	12	2
		1	$\equiv 0(2) \not\equiv 0(3)$	$w_0s_4s_{19}$	$(s_4s_{19})'$	1	7	S_3
		2	$\equiv 0(2) \not\equiv 0(3)$	$w_0s_4s_{19}w_\Gamma$	$(s_4s_{19}w_\Gamma)'$	1	14	2
		1	$\not\equiv 0(2) \equiv 0(3)$	$w_0s_4s_{19}w_\Gamma$	w_0w_Γ	1	13	2
		2	$\not\equiv 0(2) \equiv 0(3)$	$w_0s_4s_{19}$	w_0	1	2	S_3
		1	$\gcd(6,f/m)=1$	$w_0s_4s_{19}w_\Gamma$	$(w_0s_4s_{19}w_\Gamma)'$	1	15	2
		2	$\gcd(6,f/m)=1$	$w_0s_4s_{19}$	$(w_0s_4s_{19})'$	1	8	S_3

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$1,4$	$(14,1)$	1	$-$	1	1	1	1	2^2
		2	$-$	1	1	w_Γ	22	
	$(14,4)$	1	$\equiv 0(2)$	$s_{17}s_{22}$	1	1	1	
		2	$\equiv 0(2)$	$s_{17}s_{22}$	1	w_Γ	22	
		1	$\not\equiv 0(2)$	$s_{17}s_{22}$	$s_{17}s_{22}$	1	22	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
	$(14,2)$	1	$\equiv 0(2)$	s_{17}	1	1	1	
		2	$\equiv 0(2)$	s_{17}	1	w_Γ	22	
		1	$\not\equiv 0(2)$	s_{17}	s_{17}	1	17	
		2	$\not\equiv 0(2)$	w_0s_{22}	w_0s_{22}	1	13	
	$(14,3)$	1	$\equiv 0(2)$	s_{22}	1	1	1	
		2	$\equiv 0(2)$	s_{22}	1	w_Γ	22	
		1	$\not\equiv 0(2)$	s_{22}	s_{22}	1	12	
		2	$\not\equiv 0(2)$	w_0s_{17}	w_0s_{17}	1	18	
$2,3$	$(15,1)$	1	$-$	1	1	1	1	D_8
		2	$-$	1	1	w_Γ	3	
	$(15,3)$	1	$\equiv 0(2)$	w_0w_Γ	1	1	1	
		2	$\equiv 0(2)$	w_0w_Γ	1	w_Γ	3	
		1	$\not\equiv 0(2)$	w_0w_Γ	w_0w_Γ	1	3	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
	$(15,2)$	1	$\equiv 0(2)$	s_{16}	1	1	1	
		2	$\equiv 0(2)$	s_{16}	1	w_Γ	3	
		1	$\not\equiv 0(2)$	s_{16}	s_{16}	1	12	
		2	$\not\equiv 0(2)$	w_0s_{16}	w_0s_{16}	1	13	
	$(15,4)$	1	$\equiv 0(2)$	s_8	1	1	1	
		2	$\equiv 0(2)$	s_8	1	w_Γ	3	
		1	$\not\equiv 0(2)$	s_8	s_8	1	17	
		2	$\not\equiv 0(2)$	w_0s_8	w_0s_8	1	18	
	$(15,5)$	1	$\equiv 0(4)$	s_8s_{16}	1	1	1	
		2	$\equiv 0(4)$	s_8s_{16}	1	w_Γ	3	
		1	$\equiv 2(4)$	s_8s_{16}	$(s_8s_{16})^2$	1	3	
		2	$\equiv 2(4)$	s_8s_{16}	$(s_8s_{16})^2$	w_Γ	2	
		1	$\not\equiv 0(2)$	s_8s_{16}	$(s_8s_{16})'$	1	23	
		2	$\not\equiv 0(2)$	$w_0s_8s_{16}$	$(w_0s_8s_{16})'$	1	24	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$2, 48$	$(16,1)$	1	$-$	1	1	1	1	2^2
		2	$-$	1	1	w_Γ	3	
	$(16,9)$	1	$\equiv 0(2)$	$(s_4s_9)^2$	1	1	1	
		2	$\equiv 0(2)$	$(s_4s_9)^2$	1	w_Γ	3	
		1	$\not\equiv 0(2)$	$(s_4s_9)^2$	$(s_4s_9)^2$	1	3	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
	$(16,2)$	1	$\equiv 0(2)$	s_4	1	1	1	
		2	$\equiv 0(2)$	s_4	1	w_Γ	3	
		1	$\not\equiv 0(2)$	s_4	s_4	1	17	
		2	$\not\equiv 0(2)$	w_0s_4	w_0s_4	1	18	
	$(16,6)$	1	$\equiv 0(2)$	s_9	1	1	1	
		2	$\equiv 0(2)$	s_9	1	w_Γ	3	
		1	$\not\equiv 0(2)$	s_9	s_9	1	12	
		2	$\not\equiv 0(2)$	w_0s_9	w_0s_9	1	13	
	$(16,5)$	1	$\equiv 0(4)$	s_4s_9	1	1	1	
		2	$\equiv 0(4)$	s_4s_9	1	w_Γ	3	
		1	$\equiv 2(4)$	s_4s_9	$(s_4s_9)^2$	1	3	
		2	$\equiv 2(4)$	s_4s_9	$(s_4s_9)^2$	w_Γ	2	
		1	$\not\equiv 0(2)$	s_4s_9	$(s_4s_9)'$	1	23	
		2	$\not\equiv 0(2)$	$w_0s_4s_9$	$(w_0s_4s_9)'$	1	24	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
	$(16,4)$	1	$\equiv 0(2)$	s_4s_{17}	1	1	1	2^2
		2	$\equiv 0(2)$	s_4s_{17}	1	w_Γ	3	2^2
		1	$\not\equiv 0(2)$	s_4s_{17}	s_4s_{17}	1	3	2
		2	$\not\equiv 0(2)$	s_4s_{17}	s_4s_{17}	1	3	2
	$(16,7)$	1	$\equiv 0(2)$	s_9s_{17}	1	1	1	2^2
		2	$\equiv 0(2)$	s_9s_{17}	1	w_Γ	3	2^2
		1	$\not\equiv 0(2)$	s_9s_{17}	s_9s_{17}	1	22	2
		2	$\not\equiv 0(2)$	s_9s_{17}	s_9s_{17}	1	22	2
	$(16,3)$	1	$\equiv 0(2)$	s_{17}	1	1	1	2^2
		2	$\equiv 0(2)$	s_{17}	1	w_Γ	3	2^2
		1	$\not\equiv 0(2)$	s_{17}	s_{17}	1	17	2
		2	$\not\equiv 0(2)$	s_{17}	s_{17}	1	17	2
	$(16,10)$	1	$\equiv 0(2)$	w_0s_{17}	1	1	1	2^2
		2	$\equiv 0(2)$	w_0s_{17}	1	w_Γ	3	2^2
		1	$\not\equiv 0(2)$	w_0s_{17}	w_0s_{17}	1	18	2
		2	$\not\equiv 0(2)$	w_0s_{17}	w_0s_{17}	1	18	2
	$(16,8)$	1	$\equiv 0(4)$	$s_4s_9s_{17}$	1	1	1	2^2
		2	$\equiv 0(4)$	$s_4s_9s_{17}$	1	w_Γ	3	2^2
		1	$\equiv 2(4)$	$s_4s_9s_{17}$	$(s_4s_9)^2$	1	3	2^2
		2	$\equiv 2(4)$	$s_4s_9s_{17}$	$(s_4s_9)^2$	w_Γ	2	2^2
		1	$\not\equiv 0(2)$	$s_4s_9s_{17}$	$(s_4s_9s_{17})'$	1	16	2
		2	$\not\equiv 0(2)$	$s_4s_9s_{17}$	$(s_4s_9s_{17})'$	1	16	2

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
$3,4$	$(17,1)$	1	$-$	1	1	1	1	S_3
		2	$-$	1	1	w_Γ	17	2
	$(17,6)$	1	$\equiv 0(2)$	w_0	1	1	1	S_3
		2	$\equiv 0(2)$	w_0w_Γ	1	w_Γ	17	2
		1	$\not\equiv 0(2)$	w_0w_Γ	w_0w_Γ	1	18	2
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	S_3
	$(17,3)$	1	$\equiv 0(2)$	s_1	1	1	1	S_3
		2	$\equiv 0(2)$	s_1	1	w_Γ	17	2
		1	$\not\equiv 0(2)$	s_1	s_1	1	12	S_3
		2	$\not\equiv 0(2)$	s_1w_Γ	s_1w_Γ	1	22	2
	$(17,2)$	1	$\equiv 0(2)$	w_0s_1	1	1	1	S_3
		2	$\equiv 0(2)$	w_0s_1	1	w_Γ	17	2
		1	$\not\equiv 0(2)$	$w_0s_1w_\Gamma$	$w_0s_1w_\Gamma$	1	22	2
		2	$\not\equiv 0(2)$	w_0s_1	w_0s_1	1	13	S_3
	$(17,5)$	1	$\equiv 0(3)$	s_1s_{23}	1	1	1	S_3
		2	$\equiv 0(3)$	s_1s_{23}	1	w_Γ	17	2
		1	$\not\equiv 0(3)$	s_1s_{23}	$(s_1s_{23})'$	1	4	S_3
		2	$\not\equiv 0(3)$	$s_1s_{23}w_\Gamma$	$(s_1s_{23}w_\Gamma)'$	1	19	2
	$(17,4)$	1	$\equiv 0(6)$	$w_0s_1s_{23}$	1	1	1	S_3
		2	$\equiv 0(6)$	$w_0s_1s_{23}$	1	w_Γ	17	2
		1	$\equiv 0(2)\not\equiv 0(3)$	$w_0s_1s_{23}$	$(s_1s_{23})'$	1	4	S_3
		2	$\equiv 0(2)\not\equiv 0(3)$	$w_0s_1s_{23}w_\Gamma$	$(s_1s_{23}w_\Gamma)'$	1	19	2
		1	$\not\equiv 0(2)\equiv 0(3)$	$w_0s_1s_{23}w_\Gamma$	w_0w_Γ	1	18	2
		2	$\not\equiv 0(2)\equiv 0(3)$	$w_0s_1s_{23}$	w_0	1	2	S_3
		1	$\gcd(6,f/m)=1$	$w_0s_1s_{23}w_\Gamma$	$(w_0s_1s_{23}w_\Gamma)'$	1	20	2
		2	$\gcd(6,f/m)=1$	$w_0s_1s_{23}$	$(w_0s_1s_{23})'$	1	5	S_3

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
1	$(18,1)$	1	$-$	1	1	1	1	2
		2	$-$	1	1	w_Γ	12	
	$(18,10)$	1	$\equiv 0(2)$	w_0w_Γ	1	1	1	
		2	$\equiv 0(2)$	w_0w_Γ	1	w_Γ	12	
		1	$\not\equiv 0(2)$	w_0w_Γ	w_0w_Γ	1	13	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
	$(18,2)$	1	$\equiv 0(2)$	s_4	1	1	1	
		2	$\equiv 0(2)$	s_4	1	w_Γ	12	
		1	$\not\equiv 0(2)$	s_4	s_4	1	17	
		2	$\not\equiv 0(2)$	s_4w_Γ	s_4w_Γ	1	22	
	$(18,8)$	1	$\equiv 0(2)$	w_0s_4	1	1	1	
		2	$\equiv 0(2)$	w_0s_4	1	w_Γ	12	
		1	$\not\equiv 0(2)$	$w_0s_4w_\Gamma$	$w_0s_4w_\Gamma$	1	22	
		2	$\not\equiv 0(2)$	w_0s_4	w_0s_4	1	18	
	$(18,9)$	1	$\equiv 0(2)$	s_{14}	1	1	1	
		2	$\equiv 0(2)$	s_{14}	1	w_Γ	12	
		1	$\not\equiv 0(2)$	s_{14}	s_{14}	1	12	
		2	$\not\equiv 0(2)$	$s_{14}w_\Gamma$	$s_{14}w_\Gamma$	1	3	
	$(18,5)$	1	$\equiv 0(2)$	w_0s_{14}	1	1	1	
		2	$\equiv 0(2)$	w_0s_{14}	1	w_Γ	12	
		1	$\not\equiv 0(2)$	$w_0s_{14}w_\Gamma$	$w_0s_{14}w_\Gamma$	1	3	
		2	$\not\equiv 0(2)$	w_0s_{14}	w_0s_{14}	1	13	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
	$(18,6)$	1	$\equiv 0(4)$	$s_{14}s_4$	1	1	1	2
		2	$\equiv 0(4)$	$s_{14}s_4$	1	w_Γ	12	
		1	$\equiv 2(4)$	$s_{14}s_4$	$(s_{14}s_4)^2$	1	3	
		2	$\equiv 2(4)$	$s_{14}s_4$	$(s_{14}s_4)^2$	w_Γ	13	
		1	$\not\equiv 0(2)$	$s_{14}s_4$	$(s_{14}s_4)'$	1	23	
		2	$\not\equiv 0(2)$	$s_{14}s_4w_\Gamma$	$(s_{14}s_4w_\Gamma)'$	1	21	
	$(18,4)$	1	$\equiv 0(4)$	$w_0s_{14}s_4$	1	1	1	
		2	$\equiv 0(4)$	$w_0s_{14}s_4$	1	w_Γ	12	
		1	$\equiv 2(4)$	$w_0s_{14}s_4$	$(s_{14}s_4)^2$	1	3	
		2	$\equiv 2(4)$	$w_0s_{14}s_4$	$(s_{14}s_4)^2$	w_Γ	13	
		1	$\not\equiv 0(2)$	$w_0s_{14}s_4w_\Gamma$	$(w_0s_{14}s_4w_\Gamma)'$	1	21	
		2	$\not\equiv 0(2)$	$w_0s_{14}s_4$	$(w_0s_{14}s_4)'$	1	24	
	$(18,3)$	1	$\equiv 0(3)$	s_4s_3	1	1	1	
		2	$\equiv 0(3)$	s_4s_3	1	w_Γ	12	
		1	$\not\equiv 0(3)$	s_4s_3	$(s_4s_3)'$	1	7	
		2	$\not\equiv 0(3)$	$s_4s_3w_\Gamma$	$(s_4s_3w_\Gamma)'$	1	14	
	$(18,7)$	1	$\equiv 0(6)$	$w_0s_4s_3$	1	1	1	
		2	$\equiv 0(6)$	$w_0s_4s_3$	1	w_Γ	12	
		1	$\equiv 0(2) \not\equiv 0(3)$	$w_0s_4s_3$	$(s_4s_3)'$	1	7	
		2	$\equiv 0(2) \not\equiv 0(3)$	$w_0s_4s_3w_\Gamma$	$(s_4s_3w_\Gamma)'$	1	14	
		1	$\not\equiv 0(2) \equiv 0(3)$	$w_0s_4s_3w_\Gamma$	w_0w_Γ	1	13	
		2	$\not\equiv 0(2) \equiv 0(3)$	$w_0s_4s_3$	w_0	1	2	
		1	$\gcd(6, f/m) = 1$	$w_0s_4s_3w_\Gamma$	$(w_0s_4s_3w_\Gamma)'$	1	15	
		2	$\gcd(6, f/m) = 1$	$w_0s_4s_3$	$(w_0s_4s_3)'$	1	8	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
4	$(19,1)$	1	$-$	1	1	1	1	2
		2	$-$	1	1	w_Γ	17	
	$(19,10)$	1	$\equiv 0(2)$	w_0w_Γ	1	1	1	
		2	$\equiv 0(2)$	w_0w_Γ	1	w_Γ	17	
		1	$\not\equiv 0(2)$	w_0w_Γ	w_0w_Γ	1	18	
		2	$\not\equiv 0(2)$	w_0	w_0	1	2	
	$(19,2)$	1	$\equiv 0(2)$	s_{13}	1	1	1	
		2	$\equiv 0(2)$	s_{13}	1	w_Γ	17	
		1	$\not\equiv 0(2)$	s_{13}	s_{13}	1	17	
		2	$\not\equiv 0(2)$	$s_{13}w_\Gamma$	$s_{13}w_\Gamma$	1	3	
	$(19,5)$	1	$\equiv 0(2)$	w_0s_{13}	1	1	1	
		2	$\equiv 0(2)$	w_0s_{13}	1	w_Γ	17	
		1	$\not\equiv 0(2)$	$w_0s_{13}w_\Gamma$	$w_0s_{13}w_\Gamma$	1	3	
		2	$\not\equiv 0(2)$	w_0s_{13}	w_0s_{13}	1	18	
	$(19,3)$	1	$\equiv 0(2)$	s_1	1	1	1	
		2	$\equiv 0(2)$	s_1	1	w_Γ	17	
		1	$\not\equiv 0(2)$	s_1	s_1	1	12	
		2	$\not\equiv 0(2)$	s_1w_Γ	s_1w_Γ	1	22	
	$(19,4)$	1	$\equiv 0(2)$	w_0s_1	1	1	1	
		2	$\equiv 0(2)$	w_0s_1	1	w_Γ	17	
		1	$\not\equiv 0(2)$	$w_0s_1w_\Gamma$	$w_0s_1w_\Gamma$	1	22	
		2	$\not\equiv 0(2)$	w_0s_1	w_0s_1	1	13	

Table 23. Construction of Centralizers (continued)

Γ	(i,k)	e	f/m	v	$v^{f/m}$	w	cl	$C_\Gamma(v^{f/m}w)$
	$(19,6)$	1	$\equiv 0(4)$	$s_{13}s_1$	1	1	1	2
		2	$\equiv 0(4)$	$s_{13}s_1$	1	w_Γ	17	
		1	$\equiv 2(4)$	$s_{13}s_1$	$(s_{13}s_1)^2$	1	3	
		2	$\equiv 2(4)$	$s_{13}s_1$	$(s_{13}s_1)^2$	w_Γ	18	
		1	$\not\equiv 0(2)$	$s_{13}s_1$	$(s_{13}s_1)'$	1	23	
		2	$\not\equiv 0(2)$	$s_{13}s_1w_\Gamma$	$(s_{13}s_1w_\Gamma)'$	1	16	
	$(19,7)$	1	$\equiv 0(1)$	$w_0s_{13}s_1$	1	1	1	
		2	$\equiv 0(4)$	$w_0s_{13}s_1$	1	w_Γ	17	
		1	$\equiv 2(4)$	$w_0s_{13}s_1$	$(s_{13}s_1)^2$	1	3	
		2	$\equiv 2(4)$	$w_0s_{13}s_1$	$(s_{13}s_1)^2$	w_Γ	18	
		1	$\not\equiv 0(2)$	$w_0s_{13}s_1w_\Gamma$	$(w_0s_{13}s_1w_\Gamma)'$	1	16	
		2	$\not\equiv 0(2)$	$w_0s_{13}s_1$	$(w_0s_{13}s_1)'$	1	24	
	$(19,8)$	1	$\equiv 0(3)$	s_1s_2	1	1	1	
		2	$\equiv 0(3)$	s_1s_2	1	w_Γ	17	
		1	$\not\equiv 0(3)$	s_1s_2	$(s_1s_2)'$	1	4	
		2	$\not\equiv 0(3)$	$s_1s_2w_\Gamma$	$(s_1s_2w_\Gamma)'$	1	19	
	$(19,9)$	1	$\equiv 0(6)$	$w_0s_1s_2$	1	1	1	
		2	$\equiv 0(6)$	$w_0s_1s_2$	1	w_Γ	17	
		1	$\equiv 0(2) \not\equiv 0(3)$	$w_0s_1s_2$	$(s_1s_2)'$	1	4	
		2	$\equiv 0(2) \not\equiv 0(3)$	$w_0s_1s_2w_\Gamma$	$(s_1s_2w_\Gamma)'$	1	19	
		1	$\not\equiv 0(2) \equiv 0(3)$	$w_0s_1s_2w_\Gamma$	w_0w_Γ	1	18	
		2	$\not\equiv 0(2) \equiv 0(3)$	$w_0s_1s_2$	w_0	1	2	
		1	$\gcd(6,f/m)=1$	$w_0s_1s_2w_\Gamma$	$(w_0s_1s_2w_\Gamma)'$	1	20	
		2	$\gcd(6,f/m)=1$	$w_0s_1s_2$	$(w_0s_1s_2)'$	1	5	

9.5. **Fusion of maximal tori and central elements of centralizers.** We say that an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ *fuses* into an F-stable closed connected reductive subgroup $\mathbf{M} \leq \mathbf{G}$ of maximal rank, if some G-conjugate of $\mathbf{T}$ lies in $\mathbf{M}$. Table 24, contains, for each class type (i, k), the F-stable maximal tori of $\mathbf{G}$, identified by their number, fusing into $\mathbf{M}_{i,k}$. It also gives the class types of those elements $z \in Z(M_{i,k})$ which belong to a different class type. This table is an excerpt of [58].

Table 24: Tori and other class types in centralizers

(i,k)	Tori in $\mathbf{M}_{i,k}$	$(i',k') \in Z(M_{i,k})$
(1,1)	{ 1, …, 25 }	
(2,1)	1, 2, 3, 3, 4, 5, 6, 12, 13, 16, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25	
(3,1)	1, 3, 4, 12, 16, 17, 18, 19, 22, 24	(2, 1)
(3,2)	2, 3, 5, 13, 16, 17, 18, 20, 22, 23	(2, 1)
(4,1)	1, 4, 7, 9, 12, 14, 17, 19, 22	
(4,2)	2, 5, 8, 10, 13, 15, 18, 20, 22	
(5,1)	1, 2, 3, 3, 7, 8, 12, 12, 13, 13, 14, 15, 17, 18, 21, 21, 22, 22, 23, 24	
(6,1)	1, 3, 4, 12, 16, 17, 18, 20, 22, 23	(2, 1)
(6,2)	2, 3, 5, 13, 16, 17, 18, 19, 22, 24	(2, 1)
(7,1)	1, 4, 12, 17, 19, 22	(2,1), (3,1), (4, 1)
(7,2)	2, 5, 13, 18, 20, 22	(2,1), (3,2), (4, 2)
(8,1)	1, 3, 4, 12, 16	(2,1), (3,1), (6, 1)
(8,2)	17, 18, 19, 22, 24	(2,1), (3,1), (6, 2)
(8,3)	17, 18, 20, 22, 23	(2,1), (3,2), (6, 1)
(8,4)	2, 3, 5, 13, 16	(2,1), (3,2), (6, 2)
(9,1)	1, 7, 12, 14, 17, 22	(4,1), (5, 1)
(9,2)	2, 8, 13, 15, 18, 22	(4,2), (5, 1)
(10,1)	1, 3, 7, 12, 13, 15, 17, 21, 22, 23	(5, 1)
(10,2)	2, 3, 8, 12, 13, 14, 18, 21, 22, 24	(5, 1)

Table 24: continued

(i,k)	Tori in $\mathbf{M}_{i,k}$	$(i',k') \in Z(M_{i,k})$
(11,1)	1, 3, 3, 12, 12, 13, 17, 21, 22, 23	(2,1), (5,1), (10, 1)
(11,2)	2, 3, 3, 12, 13, 13, 18, 21, 22, 24	(2,1), (5,1), (10, 2)
(12,1)	1, 3, 12, 12, 17, 18, 22, 22	(2,1), (3,1), (5,1), (6, 1)
(12,2)	3, 16, 17, 23	(2,1), (3,2), (6, 1)
(12,3)	2, 3, 13, 13, 17, 18, 22, 22	(2,1), (3,2), (5,1), (6, 2)
(12,4)	3, 16, 18, 24	(2,1), (3,1), (6, 2)
(13,1)	1, 4, 12	(2,1), (3,1), (4,1), (6,1), (7,1), (8, 1)
(13,2)	17, 19, 22	(2,1), (3,1), (4,1), (6,2), (7,1), (8, 2)
(13,3)	18, 20, 22	(2,1), (3,2), (4,2), (6,1), (7,2), (8, 3)
(13,4)	8, 10, 15	(4, 2)
(13,5)	7, 9, 14	(4, 1)
(13,6)	2, 5, 13	(2,1), (3,2), (4,2), (6,2), (7,2), (8, 4)
(14,1)	1, 12, 17, 22	(2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (9,1), (10,1), (11,1), (12, 1)
(14,2)	3, 13, 17, 22	(2,1), (3,2), (5,1), (6,2), (10,1), (11,1), (12, 3)
(14,3)	3, 12, 18, 22	(2,1), (3,1), (5,1), (6,1), (10,2), (11,2), (12, 1)
(14,4)	2, 13, 18, 22	(2,1), (3,2), (4,2), (5,1), (6,2), (7,2), (9,2), (10,2), (11,2), (12, 3)
(15,1)	1, 3, 12, 17, 23	(2,1), (5,1), (6,1), (10,1), (11, 1)
(15,2)	3, 12, 13, 21, 22	(2,1), (5,1), (10,1), (10,2), (11,1), (11, 2)

Table 24: continued

(i,k)	Tori in $\mathbf{M}_{i,k}$	$(i',k') \in Z(M_{i,k})$
(15,3)	2, 3, 13, 18, 24	(2,1), (5,1), (6,2), (10,2), (11, 2)
(15,4)	3, 16, 17, 18, 22	(2,1), (6,1), (6, 2)
(15,5)	6, 16, 21, 23, 24	(2, 1)
(16,1)	1, 3, 12, 12	(2,1), (3,1), (5,1), (6,1), (8,1), (10,1), (11,1), (12,1), (15, 1)
(16,2)	17, 18, 22, 22	(2,1), (3,1), (3,2), (5,1), (6,1), (6,2), (8,2), (8,3), (12,1), (12,3), (15, 4)
(16,3)	17, 23	(2,1), (3,2), (5,1), (6,1), (8,3), (10,1), (11,1), (12,2), (15, 1)
(16,4)	3, 16	(2,1), (3,1), (3,2), (6,1), (6,2), (8,1), (8,4), (12,2), (12,4), (15, 4)
(16,5)	21, 21, 23, 24	(2,1), (5,1), (15, 5)
(16,6)	3, 3, 12, 13	(2,1), (5,1), (10,1), (10,2), (11,1), (11,2), (15, 2)
(16,7)	21, 22	(2,1), (5,1), (10,1), (10,2), (11,1), (11,2), (15, 2)
(16,8)	6, 16	(2,1), (15, 5)
(16,9)	2, 3, 13, 13	(2,1), (3,2), (5,1), (6,2), (8,4), (10,2), (11,2), (12,3), (15, 3)
(16,10)	18, 24	(2,1), (3,1), (5,1), (6,2), (8,2), (10,2), (11,2), (12,4), (15, 3)
(17,1)	1, 7, 17	(4,1), (5,1), (9,1), (10, 1)
(17,2)	13, 15, 22	(4,2), (5,1), (9,2), (10, 1)
(17,3)	12, 14, 22	(4,1), (5,1), (9,1), (10, 2)
(17,4)	5, 10, 20	(4, 2)
(17,5)	4, 9, 19	(4, 1)
(17,6)	2, 8, 18	(4,2), (5,1), (9,2), (10, 2)

Table 24: continued

(i,k)	Tori in $\mathbf{M}_{i,k}$	$(i',k') \in Z(M_{i,k})$
(18,1)	1, 12	(2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (8,1), (9,1), (10,1), (11,1), (12,1), (13,1), (14,1), (15,1), (16, 1)
(18,2)	17, 22	(2,1), (3,1), (3,2), (4,1), (5,1), (6,1), (6,2), (7,1), (8,2), (8,3), (9,1), (10,1), (11,1), (12,1), (12,3), (13,2), (14,1), (14,2), (15,4), (16, 2)
(18,3)	7, 14	(4,1), (5,1), (9,1), (13, 5)
(18,4)	21, 24	(2,1), (5,1), (10,2), (11,2), (15,5), (16, 5)
(18,5)	3, 13	(2,1), (3,2), (5,1), (6,2), (8,4), (10,1), (10,2), (11,1), (11,2), (12,3), (14,2), (15,2), (15,3), (16,6), (16, 9)
(18,6)	21, 23	(2,1), (5,1), (10,1), (11,1), (15,5), (16, 5)
(18,7)	8, 15	(4,2), (5,1), (9,2), (13, 4)
(18,8)	18, 22	(2,1), (3,1), (3,2), (4,2), (5,1), (6,1), (6,2), (7,2), (8,2), (8,3), (9,2), (10,2), (11,2), (12,1), (12,3), (13,3), (14,3), (14,4), (15,4), (16, 2)
(18,9)	3, 12	(2,1), (3,1), (5,1), (6,1), (8,1), (10,1), (10,2), (11,1), (11,2), (12,1), (14,3), (15,1), (15,2), (16,1), (16, 6)
(18,10)	2, 13	(2,1), (3,2), (4,2), (5,1), (6,2), (7,2), (8,4), (9,2), (10,2), (11,2), (12,3), (13,6), (14,4), (15,3), (16, 9)

Table 24: continued

(i,k)	Tori in $\mathbf{M}_{i,k}$	$(i',k') \in Z(M_{i,k})$
(19,1)	1, 17	(2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (9,1), (10,1), (11,1), (12,1), (14,1), (15,1), (17, 1)
(19,2)	3, 17	(2,1), (3,2), (5,1), (6,1), (6,2), (10,1), (11,1), (12,2), (12,3), (14,2), (15,1), (15, 4)
(19,3)	12, 22	(2,1), (3,1), (4,1), (5,1), (6,1), (7,1), (9,1), (10,1), (10,2), (11,1), (11,2), (12,1), (14,1), (14,3), (15,2), (17, 3)
(19,4)	13, 22	(2,1), (3,2), (4,2), (5,1), (6,2), (7,2), (9,2), (10,1), (10,2), (11,1), (11,2), (12,3), (14,2), (14,4), (15,2), (17, 2)
(19,5)	3, 18	(2,1), (3,1), (5,1), (6,1), (6,2), (10,2), (11,2), (12,1), (12,4), (14,3), (15,3), (15, 4)
(19,6)	16, 23	(2,1), (3,2), (6,1), (12,2), (15, 5)
(19,7)	16, 24	(2,1), (3,1), (6,2), (12,4), (15, 5)
(19,8)	4, 19	(2,1), (3,1), (4,1), (7,1), (17, 5)
(19,9)	5, 20	(2,1), (3,2), (4,2), (7,2), (17, 4)
(19,10)	2, 18	(2,1), (3,2), (4,2), (5,1), (6,2), (7,2), (9,2), (10,2), (11,2), (12,3), (14,4), (15,3), (17, 6)

9.6. **Sylow 3-subgroups of centralizers.** Table 25 collects information on the G-class types and their centralizers. The numbering of the class types follows the numbering in the Tables in [58], with slight adjustments explained in Subsection 4.9. For each class type with label (i,k), we let $\mathbf{M} := \mathbf{M}_{i,k}$ denote a corresponding F-stable regular subgroup of maximal rank of $\mathbf{G}$, such that $\mathbf{M}_{i,k}$ is G-conjugate to the centralizers in $\mathbf{G}$ of the elements of G-class type (i,k); see also Subsection 9.3. Table 25 gives the rough structure of M following the conventions introduced in Subsection 2.1 and 2.9. The structure of these groups can be determined with the methods utilized in Subsection 4.13. The structure of the groups for $i \in \{4,7,9,13,17\}$ assumes $3 \nmid q$. (In case $3 \mid q$, these groups are direct products; see Propositions 4.14 and 4.17.) We also give, in case $3 \nmid q$, a representative of the conjugacy class of radical 3-subgroups of G containing a Sylow 3-subgroup of M; cf. Proposition 6.6 and Table 26. Finally, the last column of this table gives the conditions for the existence of a semisimple element $s \in G$ with $C_{\mathbf{G}}(s) = \mathbf{M}$. The other notational conventions used in the table have been in effect throughout this paper. The integer $\varepsilon \in \{1,-1\}$ is defined by $3 \mid q-\varepsilon$, provided $3 \nmid q$. Also $d = \gcd(2, q-1) \in \{1,2\}$. The symbol $\circ_1$ denotes the direct product of two groups. Any number in square brackets denotes a cyclic group of that order. The notational conventions used to distinguish the cases $\varepsilon = 1$ and $\varepsilon = -1$, are the same as in Tables 1–19; see the introduction to Subsection 9.1. The information given in Table 25, apart from that on the Sylow 3-subgroups, is contained in [58].

Table 25. Sylow 3-subgroups of centralizers

i	k	M	$\mathrm{Syl}_3(M)$	Condition
1	1	$F_4(q)$	R_{38}	
2	1	$\mathrm{Spin}_9(q)$	R_{34}	q odd
3	1	$(\mathrm{SL}_4^{\varepsilon}(q) \circ_2 \mathrm{SL}_2(q)).2$	R_{34}	$4 \mid q-1$
		$(\mathrm{SL}_4^{-\varepsilon}(q) \circ_2 \mathrm{SL}_2(q)).2$	R_{17}	$4 \mid q-1$
3	2	$(\mathrm{SL}_4^{\varepsilon}(q) \circ_2 \mathrm{SL}_2(q)).2$	R_{34}	$4 \mid q+1$
		$(\mathrm{SL}_4^{-\varepsilon}(q) \circ_2 \mathrm{SL}_2(q)).2$	R_{17}	$4 \mid q+1$
4	$1,2$	$(\mathrm{SL}_3^{\varepsilon}(q) \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{38}	
5	1	$(\mathrm{Sp}_6(q) \circ_2 \mathrm{SL}_2(q)).2$	R_{33}	q odd
6	$1,2$	$([q-\varepsilon] \circ_d \mathrm{Spin}_7(q)).d$	R_{34}	$(q,k) \neq (2,1)$
		$([q+\varepsilon] \circ_d \mathrm{Spin}_7(q)).d$	R_{32}	$(q,k) \neq (2,1)$
7	$1,2$	$(\mathrm{SL}_3^{\varepsilon}(q) \circ_3 \mathrm{GL}_2^{\varepsilon}(q)).3$	R_{34}	$q \neq 2, (q,k) \neq (4,1)$
		$(\mathrm{SL}_3^{-\varepsilon}(q) \circ_3 \mathrm{GL}_2^{-\varepsilon}(q)).3$	R_9	$q \neq 2, (q,k) \neq (4,1)$
8	$1,4$	$([q-\varepsilon] \circ_2 \mathrm{SL}_4^{\varepsilon}(q)).2$	R_{34}	q odd
		$([q+\varepsilon] \circ_2 \mathrm{SL}_4^{-\varepsilon}(q)).2$	R_{10}	q odd
	$2,3$	$([q+\varepsilon] \circ_2 \mathrm{SL}_4^{\varepsilon}(q)).2$	R_{32}	q odd
		$([q-\varepsilon] \circ_2 \mathrm{SL}_4^{-\varepsilon}(q)).2$	R_{17}	q odd
9	$1,2$	$(\mathrm{GL}_2^{\varepsilon}(q) \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{33}	$q \neq 2, (q,k) \neq (4,1)$
		$(\mathrm{GL}_2^{-\varepsilon}(q) \circ_3 (\mathrm{SL}_3^{-\varepsilon}(q)).3$	R_9	$q \neq 2, (q,k) \neq (4,1)$
10	$1,2$	$([q-\varepsilon] \circ_d \mathrm{Sp}_6(q)).d$	R_{33}	$(q,k) \neq (2,1)$
		$([q+\varepsilon] \circ_d \mathrm{Sp}_6(q)).d$	R_{31}	$(q,k) \neq (2,1)$
11	$1,2$	$([q-\varepsilon] \circ_2 (\mathrm{Sp}_4(q) \times \mathrm{SL}_2(q))).2$	R_{18}	q odd
		$([q+\varepsilon] \circ_2 (\mathrm{Sp}_4(q) \times \mathrm{SL}_2(q))).2$	R_{16}	q odd
12	$1,3$	$([q-\varepsilon] \circ_2 (\mathrm{SL}_2(q)^2 \circ_2 \mathrm{SL}_2(q)).2).2$	R_{18}	q odd
		$([q+\varepsilon] \circ_2 (\mathrm{SL}_2(q)^2 \circ_2 \mathrm{SL}_2(q)).2).2$	R_{17}	q odd
	$2,4$	$([q-\varepsilon] \circ_2 (\mathrm{SL}_2(q^2) \circ_2 \mathrm{SL}_2(q)).2).2$	R_{17}	q odd
		$([q+\varepsilon] \circ_2 (\mathrm{SL}_2(q^2) \circ_2 \mathrm{SL}_2(q)).2).2$	R_{10}	q odd
13	$1,6$	$([q-\varepsilon]^2 \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{34}	$q \neq 2,4$
		$([q+\varepsilon]^2 \circ_3 \mathrm{SL}_3^{-\varepsilon}(q)).3$	R_2	$q \neq 2,4$
	$2,3$	$([q^2-1] \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{32}	$q \neq 2$
		$([q^2-1] \circ_3 \mathrm{SL}_3^{-\varepsilon}(q)).3$	R_9	$q \neq 2$
	$5,4$	$([q^2+\varepsilon q+1] \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{30}	$(q,k) \neq (2,4)$
		$([q^2-\varepsilon q+1] \circ_3 \mathrm{SL}_3^{-\varepsilon}(q)).3$	R_2	$(q,k) \neq (2,4)$

Table 25. Sylow 3-subgroups of centralizers (continued)

i	k	M	$\mathrm{Syl}_3(M)$	Condition
14	$1, 4$	$([q-\varepsilon]^2 \circ_{d^2} \mathrm{SL}_2(q)^2).d^2$	R_{18}	$q \neq 2, 4$
		$([q+\varepsilon]^2 \circ_{d^2} \mathrm{SL}_2(q)^2).d^2$	R_9	$q \neq 2, 4$
	2	$([q^2-1] \circ_d \mathrm{SL}_2(q)^2).d$	R_{17}, R_{16}	$q \neq 2$
	3	$([q^2-1] \circ_d \mathrm{SL}_2(q)^2).d$	R_{16}, R_{17}	$q \neq 2$
15	$1, 3$	$([q-\varepsilon]^2 \circ_d \mathrm{Sp}_4(q)).d$	R_{18}	$q \neq 2, (q,k) \neq (4,1)$
		$([q+\varepsilon]^2 \circ_d \mathrm{Sp}_4(q)).d$	R_{10}	$q \neq 2, (q,k) \neq (4,1)$
	$2/4$	$([q^2-1] \circ_d \mathrm{Sp}_4(q)).d$	R_{16}/R_{17}	$q \neq 2$
	5	$([q^2+1] \circ_d \mathrm{Sp}_4(q)).d$	R_{10}	
16	$1, 9$	$([q-\varepsilon]^2 \circ_2 \mathrm{SL}_2(q)^2).2$	R_{18}	q odd
		$([q+\varepsilon]^2 \circ_2 \mathrm{SL}_2(q)^2).2$	R_{10}	q odd
	$2/6$	$([q^2-1] \circ_2 \mathrm{SL}_2(q)^2).2$	R_{17}/R_{16}	q odd
	5	$([q^2+1] \circ_2 \mathrm{SL}_2(q)^2).2$	R_{10}	q odd
	$3, 10$	$([q-\varepsilon]^2 \circ_2 \mathrm{SL}_2(q^2)).2$	R_{17}	q odd
		$([q+\varepsilon]^2 \circ_2 \mathrm{SL}_2(q^2)).2$	R_3	q odd
	$4/7$	$([q^2-1] \circ_2 \mathrm{SL}_2(q^2)).2$	R_{10}/R_9	q odd
	8	$([q^2+1] \circ_2 \mathrm{SL}_2(q^2)).2$	R_3	q odd
17	$1, 6$	$([q-\varepsilon]^2 \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{33}	$q \neq 2, 4$
		$([q+\varepsilon]^2 \circ_3 \mathrm{SL}_3^{-\varepsilon}(q)).3$	R_3	$q \neq 2, 4$
	$3, 2$	$([q^2-1] \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{31}	$q \neq 2$
		$([q^2-1] \circ_3 \mathrm{SL}_3^{-\varepsilon}(q)).3$	R_9	$q \neq 2$
	$5, 4$	$([q^2+\varepsilon q+1] \circ_3 \mathrm{SL}_3^{\varepsilon}(q)).3$	R_{29}	$(q,k) \neq (2,4)$
		$([q^2-\varepsilon q+1] \circ_3 \mathrm{SL}_3^{-\varepsilon}(q)).3$	R_3	$(q,k) \neq (2,4)$
18	$1, 10$	$([q-\varepsilon]^3 \circ_d \mathrm{SL}_2(q)).d$	R_{18}	$q \neq 2, 4$
		$([q+\varepsilon]^3 \circ_d \mathrm{SL}_2(q)).d$	R_2	$q \neq 2, 4$
	$2, 8/9, 5$	$(([q^2-1] \times [q-\varepsilon]) \circ_d \mathrm{SL}_2(q)).d$	R_{17}/R_{16}	$q \neq 2, (q,k) \neq (4,2), (4,9)$
		$(([q^2-1] \times [q+\varepsilon]) \circ_d \mathrm{SL}_2(q)).d$	R_9/R_{10}	$q \neq 2, (q,k) \neq (4,2), (4,9)$
	$3, 7$	$([q^3-\varepsilon] \circ_d \mathrm{SL}_2(q)).d$	R_{11}	$(q,k) \neq (2,3)$
		$([q^3+\varepsilon] \circ_d \mathrm{SL}_2(q)).d$	R_2	$(q,k) \neq (2,3)$
	$6, 4$	$(([q-\varepsilon] \times [q^2+1]) \circ_d \mathrm{SL}_2(q)).d$	R_{10}	$(q,k) \neq (2,6)$
		$(([q+\varepsilon] \times [q^2+1]) \circ_d \mathrm{SL}_2(q)).d$	R_2	$(q,k) \neq (2,6)$

Table 25. Sylow 3-subgroups of centralizers (continued)

i	k	M	$\mathrm{Syl}_3(M)$	Condition
19	$1, 10$	$([q-\varepsilon]^3 \circ_d \mathrm{SL}_2(q)).d$	R_{18}	$q \neq 2, 4$
		$([q+\varepsilon]^3 \circ_d \mathrm{SL}_2(q)).d$	R_3	$q \neq 2, 4$
	$2, 5/3, 4$	$(([q^2-1] \times [q-\varepsilon]) \circ_d \mathrm{SL}_2(q)).d$	R_{17}/R_{16}	$q \neq 2$
		$(([q^2-1] \times [q+\varepsilon]) \circ_d \mathrm{SL}_2(q)).d$	R_{10}/R_9	$q \neq 2$
	$8, 9$	$([q^3-\varepsilon] \circ_d \mathrm{SL}_2(q)).d$	R_{12}	$(q, k) \neq (2, 8)$
		$([q^3+\varepsilon] \circ_d \mathrm{SL}_2(q)).d$	R_3	$(q, k) \neq (2, 8)$
	$6, 7$	$(([q-\varepsilon] \times [q^2+1]) \circ_d \mathrm{SL}_2(q)).d$	R_{10}	$(q, k) \neq (2, 6)$
		$(([q+\varepsilon] \times [q^2+1]) \circ_d \mathrm{SL}_2(q)).d$	R_3	$(q, k) \neq (2, 6)$
20	$1, 2$	$[q-\varepsilon]^4$	R_{18}	$q \neq 2$
		$[q+\varepsilon]^4$	1	$q \neq 2$
	3	$[d] \times [(q^2-1)/d] \times [q^2-1]$	R_{10}	$q \neq 2$
	$4, 5/7, 8$	$[q-\varepsilon] \times [q^3-\varepsilon]$	R_{12}/R_{11}	$q \neq 2$
		$[q+\varepsilon] \times [q^3+\varepsilon]$	1	$q \neq 2$
	6	$[q^2+1]^2$	1	$q \neq 2$
	$9, 10$	$[q^2+\varepsilon q+1]^2$	R_8	$q \neq 2$
		$[q^2-\varepsilon q+1]^2$	1	$q \neq 2$
	11	$[q^4-q^2+1]$	1	
	$12, 13/17, 18$	$[q-\varepsilon]^2 \times [q^2-1]$	R_{16}/R_{17}	$q \neq 2$
		$[q+\varepsilon]^2 \times [q^2-1]$	R_2/R_3	$q \neq 2$
	$14, 15/19, 20$	$[(q^3-\varepsilon)(q+\varepsilon)]$	R_6/R_7	$(q, k) \neq (2, 15), (2, 20)$
		$[(q^3+\varepsilon)(q-\varepsilon)]$	R_2/R_3	$(q, k) \neq (2, 15), (2, 20)$
	$16/21$	$[d] \times [(q^4-1)/d]$	R_3/R_2	$q \neq 2$
	22	$[q^2-1]^2$	R_9	$q \neq 2$
	$23, 24$	$[(q^2+1)(q-\varepsilon)] \times [q-\varepsilon]$	R_{10}	$q \neq 2$
		$[(q^2+1)(q+\varepsilon)] \times [q+\varepsilon]$	1	$q \neq 2$
	25	$[q^4+1]$	1	

9.7. **The radical 3-subgroups of $F_4(q)$.** The table in this subsection lists the non-trivial radical 3-subgroups of $G = F_4(q)$ and some of their properties in case $3 \nmid q$. The displayed information has been determined in [6] for odd q, and in [4] for even q.

The parameters ε, e, a and d have the usual meaning: $\varepsilon \in \{-1, 1\}$ is such that $3 \mid q - \varepsilon$, and 3^a is the highest power of 3 dividing $q - \varepsilon$; moreover, $e = 1$ if $\varepsilon = 1$, and $e = 2$, otherwise, and $d = \gcd(2, q - 1)$.

Let R be one of the listed radical 3-subgroups. For easier reference, the first column of Table 26 contains a name for R, or rather for a representative of the G-conjugacy class containing R. This numbering is different from the one given in [4]. The second column gives the rough structure of the groups, using the conventions naming cyclic groups, extensions and central products as introduced in Subsections 2.1 and 2.9. These conventions are also used in Columns 5–7, where $C_G(R)$, $N_G(R)$ and $\mathrm{Out}_G(R) = N_G(R)/RC_G(R)$ are described. The groups $L^i \cong \mathrm{SL}_3^{\varepsilon}(q)$, $i = 1, 2$ have the same significance as in Subsection 6.4, and T_i and D_i, $i = 1, 2$, denote a maximal e-split torus and a Sylow 3-subgroup of L^i, respectively, constructed in Subsection 6.2.

The *characteristic* of R is defined to be the elementary abelian 3-subgroup $\Omega_1(Z(R))$, unless $R \in_G \{R_{29}–R_{34}\}$, where the characteristic is defined to be $\Omega_1([R, R])$. In the latter cases, $[R, R]$ is abelian. Recall that if Q is a finite r-group for some prime r, then $\Omega_1(Q)$ denotes the subgroup of Q generated by its elements of order r. This is elementary abelian if Q is abelian. The characteristics of the radical subgroups of G are worked out in [6, Lemma 4.2] and in [4, Proposition 4.5 and Theorem 4.8], respectively. The fourth column of Table 26 gives the names of the characteristics in the classification of [5, Table 4] (which holds for all q). The third column of Table 26 gives the types of the characteristics. If E is a characteristic, the *type* of E is the triple (i, j, k) of non-negative integers adding up to $|E| - 1$, indicating that E has exactly i, j and k non-trivial elements lying in the conjugacy classes 3A, 3B and 3C of G, respectively. The type of E is written as $3\mathrm{A}_i\mathrm{B}_j\mathrm{C}_k$ (with obvious modifications if one or two of the i, j, k are equal to 0). Column 8 contains the conditions on q for which the corresponding radical subgroup exists. Finally, Column 9 gives the name of the group $R^{\dagger}$, defined in Subsection 6.10.

	R	Type	Char	$C_G(R)$	$N_G(R)$	$\mathrm{Out}_G(R)$	Cond.	$R^\dagger$
R_1	3	3C	E_3	$(\mathrm{SL}_3^\varepsilon(q)\circ_3\mathrm{SL}_3^\varepsilon(q)).3$	$((\mathrm{SL}_3^\varepsilon(q)\circ_3\mathrm{SL}_3^\varepsilon(q)).3).2$	2	$q\neq 2$	R_1
R_2	3^a	3A	E_1	$([q-\varepsilon]\circ_d\mathrm{Sp}_6(q)).d$	$(([q-\varepsilon]\circ_d\mathrm{Sp}_6(q)).d).2$	2		R_3
R_3	3^a	3B	E_2	$([q-\varepsilon]\circ_d\mathrm{Spin}_7(q)).d$	$(([q-\varepsilon]\circ_d\mathrm{Spin}_7(q)).d).2$	2		R_2
R_4	3^a	3C	E_3	$(\mathrm{GL}_2^\varepsilon(q)\circ_3\mathrm{SL}_3^\varepsilon(q)).3$	$((\mathrm{GL}_2^\varepsilon(q)\circ_3\mathrm{SL}_3^\varepsilon(q)).3).2$	2	$a\geq 2$	R_5
R_5	3^a	3C	E_3	$(\mathrm{SL}_3^\varepsilon(q)\circ_3\mathrm{GL}_2^\varepsilon(q)).3$	$((\mathrm{SL}_3^\varepsilon(q)\circ_3\mathrm{GL}_2^\varepsilon(q)).3).2$	2	$a\geq 2$	R_4
R_6	3^{a+1}	3C	E_3	$(\mathrm{GL}_2^\varepsilon(q)\circ_3[q^2+\varepsilon q+1]).3$	$((\mathrm{GL}_2^\varepsilon(q)\circ_3[q^2+\varepsilon q+1]).3).6$	6	$q\neq 2$	R_7
R_7	3^{a+1}	3C	E_3	$([q^2+\varepsilon q+1]\circ_3\mathrm{GL}_2^\varepsilon(q)).3$	$(([q^2+\varepsilon q+1]\circ_3\mathrm{GL}_2^\varepsilon(q)).3).6$	6	$q\neq 2$	R_6
R_8	3^2	$(3\mathrm{C}^2)_2$	E_9	$[q^2+\varepsilon q+1]^2.3$	$([q^2+\varepsilon q+1]^2.3).\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	$q\neq 2$	R_8
R_9	$[3^a]^2$	$3\mathrm{A}_2\mathrm{B}_2\mathrm{C}_4$	E_5	$(\mathrm{GL}_2^\varepsilon(q)\circ_3\mathrm{GL}_2^\varepsilon(q)).3$	$((\mathrm{GL}_2^\varepsilon(q)\circ_3\mathrm{GL}_2^\varepsilon(q)).3).2^2$	2^2	$q\neq 2$	R_9
R_{10}	$[3^a]^2$	$3\mathrm{A}_4\mathrm{B}_4$	E_4	$([q-\varepsilon]^2\circ_d\mathrm{Sp}_4(q)).d$	$(([q-\varepsilon]^2\circ_d\mathrm{Sp}_4(q)).d).D_8$	D_8		R_{10}
R_{11}	$[3^a]\times[3^{a+1}]$	$3\mathrm{A}_6\mathrm{C}_2$	E_6	$[q-\varepsilon]\times[q^3-\varepsilon]$	$([q-\varepsilon]\times[q^3-\varepsilon]).(S_3\times 6)$	$S_3\times 6$	$q\neq 2,4,8$	R_{12}
R_{12}	$[3^{a+1}]\times[3^a]$	$3\mathrm{B}_6\mathrm{C}_2$	E_7	$[q^3-\varepsilon]\times[q-\varepsilon]$	$([q^3-\varepsilon]\times[q-\varepsilon]).(6\times S_3)$	$6\times S_3$	$q\neq 2,4,8$	R_{11}
R_{13}	$[3^a]^2$	$3\mathrm{A}_6\mathrm{C}_2$	E_6	$(T_1\circ_3 L^2).3$	$((T_1\circ_3 L^2).3).(S_3\times 2)$	$S_3\times 2$	$q\neq 2,4,8$	R_{14}
R_{14}	$[3^a]^2$	$3\mathrm{B}_6\mathrm{C}_2$	E_7	$(L^1\circ_3 T_2).3$	$((L^1\circ_3 T_2).3).(2\times S_3)$	$2\times S_3$	$q\neq 2,4,8$	R_{13}
R_{15}	3^3	$(3\mathrm{C}^3)_1$	E_{13}	3^3	$3^3.\mathrm{SL}_3(3)$	$\mathrm{SL}_3(3)$		R_{15}
R_{16}	$[3^a]^3$	$3\mathrm{A}_6\mathrm{B}_{12}\mathrm{C}_8$	E_{11}	$(\mathrm{SL}_2(q)\circ_d[q-\varepsilon]^3).d$	$((\mathrm{SL}_2(q)\circ_d[q-\varepsilon]^3).d).2^3.S_3$	$2^3.S_3$	$q\neq 2$	R_{17}
R_{17}	$[3^a]^3$	$3\mathrm{A}_{12}\mathrm{B}_6\mathrm{C}_8$	E_{10}	$([q-\varepsilon]^3\circ_d\mathrm{SL}_2(q)).d$	$(([q-\varepsilon]^3\circ_d\mathrm{SL}_2(q)).d).2^3.S_3$	$2^3.S_3$	$q\neq 2$	R_{16}
R_{18}	$[3^a]^4$	$3\mathrm{A}_{24}\mathrm{B}_{24}\mathrm{C}_{32}$	E_{15}	$T=[q-\varepsilon]^4$	$T.W(F_4)$	$W(F_4)$		R_{18}

TABLE 26. Non-trivial radical 3-subgroups of $G=F_4(q)$

	R	Type	Char	$C_G(R)$	$N_G(R)$	$\mathrm{Out}_G(R)$	Cond.	$R^\dagger$
R_{19}	3_+^{1+2}	3C	E_3	$L^1 = \mathrm{SL}_3^\varepsilon(q)$	$(L^1 \circ_3 (3_+^{1+2}.\mathrm{SL}_2(3))).2$	$\mathrm{SL}_2(3).2$	$q \neq 2$	R_{19}
R_{20}	3_+^{1+2}	3C	E_3	$L^2 = \mathrm{SL}_3^\varepsilon(q)$	$((3_+^{1+2}.\mathrm{SL}_2(3)) \circ_3 L^2).2$	$\mathrm{SL}_2(3).2$	$q \neq 2$	R_{20}
R_{21}	3_+^{1+4}	3C	E_3	3	$((3_+^{1+2}.Q_8) \circ_3 (3_+^{1+2}.Q_8)).S_3$	$(Q_8 \times Q_8).S_3$	$a = 1$	R_{21}
R_{21}	3_+^{1+4}	3C	E_3	3	$((3_+^{1+2}.\mathrm{SL}_2(3)) \circ_3 (3_+^{1+2}.\mathrm{SL}_2(3))).2$	$(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)).2$	$a \geq 2$	R_{21}
R_{22}	3_+^{1+4}	3C	E_3	3	$(3_+^{1+2}.\mathrm{SL}_2(3)) \circ_3 (3_+^{1+2}.\mathrm{SL}_2(3))$	$\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)$	$a \geq 2$	R_{22}
R_{23}	$[3^a] \circ_3 3_+^{1+2}$	3C	E_3	$\mathrm{GL}_2^\varepsilon(q) \leq L^1$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 (3_+^{1+2}.\mathrm{SL}_2(3))).2$	$\mathrm{SL}_2(3).2$	$a \geq 2$	R_{24}
R_{24}	$3_+^{1+2} \circ_3 [3^a]$	3C	E_3	$\mathrm{GL}_2^\varepsilon(q) \leq L^2$	$((3_+^{1+2}.\mathrm{SL}_2(3)) \circ_3 \mathrm{GL}_2^\varepsilon(q)).2$	$\mathrm{SL}_2(3).2$	$a \geq 2$	R_{23}
R_{25}	$[3^a]^2 \circ_3 3_+^{1+2}$	$3\mathrm{A}_6\mathrm{C}_2$	E_6	$T_1 = [q-\varepsilon]^2$	$((T_1.S_3) \circ_3 (3_+^{1+2}.\mathrm{SL}_2(3))).2$	$(S_3 \times \mathrm{SL}_2(3)).2$	$q \neq 2, 4, 8$	R_{26}
R_{26}	$3_+^{1+2} \circ_3 [3^a]^2$	$3\mathrm{B}_6\mathrm{C}_2$	E_7	$T_2 = [q-\varepsilon]^2$	$((3_+^{1+2}.\mathrm{SL}_2(3)) \circ_3 (T_2.S_3))).2$	$(\mathrm{SL}_2(3) \times S_3).2$	$q \neq 2, 4, 8$	R_{25}
R_{27}	$D_2 = [3^a]^2.3$	3C	E_3	L^1	$(L^1 \circ_3 (D_2.2)).S_3$	$2 \times S_3$	$a \geq 2$	R_{27}
R_{28}	$D_1 = [3^a]^2.3$	3C	E_3	L^2	$((D_1.2) \circ_3 L^2).S_3$	$2 \times S_3$	$a \geq 2$	R_{28}
R_{29}	$D_2.3$	$3\mathrm{B}_6\mathrm{C}_2$	E_7	$[q^2+\varepsilon q+1] \leq L^1$	$(([q^2+\varepsilon q+1] \circ_3 (D_2.2)).3).S_3$	$2 \times S_3$	$q \neq 2$	R_{30}
R_{30}	$D_1.3$	$3\mathrm{A}_6\mathrm{C}_2$	E_6	$[q^2+\varepsilon q+1] \leq L^2$	$(((D_1.2) \circ_3 [q^2+\varepsilon q+1]).3).S_3$	$2 \times S_3$	$q \neq 2$	R_{29}
R_{31}	$([3^a] \circ_3 D_2).3$	$3\mathrm{B}_6\mathrm{C}_2$	E_7	$\mathrm{GL}_2^\varepsilon(q) \leq L^1$	$(\mathrm{GL}_2^\varepsilon(q) \circ_3 (D_2.2)).S_3$	2^2	$q \neq 2$	R_{32}
R_{32}	$(D_1 \circ_3 [3^a]).3$	$3\mathrm{A}_6\mathrm{C}_2$	E_6	$\mathrm{GL}_2^\varepsilon(q) \leq L^2$	$((D_1.2) \circ_3 \mathrm{GL}_2^\varepsilon(q)).S_3$	2^2	$q \neq 2$	R_{31}
R_{33}	$([3^a]^2 \circ_3 D_2).3$	$3\mathrm{B}_6\mathrm{C}_2$	E_7	$T_1 = [q-\varepsilon]^2$	$((T_1.2) \circ_3 (D_2.2)).S_3$	2^3	$q \neq 2, 4, 8$	R_{34}
R_{34}	$(D_1 \circ_3 [3^a]^2).3$	$3\mathrm{A}_6\mathrm{C}_2$	E_6	$T_2 = [q-\varepsilon]^2$	$((D_1.2) \circ_3 (T_2.2)).S_3$	2^3	$q \neq 2, 4, 8$	R_{33}
R_{35}	$3_+^{1+2} \circ_3 D_2$	3C	E_3	3	$((3_+^{1+2}.\mathrm{SL}_2(3) \circ_3 (D_2.2)).2$	$(\mathrm{SL}_2(3) \times 2).2$	$a \geq 2$	R_{35}
R_{36}	$D_1 \circ_3 3_+^{1+2}$	3C	E_3	3	$((D_1.2) \circ_3 (3_+^{1+2}.\mathrm{SL}_2(3))).2$	$(2 \times \mathrm{SL}_2(3)).2$	$a \geq 2$	R_{36}
R_{37}	$[3^a]^4.3$	$(3\mathrm{C}^2)_1$	E_8	3^2	$[3^a]^4.3.\mathrm{SL}_2(3).2$	$\mathrm{SL}_2(3).2$		R_{37}
R_{38}	$\mathrm{Syl}_3(G)$	3C	E_3	3	$\mathrm{Syl}_3(G).2^3$	2^3		R_{38}

TABLE 26. Non-trivial radical 3-subgroups of $G = F_4(q)$ (continued)

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