A NOTE ON THE CONSTRUCTION OF RIGHT CONJUGACY CLOSED LOOPS

GERHARD HISS AND LUCIA ORTJOHANN

ABSTRACT. We describe a group theoretical construction of non-associative right conjugacy closed loops with abelian inner mapping groups.

1. Introduction

A *loop* is a quasigroup with an identity element. If the multiplication of the loop is associative, it is a group. In the following, every loop, and in particular every group, will be assumed to be finite.

Let $(\mathcal{L}, *)$ be a loop with identity element $1_{\mathcal{L}}$. For every $x \in \mathcal{L}$, we denote by R_x the *right multiplication* by x in $\mathcal{L}$, i.e. $R_x : \mathcal{L} \to \mathcal{L}, y \mapsto y * x$, and we set $R_{\mathcal{L}} := \{R_x \mid x \in \mathcal{L}\}$. Then $\mathrm{RM}(\mathcal{L}) := \langle R_{\mathcal{L}} \rangle \leq \mathrm{Sym}(\mathcal{L})$ and its subgroup $\mathrm{Stab}_{\mathrm{RM}(\mathcal{L})}(1_{\mathcal{L}})$ are called the *right multiplication group*, and the *inner mapping group* of $\mathcal{L}$, respectively. The *envelope* of $\mathcal{L}$ consists of the triple $(\mathrm{RM}(\mathcal{L}), \mathrm{Stab}_{\mathrm{RM}(\mathcal{L})}(1_{\mathcal{L}}), R_{\mathcal{L}})$. To simplify notation, let us put $G := \mathrm{RM}(\mathcal{L})$, $H := \mathrm{Stab}_{\mathrm{RM}(\mathcal{L})}(1_{\mathcal{L}})$ and $T := R_{\mathcal{L}}$. Clearly, G acts faithfully and transitively on $\mathcal{L}$, which may hence be identified with the set of right cosets of H in G. Notice that $\mathcal{L}$ is a group if and only if $|G| = |\mathcal{L}|$, or, equivalently, $H = \{1\}$. By definition, T generates G, and one can check that T is a transversal for the set of right cosets of H^g in G for every $g \in G$. Envelopes of loops are generalized to *loop folders*.

The connection between loops and loop folders, summarized below, goes back to Baer [3], and is described in detail by Aschbacher in [2, Section 1]. In the following, G denotes a finite group and H a subgroup of G; we write $H\backslash G$ for the set of right cosets of H in G. The triple (G, H, T) is called a *loop folder* if $T \subseteq G$ is a transversal for $H^g\backslash G$ for every $g \in G$, and if $1 \in T$. We call (G, H, T) *faithful* if G acts faithfully on $H\backslash G$, i.e. if $\mathrm{core}_G(H) = \{1\}$.

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By construction, the envelope (G, H, T) of a loop $\mathcal{L}$ is a faithful loop folder with $G = \langle T \rangle$, and there is a natural bijection between T and $\mathcal{L}$. Conversely, given a loop folder (G, H, T), one can construct a loop $(T, *)$ on the set T in such a way that (G, H, T) is isomorphic to the envelope of $(T, *)$, provided (G, H, T) is faithful and $G = \langle T \rangle$. This motivates the following definition. A transversal T for $H \backslash G$ is called a *generating transversal* if $G = \langle T \rangle$.

A loop $\mathcal{L}$ is called *right conjugacy closed* or an RCC-*loop* if the set $R_{\mathcal{L}}$ is closed under conjugation, i.e. if $R_x^{-1} R_y R_x \in R_{\mathcal{L}}$ for all $x, y \in \mathcal{L}$. Analogously, a loop folder (G, H, T) is called *right conjugacy closed*, or an RCC-*loop folder* if T is G-invariant under conjugation, i.e. $g^{-1}tg \in T$ for all $g \in G, t \in T$. Clearly, a loop is right conjugacy closed if and only if its envelope is an RCC-loop folder.

In this paper we construct envelopes of RCC-loops with abelian inner mapping groups. The following trivial observations form the starting point of our construction.

Proposition 1.1. Let G be a finite group, $Q \trianglelefteq G$ and $H \leq G$ with $H \cap Q = \{1\}$. Let $\hat{T} = \{t_1, \ldots, t_n\}$ be a transversal for HQ in G. Then $T := \hat{T}Q$ is a transversal for H in G and $\{t_1 Q, \ldots, t_n Q\}$ is a transversal for HQ/Q in G/Q. Furthermore, we have the following two statements.

(a) The transversal $\{t_1 Q, \ldots, t_n Q\}$ is G/Q-invariant if and only if T is G-invariant.

(b) The transversal $\{t_1 Q, \ldots, t_n Q\}$ generates G/Q if and only if T generates G.

Thus if $\mathrm{core}_H(G) = \{1\}$ and the transversal $\{t_1 Q, \ldots, t_n Q\}$ is G/Q-invariant and generates G/Q, then (G, H, T) is an envelope of an RCC-loop, which is non-associative if $\{1\} \lneq H \lneq G$. □

Notice that $\mathrm{core}_H(G) \leq C_H(Q)$ under our assumption $H \cap Q = \{1\}$, so that $C_H(Q) = \{1\}$ implies $\mathrm{core}_H(G) = \{1\}$. If G/Q is abelian, then H is abelian and T is G-invariant by part (a) of Proposition 1.1. This holds in particular for Q equal to the commutator subgroup $[G, G]$ of G. We conjecture that the converse of this statement holds.

Conjecture 1.2. *Let G be a finite group, $H \leq G$ an abelian subgroup such that there exists a G-invariant transversal T for $H \backslash G$ with $1 \in T$, i.e. (G, H, T) is an* RCC-*loop folder. Then $[G, G] \cap H = \{1\}$.* □

In Section 3 we prove this conjecture in special cases.

2. Construction of generating transversals for abelian groups

In this section we investigate the existence of generating transversals in abelian groups. Let p be a prime. We first show that if G is an abelian p-group, and the index of H in G is larger than the minimal size of a generating set of G, there exists a generating transversal for $H\backslash G$ containing 1. We generalize this result for an arbitrary abelian group G, however with a stronger condition on the index of H in G.

The minimal size of a generating set of G is called the *rank* of G, i.e.

$$\mathrm{rk}(G) := \min\{|S| \mid S \subseteq G, G = \langle S\rangle\}.$$

A cyclic group of order n is denoted by C_n. If T is a generating transversal for $H\backslash G$ containing 1, then necessarily, $|G : H| > \mathrm{rk}(G)$. For the sake of clarity in the proofs to follow, we write the elements of a direct product $A \times B$ of groups as pairs (a, b) with $a \in A$, $b \in B$.

Proposition 2.1. *Let G be an abelian p-group. Suppose that $H \leq G$ is a subgroup of G such that $|G : H| > \mathrm{rk}(G)$. Then there exists a generating transversal T for $H\backslash G$ with $1 \in T$.*

Proof. We proceed by induction on the order of G, where the base case is trivial. For the induction step, we assume that $G \neq \{1\}$, that the statement holds for every abelian p-group of order less than $|G|$, and distinguish two cases.

Case 1: For every decomposition

$$G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}, \tag{1}$$

with $\{1\} \neq C_{m_i} \leq G$ for $1 \leq i \leq r$, we have $C_{m_i} \not\leq H$ for all $1 \leq i \leq r$.

Consider an arbitrary decomposition of G as in (1), and let a_i be a generator of C_{m_i} for every $1 \leq i \leq r$. Then $G = \langle a_1, \ldots, a_r\rangle$ and our assumption implies that $a_i \notin H$ for all $1 \leq i \leq r$. Suppose that for any $1 \leq i \neq j \leq r$, the generators a_i and a_j of G lie in distinct cosets of H in G. Then there is a transversal for $H\backslash G$ containing $\{1, a_1, \ldots, a_r\}$ and we are done. Otherwise, $Ha_i = Ha_j$ for some $1 \leq i \neq j \leq r$. Without loss of generality, we may assume that $|a_j| \geq |a_i|$. Then

$$G = \langle a_1\rangle \times \cdots \times \langle a_{j-1}\rangle \times \langle a_j a_i^{-1}\rangle \times \langle a_{j+1}\rangle \times \cdots \times \langle a_r\rangle,$$

and we have $\langle a_j a_i^{-1}\rangle \leq H$. We have thus reduced the assertion to the following situation.

Case 2: There exist $\{1\} \neq C_{m_i} \leq G$ for $1 \leq i \leq r$ such that

$$G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r},$$

and $C_{m_j} \leq H$ for some $1 \leq j \leq r$.

Note that the generators of these cyclic groups form a minimal generating set of G of size r. Thus, it follows from Burnside's basis theorem [5, III, Satz 3.15] that $r = \mathrm{rk}(G)$.

Set $U := C_{m_j}$,

$$\widetilde{G} := C_{m_1} \times \cdots \times C_{m_{j-1}} \times C_{m_{j+1}} \times \cdots \times C_{m_r},$$

and $\widetilde{H} := \widetilde{G} \cap H$. Clearly, $\widetilde{H}$ is a complement to U in H and thus, without loss of generality, we may assume that

$$G = \widetilde{G} \times U \quad \text{and} \quad H = \widetilde{H} \times U.$$

By construction,

$$\mathrm{rk}(\widetilde{G}) = r - 1 < r = \mathrm{rk}(G).$$

Since $\left|\widetilde{G}\right| < |G|$ and

$$\left|\widetilde{G} : \widetilde{H}\right| = |G : H| > \mathrm{rk}(G) > \mathrm{rk}(\widetilde{G}), \tag{2}$$

we can apply the induction hypothesis to $\widetilde{G}$ and hence there exists a generating transversal $\widetilde{T}$ for $\widetilde{H}\backslash\widetilde{G}$ with $1 \in \widetilde{T}$. Moreover, from Equation (2) we obtain

$$\left|\widetilde{T} - \{1\}\right| = \left|\widetilde{G} : \widetilde{H}\right| - 1 > \mathrm{rk}(\widetilde{G}). \tag{3}$$

Suppose that $\widetilde{T} - \{1\}$ is a minimal generating set for $\widetilde{G}$. Then Burnside's basis theorem [5, III, Satz 3.15] implies that $\left|\widetilde{T} - \{1\}\right| = \mathrm{rk}(\widetilde{G})$, contradicting Equation (3). Thus there exists $1 \neq t \in \widetilde{T}$ such that $t = t_1 \cdots t_k$ for certain $t_1, \ldots, t_k \in \langle \widetilde{T} \setminus \{1, t\} \rangle$.

Now $\widetilde{T} \times \{1\}$ is a transversal for $H\backslash G$ and we set

$$T := (\widetilde{T} \times \{1\} \setminus \{(t, 1)\}) \cup \{(t, u)\},$$

where u is a generator of U. Clearly, $(1, 1) \in T$, and T is a transversal for $H\backslash G$ since $(t, 1)$ and (t, u) lie in the same coset of H in G. It remains to show that T generates G. Recall that $t = t_1 \cdots t_k$ with $t_1, \ldots, t_k \in \langle \widetilde{T} \setminus \{1, t\} \rangle$. As $(t_1, 1), \ldots, (t_k, 1) \in \langle T \rangle$, we also have $(t^{-1}, 1) \in \langle T \rangle$. Hence $(1, u) = (t^{-1}, 1)(t, u) \in \langle T \rangle$ and then $(t, 1) = (t, u)(1, u^{-1}) \in \langle T \rangle$. We conclude that $\langle T \rangle \geq \langle \widetilde{T} \rangle \times \langle u \rangle = \widetilde{G} \times U = G$ and we are done. □

Let G be an abelian group and let $p_1, \ldots, p_n$ be the distinct prime divisors of G. Assume that $G = G_1 \times \cdots \times G_n$ with $G_i := O_{p_i}(G)$ for all $1 \leq i \leq n$. Then an easy induction on n shows that

$$\mathrm{rk}(G) = \max\{\mathrm{rk}(G_i) \mid 1 \leq i \leq n\}. \tag{4}$$

We now transfer the result of Proposition 2.1 to an arbitrary abelian group.

Theorem 2.2. *Let G be an abelian group, let $p_1, \dots, p_n$ be the distinct prime divisors of G and let $H \leq G$. Then*

$$G = G_1 \times \cdots \times G_n \quad \text{and} \quad H = H_1 \times \cdots \times H_n,$$

with $G_i := O_{p_i}(G)$ and $H_i := O_{p_i}(H)$. If

$$\max\{|G_i : H_i| \mid 1 \leq i \leq n\} > \mathrm{rk}(G),$$

then there exists a generating transversal for $H\backslash G$ containing 1.

Proof. Without loss of generality, we assume that

$$|G_1 : H_1| = \max\{|G_i : H_i| \mid 1 \leq i \leq n\}$$

and we set $\widetilde{G} := G_2 \times \cdots \times G_n$ and $\widetilde{H} := H_2 \times \cdots \times H_n$. Then $G = G_1 \times \widetilde{G}$ and $H = H_1 \times \widetilde{H}$. Equation (4) yields

$$\begin{aligned} m := |G_1 : H_1| = \max\{|G_i : H_i| \mid 1 \leq i \leq n\} &> \mathrm{rk}(G) \\ = \max\{\ \mathrm{rk}(G_i)\ \mid 1 \leq i \leq n\} &\geq \mathrm{rk}(G_1). \end{aligned}$$

Since G_1 is an abelian p_1-group with $|G_1 : H_1| > \mathrm{rk}(G_1)$, it follows from Proposition 2.1 that there exists a transversal $T_1 = \{t_1, \dots, t_m\}$ for $H_1\backslash G_1$ with $t_1 = 1$ and $G_1 = \langle T_1 \rangle$. We are done if $n = 1$. Assume from now on that $n > 1$.

Put $K := H_1 \times \widetilde{G}$. Then $H \leq K \leq G$ and $|G : K| = |G_1 : H_1| = m$. We next construct a generating transversal for $K\backslash G$ containing 1. Our hypothesis and Equation (4) imply that

$$\begin{aligned} k := \mathrm{rk}(\widetilde{G}) = &\max\{\mathrm{rk}(G_i) \mid 2 \leq i \leq n\} \leq \mathrm{rk}(G) \\ &< \max\{|G_i : H_i| \mid 1 \leq i \leq n\} = |G_1 : H_1| = m. \end{aligned}$$

Let S be a generating set of $\widetilde{G}$ with $|S| = k$. Then S is a minimal generating set and thus $1 \notin S$. Write $S \cup \{1\} := \{s_1, \dots, s_{k+1}\}$ with $s_1 = 1$. Now $|S \cup \{1\}| = k + 1 \leq m$, and we set

$$R := \bigcup_{i=1}^{k+1} (t_i, s_i) \cup \bigcup_{j=k+2}^{m} (t_j, s_1).$$

As $t_1 = 1$ and $s_1 = 1$, we have $(1, 1) \in R$ and $|R| = m = |G : K|$. We proceed to show that R is a generating transversal for $K\backslash G$. Suppose that $(t_i, s_j), (t_k, s_l) \in R$ such that $(t_i, s_j)(t_k, t_l)^{-1} \in H_1 \times \widetilde{G}$. Then $t_i t_k^{-1} \in H_1$ and as T_1 is a transversal for $H_1\backslash G_1$, it follows that $i = k$. This implies that $j = l$. We conclude that R is a transversal for $K\backslash G$. The fact $\gcd(|G_1|, |\widetilde{G}|) = 1$ yields that for every $(t, s) \in R$ there exist $a, b \in \mathbb{Z}$ such that $(t, s)^a = (1, s)$ and $(t, s)^b = (t, 1)$. Hence

$$\langle R \rangle \geq \langle T_1 \rangle \times \langle S \rangle = G_1 \times \widetilde{G} = G$$

and thus, R is a generating transversal for $K\backslash G$ with $1 \in R$.

Let V be a transversal for $H\backslash K$ with $1 \in V$. Then $T := VR$ is a transversal for $H\backslash G$. Since $1 \in V$, we have $R \subseteq T$ and it follows that $\langle T\rangle \geq \langle R\rangle = G$. This implies that T is a generating transversal for $H\backslash G$ with $1 \in T$. $\square$

With this result and Proposition 1.1 we can construct envelopes of RCC-loops.

Corollary 2.3. *Let G be a group and let H be a subgroup of G. Let Q be a normal subgroup of G such that G/Q is abelian, $H \cap Q = \{1\}$, $C_H(Q) = \{1\}$ and*

$$\max\{|O_p(G/Q) : O_p(HQ/Q)| \mid p \text{ prime divisor of } G/Q\} > \mathrm{rk}(G/Q).$$

Then there exists a G-invariant generating transversal T for $H\backslash G$ with $1 \in T$, and G acts faithfully on $H\backslash G$; thus (G, H, T) is an envelope of a non-associative RCC*-loop.* $\square$

Recall that T in Corollary 2.3 arises from combining a generating transversal for $HQ\backslash G$ with Q; see Proposition 1.1. If G is a Frobenius group with kernel Q (in which case $Q = [G, G]$, the commutator subgroup of G), every G-invariant transversal for $H\backslash G$ has this form (see [7, Theorem 3.6]). In general, there may be G-invariant transversals, which are not obtained in this way. Since $[G, G]$ is the smallest normal subgroup of G with abelian quotient, we can replace Q by $[G, G]$.

Finally, notice that the construction of RCC-loops arising from Proposition 1.1 is, of course, not restricted to the case G/Q abelian.

3. A conjecture for RCC-loop folders

In this final section we discuss Conjecture 1.2. Using GAP [4], this conjecture has been verified for all non-abelian groups of order smaller than 40 by the second author in her master thesis [7], and for the multiplication groups of RCC-loops of order up to 30, by Artic in her dissertation [1].

It follows from a result of Zappa, that Conjecture 1.2 holds in case H is a Hall subgroup of G. Indeed, Zappa shows that if H is a nilpotent Hall subgroup of G such that there exists a transversal for $H\backslash G$ which is invariant under conjugation by H, then H has a normal complement; see [8, Proposizione XIV 12.1]. Now if H is abelian, the commutator subgroup of G is contained in this normal complement. In [6], Kochendörffer generalizes Zappa's result. We present the essence of Zappa's and Kochendörffer's argument in the following theorem.

Theorem 3.1. *Let G be a finite group and let H be an abelian Hall subgroup of G. Suppose that there exists transversal T for $H\backslash G$ which is invariant under conjugation by H. Then $[G,G]\cap H=\{1\}$.*

Proof. This is very much inspired by the proof of [6, Theorem]. The transfer map

$$\tau : G \to H, x \mapsto \prod_{t\in T} \lambda_x^T(t),$$

where $\lambda_x^T(t)$ is the unique element in H such that $tx=\lambda_x^T(t)t'$ for some $t'\in T$, is a group homomorphism; see [5, IV, Hauptsatz 1.4].

Let $h\in H$ and let $h':=\lambda_h^T(t)$ for some $t\in T$. Then $th=h't'$ for some $t'\in T$. It follows that $hh'^{-1}=t^{-1}h't'h'^{-1}\in H$ and since T is H-invariant, we have $h't'h'^{-1}\in T$. Thus, $t=h't'h'^{-1}$. This yields that $h=h'=\lambda_h^T(t)$ and hence

$$\tau(h)=\prod_{t\in T}\lambda_h^T(t)=\prod_{t\in T} h = h^{|G:H|}.$$

As H is an abelian Hall subgroup of G, the map $f : H\to H, h\mapsto h^{|G:H|}$ is an isomorphism. Thus $\ker\tau\cap H=\{1\}$. Furthermore, $G/\ker\tau$ is abelian, because the image of τ is abelian as subgroup of H. Hence, $[G,G]\leq\ker\tau$. We conclude that $[G,G]\cap H\leq\ker\tau\cap H=\{1\}$. □

In the next example we show that for the conclusion of Conjecture 1.2 to be true, it is not enough to require the existence of an H-invariant transversal for $H\backslash G$.

Example 3.2. *Let $G:=Q_8$ and let $H:=Z(G)$. Then H is abelian and every transversal of $H\backslash G$ is H-invariant. However, $H=Z(G)=[G,G]$. Notice that there does not exist any G-invariant transversal for $H\backslash G$.*

However, if p is a prime, G is a group of order p^3 and there exists a G-invariant transversal for $H\backslash G$, then Conjecture 1.2 holds.

Lemma 3.3. *Let G be a p-group with $[G,G]=Z(G)$ and $|Z(G)|=p$. Suppose that $H\leq G$ is abelian and that there exists a G-invariant transversal T of $H\backslash G$ containing 1. Then $[G,G]\cap H=\{1\}$.*

Proof. Since T is G-invariant, T is a union of conjugacy classes of G. As $1\in T$, we conclude that T contains at least p conjugacy classes with exactly one element, i.e. T contains at least p elements of $Z(G)$. Hence $[G,G]=Z(G)\subseteq T$ and thus $[G,G]\cap H\subseteq T\cap H=\{1\}$. □

This lemma shows that Conjecture 1.2 holds for groups of order p^3.

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Lehrstuhl für Algebra und Zahlentheorie, RWTH Aachen University, 52056 Aachen, Germany
Email address: G.H.: gerhard.hiss@math.rwth-aachen.de
Email address: L.O.: lucia.ortjohann@rwth-aachen.de