The R_∞-property of flat manifolds: Toward the eigenvalue one property of finite groups

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Dedicated to Otto Kegel

Abstract We introduce a conjecture of Dekimpe, De Rock and Penninckx and sketch some major steps in its proof. This text presents an extended account of the talk of the first author given at the conference.

1 Introduction

The purpose of this note is to outline the proof, recently obtained by the authors in [7], of a conjecture of Dekimpe, De Rock and Penninckx. This yields a sufficient condition for a flat manifold to be an R_∞-manifold. Here, we will sketch the main steps of our proof, giving details for some of the more elementary arguments. We show how to reduce the conjecture to the finite simple groups. Then, using extensive, detailed knowledge about the automorphism groups, the subgroup structure and the character theory of the non-abelian finite simple groups, these can be ruled out as minimal counterexamples to the conjecture. For the necessary information we rely largely on [6]. Our arguments work for the majority of these groups, but they may fail for particular small instances. These are treated with computational methods, using the systems Chevie [4, 13] and GAP [3].

With respect to groups and characters we use standard notation. In particular, $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of the finite group G. Characters of $\mathbb{R}G$-modules are tacitly viewed as complex characters.

2 Motivation

Let M be a real closed manifold with fundamental group $\pi_1(M)$ and let $f\colon M \to M$ be a homeomorphism. Several invariants can be attached to f. For example, the Reidemeister number $R(f)$ of f is the number of $f_\#$-conjugacy classes on $\pi_1(M)$, where $f_\#$ is the induced map on $\pi_1(M)$. By definition, $R(f)$ is a positive integer or infinity. Other invariants are the Lefshetz number $L(f)$, defined as an alternating sum of traces of the maps induced by f on the homology groups of M, or the Nielsen number $N(f)$, the number of fixed point classes of f on M under a certain equivalence relation. In particular, f has at least $N(f)$ fixed points.

The manifold M is called an R_∞-manifold, if $R(f) = \infty$ for every homeomorphism f of M.

Let us consider the special case where M is an *infra-nilmanifold*, i.e., $M = \Gamma\backslash L$, where L is a connected, simply connected, nilpotent Lie group, and $\Gamma \leq L \rtimes C$ is discrete, cocompact and torsion-free for some maximal compact subgroup $C \leq \mathrm{Aut}(L)$. Then $\pi_1(M) \cong \Gamma$. In the case when $\Gamma \leq L$, i.e. M is a *nilmanifold*, $R(f) = \infty$ implies that $L(f) = N(f) = 0$; see the introduction of [2].

Let us even further specialize an infra-nilmanifold to the case when $L = \mathbb{R}^m$. Then M is a *flat manifold*. Here, $C = O(m)$, $\Gamma \cap \mathbb{R}^m \cong \mathbb{Z}^m$, and there is a finite group G such that

$$1 \longrightarrow \mathbb{Z}^m \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

is a short exact sequence. Conjugation of Γ on $\mathbb{Z}^m$ induces a homomorphism $\rho\colon G \to \mathrm{GL}_m(\mathbb{Z})$, the corresponding *holonomy representation.*

Let M be a flat manifold, and ρ the corresponding holonomy representation. A $\mathbb{Z}$-subrepresentation of ρ is a representation $\rho'\colon G \to \mathrm{GL}_d(\mathbb{Z})$ arising from a $\rho(G)$-invariant, pure sublattice $Y \leq \mathbb{Z}^m$ of rank d; here, Y is called pure, if some $\mathbb{Z}$-basis of Y extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^m$. We may optionally view ρ or ρ' as a $\mathbb{Q}$-representation or an $\mathbb{R}$-representation of G in a natural way.

Theorem 1 (Dekimpe, De Rock, Penninckx, 2009, [2]) *Suppose there is a $\mathbb{Z}$-subrepresentation $\rho'\colon G \to \mathrm{GL}_d(\mathbb{Z})$, which is an irreducible component of ρ of multiplicity one as a $\mathbb{Q}$-subrepresentation, and such that the following two conditions are satisfied:*

(i) *If ρ'' is a $\mathbb{Q}$-subrepresentation of ρ of degree d such that $\rho'(G)$ and $\rho''(G)$ are conjugate in $\mathrm{GL}_d(\mathbb{Q})$, then ρ' and ρ'' are equivalent.*
(ii) *For all $D \in N_{\mathrm{GL}_d(\mathbb{Z})}(\rho'(G))$ there is $g \in G$ such that $\rho'(g)D$ has eigenvalue* 1.

Then M is an R_∞ manifold. □

Condition (ii) on ρ' in Theorem 1 implies:

- The $\mathbb{Q}$-representation ρ' is $\mathbb{R}$-irreducible.
- The normalizer $N_{\mathrm{GL}_d(\mathbb{Z})}(\rho'(G))$ has finite order.

These observations follow, e.g., from the proof of [14, Theorem A].

3 The eigenvalue one condition

Motivated by Theorem 1, its authors formulated a conjecture which we are now going to introduce. Before doing so, we fix some notation which will be kept throughout this article.

Let G be a finite group, V a finite-dimensional $\mathbb{R}G$-module and $\rho\colon G \to \mathrm{GL}(V)$ the representation afforded by V. Moreover, $n \in \mathrm{GL}(V)$ is an element of finite order normalizing $\rho(G)$.

Definition 1 Let the notation be as introduced above. We then say that:

1) The triple (G, V, n) has the *$E1$-property*, if there is $g \in G$ such that $\rho(g)n$ has eigenvalue 1.
2) The pair (G, V) has the *$E1$-property*, if (G, V, n') has the $E1$-property for all $n' \in \mathrm{GL}(V)$ of finite order normalizing $\rho(G)$.
3) The group G has the *$E1$-property*, if (G, V') has the $E1$-property for all **irreducible**, **non-trivial** $\mathbb{R}G$-modules V' of **odd dimension**.

Let us give some examples.

Example 1 If $V = \mathbb{R}$ with trivial action of G, i.e. V is the trivial $\mathbb{R}G$-module, then V does not have the $E1$-property. Indeed, $\rho(G) = \{1\}$ in this case, and $n = -1$ violates the eigenvalue one condition.

Example 2 If G is an elementary abelian p-group, then G has the $E1$-property. Indeed, if p is odd, then G does not have any non-trivial, irreducible, odd-dimensional module over $\mathbb{R}$. If $p = 2$ and V is non-trivial and irreducible, then $\dim(V) = 1$ and $\rho(G) = \{\pm 1\}$, which proves our claim, as the only elements of finite order in $\mathbb{R}^* = \mathrm{GL}(V)$ are ± 1.

Example 3 (Dekimpe, De Rock, Penninckx, 2009, [2]) Let G be the extraspecial 2 group 2_+^{1+4} of order 32, and let V be the irreducible $\mathbb{R}G$-module of dimension 4. Then (G, V) does not have the $E1$-property.

With these examples in mind, we can now formulate the following conjecture, which is a slight generalization of the original conjecture in [2, Conjecture 4.8].

Conjecture 1 (Dekimpe, De Rock, Penninckx, 2009, [2]) Every finite group has the $E1$-property. □

4 The main theorem

The purpose of this article is to announce the proof of Conjecture 1.

Theorem 2 ([7]) *Every finite group has the $E1$-property.* □

In view of Theorem 1, this has the following consequence.

Corollary 1 *Let M be a flat manifold with holonomy representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{Z})$. Suppose there is a non-trivial $\mathbb{Z}$-subrepresentation $\rho'\colon G \to \mathrm{GL}_d(\mathbb{Z})$ of odd degree d, which is irreducible and of multiplicity one as an $\mathbb{R}$-subrepresentation of ρ. Suppose further that ρ' satisfies Condition* (i) *of* Theorem 1.
Then M is an R_∞-manifold. □

Corollary 1 for solvable groups G has been proved by Lutowski and Szczepański in [10, Theorem 1.4].

The proof of Theorem 2 uses the classification of the finite simple groups. We are now going to sketch the main steps, mostly without proofs.

4.1 The restriction method

Let us begin to set up our notation, which will be valid throughout this subsection. Let G be a finite group and V a non-trivial, odd-dimensional $\mathbb{R}G$-module. Write $\rho\colon G \to \mathrm{GL}(V)$ for the representation afforded by V. The assertion of the conjecture only concerns the image $\rho(G)$, so we may and will, for the rest of this subsection, assume that ρ is faithful. We identify G with $\rho(G) \leq \mathrm{GL}(V)$. Also, $n \in N_{\mathrm{GL}(V)}(G)$ is an element of finite order.

Remark 1 If V is irreducible, it is absolutely irreducible. □

It is natural to search for elements $g \in G$ such that gn has eigenvalue 1 in suitable subgroups of G, to which we can apply an inductive hypothesis.

Lemma 1 *Let $H \leq G$. Suppose that the following conditions are satisfied.*

- *The group H is n-invariant.*
- *There is $V_1 \leq V$, H-invariant and n-invariant.*
- *The triple (H, V_1, n) has the E_1-property.*

Then (G, V, n) has the $E1$-property.

Proof. Choosing a basis of V through V_1, the elements of H and n are represented by matrices of the following shape:

$$H = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right\}, \qquad n = \left(\begin{array}{c|c} n_1 & * \\ \hline 0 & * \end{array} \right).$$

Since (H, V_1, n_1) has the $E1$-property, there is

$$h = \left(\begin{array}{c|c} h_1 & * \\ \hline 0 & * \end{array} \right) \in H,$$

such that $h_1 n_1$ has eigenvalue 1. Thus hn has eigenvalue 1. □

In the following two lemmas, we present two important applications of the restriction method.

Lemma 2 *Let $S \leq V$ be an irreducible $\mathbb{R}G$-submodule such that the following conditions hold.*

- *The module V is S-homogeneous.*
- *The pair (G, S) has the $E1$-property.*

Then (G, V) has the $E1$-property.

Proof. Recall that $n \in \mathrm{GL}(V)$ is an arbitrary element of finite order normalizing G. Put $A := \langle G, n \rangle \leq \mathrm{GL}(V)$. Let $V_1' \leq V$ be an irreducible $\mathbb{R}A$-submodule of V of odd dimension and let $V_1 \leq V_1'$ be an irreducible $\mathbb{R}G$-submodule of V_1'. Then V_1 and V_1' are absolutely irreducible by Remark 1.

By hypothesis, $V_1 \cong S \cong nV_1$ as $\mathbb{R}G$-modules. As A/G is cyclic and V_1 is absolutely irreducible, the character of V_1 extends to A. Then, by Clifford theory, every absolutely irreducible $\mathbb{R}A$-submodule of $\mathrm{Ind}_G^A(V_1)$ has dimension $\dim(V_1)$. Hence $V_1 = V_1'$, and thus V_1 is n-invariant. The claim follows from Lemma 1, applied with $H = G$. □

Lemma 3 *Let $H \trianglelefteq G$ be characteristic in G. Suppose that V is irreducible and let S be an irreducible $\mathbb{R}H$-submodule of V.*

If (H, S) has the $E1$-property, so does (G, V).

Proof. The group G permutes the homogeneous components of $\mathrm{Res}^G_H(V)$ transitively. Hence the S-homogeneous component V_1 of $\mathrm{Res}^G_H(V)$ has odd dimension, and there exists $g \in G$ such that $gnV_1 = V_1$.

Thus $gn \in \mathrm{GL}(V_1)$ has finite order and normalizes H. By Lemma 2, the triple (H, V_1, gn) has the $E1$-property. In turn, (G, V, gn) has the $E1$-property by Lemma 1. As n was arbitrary, this proves our assertion. □

4.2 Reduction to the finite simple groups

We indicate how the following proposition, whose proof we omit, and the restriction method yield a reduction of the main theorem to the finite simple groups.

Proposition 1 *Let $G = L \times \cdots \times L$, where L is a non-abelian finite simple group, i.e. G is non-abelian and characteristically simple.*

If L has the $E1$-property, then G has. □

Corollary 2 *A minimal counterexample to* Conjecture 1 *is a non-abelian finite simple group.*

Proof. Let G be a minimal counterexample. If $H \leq G$ is characteristic, H has the $E1$-property by hypothesis. Then G has the $E1$-property by Lemma 3.

Thus G is characteristically simple. But G is non-abelian by Example 2. Hence G is simple by Proposition 1. □

Corollary 3 *A solvable group has the $E1$-property.*

4.3 On the structure of the problem

Here, we will present some elementary observations which help to identify the structure of the problem, and which prepare for a further method to be introduced below.

We return to the hypotheses of Subsection 4.1. Thus $G \leq \mathrm{GL}(V)$ is a finite group, where V is a non-trivial $\mathbb{R}G$-module of odd dimension. Moreover, $n \in N_{\mathrm{GL}(V)}(G)$ is an element of finite order. In addition, we assume that V is irreducible. Recall from Remark 1 that V is absolutely irreducible.

With these notations, we note the following easy facts.

Lemma 4 *The following statements hold.*

- *We have $C_{\mathrm{GL}(V)}(G) = \{x \cdot \mathrm{id}_V \mid x \in \mathbb{R}\}$, and $N_{\mathrm{SL}(V)}(G)$ embeds into* $\mathrm{Aut}(G)$.

- *The set of elements of finite order in* $N_{\mathrm{GL}(V)}(G)$ *equals* $N_{\mathrm{SL}(V)}(G) \times \langle -\mathrm{id}_V \rangle$.
- *If* $\det(n) = 1$*, then* n *has eigenvalue* 1. □

The first item of this lemma implies that $Z(G) \leq \langle \pm\mathrm{id}_V \rangle$. With respect to the last item of this lemma, we emphasize that $-n$ is an element of finite order normalizing G, and $\det(-n) = -1$, as $\dim(V)$ is odd. In particular, if $-\mathrm{id}_V \in G$, then (G, V, n) has the $E1$-property. So let us assume that $-\mathrm{id}_V \notin G$ in the following. Then $Z(G)$ is trivial.

Let us introduce further notation. For $g \in N_{\mathrm{GL}(V)}(G)$, we write $\mathrm{ad}_g \in \mathrm{Aut}(G)$ for the automorphism induced by conjugation with g, and put $\nu := \mathrm{ad}_n$. Notice that $\nu = \mathrm{ad}_{-n}$. As in the proof of Lemma 2, set $A := \langle G, n \rangle \leq \mathrm{GL}(V)$. Finally, let $A_1 := A \cap \mathrm{SL}(V)$. We distinguish two cases:

Case 1: We have $-\mathrm{id}_V \notin A$.
Case 2: We have $-\mathrm{id}_V \in A$.

Notice that $A = A_1 \times \langle -\mathrm{id}_V \rangle$ in Case 2. It is helpful to take a more abstract point of view. Put

$$A' := \begin{cases} A, & \text{in Case 1,} \\ A_1, & \text{in Case 2,} \end{cases}$$

and

$$G' := \langle \mathrm{Inn}(G), \nu \rangle \leq \mathrm{Aut}(G).$$

Remark 2 There is a surjective homomorphism

$$\rho' \colon A \to G'$$

with

$$gn^i \mapsto \mathrm{ad}_g \circ \nu^i \text{ for } g \in G \text{ and } i \in \mathbb{Z}.$$

Moreover, ρ' restricts to an isomorphism $A' \to G'$. □

Let $\chi \in \mathrm{Irr}(G)$ and $\chi' \in \mathrm{Irr}(A)$ denote the irreducible characters of G, respectively A, afforded by V. We also write χ' for the restriction of χ' to A'. The isomorphism $(\rho'|_{A'})^{-1} \colon G' \to A'$ from Remark 2 makes V into an $\mathbb{R}G'$-module, and, by a slight abuse of notation, we also let χ' denote the character of G' afforded by V. Thus $\chi'(\rho'(a')) = \chi'(a')$ for all $a' \in A'$.

4.4 The large degree method

Keep the hypotheses and notation of Subsection 4.3. Notice that a cyclic group of even order contains exactly two real absolutely irreducible characters.

Proposition 2 *Suppose there is* $g \in G$ *such that* $\alpha := \mathrm{ad}_g \circ \nu$ *has even order and* $(\mathrm{Res}^{G'}_{\langle \alpha \rangle}(\chi'), \lambda) > 0$ *for every real* $\lambda \in \mathrm{Irr}(\langle \alpha \rangle)$*. Then* (G, V, n) *has the* $E1$*-property.*

Proof. Let ρ' denote the homomorphism from Remark 2.

Suppose that we are in Case 1. Then $\rho'(gn) = \alpha$. By hypothesis, $\mathrm{Res}^{A'}_{\langle gn \rangle}(\chi')$ contains the trivial character of $\langle gn \rangle$, and thus gn has eigenvalue 1.

Suppose that we are in Case 2. Let $n_1 \in \{\pm n\}$ such that $gn_1 \in A'$. Then $\rho'(gn_1) = \alpha$. By hypothesis, $\mathrm{Res}^{A'}_{\langle gn_1 \rangle}(\chi')$ contains each of the two real irreducible characters of $\langle gn_1 \rangle$, and thus gn_1 has eigenvalue 1 and -1. Hence $gn \in \{gn_1, -gn_1\}$ has eigenvalue 1. □

Corollary 4 *Suppose there is $g \in G$ such that $\mathrm{ad}_g \circ \nu$ has order 2. Then (G, V, n) has the $E1$-property.* □

These observations lead to the *Large Degree Method*, which is formulated in the following proposition.

Proposition 3 *Suppose there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \nu$ has even order and*

$$\dim(V) > (|\alpha| - 1)|C_G(\alpha')|^{1/2}$$

for every $\alpha' \in \langle \alpha \rangle$ of prime order. Then (G, V, n) has the $E1$-property.

Proof. The second orthogonality relation implies $|\chi'(\alpha')| \leq |C_G(\alpha')|^{1/2}$ for every nontrivial $\alpha' \in \langle \alpha \rangle$. Our hypothesis implies $(\mathrm{Res}^{G'}_{\langle \alpha \rangle}(\chi'), \lambda) > 0$ for every $\lambda \in \mathrm{Irr}(\langle \alpha \rangle)$. Now use Proposition 2. □

Corollary 5 *Let G b one of the following simple groups:*

- *the Tits group;*
- *a sporadic simple group;*
- *an alternating group A_n with $n \geq 5$ and $n \neq 6$.*

Then G has the $E1$-property.

Proof. In these cases, $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \Phi$ with $|\Phi| \leq 2$. The claim follows from Corollary 4. □

The group A_6 omitted here will be treated as the Chevalley group $\mathrm{PSL}_2(9)$.

Example 4 Let $G = G_2(q)$, the simple Chevalley group type G_2 with $q = 3^f$, and let V be the the Steinberg module of G over $\mathbb{R}$.

Then $|G| = q^6(q^2 - 1)(q^6 - 1)$ and $\dim(V) = q^6$. It is known that $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \Phi$, where Φ is cyclic of order $2f$. (This is the reason for taking q to be a 3-power, as otherwise Φ is cyclic of order f and the arguments become easier.) There is $h \in G$ such that $\mathrm{ad}_h \circ \nu = \mu \in \Phi$.

If $|\mu|$ is even, put $\alpha := \mu$. Otherwise, let $u \in G$ be a μ-stable involution, and put $\alpha := \mathrm{ad}_u \circ \mu = \mathrm{ad}_{uh} \circ \nu$. Such a μ-stable involution exists, as the set of μ-fixed points in G is a Chevalley group of type G_2 or a twisted group ${}^3G_2(3^{2m+1})$ for some non-negative integer m; see [6, Proposition 4.9.1(a)].

Then $|\alpha|$ is even, $|\alpha| \leq 2f$, and $|C_G(\alpha')| \leq q^7$ for every $\alpha' \in \langle\alpha\rangle$ of prime order. This follows from the known fixed point groups of elements of $\mathrm{Aut}(G)$; see [6, Propositions 4.9.1, 4.9.2]. Since

$$\dim(V) = q^6 > (|\alpha| - 1)q^{7/2},$$

(G, V) has the $E1$-property.

4.5 The simple groups of Lie type

By Corollary 5 and the classification of the finite simple groups, we have to rule out the simple groups of Lie type as minimal counterexamples to Conjecture 1. Here is a list of these groups, omitting the conditions on simplicity:

- *classical groups*: $\mathrm{PSL}_d(q)$, $\mathrm{PSU}_d(q)$, $P\Omega_{2d+1}(q)$, $\mathrm{PSp}_{2d}(q)$, $P\Omega^+_{2d}(q)$, $P\Omega^-_{2d}(q)$;
- *exceptional Chevalley groups*: $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$;
- *twisted groups*: ${}^3D_4(q)$, ${}^2E_6(q)$.

In all these cases, q is a power of a prime r, the *characteristic* of G. We further have the following series of groups:

- *Ree groups*: ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$;
- *Suzuki groups*: ${}^2B_2(2^{2m+1})$.

4.6 Groups of Lie type of odd characteristic

Let G be a finite group of Lie type, of odd characteristic, i.e. r is odd in the notation of Subsection 4.5, or G is a Ree group ${}^2G_2(3^{2m+1})$. Let $r = 3$ in the latter case. Then G is a group with a split BN-pair of characteristic r; see, e.g., [1, Subsection 2.5].

In particular, there are distinguished subgroups B, U and T, with $B = U \rtimes T$, where $U = O_r(B)$. The groups B and T are the *standard Borel subgroup*, respectively the *standard maximal torus* of G. In classical groups, B arises from the group of upper triangular matrices, and T from the group of diagonal matrices. More generally, a group P with $B \leq P$ is a *standard parabolic subgroup*. This has a *Levi decomposition* $P = O_r(P) \rtimes L$, where L is a *standard Levi subgroup* of G, and $O_r(P)$ is the *unipotent radical* of P. The following lemma is the main tool to deal with the groups of Lie type of odd characteristic. It will be applied to the character $\chi := \chi_V$ of the $\mathbb{R}G$-module V under consideration; see the introduction to Subsection 4.1. Notice that χ is real valued, non-trivial and of odd degree.

Lemma 5 *Let $P = O_r(P) \rtimes L$ be a standard parabolic subgroup of G with standard Levi subgroup L. Let $\chi \in \mathrm{Irr}(G)$ with χ real and $\chi(1)$ odd. Then there exists $\lambda \in \mathrm{Irr}(P)$ such that the following conditions hold.*

- *The unipotent radical $O_r(P)$ is in the kernel of λ (i.e. $\lambda \in \mathrm{Irr}(L)$).*
- *The scalar product $(\lambda, \mathrm{Res}_P^G(\chi))$ is odd.*
- *If $P = B$, i.e. $L = T$, then $\lambda^2 = 1_T$ (in the character group of T).*

Proof. Let V be an $\mathbb{R}G$-module with character χ. Let $V_1 \leq \mathrm{Res}_P^G(V)$ be a homogeneous component of odd dimension. Let $S \leq V$ be a simple $\mathbb{R}P$-submodule of V_1. Then $\dim(S)$ is odd, hence S is absolutely irreducible by Remark 1. Let λ be the character of S. Then the scalar product $(\lambda, \mathrm{Res}_P^G(\chi))$ divides the dimension of V_1, hence is odd.

Now consider the restriction of S to $O_r(P)$. Let S' denote a simple $\mathbb{R}O_r(P)$-submodule of $\mathrm{Res}^P_{O_r(P)}(S)$. Then $\dim(S')$ is odd, S' is absolutely irreducible, and its character is real. Hence S' is the trivial module and $O_r(P)$ is in the kernel of λ.

If $L = T$, which is abelian, then $\lambda(1) = 1$ and $\lambda^2 = 1_T$. □

The proof uses the fact that r is odd, i.e. that $|O_r(P)|$ is odd, in an essential way. Namely, the only irreducible $\mathbb{R}H$-module of odd dimension of a group H of odd order is the trivial module.

We now sketch how Lemma 5 is applied. Assume first that $P = B$, hence $L = T$ and $O_r(P) = U$, and let χ and λ be as in this lemma. Then $(\lambda, \mathrm{Res}_B^G(\chi))$ is odd, in particular non-zero. By Frobenius reciprocity, $(\mathrm{Ind}_B^G(\lambda), \chi) > 0$. Moreover, U is in the kernel of λ, so that λ may be thought of as the inflation $\mathrm{Infl}_T^B(\lambda)$, where λ now stands for the restriction of λ to T. Thus χ is a constituent of

$$R_T^G(\lambda) := \mathrm{Ind}_B^G(\mathrm{Infl}_T^B(\lambda)).$$

The map R_T^G defined in the above equation is called *Harish-Chandra induction*. The constituents of $R_T^G(\lambda)$ for $\lambda \in \mathrm{Irr}(T)$ are classified by Harish-Chandra theory; see [1, Section 9, 10]. Those arising for $\lambda \in \mathrm{Irr}(T)$ with $\lambda^2 = 1_T$ are rare. For example, if $G = E_6(q)$ with q odd, there are 8 irreducible, real characters of odd degree, whereas $|\mathrm{Irr}(G)| = q^6 + q^5 +$ (lower terms in q). The constituents of $R_T^G(1_T)$, i.e. for $\lambda = 1_T$, are called *principal series characters*. These are, in particular, *unipotent characters* of G. Unipotent characters of odd degree are known in each case. The Steinberg character is one of these. Thus a first application of Lemma 5 reduces the number of $\mathbb{R}G$-modules V to be investigated drastically.

We aim to apply the restriction method of Lemma 1 with $H = P$, a standard parabolic subgroup of G. If possible, we choose P in such a way that for every $\alpha \in \mathrm{Aut}(G)$, the pair $(\alpha(P), \alpha(L))$, where L is the standard Levi subgroup of P, is conjugate to (P, L) in G. This condition, which is always satisfied for $P = B$ and $L = T$, can be verified by using the description of $\mathrm{Aut}(G)$ (see [6, Theorem 2.5.12]) and the construction of the standard BN-pair of G (see [6, Subsections 1.11, 2.3]).

To continue, assume that $G \leq \mathrm{GL}(V)$, where V is a non-trivial irreducible $\mathbb{R}G$-module of odd dimension. Let $n \in \mathrm{GL}(V)$ be of finite order normalizing G. Let $\chi = \chi_V$ be the character of G afforded by V. Suppose that P and L are chosen as above. Then, by replacing n with gn for a suitable $g \in G$, we may assume that n fixes P and L.

Assume first that χ is not a principal series character. In this case, choose $P = B$ and $L = T$, and let λ be as in Lemma 5. Let $V_1 \leq V$ denote the λ-homogeneous component of $\mathrm{Res}^G_B(V)$. Then $\lambda \neq 1_T$ and $\dim(V_1)$ is odd. Hence V_1 has the $E1$-property by Lemma 2 and Corollary 3, as B is solvable. However, V_1 is not, a priori, invariant under n. In this case, we can usually replace V_1 by an n-invariant N-conjugate V_1' in V, where $N \leq N_G(T)$ is the group N from the (B, N)-pair of G. Then the restriction method applies with $H = B$ and V_1'. The few instances, where there is no n-invariant N-conjugate of V_1 only occur for $G = \mathrm{PSL}_d(q)$, and are treated by replacing B by a slightly larger parabolic subgroup.

Assume now that χ is a principal series character. Here, we cannot use the approach from the previous paragraph, as then V_1, being a direct sum of trivial modules, does not have the $E1$-property. Instead, we choose a parabolic subgroup P as above, such that $\mathrm{Ind}^G_P(\mathbb{R})$ contains V with even multiplicity (including 0), where $\mathbb{R}$ denotes the trivial $\mathbb{R}P$-module. Then $\mathrm{Res}^G_P(V)$ contains a homogeneous component V_1 of odd dimension, which is not a direct sum of trivial modules. A simple $\mathbb{R}P$-submodule $S \leq V$ has $O_r(P)$ in its kernel by Lemma 5, and, viewed as an $\mathbb{R}L$-module, S is in the principal series of L. The n-invariance of V_1 is satisfied, since, for the chosen groups P, the principal series $\mathbb{R}L$-modules are invariant under automorphisms of L; see [12, Theorem 2.5].

This approach fails for groups of small rank and V the Steinberg module. The worst case is $G = \mathrm{PSL}_2(q)$, which has to be treated in an ad hoc manner. In other cases, the large degree method as in Example 4 can be applied.

This way we can rule out the simple groups of Lie type of odd characteristic as minimal counterexamples to Conjecture 1.

4.7 Groups of Lie type of even characteristic

Now let G be a finite simple group of Lie type of even characteristic, i.e. q is even in the notation of Subsection 4.5 or G is a Ree group ${}^2F_4(2^{2m+1})$ or a Suzuki group ${}^2B_2(2^{2m+1})$. In this case, most of the irreducible characters of G have odd degree. For example, if $G = E_6(q)$, with q even, there are $q^6 + 8q^2$ of them.

One of the issues in this case is to parametrize the odd degree irreducible characters and to find the reals among them. This is achieved with Lusztig's generalized Jordan decomposition of characters. Before we discuss this in more detail, we consider a special situation which simplifies the problem.

As always, assume that $G \leq \mathrm{GL}(V)$ for some non-trivial irreducible $\mathbb{R}G$-module V of odd dimension. Let $n \in \mathrm{GL}(V)$ be of finite order normalizing G. Let ν denote the automorphism of G induced by conjugation with n. Let $\chi = \chi_V$ be the character of G afforded by V. Again, let B denote the standard Borel subgroup of G and T its standard maximal torus. Then $B = U \rtimes T$ with $U = O_2(B)$. As in Subsection 4.6, we may assume that ν fixes U and T.

Put $\mathrm{Lin}(U) := \mathrm{Irr}(U/[U, U])$, the set of linear characters of U.

Proposition 4 *Suppose that ν^2 fixes every T-orbit on* $\mathrm{Lin}(U)$. *Then* (G, V, n) *has the $E1$-property.*

Proof. Since $\chi(1)$ is odd, there is a T-orbit O on $\mathrm{Lin}(U)$ such that $|O|(\lambda, \mathrm{Res}_U^G(\chi))$ is odd for every $\lambda \in O$. Moreover, the number of such orbits is odd.

By hypothesis, there is a ν-stable such orbit O. Let $\lambda \in O$. Suppose that $\lambda = 1_U$. Then the set V^U of U-fixed point on V has odd dimension $(\lambda, \mathrm{Res}_U^G(\chi))$. As V^U is an $\mathbb{R}B$-module, some irreducible $\mathbb{R}B$-constituent S of $\mathrm{Res}_B^G(V)$ with U in its kernel has odd dimension. As $|T|$ is odd, S must be the trivial module. Thus $\mathrm{Res}_B^G(\chi)$ contains a trivial constituent. In turn, χ is a principal series character. However, the only principal series character of odd degree is the trivial character by [11, Theorem 6.8]. But $\chi \neq 1_G$ by hypothesis. Thus $\lambda \neq 1_U$.

Since $U/[U, U]$ is an elementary abelian 2-group, λ is the character of an irreducible $\mathbb{R}U$-module. Let V_1 denote the λ-homogeneous component of $\mathrm{Res}_U^G(V)$. Then V_1 has odd dimension $(\lambda, \mathrm{Res}_U^G(\chi))$. Let S be a simple $\mathbb{R}U$-submodule of V_1. The character of nS is the ν-conjugate of λ. Since the T-orbit of λ is ν-invariant, there is $t \in T$ such that $tnS \cong S$ as $\mathbb{R}U$-modules. It follows that $tnV_1 = V_1$. As U is solvable, it has the $E1$-property by Corollary 3. Lemma 1 applied with $H = U$ and n replaced by tn yields our assertion. □

4.8 The remaining groups

Using Proposition 4, one can rule out the simple groups of Lie type of even characteristic as minimal counterexamples to Conjecture 1, except those in the following list (where q is even in each case):

(i) $G = \mathrm{PSL}_d(q)$ with $d \geq 3$ and $\gcd(d, q-1) > 1$;
(ii) $G = \mathrm{PSU}_d(q)$ with $d \geq 3$ and $\gcd(d, q+1) > 1$;
(iii) $G = E_6(q)$ with $3 \mid q-1$;
(iv) $G = {}^2E_6(q)$ with $3 \mid q+1$;
(v) $G = P\Omega_8^+(q)$.

For these groups, there exist non-trivial, non-involutary, inner-diagonal, respectively graph automorphisms, that do not satisfy the hypothesis of Proposition 4 in general. To rule out these remaining groups, is more than half of the entire work. Let us give a very rough sketch of the main ideas, concentrating on the groups G in the first four of the cases. The group $P\Omega_8^+(q)$ is treated in a similar way.

Choose a σ-setup for G according to [6, Definition 2.2.1]. That is, let $\overline{G}$ be a simple algebraic group, of adjoint type, over the algebraic closure $\mathbb{F}$ of the field with 2 elements. Let σ be a Steinberg morphism of $\overline{G}$ such that $O^{2'}(\overline{G}^{\sigma}) = G$. (Contrary to the usage in [6], we write $\overline{G}^{\sigma}$ for the set of σ-fixed points of $\overline{G}$, rather than $C_{\overline{G}}(\sigma)$.) In cases (i) and (ii), take $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$, in cases (iii) and (iv), take $\overline{G} = E_6(\mathbb{F})_{\mathrm{ad}}$. Then, with a suitable choice of σ, we obtain $\overline{G}^{\sigma} = \mathrm{PGL}_d(q), \mathrm{PGU}_d(q), E_6(q)_{\mathrm{ad}}$ and

${}^2E_6(q)_{\mathrm{ad}}$, respectively. These groups are not simple under the restrictions on q given above. In fact, $G \trianglelefteq \overline{G}^{\sigma}$ with cyclic factor of order $\gcd(d, q-1), \gcd(d, q+1), 3, 3$, respectively. With this setup it is easy to describe the automorphism group of G. Namely,

$$\mathrm{Aut}(G) \cong \overline{G}^{\sigma} \rtimes (\Phi_G \times \Gamma_G),$$

where $\overline{G}^{\sigma}$ acts on G by conjugation, and Φ_G and Γ_G are the groups of field, respectively graph automorphisms of G; see [6, Theorem 2.5.12]. Suppose that $q = 2^f$. In cases (i) and (iii) the group Γ_G has order 2 and Φ_G is cyclic of order f; in cases (ii) and (iv), the group Γ_G is trivial by convention, and Φ_G is cyclic of order $2f$.

We will also need to consider the groups dual to $\overline{G}$. To describe the irreducible characters of G it is more convenient to realize G as a central quotient of a covering group of G. If $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$, respectively $E_6(\mathbb{F})_{\mathrm{ad}}$, let $\overline{G}^* = \mathrm{SL}_d(\mathbb{F})$, respectively $E_6(\mathbb{F})_{\mathrm{sc}}$. Then there is a Steinberg morphism σ of $\overline{G}^*$ dual to σ; see [5, Definition 1.5.17]. In our cases we have $\overline{G}^{*\sigma} = \mathrm{SL}_d(q), \mathrm{SU}_d(q), E_6(q)_{\mathrm{sc}}$ and ${}^2E_6(q)_{\mathrm{sc}}$, respectively, and $G \cong \overline{G}^{*\sigma}/Z(\overline{G}^{*\sigma})$. (Up to finitely many exceptions, $\overline{G}^{*\sigma}$ is the universal covering group of G.) We may view the irreducible characters of G as characters of $\overline{G}^{*\sigma}$ via inflation. Starting with $\mathrm{Irr}(\overline{G}^{*\sigma})$ right away does not introduce additional aspects to be investigated, as a real irreducible character of $\overline{G}^{*\sigma}$ has $Z(\overline{G}^{*\sigma})$ in its kernel.

The set $\mathrm{Irr}(\overline{G}^{*\sigma})$ is partitioned into Lusztig series $\mathcal{E}(\overline{G}^{*\sigma}, s)$, where s runs through the $\overline{G}^{\sigma}$-conjugacy classes of semisimple elements of $\overline{G}^{\sigma}$; see [5, Definition 2.6.1]. The following lemma, whose proof can be extracted from the literature, is used to parametrize the real characters of G of odd degree. Recall that an element of a group is called *real*, if it is conjugate to its inverse.

Lemma 6 *Let $\chi \in \mathrm{Irr}(\overline{G}^{*\sigma})$ be of odd degree and let $s \in \overline{G}^{\sigma}$ be semisimple such that $\chi \in \mathcal{E}(\overline{G}^{*\sigma}, s)$. Then the following statements hold.*

(a) *We have $\chi(1) = [\overline{G}^{\sigma} : C_{\overline{G}^{\sigma}}(s)]_{2'}$.*
(b) *The characters in $\mathcal{E}(\overline{G}^{*\sigma}, s)$ of odd degree correspond, via Lusztig's generalized Jordan decomposition of characters, to the irreducible characters of $(C_{\overline{G}}(s)/C^{\circ}_{\overline{G}}(s))^{\sigma}$, and they all have the same degree. (Here, $C^{\circ}_{\overline{G}}(s)$ denotes the connected component of $C_{\overline{G}}(s)$.) In particular, if $C_{\overline{G}}(s)$ is connected, then χ is the unique character in $\mathcal{E}(\overline{G}^{*\sigma}, s)$ of odd degree.*
(c) *If χ is real, then s is real in $\overline{G}^{\sigma}$. If s is real in $\overline{G}^{\sigma}$ and $C_{\overline{G}}(s)$ is connected, then χ is real.*

Taking $s = 1$, the lemma shows that the trivial character is the unique character of odd degree in $\mathcal{E}(\overline{G}^{*\sigma}, 1)$, the set of unipotent characters of $\overline{G}^{*\sigma}$. There are analogous compatibility properties as in Lemma 6(c) for the action of certain automorphisms of G, but these are too technical to state here. By Lemma 6, in order to classify the real irreducible characters of odd degree in $\overline{G}^{*\sigma}$, it is necessary to describe the conjugacy classes of real elements and their centralizers in $\overline{G}^{\sigma}$. If $\overline{G}^{\sigma}$ is one of $\mathrm{PGL}_d(q)$ or $\mathrm{PGU}_d(q)$, this task is achieved with methods of linear algebra. If $\overline{G}^{\sigma}$ equals $E_6(q)_{\mathrm{ad}}$ or ${}^2E_6(q)_{\mathrm{ad}}$, we make use of the tables of Frank Lübeck on the website [8].

When the large degree method fails, we will use the restriction method. This is based on the following result. Here, V is an irreducible $\mathbb{R}G$-module of odd dimension, $\rho\colon G \to \mathrm{GL}(V)$ is the representation afforded by V, and $n \in \mathrm{GL}(V)$ is an element of finite order normalizing $\rho(G)$. Moreover, ν is the automorphism of G induced by n. The character $\chi = \chi_V$ of G is viewed as a character of $\overline{G}^{*\sigma}$ via inflation.

Lemma 7 *Suppose that every proper subgroup of G has the $E1$-property. Let ι denote the standard graph automorphism of $\overline{G}$ of order 2. Let $s \in \overline{G}^{\sigma}$ be such that $\chi \in \mathcal{E}(\overline{G}^{*\sigma}, s)$. Then (G, V, n) has the $E1$-property under the following hypotheses.*

There is a ι-stable, proper standard Levi subgroup $\overline{L}$ of $\overline{G}$ and a $\overline{G}^{\sigma}$-conjugate $s' \in \overline{L}^{\sigma}$ of s, such that the following three conditions hold.

(i) *The element s' is real in $\overline{L}^{\sigma}$.*
(ii) *The centralizer $C_{\overline{L}}(s')$ is connected.*
(iii) *For every $\alpha \in \mathrm{Aut}(\overline{G}^{\sigma})$ stabilizing $\overline{L}^{\sigma}$, the following holds: If $\alpha(s)$ and s are conjugate in $\overline{G}^{\sigma}$, then $\alpha(s')$ and s' are conjugate in $\overline{L}^{\sigma}$.*

Proof. (Rough sketch) There is a standard parabolic subgroup $\overline{P}^{*}$ of $\overline{G}^{*}$ containing the standard Levi subgroup $\overline{L}^{*}$ as a Levi complement, such that $\overline{L}^{*}$ is dual to $\overline{L}$. The finite group $\overline{L}^{\sigma}$ is a standard Levi subgroup of $\overline{G}^{\sigma}$, contained in the standard parabolic subgroup $\overline{P}^{\sigma}$ as a Levi complement. The fact that $\overline{L}$ is ι-stable implies that there is $g \in G$ such that $\overline{L}^{\sigma}$ and $\overline{P}^{\sigma}$ are stable under $\alpha := \mathrm{ad}_g \circ \nu$ (where α is extended to an automorphism of $\overline{G}^{\sigma}$ via [6, Theorem 2.5.14(a)]). Moreover, with a suitable choice of g, the automorphism α of G lifts to an automorphism α^* of $\overline{G}^{*\sigma}$, such that $\overline{L}^{*\sigma}$ and $\overline{P}^{*\sigma}$ are stable under α^*.

We may assume that $s = s' \in \overline{L}^{\sigma}$. In view of Lemma 6(b–c), the compatibility of Harish-Chandra restriction and the Jordan decomposition of characters implies that the restriction of V to $\overline{P}^{*\sigma}$ contains an irreducible submodule V_1 of odd dimension, of multiplicity one, and with $O_2(\overline{P}^{*\sigma})$ in its kernel. The analogue of Lemma 6(c) for the automorphisms α and α^* implies that V_1 is stable under α^*, hence under α. The claim follows from Lemma 1, applied to $(\overline{P}^{*\sigma}, V_1)$. □

To apply the large degree method based on Proposition 1, we need to estimate the orders $|C_G(\alpha')|$ for every $\alpha' \in \langle \alpha \rangle$ of prime order, for suitable $\alpha \in \mathrm{Aut}(G)$. The fixed point groups $C_G(\beta)$ for certain non-inner automorphisms $\beta \in \mathrm{Aut}(G)$ are described in [6, Propositions 4.9.1, 4.9.2]. This leads to the following results.

Proposition 5 *Let V be a non-trivial irreducible $\mathbb{R}G$-module of odd dimension with character χ. Then the following hold.*

(a) *Suppose that $G = \mathrm{PSL}_d(q)$ or $\mathrm{PGU}_d(q)$ with $d \geq 5$ and $q > 4$. If*

$$\chi(1) > q^2 \cdot q^{d(d+1)/4},$$

then (G, V) *has the* $E1$*-property.*

(b) *Suppose that* $G = E_6(q)$ *or* ${}^2E_6(q)$ *with* $q > 16$. *If*

$$\chi(1) > q \cdot q^{26},$$

then (G, V) *has the* $E1$*-property.*

Proof. (Rough sketch) Recall that $q = 2^f$. Let $\beta \in \mathrm{Aut}(G)$. Then there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \beta$ has even order and the following properties hold.

If G is as in (a), then $|\alpha| \leq 2f(q+1)$ and $|C_G(\alpha')| \leq q^{d(d+1)/2}$ for every $\alpha' \in \langle\alpha\rangle$ of prime order. Observe that $q^2 = 2^{2f} > 2f(2^f + 1)$ for $f > 2$.

If G is as in (b), then $|\alpha| \leq 6f$ and $|C_G(\alpha')| \leq q^{52}$ for every $\alpha' \in \langle\alpha\rangle$ of prime order. Observe that $q = 2^f > 6f$ for $f > 4$.

The assertions now follow from Proposition 3. □

For small values of q and for $d = 3$ in (a) above, we derive more precise estimates for the orders of the automorphisms α, to deal with more cases.

For the characters χ, which do not meet the degree estimates of Proposition 5 and their refinements, we usually can apply Lemma 7. If $G = \mathrm{PGL}_d(q)$, let $\hat{s} \in \mathrm{GL}_d(q)$ be a real lift of an element $s \in \overline{G}^{\sigma}$ parametrizing χ as in Lemma 6. The fact that $\chi(1)$ is smaller than the estimate required in Proposition 5 implies strong restrictions on $\hat{s}$, acting on its natural vector space. Namely, either the fixed point space of $\hat{s}$ has dimension at least $d/3$, or $\hat{s}$ has at most three distinct eigenvalues $1, \zeta, \zeta^{-1}$. It is then not difficult to construct the Levi subgroup $\overline{L}$ as required in Lemma 7. A similar approach works for the group $G = \mathrm{PGU}_d(q)$. For the exceptional groups of type E_6, we use the lists of explicit character degrees computed by Lübeck and given in [9], as well as the Chevie system [4, 13] for extensive computations in the Weyl group of type E_6. Still, there are numerous small cases, which cannot be handled either way. These are treated with computational methods using GAP [3].

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