# THE 2-MODULAR CHARACTERS OF THE FISCHER GROUP $\mathbf{Fi_{23}}$

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Abstract. We determine the 2-modular character table of the Fischer group $\mathrm{Fi}_{23}$.

## 1. Introduction

Using computational methods, we construct the 2-modular character table of Fischer's second sporadic simple group $\mathrm{Fi}_{23}$. By previous results published in [7, 8], this leaves only the 3-modular character table of $\mathrm{Fi}_{23}$ to be determined.

All modular character tables of the first Fischer group $\mathrm{Fi}_{22}$ and its covering groups are also known. The third author has completed this knowledge in his PhD thesis [15] by determining the 2- and 3-modular tables for these groups. The situation for the third Fischer group $\mathrm{Fi}_{24}'$ is less favorable. Here, the $p$-modular character tables for the primes $p \in \{2, 3, 5, 7\}$ are still not available.

The degrees of the irreducible 2-modular characters $\varphi_1, \ldots, \varphi_{25}$ of $\mathrm{Fi}_{23}$ are displayed in Table 1. The first twenty of these belong to the principal block, $\varphi_{21}$ is the unique irreducible character in a block of defect 1, the two characters $\varphi_{22}$ and $\varphi_{23}$ belong to a block of defect 3, and $\varphi_{24}$ and $\varphi_{25}$ are of defect 0. The decomposition matrices of the principal block and the block of defect 3 are given in Tables 2 and 3, respectively. The ordering of the columns follows the numbering of the characters in Table 1. As usual, zero entries have been replaced by dots.

Table 1. The degrees of the irreducible 2-modular characters of $\mathrm{Fi}_{23}$

| No. | Degree |
|---|---|
| 1 | 1 |
| 2 | 782 |
| 3 | 1 494 |
| 4 | 3 588 |
| 5 | 19 940 |
| 6 | 57 408 |
| 7 | 94 588 |
| 8 | 94 588 |
| 9 | 79 442 |
| 10 | 583 440 |

| No. | Degree |
|---|---|
| 11 | 1 951 872 |
| 12 | 724 776 |
| 13 | 979 132 |
| 14 | 1 997 872 |
| 15 | 1 997 872 |
| 16 | 7 821 240 |
| 17 | 8 280 208 |
| 18 | 5 812 860 |
| 19 | 17 276 520 |
| 20 | 34 744 192 |

| No. | Degree |
|---|---|
| 21 | 73 531 392 |
| 22 | 97 976 320 |
| 23 | 166 559 744 |
| 24 | 504 627 200 |
| 25 | 504 627 200 |

Table 2. The 2-decomposition matrix of the principal block of $Fi_{23}$

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 782 | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 3588 | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 5083 | 1 | . | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 25806 | 2 | 1 | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 30888 | 2 | 1 | 2 | 2 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 60996 | . | . | . | 1 | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 106743 | 3 | 1 | 2 | 1 | 1 | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . |
| 111826 | 4 | 1 | 3 | 2 | 1 | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . |
| 274482 | . | 1 | . | 2 | 1 | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . |
| 279565 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . |
| 752675 | 5 | 1 | 3 | 2 | 1 | 1 | . | . | 1 | 1 | . | . | . | . | . | . | . | . | . | . |
| 789360 | . | . | . | 2 | . | 1 | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . |
| 812889 | 1 | . | 1 | 2 | . | . | . | . | 1 | . | . | 1 | . | . | . | . | . | . | . | . |
| 837200 | . | . | . | 2 | . | 1 | 1 | 1 | . | 1 | . | . | . | . | . | . | . | . | . | . |
| 837200 | . | . | . | 2 | . | 1 | 1 | 1 | . | 1 | . | . | . | . | . | . | . | . | . | . |
| 850850 | 4 | 1 | 3 | 3 | 1 | 1 | 1 | . | 1 | 1 | . | . | . | . | . | . | . | . | . | . |
| 850850 | 4 | 1 | 3 | 3 | 1 | 1 | . | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . |
| 1677390 | 4 | 1 | 3 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | . | . | . | . | . | . | . |
| 1951872 | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . |
| 2236520 | 6 | 3 | 4 | 6 | 2 | 2 | 1 | 1 | 2 | . | . | 1 | 1 | . | . | . | . | . | . | . |
| 2322540 | 2 | 2 | 2 | 6 | 1 | 2 | 2 | 2 | 1 | . | . | 1 | 1 | . | . | . | . | . | . | . |
| 3913910 | . | 1 | . | 3 | . | 1 | 1 | 1 | . | . | 1 | 1 | 1 | . | . | . | . | . | . | . |
| 5533110 | 4 | 3 | 3 | 9 | 2 | 3 | 3 | 3 | 1 | . | 1 | 1 | 2 | . | . | . | . | . | . | . |
| 6709560 | 4 | 1 | 3 | 6 | 2 | 1 | 1 | 1 | . | 1 | . | . | . | . | . | . | . | 1 | . | . |
| 7468032 | 2 | . | 1 | 6 | 1 | 2 | 1 | 1 | . | 1 | . | 1 | . | . | . | . | . | 1 | . | . |
| 8783424 | 10 | 4 | 7 | 16 | 3 | 5 | 4 | 4 | 2 | 3 | . | 1 | 1 | 1 | 1 | . | . | . | . | . |
| 9108736 | 10 | 4 | 7 | 12 | 3 | 3 | 3 | 3 | 2 | 2 | 1 | . | 1 | 1 | 1 | . | . | . | . | . |
| 9108736 | 10 | 4 | 7 | 12 | 3 | 3 | 3 | 3 | 2 | 2 | 1 | . | 1 | 1 | 1 | . | . | . | . | . |
| 10567557 | 5 | 4 | 4 | 10 | 3 | 3 | 3 | 3 | . | . | 1 | . | 2 | . | . | . | . | 1 | . | . |
| 10674300 | 10 | 4 | 7 | 15 | 3 | 4 | 4 | 4 | 2 | 3 | 1 | 1 | 1 | 1 | 1 | . | . | . | . | . |
| 12077208 | 14 | 6 | 10 | 19 | 4 | 6 | 5 | 5 | 3 | 3 | 1 | 1 | 2 | 1 | 1 | . | . | . | . | . |
| 15096510 | 14 | 7 | 10 | 20 | 4 | 5 | 5 | 5 | 3 | 2 | 2 | 2 | 3 | 1 | 1 | . | . | . | . | . |
| 17892160 | 20 | 8 | 14 | 28 | 6 | 8 | 7 | 7 | 4 | 5 | 1 | 1 | 2 | 2 | 2 | . | . | . | . | . |
| 18812574 | 4 | 3 | 2 | 11 | 1 | 3 | 3 | 3 | . | 1 | 2 | 1 | 1 | 1 | 1 | 1 | . | . | . | . |
| 20322225 | 19 | 7 | 13 | 20 | 6 | 7 | 5 | 5 | 3 | 4 | 1 | . | 2 | 1 | 1 | . | 1 | . | . | . |
| 21135114 | 6 | 5 | 4 | 17 | 2 | 5 | 5 | 5 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | . | . | . | . |
| 21348600 | 2 | 4 | 1 | 11 | 1 | 3 | 4 | 4 | . | . | 3 | 1 | 2 | 1 | 1 | 1 | . | . | . | . |
| 22644765 | 21 | 9 | 15 | 26 | 7 | 9 | 7 | 7 | 4 | 4 | 1 | 1 | 3 | 1 | 1 | . | 1 | . | . | . |
| 26838240 | 10 | 5 | 7 | 12 | 4 | 4 | 3 | 3 | 1 | . | 1 | . | 2 | . | . | 1 | 1 | 1 | . | . |
| 28464800 | 10 | 5 | 7 | 16 | 4 | 6 | 4 | 4 | 1 | 1 | 1 | 1 | 2 | . | . | 1 | 1 | 1 | . | . |
| 29354325 | 9 | 5 | 6 | 17 | 3 | 6 | 5 | 5 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | . | . | . |
| 35225190 | 4 | 3 | 2 | 13 | 1 | 5 | 4 | 4 | . | 1 | 2 | 1 | 1 | 1 | 1 | 2 | 1 | . | . | . |
| 37573536 | 20 | 9 | 14 | 28 | 7 | 9 | 7 | 7 | 3 | 3 | 2 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | . | . |
| 40840800 | 28 | 13 | 20 | 38 | 9 | 12 | 10 | 10 | 5 | 5 | 3 | 1 | 4 | 2 | 2 | 1 | 1 | . | . | . |

Table 2. The 2-decomposition matrix of the principal block of $Fi_{23}$, continued

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 42270228 | 26 | 12 | 18 | 39 | 8 | 12 | 10 | 10 | 5 | 5 | 3 | 3 | 4 | 2 | 2 | 1 | 1 | . | . | . |
| 48034350 | 18 | 9 | 12 | 34 | 5 | 11 | 9 | 9 | 3 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 1 | . | . | . |
| 48308832 | 30 | 13 | 21 | 44 | 10 | 14 | 11 | 11 | 5 | 6 | 3 | 2 | 4 | 2 | 2 | 1 | 1 | 1 | . | . |
| 55740960 | 8 | 5 | 6 | 10 | 3 | 4 | 3 | 3 | 1 | . | 1 | . | 2 | . | . | 1 | 1 | . | . | 1 |
| 56360304 | 12 | 10 | 9 | 32 | 7 | 11 | 10 | 10 | 1 | 3 | 2 | 1 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | . |
| 56360304 | 10 | 8 | 6 | 30 | 3 | 10 | 9 | 9 | 1 | 2 | 4 | 3 | 3 | 2 | 2 | 3 | 1 | . | . | . |
| 57254912 | 20 | 10 | 14 | 28 | 8 | 10 | 7 | 7 | 2 | 1 | 3 | 1 | 4 | . | . | 2 | 2 | 2 | . | . |
| 58708650 | 16 | 11 | 11 | 30 | 7 | 11 | 9 | 9 | 1 | 1 | 4 | 1 | 5 | 1 | 1 | 2 | 2 | 1 | . | . |
| 58708650 | 12 | 11 | 9 | 32 | 7 | 11 | 10 | 10 | 1 | 2 | 3 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | . |
| 65875680 | 30 | 14 | 21 | 48 | 10 | 17 | 13 | 13 | 4 | 6 | 3 | 2 | 5 | 2 | 2 | 2 | 2 | 1 | . | . |
| 78278200 | 40 | 20 | 28 | 63 | 14 | 20 | 17 | 17 | 6 | 7 | 5 | 3 | 7 | 3 | 3 | 2 | 2 | 1 | . | . |
| 93933840 | 38 | 18 | 26 | 62 | 13 | 21 | 17 | 17 | 5 | 8 | 5 | 3 | 6 | 3 | 3 | 3 | 3 | 1 | . | . |
| 133398252 | 28 | 20 | 20 | 60 | 13 | 21 | 18 | 18 | 4 | 4 | 5 | 3 | 8 | 2 | 2 | 3 | 2 | 1 | 1 | 1 |
| 153014400 | 24 | 18 | 16 | 62 | 10 | 23 | 19 | 19 | 2 | 3 | 7 | 4 | 7 | 3 | 2 | 6 | 3 | 1 | . | 1 |
| 153014400 | 24 | 18 | 16 | 62 | 10 | 23 | 19 | 19 | 2 | 3 | 7 | 4 | 7 | 2 | 3 | 6 | 3 | 1 | . | 1 |
| 176125950 | 34 | 23 | 24 | 74 | 16 | 28 | 22 | 22 | 3 | 5 | 6 | 3 | 9 | 2 | 2 | 5 | 4 | 2 | 1 | 1 |
| 176125950 | 32 | 21 | 21 | 72 | 12 | 27 | 21 | 21 | 3 | 4 | 8 | 5 | 8 | 3 | 3 | 7 | 4 | 1 | . | 1 |
| 203802885 | 49 | 30 | 34 | 101 | 20 | 36 | 29 | 29 | 6 | 9 | 8 | 5 | 11 | 4 | 4 | 6 | 4 | 2 | 1 | 1 |
| 207793431 | 57 | 34 | 39 | 111 | 22 | 39 | 32 | 32 | 8 | 11 | 9 | 6 | 12 | 5 | 5 | 6 | 4 | 1 | 1 | 1 |
| 211351140 | 32 | 24 | 21 | 87 | 13 | 32 | 26 | 26 | 3 | 6 | 9 | 6 | 9 | 4 | 4 | 8 | 4 | 1 | 1 | 1 |
| 216154575 | 45 | 31 | 31 | 102 | 19 | 36 | 30 | 30 | 5 | 7 | 10 | 6 | 12 | 4 | 4 | 7 | 4 | 2 | 1 | 1 |
| 216770400 | 60 | 36 | 42 | 116 | 23 | 39 | 33 | 33 | 8 | 11 | 10 | 6 | 13 | 5 | 5 | 6 | 4 | 2 | 1 | 1 |
| 244563462 | 58 | 37 | 40 | 123 | 23 | 44 | 36 | 36 | 7 | 10 | 11 | 7 | 14 | 5 | 5 | 8 | 5 | 2 | 1 | 1 |
| 263376036 | 62 | 40 | 42 | 134 | 24 | 47 | 39 | 39 | 7 | 11 | 13 | 8 | 15 | 6 | 6 | 9 | 5 | 2 | 1 | 1 |
| 264188925 | 69 | 42 | 48 | 136 | 28 | 48 | 39 | 39 | 8 | 11 | 12 | 7 | 16 | 5 | 5 | 8 | 6 | 3 | 1 | 1 |
| 286274560 | 54 | 34 | 37 | 122 | 22 | 46 | 36 | 36 | 6 | 11 | 10 | 7 | 12 | 5 | 5 | 9 | 6 | 1 | 1 | 2 |
| 287721720 | 60 | 37 | 41 | 126 | 24 | 47 | 37 | 37 | 8 | 11 | 10 | 7 | 13 | 5 | 5 | 9 | 6 | 1 | 1 | 2 |
| 289027200 | 42 | 32 | 29 | 110 | 20 | 42 | 33 | 33 | 4 | 7 | 9 | 6 | 12 | 3 | 4 | 9 | 5 | 2 | 2 | 2 |
| 289027200 | 42 | 32 | 29 | 110 | 20 | 42 | 33 | 33 | 4 | 7 | 9 | 6 | 12 | 4 | 3 | 9 | 5 | 2 | 2 | 2 |
| 289103904 | 74 | 43 | 51 | 150 | 29 | 55 | 43 | 43 | 9 | 14 | 12 | 8 | 15 | 6 | 6 | 10 | 7 | 2 | 1 | 1 |
| 313112800 | 64 | 39 | 44 | 134 | 26 | 50 | 39 | 39 | 7 | 11 | 11 | 7 | 14 | 5 | 5 | 10 | 7 | 2 | 1 | 2 |
| 313112800 | 58 | 40 | 41 | 132 | 27 | 49 | 39 | 39 | 6 | 9 | 10 | 6 | 15 | 4 | 4 | 9 | 6 | 3 | 2 | 2 |
| 313112800 | 56 | 38 | 38 | 130 | 23 | 48 | 38 | 38 | 6 | 8 | 12 | 8 | 14 | 5 | 5 | 11 | 6 | 2 | 1 | 2 |
| 322058880 | 48 | 36 | 32 | 126 | 20 | 48 | 38 | 38 | 4 | 7 | 13 | 8 | 14 | 5 | 5 | 12 | 6 | 2 | 1 | 2 |
| 336061440 | 72 | 44 | 50 | 144 | 29 | 54 | 42 | 42 | 8 | 11 | 13 | 7 | 16 | 5 | 5 | 11 | 8 | 2 | 1 | 2 |
| 341577600 | 74 | 44 | 51 | 150 | 30 | 56 | 43 | 43 | 8 | 12 | 12 | 8 | 16 | 5 | 5 | 11 | 8 | 3 | 1 | 2 |
| 343529472 | 74 | 44 | 51 | 150 | 30 | 56 | 43 | 43 | 8 | 12 | 13 | 8 | 16 | 5 | 5 | 11 | 8 | 3 | 1 | 2 |
| 352251900 | 62 | 44 | 43 | 148 | 28 | 55 | 44 | 44 | 6 | 10 | 13 | 8 | 17 | 5 | 5 | 11 | 7 | 3 | 2 | 2 |
| 362316240 | 58 | 44 | 40 | 150 | 27 | 56 | 45 | 45 | 5 | 9 | 14 | 9 | 17 | 5 | 5 | 12 | 7 | 3 | 2 | 2 |
| 476702577 | 93 | 63 | 64 | 209 | 39 | 76 | 62 | 62 | 11 | 15 | 19 | 12 | 24 | 8 | 8 | 15 | 9 | 3 | 2 | 3 |
| 496897335 | 81 | 58 | 56 | 199 | 36 | 74 | 59 | 59 | 7 | 12 | 19 | 11 | 22 | 7 | 7 | 17 | 10 | 4 | 2 | 3 |
| 526752072 | 94 | 66 | 64 | 223 | 39 | 81 | 66 | 66 | 9 | 15 | 22 | 13 | 25 | 9 | 9 | 18 | 10 | 4 | 2 | 3 |
| 528377850 | 90 | 65 | 62 | 222 | 40 | 82 | 66 | 66 | 9 | 15 | 20 | 13 | 25 | 8 | 8 | 17 | 10 | 4 | 3 | 3 |
| 559458900 | 110 | 74 | 75 | 251 | 45 | 91 | 74 | 74 | 12 | 18 | 24 | 15 | 28 | 10 | 10 | 19 | 11 | 4 | 2 | 3 |

Table 3. The decomposition matrix of Block 3

| | | |
|---|---|---|
| 97 976 320 | 1 | . |
| 166 559 744 | . | 1 |
| 166 559 744 | . | 1 |
| 264 536 064 | 1 | 1 |
| 264 536 064 | 1 | 1 |

Our work adds to the program of computing the modular character tables for all sporadic groups. The methods used are by now standard: The Meat-Axe enhanced by condensation. An excellent survey on these techniques can be found in the article by Lux and Pahlings [12, Section 4]. To actually perform the computations described in the sequel, we had to use several new tuning tricks and reimplement major parts of the programs for tensor condensation and induced condensation. This work will be published elsewhere.

A novel feature of our approach is the fact that we work with a condensation algebra which we know to be Morita equivalent to the group algebra. More precisely, we produce an idempotent $e \in \mathbb{F}_2\mathrm{Fi}_{23}$ such that $e(\mathbb{F}_2\mathrm{Fi}_{23})e$ is Morita equivalent to $\mathbb{F}_2\mathrm{Fi}_{23}$, and group elements $g_1, \ldots, g_{38}$ such that $e(\mathbb{F}_2\mathrm{Fi}_{23})e$ is generated, as an $\mathbb{F}_2$-algebra, by $eg_1e, \ldots, eg_{38}e$. The idempotent $e$ is of the form $e = 1/|K| \sum_{y \in K} y$ for a subgroup $K$ of order $3^9$.

## 2. The blocks

Using GAP [4], it is easy to determine the invariants of the 2-blocks of $\mathrm{Fi}_{23}$ and the distribution of the ordinary irreducible characters into blocks. The invariants of the blocks are collected in Table 4 (we use a standard convention where $d(B)$, $k(B)$, and $\ell(B)$ denote the defect, the number of ordinary irreducible characters, and the number of modular irreducible characters of the block $B$, respectively). Block 2 contains the two ordinary characters $\chi_{56}$ and $\chi_{57}$, Block 3 contains the five irreducible characters $\chi_{60}$, $\chi_{64}$, $\chi_{65}$, $\chi_{76}$, and $\chi_{77}$, and the two characters $\chi_{94}$ and $\chi_{95}$ are of 2-defect 0.

Each of the Blocks 2, 4 and 5 only has one irreducible Brauer character and thus their decomposition matrices are trivial. Peter Landrock has shown [9, Section 8], that the defect group of Block 3 is dihedral of order 8. Karin Erdmann gives three possible decompositon matrices for blocks $B$ with this defect group, $k(B) = 5$ and $\ell(B) = 2$ (see [3, Propositions (3.2), (3.3), (3.6)]). In all of these, at least one of the two irreducible Brauer characters is liftable. It follows that the ordinary character $\chi_{60}$, the one with the smallest degree in Block 3, remains irreducible modulo 2. Since it occurs only once in Block 3, whereas in the possibility described in [3, Proposition (3.3)] there are two ordinary characters of the same smallest degree, the decompositon matrix of Block 3 is described by one of the remaining possibilities. In each of these, both irreducible Brauer characters are liftable. This, together with the relations of the ordinary characters in Block 3 on elements of odd order, determines the decompositon matrix of Block 3 as displayed in Table 3.

Table 4. Blocks of $\mathrm{Fi}_{23}$

| $B$ | $d(B)$ | $k(B)$ | $\ell(B)$ |
|---|---|---|---|
| 1 | 18 | 89 | 20 |
| 2 | 1 | 2 | 1 |
| 3 | 3 | 5 | 2 |
| 4 | 0 | 1 | 1 |
| 5 | 0 | 1 | 1 |

## 3. Irrationalities

There are 25 conjugacy classes of elements of odd order in $\mathrm{Fi}_{23}$. Exactly one pair of these, $23A$, $23B$ is not real (i.e., the inverses of the elements of $23A$ lie in $23B$). Hence, by Brauer's permutation lemma, there is exactly one pair of complex conjugate irreducible Brauer characters. This must belong to the principal block, since the irreducible Brauer characters in the other blocks are real valued.

The principal block contains ordinary characters with values $b13 = (-1+\sqrt{13})/2$ on some elements of odd order. Thus there is an irreducible Brauer character $\varphi$ taking non-rational values in the field $\mathbb{Q}(b13)$. The minimal polynomial of $b13$ equals $x^2 + x - 3$. Since this is irreducible modulo 2, the simple module (over an algebraically closed field of characteristic 2) with Brauer character $\varphi$ is not realizable over $\mathbb{F}_2$. Hence there is at least one pair of irreducible Brauer characters in the principal block, whose underlying simple modules are conjugate by the non-trivial Galois automorphism of $\mathbb{F}_4/\mathbb{F}_2$.

## 4. Applying the Meat-Axe

To begin with, we consider the permutation representation of $\mathrm{Fi}_{23}$ of degree 31 671 on the cosets of $2.\mathrm{Fi}_{22}$. This representation is available from Robert Wilson's Atlas of Finite Group Representations [16].

Using the Meat-Axe (version 2.4.3 of Ringe's C-Meat-Axe) we find that the corresponding matrix representation over $\mathbb{F}_2$ has 10 irreducible constituents, namely

$$1a, 1a, 1a, 782a, 782a, 1\,494a, 1\,494a, 3\,588a, 3\,588a, 19\,940a. \tag{1}$$

## 5. Applying condensation

From now on we write $G$ for $\mathrm{Fi}_{23}$. Let $N$ denote the seventh maximal subgroup of $G$, which is the normalizer of a subgroup of order 3 containing an element of the conjugacy class $3B$ (see [2, p. 177]).

The permutation character of $G$ on the cosets of $N$ is described in Table 5. The norm of this character, and hence the number of $N$-$N$-double cosets of $G$, equals 36.

Let $K$ denote the largest normal 3-subgroup of $N$. Then $K$ is an extraspecial group of order $3^9$ (see [2, p. 177]). We take $K$ as our condensation subgroup. Thus let

$$e := \frac{1}{|K|} \sum_{y \in K} y \in \mathbb{F}_2 G,$$

Table 5. The permutation character on the cosets of $N$

| No. | Degree | Mult. |
|---|---|---|
| 1 | 1 | 1 |
| 6 | 30 888 | 2 |
| 8 | 106 743 | 2 |
| 14 | 812 889 | 1 |
| 24 | 5 533 110 | 1 |
| 26 | 7 468 032 | 1 |
| 27 | 8 783 424 | 1 |
| 32 | 12 077 208 | 3 |

| No. | Degree | Mult. |
|---|---|---|
| 36 | 20 322 225 | 1 |
| 40 | 26 838 240 | 1 |
| 44 | 37 573 536 | 2 |
| 53 | 58 708 650 | 1 |
| 56 | 73 531 392 | 1 |
| 71 | 216 154 575 | 1 |
| 72 | 216 770 400 | 2 |
| 82 | 289 103 904 | 1 |

end put $\mathcal{H} := e\mathbb{F}_2Ge$. Then $\mathcal{H}$ is isomorphic to the endomorphism ring of the $\mathbb{F}_2$-permutation module of $G$ on the right cosets of $K$. We obtain a functor (called condensation)

$$\text{mod-}\mathbb{F}_2G \to \text{mod-}\mathcal{H},$$

sending an $\mathbb{F}_2G$-module $M$ to the $\mathcal{H}$-module $Me$. It will turn out later that $K$ is a faithful condensation subgroup for $G$, i.e., that $Se \neq 0$ for every simple $\mathbb{F}_2G$-module $S$. This implies that $\mathcal{H}$ is Morita equivalent to $\mathbb{F}_2G$ (see [5, Section 6]).

We shall condense the $\mathbb{F}_2G$-modules $S_i \otimes_{\mathbb{F}_2} S_j$, for $2 \leq i < j \leq 5$. Here, $S_2, \ldots, S_5$ denote the simple $\mathbb{F}_2G$-modules $782a$, $1\,494a$, $3\,588a$, and $19\,940a$, respectivly. (See Section 4, where we obtained these as constituents of the permutation representation of $G$ on the cosets of the maximal subgroup $2.\text{Fi}_{22}$.) The second maximal subgroup $M_2 := O_8^+(3)\colon S_3$ (see [2, p. 177]) of $G$ has an absolutely irreducible module of degree 596 over $\mathbb{F}_2$. We construct it as a tensor product of a 2-dimensional by a 298-dimensional $\mathbb{F}_2M_2$-module. The first is a constituent of the restriction to $M_2$ of $782a$, the second of the restriction of 1
$ii494a$. Generators of $M_2$ as words in the standard generators of $G$ were taken from Rob Wilson's Web Atlas [16].

The algorithms to condense tensor products have been developed by Lux and Wiegelmann in [13]. The condensation of induced modules is described in a paper [14] by Müller and Rosenboom.

To give the reader an idea of the computational magnitude involved, we comment on some timings. Let $M$ denote the $\mathbb{F}_2G$-module $19\,940a \otimes 19\,940a$. Then the dimension of $Me$ equals $25\,542$. The computation of a single matrix describing the action of $ege$ on $Me$ for a group element $g$ took about a week on a machine with a Pentium 4 processor at 3.2 GHz. If $M$ is the $\mathbb{F}_2G$-module obtained by inducing the 596-dimensional $\mathbb{F}_2M_2$-module to $G$, the dimension of $Me$ equals $3\,694$. Here, the condensation took about two hours per matrix.

In order to exploit the full strength of the Morita equivalence between $\mathbb{F}_2G$ and $\mathcal{H}$, we need a set of algebra generators for $\mathcal{H}$. The following argument is taken from the PhD-thesis of the third author [15]. Since $K$ is a normal subgroup of $N$, the idempotent $e$ is in the center of $\mathbb{F}_2N$, and so $e\mathbb{F}_2Ge$ is generated, as an $e\mathbb{F}_2Ne$-$\mathbb{F}_2Ne$-bimodule, by the elements $eg_ie$, where $\{g_1, \ldots, g_{36}\}$ denotes a set of representatives for the $N$-$N$-double cosets of $G$. Also, $e\mathbb{F}_2Ne \cong \mathbb{F}_2(N/K)$, again since $K$ is normal in $N$. Hence $e\mathbb{F}_2Ne$ is generated, as an $\mathbb{F}_2$-algebra, by $\{en_1e, \ldots, en_re\}$, for every collection of elements $n_1, \ldots, n_r \in N$ generating $N$ as a group. In our actual computation we had $r = 3$ and $g_1 = 1$, so that we had 38 elements generating $e\mathbb{F}_2Ge$.

## 6. The construction of the double coset representatives

For our computations we need a set $\{g_1, \ldots, g_{36}\}$ of $N$-$N$-double coset representatives in different representations. Therefore we look for a straight line program that takes as input a pair of standard generators for $G$ and produces the elements $g_1$ to $g_{36}$.

The first step is to find a straight line program that takes as input a pair of standard generators of $G$ and produces generators of $N$. To this end, we choose an element $b$ from class $3B$ in $G$ and an element $c$ from class $2C$ (available as straight line programs from [16]). We then seek to find generators of $N = N_G(\langle b \rangle)$. Using the information about conjugacy class fusions available in the character table library (see [1]) we use GAP to compute that class $2C$ of $G$ has 12 839 581 755 elements, intersects $N$ non-trivially and that the intersection is the disjoint union of three conjugacy classes of $N$ with 6 561, 26 244, and 419 904 elements, respectively. This shows that if we conjugate $c$ with pseudo random (uniformly distributed) elements of $G$, the probability of reaching an element of $N$ is $(6\,561 + 26\,244 + 419\,904)/12\,839\,581\,755 \approx 3.53 \cdot 10^{-5}$ and in most cases we will reach the biggest conjugacy class of $N$, which happens to lie outside the normal subgroup $C_G(b)$ of $N$ of index 2. Thus we have good chances to find generators for $N$. Of course, we check whether an element $c^x$ lies in $N$ by checking whether it centralizes $b$ or conjugates $b$ to $b^{-1}$.

With this random approach we find three elements $n_1$, $n_2$, and $n_3$ of $N$ as straight line programs in the standard generators of $G$. By computing with GAP in the permutation representation of $G$ on 31 671 points, using standard permutation group methods, we verify that these elements generate $N$ by comparing group orders.

To find $N$-$N$-double coset representatives we consider the permutation action of $G$ on the set of right $N$-cosets of $G$, restricted to $N$, and find elements $\{g_1, \ldots, g_{36}\}$ in $G$ to reach the 36 $N$-suborbits. We compute in this permutation representation in the following way: We restrict the irreducible $\mathbb{F}_2G$-module $1\,494a$ from above to $N$, and find a non-zero invariant vector $v_0$. Its $G$-orbit $v_0G$ has length $[G : N]$ because $N$ is maximal in $G$. Thus $v_0G$ is isomorphic as a $G$-set to the set of right $N$-cosets in $G$. We can act with generators of $G$ or $N$ on vectors by vector-matrix-multiplication and can compare and store points, without writing down permutations on $[G : N]$ elements.

One vector in $v_0G$ needs about 200 bytes of storage. Therefore storing $v_0G$ completely would require $200 \cdot [G : N] \approx 250$ GB of main memory. Also, enumerating that many vectors would take much too long.

By using the methods described in [10] we can enumerate $N$-suborbits "by $U$-orbits" for a helper subgroup $U < N$ with $|U| = 6\,561$, that is, we can archive all vectors in an $N$-suborbit by storing the so called "$U$-minimal" vectors of each $U$-suborbit. Given any vector, we can then recognize whether it lies in one of the

Table 6. Lengths of $N$-suborbits in $v_0G$

| | | | |
|---|---|---|---|
| 1 | 186 624 | 5 038 848 | 30 233 088 |
| 768 | 944 784 | 7 558 272 | 34 012 224 |
| 3 888 | 944 784 | 10 077 696 | 68 024 448 |
| 15 552 | 1 679 616 | 10 077 696 | 90 699 264 |
| 15 552 | 1 679 616 | 10 077 696 | 90 699 264 |
| 19 683 | 3 359 232 | 20 155 392 | 90 699 264 |
| 62 208 | 3 779 136 | 22 674 816 | 136 048 896 |
| 78 732 | 3 779 136 | 30 233 088 | 272 097 792 |
| 124 416 | 5 038 848 | 30 233 088 | 272 097 792 |

stored $U$-suborbits: we just "$U$-minimalize" it by applying an element of $U$ and look up the result in our database.

Using these techniques we can enumerate all 36 $N$-suborbits of $v_0G$ on a machine with 2 GB of main memory. Thus, once we have candidates for $\{g_1, \ldots, g_{36}\}$, written in terms of $n_1$, $n_2$, and $n_3$, we can verify that they are in fact $N$-$N$-double coset representatives by enumerating the $N$-suborbits $v_0g_iN$ and checking that they are disjoint and their lengths sum up to $[G : N]$.

However, finding such candidates is not completely trivial, because it takes a very long time to find the short suborbits. Especially the smallest non-trivial suborbit of length 768 (see Table 6) could not be found by using standard orbit enumeration or randomized methods.

Therefore we took another approach: once we had the other 35 suborbits and knew that we were looking for a suborbit of length 768, we first guessed the stabilizer $T$ in $N$ of a vector in the last suborbit, a subgroup of index 768 in $N$. For this subgroup $T$ we computed the subspace of invariant vectors. To all of these invariant vectors we applied a generator $g$ of $G$. Using the stored knowledge about the first 35 suborbits we found at last a $T$-invariant vector $v$ not lying in one of these suborbits, such that $v$ is mapped into $v_0G$ by $g$, thereby proving that $v \in v_0G$.

Once we had one and thus all vectors in the last suborbit, we found a straight line program in our standard generators for $G$ mapping $v_0$ to $v$ by performing a breadth-first search with storing all vectors starting at $v_0$ until the main memory was full, and then doing a depth-first backward search starting at $v$.

The verified $N$-suborbit lengths in $v_0G$ can be found in Table 6.

## 7. Results of the condensation

The results of the condensation are reported in Table 7. The first column of this table just numbers the composition factors, the second columns names these by their degrees and a letter. The remaining eleven columns correspond to the tensor products of (in this order) $782a \otimes 782a$, $782a \otimes 1\,494a$, $782a \otimes 3\,588a$, $782a \otimes 19\,940a$, $1\,494a \otimes 1\,494a$, $1\,494a \otimes 3\,588a$, $1\,494a \otimes 19\,940a$, $3\,588a \otimes 3\,588a$, $3\,588a \otimes 19\,940a$, $19\,940a \otimes 19\,940a$, and the induced module of degree 596 of $M_2$, the second maximal subgroup of $G$. As usual, zero entries are replaced by dots. Except for the two pairs of modules of dimensions 6 and 22, the simple $\mathcal{H}$-modules occurring in the condensed

Table 7. Condensation results

| Nr. | Name | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $1a$ | 6 | . | . | 8 | 14 | 2 | 10 | 16 | 8 | 120 | 6 |
| 2 | $8a$ | 5 | . | . | 4 | 4 | . | 8 | 10 | 6 | 70 | 4 |
| 3 | $10a$ | 4 | . | . | 7 | 9 | 1 | 7 | 12 | 4 | 88 | 5 |
| 4 | $6a$ | 8 | . | 4 | 16 | 12 | 2 | 18 | 27 | 28 | 206 | 14 |
| 5 | $34a$ | 4 | . | . | 5 | 4 | . | 6 | 6 | 1 | 53 | 2 |
| 6 | $6b$ | 2 | . | 2 | 4 | 4 | . | 7 | 8 | 9 | 76 | 7 |
| 7 | $22a$ | 2 | 1 | . | 5 | 2 | 1 | 5 | 6 | 10 | 56 | 5 |
| 8 | $22b$ | 2 | 1 | . | 5 | 2 | 1 | 5 | 6 | 10 | 56 | 5 |
| 9 | $36a$ | . | . | . | 1 | 4 | . | . | 4 | . | 16 | . |
| 10 | $42a$ | . | . | . | 1 | 2 | 2 | . | . | 4 | 14 | . |
| 11 | $48a$ | . | . | 1 | . | . | . | 1 | 2 | 6 | 10 | 2 |
| 12 | $54a$ | . | . | 1 | . | . | . | . | 4 | 2 | 6 | . |
| 13 | $72a$ | . | 1 | . | 2 | . | . | 4 | 4 | 2 | 26 | 4 |
| 14/15 | $88a$ | . | . | . | . | . | 1 | . | . | 3 | 2 | . |
| 16 | $124a$ | . | . | . | . | . | . | . | . | 4 | 8 | 2 |
| 17 | $374a$ | . | . | . | . | . | . | 2 | . | 1 | 8 | 2 |
| 18 | $534a$ | . | . | . | 2 | . | . | 1 | . | . | 8 | 1 |
| 19 | $814a$ | . | . | . | . | . | . | . | . | . | 4 | . |
| 20 | $1248a$ | . | . | . | . | . | . | . | . | . | 2 | 1 |

modules have pairwise different dimension. The two 6-dimensional modules can easily be distinguished by their fingerprints, i.e., a vector of nullities of certain elements of $\mathcal{H}$. All simple $\mathcal{H}$-modules are absolutely simple, except the one of dimension 88, which splits over the field with four elements, and thus gives rise to two absolutely simple modules.

We have already mentioned in Section 2, that the irreducible Brauer characters of the non-principal blocks are all liftable to characteristic 0. From Table 8 we can thus read off the dimensions of the condensed simple modules lying in non-principal blocks. We find that all these are larger than 1 248. Hence all $\mathcal{H}$-modules occurring in Table 7 are condensed principal block modules.

## 8. A basic set of Brauer characters

It is easy to find a basic set of Brauer characters consisting of ordinary characters restricted to the elements of odd order. Table 8 lists the Atlas numbers, the degrees and the condensed degrees of such a basic set. The first twenty of these characters form a basic set for the principal block. The basic set for the other blocks consists of their irreducible Brauer characters, which are all liftable.

Table 8. A basic set of Brauer characters

| $i$ | $\chi_i(1)$ | $\chi_i^c(1)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 782 | 8 |
| 3 | 3 588 | 6 |
| 4 | 5 083 | 17 |
| 5 | 25 806 | 60 |
| 7 | 60 996 | 12 |
| 8 | 106 743 | 107 |
| 10 | 274 482 | 104 |
| 12 | 752 675 | 173 |
| 13 | 789 360 | 72 |

| $i$ | $\chi_i(1)$ | $\chi_i^c(1)$ |
|---|---|---|
| 17 | 850 850 | 200 |
| 20 | 1 951 872 | 48 |
| 21 | 2 236 520 | 428 |
| 25 | 6 709 560 | 772 |
| 27 | 8 783 424 | 928 |
| 35 | 18 812 574 | 774 |
| 36 | 20 322 225 | 1721 |
| 49 | 55 740 960 | 2400 |
| 50 | 56 360 304 | 3652 |
| 62 | 153 014 400 | 7136 |

| $i$ | $\chi_i(1)$ | $\chi_i^c(1)$ |
|---|---|---|
| 56 | 73 531 392 | 5 120 |
| 60 | 97 976 320 | 4 608 |
| 64 | 166 559 744 | 9 472 |
| 94 | 504 627 200 | 24 576 |
| 95 | 504 627 200 | 24 576 |

If $\varphi$ is a Brauer character, its condensed degree $\varphi^c(1)$ is the dimension of $Me$ for any module $M$ with Brauer character $\varphi$. It can be computed as the scalar product $\varphi^c(1) = (\varphi_K, 1_K)$.

It can easily be checked with GAP that the displayed set of characters is indeed a basic set. Extract the indicated characters from the ordinary character table, restrict them to the conjugacy classes of elements of odd order, and test whether every restricted ordinary character of the principal block is a $\mathbb{Z}$-linear combination of these.

## 9. Computing the irreducible Brauer characters

We now indicate how to compute the set of irreducible Brauer characters from the results of the condensation.

We begin by computing the irreducible Brauer characters contained in the permutation module on the cosets of the first maximal subgroup $2.\mathrm{Fi}_{22}$. The ordinary character of this permutation module equals

$$1a + 782a + 30\,888a \tag{2}$$

(see [2, p. 177]). The list (1) of modulo 2 compositon factors of this permutation module shows that the ordinary character of degree 782 remains irreducible modulo 2.

Every non-trivial simple module of $\mathbb{F}_2G$ condenses to a module of degree at least 6. This implies that the ordinary character of degree 3 588 remains irreducible modulo 2 (cf. Table 8), giving an irreducible $\mathbb{F}_2G$-module of degree 3 588 which condenses to an $\mathcal{H}$-module of dimension 6. We shall show below that this $\mathcal{H}$-module is in fact $6a$.

As ordinary characters, the tensor product $782a \otimes 3\,588a$ decomposes as

$$782a \otimes 3\,588a = 3\,588a + 60\,996a + 789\,360a + 1\,951\,872a. \tag{3}$$

By Table 8, the ordinary character $60\,996a$ condenses to a module of dimension 12, and by Table 7 (the column labelled 3), its reduction modulo 2 splits into two modules, each of which condenses to a 6-dimensional $\mathcal{H}$-module. Using GAP we find that

$$(60\,996a, 106\,743a \otimes 504\,627\,200a) = 1. \tag{4}$$

Since $504\,627\,200a$ is of 2-defect zero, the product $106\,743a \otimes 504\,627\,200a$ is projective, and so the scalar product (4) implies that the reduction modulo 2 of $60\,996a$ is not twice an irreducible Brauer character. Thus there is exactly one irreducible Brauer character of $G$ of degree $3\,588$ which condenses to a 6-dimensional $\mathcal{H}$-module. Moreover, there is an irreducible Brauer character $57\,408a := 60\,996a - 3\,588a$ which condenses to the other 6-dimensional $\mathcal{H}$-module.

We have, on elements of odd order, the relation

$$1a + 782a + 30\,888a = 782a + 5\,083a + 25\,806a \tag{5}$$

between restricted ordinary characters. Now $5\,083a$ condenses to a module of dimension 17 (see Table 8). By (1), (2), and (5), the reduction modulo 2 of $5\,083a$ has at most three trivial composition factors. Looking at the degrees of the condensed modules in Table 7, and using the fact that $8a$ is the condensed module of $782a$, it follows that $5\,083a$ has a composition factor condensing to a 6-dimensional $\mathcal{H}$-module. By what we have said above, this must be the condensation of $3\,588a$. Hence $3\,588a$ is a composition factor, with multiplicity 1, of the reduction modulo 2 of $5\,083a$. By (1) and (5), the Brauer character of $1\,494a$ equals

$$1\,494a = 5\,083a - 3\,588a - 1a,$$

where on the right hand side we understand the restricted ordinary characters of these degrees. From this, (1) and (2), it is clear how to compute the Brauer character of $19\,940a$.

In order to show that the irreducible $\mathbb{F}_2G$-module $3\,588a$ condenses to the $\mathcal{H}$-module $6a$, we consider the product of the Brauer character $782a$ with itself. We find that

$$782a \otimes 782a = 6\times 1a + 3\times 782a + 4\times 1\,494a + 4\times 3\,588a + 2\times 19\,940a + 2\times 274\,482a, \tag{6}$$

as Brauer characters, where the last of these characters on the right hand side is the basic set character obtained by restricting the ordinary irreducible character of this degree to the elements of odd order (cf. Table 8). Since $6b$ occurs only twice in the condensed module of $782a \otimes 782a$ (cf. Table 7), it follows from (6) that $3\,588a$ indeed condenses to $6a$.

By Table 7, the tensor product $782a \otimes 3\,588a$ contains a unique condensed constituent of degree 48. Thus it follows from (3), together with Tables 7 and 8, that the ordinary character $1\,951\,872a$ remains irreducible modulo 2.

For the computations below we modify our original basic set, described in Table 8, by replacing the restricted ordinary characters $5\,083a$ and $25\,806a$ with the Brauer characters of $1\,494a$ and $19\,940a$, respectively. Moreover, the first of these two characters is moved to position 3, and the Brauer character of $3\,588a$ to position 4 in the new basic set, whose elements will be denoted by $\vartheta_1, \ldots, \vartheta_{20}$.

Having found the irreducible Brauer characters of the simple modules $1a$, $782a$, $1\,494a$, $3\,588a$, and $19\,940a$, we can compute the products of these and express them as $\mathbb{Z}$-linear combinations of the (new) basic set characters. The result is displayed in Table 9. The columns numbered 1–10 contain the ten products of characters in the same order as in Table 7. Column 11 gives the multiplicities in the basic set of Brauer character obtained by inducing the character of degree 596 of the second maximal subgroup. For later use, columns 12 and 13 give the basic set expressions of the two restricted ordinary characters 850 850$b$ and 153 014 400$b$, algebraically conjugate to 850 850$a$ and 153 014 400$a$ (which are contained in the basic set), respectively.

The remaining irreducible Brauer characters can be found by solving a system of linear equations. Namely, let $\varphi_1, \ldots, \varphi_{20}$ denote the set of absolutely irreducible 2-modular characters of $G$, where the numbering corresponds to the one of the rows in Table 7. Let us choose an algebraic closure $\bar{\mathbb{F}}_2$ of $\mathbb{F}_2$ and write $S_i$ for the simple $\bar{\mathbb{F}}_2G$-module with Brauer character $\varphi_i$, $i = 1, \ldots, 20$.

Let $\varphi$, $\varphi'$ denote one of the two pairs of non-rational valued irreducible Brauer characters (see Section 3). Since the only possible non-rational value of $\varphi$ and $\varphi'$ are the quadratic irrationalities $b23 = (-1+\sqrt{-23})/2$ or $b13 = (-1+\sqrt{13})/2$, it follows that $\varphi$ and $\varphi'$ have the same scalar product with the trivial character of the condensation subgroup $K$. This implies that the underlying $\bar{\mathbb{F}}_2G$-modules condense to $\bar{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} \mathcal{H}$-modules of the same degree.

Since the two $\bar{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} \mathcal{H}$-modules of degrees 6 arise from $\bar{\mathbb{F}}_2G$-modules of different degrees, and since 88$a$ is the only $\mathcal{H}$-module which is not absolutely irreducible, it follows that $\varphi_7$ and $\varphi_8$ are complex conjugate, and that $\varphi_{14}$ and $\varphi_{15}$ are conjugate under the Galois automorphism of the field $\mathbb{Q}((-1+\sqrt{13})/2)$. The $\bar{\mathbb{F}}_2G$-modules $S_{14}$ and $S_{15}$ with Brauer characters $\varphi_{14}$ and $\varphi_{15}$ condense to two modules 44$a$ and 44$b$ of $\bar{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} \mathcal{H}$.

Recall that our basic set characters are denoted by $\vartheta_1, \ldots, \vartheta_{20}$, the ordering being the one indicated by the rows of Table 9. There are non-negative integers $a_{ij}$, $1 \le i, j \le 20$, such that

$$\vartheta_i = \sum_{j=1}^{20} a_{ij}\varphi_j, \qquad i = 1, \ldots, 20.$$

The matrix $(a_{ij}) \in \mathbb{Z}^{20\times 20}$ is invertible over the integers. Since we already know six of the irreducible Brauer characters, we are left with 280 unknowns.

Suppose that $M$ is an $\bar{\mathbb{F}}_2G$-module whose Brauer character $\vartheta$ is known explicitly. Suppose also that we know the composition factors of $M$, including their multiplicities. This is the case for the tensor products as well as for the induced module we have condensed.

Suppose we have

$$[M] = \sum_{j=1}^{20} s_j\,[S_j]$$

in the Grothendieck group of $\bar{\mathbb{F}}_2G$. This implies that

$$\vartheta = \sum_{j=1}^{20} s_j\,\varphi_j. \tag{7}$$

On the other hand, since $\vartheta$ is explicitly known as a class function, we can compute integers $t_i$, $1 \le i \le 20$, such that

$$\vartheta = \sum_{i=1}^{20} t_i\vartheta_i. \tag{8}$$

Equations (7) and (8) yield 20 linear equations for the unknowns $a_{ij}$, namely

$$\sum_{i=1}^{20} t_i\, a_{ij} = s_j, \qquad j = 1, \ldots, 20.$$

Thus we obtain 220 linear equations from our eleven condensed modules.

We obtain further equations from the symmetries arising from complex conjugation of Brauer characters and applying the non-trivial Galois automorphism of $\mathbb{Q}((-1+\sqrt{13})/2)$. For example, the column headed 12 of Table 9 gives the coefficients $u_i$ in the expression

$$\bar{\vartheta}_{11} = \sum_{i=1}^{20} u_i\, \vartheta_i,$$

where $\bar{\vartheta}_{11}$ is the Brauer character complex conjugate to $\vartheta_{11}$. Since $\varphi_7$ and $\varphi_8$ is the only pair of complex conjugate irreducible Brauer characters in the principal block of $\bar{\mathbb{F}}_2 G$, we obtain equations

$$\sum_{i=1}^{20} u_i\, a_{ij} - a_{11,j} = 0, \qquad j = 1, \ldots, 20, j \neq 7, 8,$$

$$\sum_{i=1}^{20} u_i\, a_{i,7} - a_{11,8} = 0, \qquad \textstyle\sum_{i=1}^{20} u_i\, a_{i,8} - a_{11,7} = 0.$$

Similar equations arise from column 13 of Table 9 which gives the coefficients in the basic set of the Galois conjugate of $\vartheta_{20}$.

The system of linear equations we obtain this way has rank 398. The solution matrix for the $a_{ij}$, with two unknowns $\alpha_1$ and $\alpha_2$, is displayed in Table 10. The determinant of this matrix equals

$$4\alpha_2\alpha_1 - 10\alpha_1 - 2\alpha_2 + 5.$$

Since the determinant of $(a_{ij})$ is $\pm 1$ and the $a_{ij}$ are non-negative integers, we conclude that $\alpha_1 \in \{0, 1\}$ and $\alpha_2 \in \{2, 3\}$. Any two solution matrices can be transformed into each other by swapping columns 7 and 8 or 14 and 15 or both. This amounts to renaming $\varphi_7$ and $\varphi_8$ or $\varphi_{14}$ and $\varphi_{15}$ or both. We choose notation such that $\alpha_1 = 1$ and $\alpha_2 = 3$.

From the inverse of the matrix $(a_{ij})$ and the table of the basic set characters $\vartheta_1, \ldots, \vartheta_{20}$, we can compute the irreducible Brauer characters $\varphi_1, \ldots, \varphi_{20}$, and from these the decomposition matrix of Table 2. We omit further details.

We should like to mention that the antique MOC-system (cf. [6, 11, 12]) was used in the first place to perform numerous experiments with products of Brauer characters, induced Brauer characters and projective characters, in order to find suitable modules to condense.

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Table 9. Some characters expressed in the basic set

| $\vartheta_i(1)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | . | . | −1 | −2 | . | . | 4 | −14 | −6 | −9 | −2 | −36 |
| 782 | 3 | −1 | . | −2 | −2 | 1 | 1 | . | −7 | −10 | −5 | −1 | −22 |
| 1 494 | 4 | . | . | . | −1 | −1 | −1 | 4 | −7 | 2 | −5 | . | −16 |
| 3 588 | 4 | −1 | 1 | −2 | 2 | −3 | . | 5 | −6 | 2 | −5 | 1 | −8 |
| 19 940 | 2 | . | . | −1 | −2 | 1 | . | . | −7 | −9 | −4 | −1 | −20 |
| 60 996 | . | −1 | 1 | . | . | . | 1 | −2 | −3 | 4 | . | −1 | −5 |
| 106 743 | . | −2 | . | −2 | 2 | 1 | −3 | −4 | 3 | −2 | −2 | . | 1 |
| 274 482 | 2 | . | . | 1 | 2 | −2 | . | 2 | 3 | 8 | . | 1 | 15 |
| 752 675 | . | . | . | −1 | 2 | −1 | −3 | . | 2 | 6 | . | 2 | 5 |
| 789 360 | . | −1 | 1 | −2 | . | . | . | . | 2 | . | . | . | 6 |
| 850 850 | . | . | . | . | . | . | . | . | . | . | . | −1 | . |
| 1 951 872 | . | . | 1 | . | . | . | −1 | 2 | −3 | −6 | −2 | . | −12 |
| 2 236 520 | . | 1 | . | 2 | . | −1 | 2 | 4 | −2 | 6 | 1 | . | 1 |
| 6 709 560 | . | . | . | 2 | . | . | 1 | . | . | 4 | 1 | . | 2 |
| 8 783 424 | . | . | . | . | . | 1 | −2 | . | −2 | −6 | −2 | . | −9 |
| 18 812 574 | . | . | . | . | . | . | . | . | 4 | 2 | 1 | . | 10 |
| 20 322 225 | . | . | . | . | . | . | 2 | . | 1 | 2 | 1 | . | 4 |
| 55 740 960 | . | . | . | . | . | . | . | . | . | 2 | 1 | . | 2 |
| 56 360 304 | . | . | . | . | . | . | . | . | . | 4 | . | . | . |
| 153 014 400 | . | . | . | . | . | . | . | . | . | . | . | . | −1 |

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TABLE 10. The matrix of the $a_{ij}$ ($\alpha'_1 = -\alpha_1 + 1$, $\alpha'_2 = -\alpha_2 + 5$)

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| . | . | . | 1 | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 3 | 1 | 2 | 1 | 1 | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . |
| . | 1 | . | 2 | 1 | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . |
| 5 | 1 | 3 | 2 | 1 | 1 | . | . | 1 | 1 | . | . | . | . | . | . | . | . | . | . |
| . | . | . | 2 | . | 1 | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . |
| 4 | 1 | 3 | 3 | 1 | 1 | $\alpha_1$ | $\alpha'_1$ | 1 | 1 | . | . | . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . |
| 6 | 3 | 4 | 6 | 2 | 2 | 1 | 1 | 2 | . | . | 1 | 1 | . | . | . | . | . | . | . |
| 4 | 1 | 3 | 6 | 2 | 1 | 1 | 1 | . | 1 | . | . | . | . | . | . | . | 1 | . | . |
| 10 | 4 | 7 | 16 | 3 | 5 | 4 | 4 | 2 | 3 | . | 1 | 1 | 1 | 1 | . | . | . | . | . |
| 4 | 3 | 2 | 11 | 1 | 3 | 3 | 3 | . | 1 | 2 | 1 | 1 | 1 | 1 | 1 | . | . | . | . |
| 19 | 7 | 13 | 20 | 6 | 7 | 5 | 5 | 3 | 4 | 1 | . | 2 | 1 | 1 | . | 1 | . | . | . |
| 8 | 5 | 6 | 10 | 3 | 4 | 3 | 3 | 1 | . | 1 | . | 2 | . | . | 1 | 1 | . | . | 1 |
| 12 | 10 | 9 | 32 | 7 | 11 | 10 | 10 | 1 | 3 | 2 | 1 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | . |
| 24 | 18 | 16 | 62 | 10 | 23 | 19 | 19 | 2 | 3 | 7 | 4 | 7 | $\alpha_2$ | $\alpha'_2$ | 6 | 3 | 1 | . | 1 |


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