THE EIGENVALUE ONE PROPERTY OF FINITE GROUPS, I

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Abstract. We prove a conjecture of Dekimpe, De Rock and Penninckx concerning the existence of eigenvalues one in certain elements of finite groups acting irreducibly on a real vector space of odd dimension. This yields a sufficient condition for a closed flat manifold to be an R_∞-manifold.

1. Introduction

The purpose of this series of two articles is to prove a conjecture of Dekimpe, De Rock and Penninckx [5, Conjecture 4.7]. Our results have been announced in [10].

1.1. **Motivation, the conjecture, and the main result.** Let M be a real closed manifold with fundamental group $\pi_1(M)$ and let $f\colon M \to M$ be a homeomorphism of M. The Reidemeister number $R(f)$ of f is the number of $f_\#$-conjugacy classes on $\pi_1(M)$, where $f_\#$ is the automorphism of $\pi_1(M)$ induced by functoriality. A priori, $R(f)$ is a positive integer or ∞. If M is a nil-manifold and $R(f) = \infty$, then $L(f) = N(f) = 0$, where $L(f)$ and $N(f)$ denote the Lefshetz number and the Nielsen number of f, respectively; see the introduction of [5]. If $R(f) = \infty$ for every homeomorphism f of M, then M is called an R_∞-manifold.

A closed flat manifold M is a space of the form $M = \Gamma\backslash\mathbb{R}^n$, where Γ is a discrete, torsion free, cocompact subgroup of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$. Then Γ is the fundamental group of M and there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \tag{1.1}$$

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where $\mathbb{Z}^n = \Gamma \cap \mathbb{R}^n$ and G is a finite group, the holonomy group of M. The sequence (1.1) gives rise to a representation

$$\gamma \colon G \to \mathrm{GL}_n(\mathbb{Z}), \tag{1.2}$$

the holonomy representation of M.

In [5, Theorem 4.7], the authors gave a necessary condition for a closed flat manifold M to be an R_∞-manifold in terms of its holonomy representation. In order to rephrase this condition, we need to introduce further terminology. A $\mathbb{Z}$-subresentation of γ is a representation $\rho \colon G \to \mathrm{GL}_d(\mathbb{Z})$ arising from a $\gamma(G)$-invariant, pure sublattice Y of $\mathbb{Z}^n$ of rank d; here, Y is called pure, if some $\mathbb{Z}$-basis of Y extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. By concatenating γ with the natural embeddings $\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Q}) \to \mathrm{GL}_d(\mathbb{R})$, we may optionally view γ as a $\mathbb{Q}$-representation or an $\mathbb{R}$-representation of G, for which we write $\gamma^{\mathbb{Q}}$ or $\gamma^{\mathbb{R}}$, respectively. An analogous convention is used for subrepresentations of γ. We can now state the criterion of Dekimpe, De Rock and Penninckx, adopting the notation used later on in this article.

Theorem 1.1.1. [5, Theorem 4.7] *Let M be a closed flat manifold with holonomy representation* (1.2)*. Suppose there is a $\mathbb{Z}$-subrepresentation $\rho \colon G \to \mathrm{GL}_d(\mathbb{Z})$, such that $\rho^{\mathbb{Q}}$ is irreducible and of multiplicity one as a composition factor of $\gamma^{\mathbb{Q}}$, and such that the following two conditions are satisfied:*

(a) *If $\tilde{\rho}$ is a $\mathbb{Q}$-subrepresentation of γ of degree d such that $\rho(G)$ and $\tilde{\rho}(G)$ are conjugate in $\mathrm{GL}_d(\mathbb{Q})$, then $\rho^{\mathbb{Q}}$ and $\tilde{\rho}^{\mathbb{Q}}$ are equivalent.*

(b) *For every $n \in N_{\mathrm{GL}_d(\mathbb{Z})}(\rho(G))$, there is $g \in G$ such that $\rho(g)n$ has eigenvalue* 1.

Then M is an R_∞-manifold. □

Subsequent to this theorem, the authors formulate the following conjecture, which we cite, up to the notation, literally.

Conjecture 1.1.2. [5, Conjecture 4.8] *Let $\rho \colon G \to \mathrm{GL}_d(\mathbb{Z})$ be a representation of a non-trivial finite group G such that $\rho^{\mathbb{R}}$ is faithful and irreducible. Suppose that d is odd. Then for every $n \in N_{\mathrm{GL}_d(\mathbb{Z})}(\rho(G))$, there is $g \in G$ such that $\rho(g)n$ has eigenvalue* 1. □

The authors give an example which shows that the condition on d to be odd is necessary.

Condition (b) in Theorem 1.1.1 implies that $\rho^{\mathbb{R}}$ is irreducible and that $N_{\mathrm{GL}_d(\mathbb{Z})}(\rho(G))$ has finite order; see the proof of [21, Theorem A]. In this paper we prove a slightly stronger version of Conjecture 1.1.2. Namely, we start with an irreducible $\mathbb{R}$-representation ρ of a finite group G of odd degree, but not necessarily realizable over the integers.

Then an element in $N_{\mathrm{GL}_d(\mathbb{R})}(\rho(G))$ need not be of finite order. We also do not insist on ρ being faithful. This relaxation is useful for inductive purposes, but does not provide a true generalization, as the eigenvalue condition only concerns the image $\rho(G)$.

Definition 1.1.3. Let G be a finite group and let V be an $\mathbb{R}G$-module affording the representation $\rho\colon G \to \mathrm{GL}(V)$. Let $n \in N_{\mathrm{GL}(V)}(\rho(G))$ be of finite order.

We say that (G, V, n) has the *eigenvalue one property*, if there is $g \in G$ such that $\rho(g)n$ has eigenvalue 1.

We say that (G, V) has the *eigenvalue one property* if (G, V, n') has the eigenvalue one property for all $n' \in N_{\mathrm{GL}(V)}(\rho(G))$ of finite order.

We say that G has the *eigenvalue one property* if (G, V') has the eigenvalue one property for every irreducible, non-trivial $\mathbb{R}G$-module V' of odd dimension. $\square$

Examples 1.1.4. (a) If V is the trivial $\mathbb{R}G$-module, then (G, V) does not have the eigenvalue one property.

(b) An elementary abelian p-group has the eigenvalue one property. $\square$

We can now present the main result of our series of two papers.

Theorem 1.1.5. *Every finite group has the eigenvalue one property.* $\square$

Clearly, Theorem 1.1.5 implies Conjecture 1.1.2. The following is a consequence of this and Theorem 1.1.1 .

Corollary 1.1.6. *Let M be a closed flat manifold with holonomy group G and holonomy representation $\gamma\colon G \to \mathrm{GL}_n(\mathbb{Z})$.*

Suppose there is a $\mathbb{Z}$-subrepresentation $\rho\colon G \to \mathrm{GL}_d(\mathbb{Z})$ of γ such that $\rho^{\mathbb{R}}$ is irreducible, non-trivial, of odd degree and of multiplicity one as a composition factor of $\gamma^{\mathbb{R}}$, and such that $\rho^{\mathbb{Q}}$ satisfies condition (a) *of* Theorem 1.1.1.

Then M is an R_∞-manifold. $\square$

Corollary 1.1.6 for solvable groups G has been proved by Lutowski and Szczepański in [15, Theorem 1.4].

1.2. **Two methods.** Let (G, V, n) and ρ be as in Definition 1.1.3. Notice that $(G, V, -n)$ is a triple with the same properties. If $g' \in G$ is such that $\rho(g')(-n)$ has eigenvalue 1, then $\rho(g')n$ has eigenvalue -1. So if (G, V) has the eigenvalue one property, there are $g, g' \in G$ such that $\rho(g)n$ and $\rho(g')n$ have eigenvalues 1 and -1, respectively.

In the course of our work, we have developed several methods to prove the eigenvalue one property for (G, V, n). Let us present the two most important ones. The first we call the *restriction method*. Suppose there is $H \leq G$ and an $\mathbb{R}H$-submodule $V_1 \leq V$ such that n normalizes $\rho(H)$, and that V_1 is n invariant. Then (G, V, n) has the eigenvalue one property if (H, V_1) has. Since $\dim(V)$ is odd, the restriction of V to H contains a homogeneous component V_1 of odd dimension. In order to proceed this way, we prove that (H, V_1) has the eigenvalue one property, if (H, S) has, where S is an irreducible $\mathbb{R}H$-submodule of V_1. In an inductive situation we may assume that the latter holds, provided S is non-trivial. The issue with this method is to find a suitable subgroup H for which, in particular, S can be chosen to be non-trivial.

The second method is the *large degree method*. Roughly speaking, if $\dim(V)$ is larger than a bound depending on group theoretical invariants derived from n, then (G, V, n) has the eigenvalue one property. Let us be more precise. If $|n|$ is odd, then n has eigenvalue 1, as $\dim(V)$ is odd. Suppose then that $|n|$ is even and let $M_G(n)$ denote the maximum of the numbers $|C_{\rho(G)}(n')|$, where n' runs through the non-trivial powers of n. If $\dim(V) > (|n| - 1)M_G(n)^{1/2}$, then (G, V, n) has the eigenvalue one property. Of course, this condition might not be satisfied right away, but can be achieved in many cases by replacing n with $\rho(g)n$ for a suitable $g \in G$. Whenever we have to choose such a g explicitly, we choose g as an involution with favorable properties. This method requires a thorough knowledge about the automorphisms of G and their fixed point subgroups.

In many instances, more than one of these methods could be applied. At any stage of our work, we have tried to apply the most elementary methods able to deal with the case under consideration.

1.3. **Survey of the paper.** Let us now give a survey on the structure of this article and the contents of the individual sections. Our strategy is by contraposition, i.e. we assume that Theorem 1.1.5 is false. By a minimal counterexample we mean a finite group of minimal order which does not have the eigenvalue one property. Section 2 is devoted to notation and preparatory material. In Section 3, following a series of reductions, we prove that a minimal counterexample is a non-abelian finite simple group. This shows in particular that a solvable group has the eigenvalue one property. The remaining sections and Part II of our series are devoted to the proof that no non-abelian finite simple group is a minimal counterexample, thereby proving Theorem 1.1.5.

In Section 4 we develop some criteria which guarantee that a triple (G, V, n) has the eigenvalue one property. These conditions are enough to rule out the sporadic simple groups, the Tits simple group and the alternating groups (with one exception) as minimal counterexamples; see Corollary 4.4.3. Our argument is based on the fact that the automorphism group of any such group is a split extension of the inner automorphism group with a group of order at most two.

It thus remains to consider the finite simple groups of Lie type. We introduce these groups and recall some of their properties in Section 5. Here, we largely follow the book [9]. Of particular relevance is the description of the automorphisms of these groups, for which we use [9, Section 2.5].

Section 6 is devoted to the simple groups G of Lie type of odd characteristic. The real irreducible characters of G of odd degrees are easily classified with Harish-Chandra theory. Such characters are very rare. For example, if $G = E_6(q)$, the simple Chevalley group of type E_6 with q odd, then G has exactly 8 real, irreducible characters of odd degree, independently of q. The relevant information on Harish-Chandra induced characters is obtained by computations inside the Weyl group of G. These computations are performed with the Chevie system based on GAP3; see [7], [17] and [18]. It should be noted, however, that some of the groups of very small Lie rank require considerable work and lengthy calculations, due to their rather restricted subgroup structure. Nevertheless, our arguments are rather elementary, only using information about the automorphism group of G and simple facts from Harish-Chandra theory. The main result of this section is Theorem 6.5.1.

By far the most difficult cases are provided by the finite groups G of Lie type of even characteristic. These will be handled in Part II of our series.

2. Notation and preliminaries

Here, we introduce our notation, which is mostly standard, and collect miscellaneous preliminary results for later reference.

2.1. **Numbers.** Let m be a non-zero integer and p a prime. We then write m_p and $m_{p'}$ for the p-part, respectively the p'-part of m.

If $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for the greatest integer smaller than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x.

2.2. **Graphs.** The *valency* of a node of an undirected graph without loops is the number of edges emanating from the node. The valency of the node of a Dynkin diagram is the valency in the underlying undirected graph. A node of valency 1 of a tree is called a *leaf*.

2.3. **Groups.** Let G be a group. For $g, x \in G$, we put ${}^x g := xgx^{-1}$. Also, ad_x denotes the inner automorphism of G corresponding to x, i.e. $\mathrm{ad}_x \colon G \to G$, $g \mapsto {}^x g$. An element $g \in G$ is called *real in* G, if g is conjugate to its inverse in G. If the surrounding group is clear form the context, we just say that g is *real*. If $X \subseteq G$ is a subset, we write $\langle X \rangle$ for the subgroup of G generated by X. The automorphism group of G is denoted by $\mathrm{Aut}(G)$, the group of inner automorphisms by $\mathrm{Inn}(G)$, and $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the group of outer automorphisms of G.

If G is finite and p is a prime, we write $O^{p'}(G)$ for the smallest normal subgroup of G of p'-index.

2.4. **Matrices.** Let Θ be a commutative ring and n a positive integer. By $\mathrm{diag}(\zeta_1, \ldots, \zeta_n)$ we denote the diagonal $(n \times n)$-matrix over Θ with entries $\zeta_i \in \Theta$ at position (i, i) for $1 \leq i \leq n$, and all other entries equal to 0. This notation is extended to block diagonal matrices, where (1×1)-blocks are identified with their entries. Thus, e.g., if A_i is an $(n_i \times n_i)$-matrix for $1 = 1, 2$ and $\zeta, \xi \in \Theta$, then $\mathrm{diag}(A_1, \zeta, \xi, A_2)$ denotes the square block diagonal matrix

$$\left(\begin{array}{c|c|c|c} A_1 & 0 & 0 & 0 \\ \hline 0 & \zeta & 0 & 0 \\ \hline 0 & 0 & \xi & 0 \\ \hline 0 & 0 & 0 & A_2 \end{array}\right)$$

with $n_1 + n_2 + 2$ rows, where the 0's indicate matrices of zeroes of the appropriate sizes.

The superscript t on a matrix indicates its transpose.

2.5. **Characters and modules.** Let G be a finite group and let K be a field. The KG-modules considered will always be left KG-modules and finite dimensional. By $\mathrm{Irr}(G)$ we denote the set of characters of the irreducible $\mathbb{C}G$-modules. A complex character of G is called *linear*, if it has degree 1. Linear characters are irreducible. The trivial character of a subgroup $H \leq G$ is denoted by 1_H. For $\chi \in \mathrm{Irr}(G)$, we write $\nu_2(\chi)$ for the Frobenius-Schur indicator of χ; see [13, p. 49, 50]. The usual inner product of two complex valued class functions χ and ψ of G is denoted by $\langle \chi, \psi \rangle$. From the context, there should not be any confusion with our notation for subgroup generation. Suppose that $H \leq G$ and that ψ and χ are K-valued class functions of H, respectively G. Then $\mathrm{Ind}_H^G(\psi)$ and $\mathrm{Res}_H^G(\chi)$ denote the induced and restricted class functions of G, respectively H.

We collect a few of well known results on real representations of odd degree.

Lemma 2.5.1. *Let V be an irreducible $\mathbb{R}G$-module of odd dimension. Then V is absolutely irreducible.*

Also, if $\chi \in \mathrm{Irr}(G)$ is real with $\chi(1)$ odd, then $\nu_2(\chi) = 1$, i.e. χ is realizable over $\mathbb{R}$.

Proof. If V is not absolutely irreducible, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is a direct sum of two $\mathbb{C}G$-modules of equal dimension, contradicting the odd-dimensionality of V.

As χ is real, $\nu_2(\chi) = 1$ or $\nu_2(\chi) = -1$; see [13, Theorem 4.5]. As $\chi(1)$ is odd, $\nu_2(\chi) = 1$ and χ is afforded by a real representation; see [13, p. 58]. □

Let V be an irreducible $\mathbb{R}G$-module of odd dimension. In view of the above lemma, the character χ of V equals the character of $\mathbb{C} \otimes_{\mathbb{R}} V$, and we view χ as an element of $\mathrm{Irr}(G)$.

Lemma 2.5.2. *Let V be an irreducible $\mathbb{R}G$-module of odd dimension and let $H \trianglelefteq G$. Suppose*

$$\mathrm{Res}^G_H(V) = V_1 \oplus \cdots \oplus V_r,$$

where the V_i are the homogeneous components of $\mathrm{Res}^G_H(V)$. Then each V_i has odd dimension. If $|H|$ is odd, H acts trivially on V.

Proof. As the V_i are conjugate by the action of G (see [13, Theorem 6.5]), they all have the same dimension.

Now suppose that $|H|$ is odd, and let S be an irreducible constituent of V_1. Then S has odd dimension and thus is an absolutely irreducible $\mathbb{R}H$-module. It follows that S is the trivial module, and thus V_1 is a direct sum of trivial modules and $r = 1$. This yields our second assertion. □

Lemma 2.5.3. *Let $H \leq G$. Let V be a non-trivial irreducible $\mathbb{R}G$-module of odd dimension. Then there is an irreducible $\mathbb{R}H$-module S of odd dimension which occurs with odd multiplicity in $\mathrm{Res}^G_H(V)$. If $C \trianglelefteq H$ has odd order, then C acts trivially on S.*

Proof. Write $\mathrm{Res}^G_H(V) = V_1 \oplus \cdots \oplus V_r$ for the decomposition of $\mathrm{Res}^G_H(V)$ into homogeneous components. Then V_i has odd dimension for some $1 \leq i \leq r$, and an irreducible constituent S of V_i satisfies the requirements. We are done by Lemma 2.5.2. □

Lemma 2.5.4. *Let G' be a finite group and let $G \trianglelefteq G'$ such that G'/G is abelian. Let $\chi' \in \mathrm{Irr}(G')$ such that $\mathrm{Res}^{G'}_G(\chi')$ is irreducible. Then*

$$|\chi'(x)| \leq |C_G(x)|^{1/2}$$

for all $x \in G'$.

Proof. Let $x \in G'$. We have $C_{G'}(x)/C_G(x) = C_{G'}(x)/(G \cap C_{G'}(x)) \cong GC_{G'}(x)/G \leq G'/G$. Thus $|C_{G'}(x)| \leq |G'/G||C_G(x)|$.

As $\mathrm{Res}_G^{G'}(\chi')$ is irreducible, $\beta\chi' \in \mathrm{Irr}(G')$ for every $\beta \in \mathrm{Irr}(G'/G)$, and $\beta\chi' \neq \beta'\chi'$ for $\beta, \beta' \in \mathrm{Irr}(G'/G)$ with $\beta \neq \beta'$. Now $|\beta\chi'(x)| = |\chi'(x)|$ for every $\beta \in \mathrm{Irr}(G'/G)$. By the second orthogonality relation we obtain $|G'/G||\chi'(x)|^2 \leq |C_{G'}(x)|$. This yields our claim. □

Lemma 2.5.5. *Let the notation and hypothesis be as in* Lemma 2.5.4. *Let $x \in G'$ and put $M := \max\{|C_G(y)| \mid y \in \langle x \rangle \setminus \{1\}\}$. Suppose that $\chi'(1) > (|x| - 1)M^{1/2}$. Then $\mathrm{Res}_{\langle x \rangle}^{G'}(\chi')$ contains every irreducible character of $\langle x \rangle$ as a constituent.*

Proof. Let $\lambda \in \mathrm{Irr}(\langle x \rangle)$. Then

$$|x|\langle \mathrm{Res}_{\langle x \rangle}^{G'}(\chi'), \lambda \rangle = \chi'(1) + \sum_{1 \neq y \in \langle x \rangle} \chi'(y)\lambda(y^{-1}).$$

Moreover,

$$\begin{aligned} \left| \sum_{1 \neq y \in \langle x \rangle} \chi'(y)\lambda(y^{-1}) \right| &\leq \sum_{1 \neq y \in \langle x \rangle} |\chi'(y)\lambda(y^{-1})| \\ &= \sum_{1 \neq y \in \langle x \rangle} |\chi'(y)| \\ &\leq \sum_{1 \neq y \in \langle x \rangle} |C_G(y)|^{1/2} \\ &\leq (|x| - 1)M^{1/2}, \end{aligned}$$

where the penultimate estimate arises from Lemma 2.5.4. This proves our assertion. □

3. The reduction to finite simple groups

In this section G is a finite group. Tensor products of $\mathbb{R}$-vector spaces are tensor products over $\mathbb{R}$, and we usually write $\otimes$ instead of $\otimes_{\mathbb{R}}$. The phrase "eigenvalue one property" in its three specifications introduced in Definition 1.1.3, will henceforth be abbreviated as "$E1$-property".

3.1. **The restriction method.** Working towards the proof of Theorem 1.1.5, we first establish some reductions. We will use the following setup. Let V be an $\mathbb{R}G$-module, and let ρ denote the representation of G afforded by V. Let $n \in \mathrm{GL}(V)$ be of finite order normalizing $\rho(G)$.

Lemma 3.1.1. *Let V_1 denote an n-invariant $\mathbb{R}G$-submodule of V. If (G, V_1) has the $E1$-property, then (G, V, n) has the $E1$-property.*

Proof. Let ρ_1 denote the representation of G afforded by V_1, and let n_1 denote the restriction of n to an automorphism of V_1. Then $n_1 \in \mathrm{GL}(V_1)$ has finite order and normalizes $\rho_1(G)$. By assumption, there exists $g \in G$ and a non-trivial vector $v \in V_1$ fixed by $\rho_1(g)n_1$. Thus $\rho(g)n$ has eigenvalue 1. □

Lemma 3.1.2. *Let S be an irreducible $\mathbb{R}G$-module of odd dimension such that (G, S) has the $E1$-property. If V is the direct sum of an odd number of copies of S, then (G, V) has the $E1$-property.*

Proof. Put $A := \langle \rho(G), n\rangle$. This is a finite subgroup of $\mathrm{GL}(V)$ and V is an $\mathbb{R}A$-module in the natural way. Let V_1 be an irreducible $\mathbb{R}A$-submodule of V of odd dimension, and let $S_1 \leq V_1$ be an irreducible $\mathbb{R}\rho(G)$-submodule of V_1. Then V_1 and S_1 are absolutely irreducible by Lemma 2.5.1. The character of $\mathbb{C} \otimes S_1$ is A-invariant, as nS_1 is an irreducible $\mathbb{R}\rho(G)$-submodule of V, and thus isomorphic to S_1.

Since $A/\rho(G)$ is cyclic, the character of $\mathbb{C} \otimes S_1$ extends to A; see [13, Corollary 11.22]. Moreover, all absolutely irreducible $\mathbb{R}A$-submodules of $\mathrm{Ind}_{\rho(G)}^{A}(S_1)$ have dimension $\dim(S)$; see [13, Corollary 6.17 (Gallagher's theorem)]. As V_1 is isomorphic to one of these, we have $V_1 = S_1$. The claim now follows from Lemma 3.1.1. □

Lemma 3.1.3. *Let $H \leq G$ be a subgroup of G such that n normalizes $\rho(H)$. Let V_1 denote an n-invariant $\mathbb{R}H$-submodule of $\mathrm{Res}_H^G(V)$ such that (H, V_1) has the $E1$-property. Then (G, V, n) has the $E1$-property.*

Proof. Apply Lemma 3.1.1 with (G, V, n) replaced by $(H, \mathrm{Res}_H^G(V), n)$. Write ρ_H for the representation of H afforded by $\mathrm{Res}_H^G(V)$. Then n normalizes $\rho_H(H) = \rho(H)$ by assumption. Moreover, V_1 is n-invariant and (H, V_1) has the $E1$-property. By Lemma 3.1.1, there is $g \in H$ such that $\rho_H(g)n = \rho(g)n$ has eigenvalue 1. □

Corollary 3.1.4. *Suppose that V is irreducible and of odd dimension.*

Let $H \trianglelefteq G$ be a normal subgroup such that $\{1\} \neq \rho(H)$ is characteristic in $\rho(G)$. Suppose that (H, S) has the $E1$-property for some irreducible submodule S of $\mathrm{Res}_H^G(V)$. Then (G, V) has the $E1$-property.

If H has the $E1$-property, then (G, V) has the $E1$-property.

Proof. As $\rho(H) \neq \{1\}$, the irreducible submodules of $\mathrm{Res}_H^G(V)$ are non-trivial. As they are also odd-dimensional, the second statement follows from the first.

Write $\mathrm{Res}_H^G(V) = V_1 \oplus \cdots \oplus V_r$, where the V_i are the homogeneous components of $\mathrm{Res}_H^G(V)$. Chose the notation so that S is a submodule of V_1. As G permutes the V_i transitively, $\dim_{\mathbb{R}}(V_1)$ is odd and there is $g \in G$ such that $\rho(g)nV_1 = V_1$. By Lemma 3.1.2 and our assumption, (H, V_1) has the $E1$-property. Since $\rho(H)$ is characteristic in $\rho(G)$, the claim follows from Lemma 3.1.3 with (G, V, n) replaced by $(G, V, \rho(g)n)$. □

3.2. **The minimal counterexamples.** Here we prove that a minimal counterexample to Theorem 1.1.5 is a non-abelian simple group.

Proposition 3.2.1. *Let H be a non-abelian finite simple group, and assume that $G = H \times \cdots \times H$ is a direct product of r copies of H. For each $1 \leq i \leq r$, let V_i be an irreducible $\mathbb{R}H$-module of odd dimension. Consider the $\mathbb{R}G$-module $V := V_1 \otimes \cdots \otimes V_r$ with the i-th factor of G acting on V_i.*

Suppose that V is not the trivial module and that for all $1 \leq i \leq r$ either V_i is the trivial $\mathbb{R}H$-module or that (H, V_i) has the $E1$-property. Then (G, V) has the $E1$-property.

Proof. For $1 \leq i \leq r$, let $\rho_i \colon H \to \mathrm{GL}(V_i)$ denote the representation of H afforded by V_i. Then $\rho := \rho_1 \otimes \cdots \otimes \rho_r$ is the representation of G afforded by V.

Let $n \in \mathrm{GL}(V)$ be of finite order normalizing $\rho(G)$. For $1 \leq i \leq r$, let $\nu_i \colon H \to G$ denote the embedding of H onto the i-th direct factor $H_i := \nu_i(H)$ of G. As conjugation by n permutes the set $\{\rho(H_i) \mid 1 \leq i \leq r\}$ of normal subgroups of $\rho(G)$, there is a permutation σ of $\{1, \ldots, r\}$, and $\alpha_i \in \mathrm{Aut}(H)$, $1 \leq i \leq r$, such that

$$\begin{aligned} n \circ (\rho_1(s_1) \otimes \cdots \otimes \rho_r(s_r)) \circ n^{-1} \\ = \rho_1(\alpha_1(s_{\sigma^{-1}(1)})) \otimes \cdots \otimes \rho_r(\alpha_r(s_{\sigma^{-1}(r)})) \end{aligned} \tag{3.1}$$

for all $s_1, \ldots, s_r \in H$. Now consider the isomorphism of $\mathbb{R}$-vector spaces

$$f_\sigma \colon V_1 \otimes \cdots \otimes V_r \to V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(r)}, v_1 \otimes \cdots \otimes v_r \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

From Equation (3.1) we obtain

$$\begin{aligned} f_\sigma \circ n \circ (\rho_1(s_1) \otimes \cdots \otimes \rho_r(s_r)) \circ n^{-1} \circ f_\sigma^{-1} \\ = (\rho_{\sigma(1)} \circ \alpha_{\sigma(1)})(s_1) \otimes \cdots \otimes (\rho_{\sigma(r)} \circ \alpha_{\sigma(r)})(s_r) \end{aligned} \tag{3.2}$$

for all $s_1, \ldots, s_r \in H$. Equation (3.2) shows that ρ is equivalent to the representation

$$(\rho_{\sigma(1)} \circ \alpha_{\sigma(1)}) \otimes \cdots \otimes (\rho_{\sigma(r)} \circ \alpha_{\sigma(r)}) \colon G \to \mathrm{GL}(V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(r)}).$$

As all the representations ρ_i are absolutely irreducible by Lemma 2.5.1, it follows that ρ_i is equivalent to $\rho_{\sigma(i)} \circ \alpha_{\sigma(i)}$ for all $1 \leq i \leq r$. Thus there are $\mathbb{R}$-vector space isomorphisms $a_i \colon V_i \to V_{\sigma(i)}$, such that

$$a_i \circ \rho_i(s) \circ a_i^{-1} = (\rho_{\sigma(i)} \circ \alpha_{\sigma(i)})(s) \tag{3.3}$$

for all $1 \leq i \leq r$ and all $s \in H$. Equations (3.2) and (3.3) and the fact that ρ is absolutely irreducible now imply that $n = cf_\sigma^{-1} \circ (a_1 \otimes \cdots \otimes a_r)$ for some $0 \neq c \in \mathbb{R}$. Replacing a_1 by ca_1 we may assume that $n = f_\sigma^{-1} \circ (a_1 \otimes \cdots \otimes a_r)$.

Suppose first that σ is an r-cycle. Then all the $\rho(H_i)$ are isomorphic, so that, in particular, V_1 is non-trivial. For $1 \leq i \leq r$ put $b_i := a_{\sigma^{i-1}(1)}$ and $\beta_i := \alpha_{\sigma^{i-1}(1)}$. Using Equation (3.3), we find

$$(b_r \circ \cdots \circ b_1) \circ \rho_1(s) \circ (b_r \circ \cdots \circ b_1)^{-1} = (\rho_1 \circ \beta_1 \circ \beta_r \circ \cdots \circ \beta_2)(s)$$

for all $s \in H$. Thus $\rho_1(H) \leq \mathrm{GL}(V_1)$ is invariant under $b := b_r \circ \cdots \circ b_1 \in \mathrm{GL}(V_1)$. As $n = f_\sigma^{-1} \circ (a_1 \otimes \cdots \otimes a_r)$ has finite order, it follows that b has finite order, since $(f_\sigma^{-1} \circ (a_1 \otimes \cdots \otimes a_r))^r = b \otimes c_2 \otimes \cdots \otimes c_r$ for suitable $c_i \in \mathrm{Aut}(V_i)$, $2 \leq i \leq r$. As (H, V_1) has the $E1$-property, there is $0 \neq v_1 \in V_1$ and $s \in H$ such that $\rho_1(s)bv_1 = v_1$. For $1 \leq i \leq r-1$, put $v_{\sigma^i(1)} := b_i v_{\sigma^{i-1}(1)}$. Then

$$\begin{aligned}
\rho(s,1,\ldots,1)n(v_1 \otimes \cdots \otimes v_r) &= \rho(s,1,\ldots,1)(f_\sigma^{-1}(a_1v_1 \otimes \cdots \otimes a_rv_r)) \\
&= \rho_1(s)a_{\sigma^{-1}(1)}v_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)}v_{\sigma^{-1}(2)} \otimes \\
&\qquad \cdots \otimes a_{\sigma^{-1}(r)}v_{\sigma^{-1}(r)} \\
&= v_1 \otimes \cdots \otimes v_r,
\end{aligned}$$

as $\rho_1(s)a_{\sigma^{-1}(1)}v_{\sigma^{-1}(1)} = \rho_1(s)b_rv_{\sigma^{r-1}(1)} = \rho_1(s)b_rb_{r-1}v_{\sigma^{r-2}(1)} = \cdots = \rho_1(s)bv_1 = v_1$, and $a_{\sigma^{-1}(i)}v_{\sigma^{-1}(i)} = v_i$ for $2 \leq i \leq r$.

If σ is not an r-cycle, we have tensor decompositions $V = W_1 \otimes W_2$ with $W_1 = V_1 \otimes \cdots \otimes V_{r'}$ and $W_2 = V_{r'+1} \otimes \cdots \otimes V_r$ for some $1 < r' < r$, and corresponding decompositions $G = G_1 \times G_2$, $\rho = \mu_1 \otimes \mu_2$, $n = n_1 \otimes n_2$, where each n_i has finite order and normalizes $\mu_i(G_i)$, for $i = 1, 2$. Without loss of generality we can assume that W_1 is non-trivial. If W_2 is non-trivial or if $n_2 = \mathrm{id}_{W_2}$, arguing by induction on r we find elements $g_i \in G_i$ and non-zero vectors $w_i \in W_i$ such that $\mu_i(g_i)n_iw_i = w_i$ for $i = 1, 2$. If W_2 is the trivial module and $n_2 = -\mathrm{id}_{W_2}$, then take $g_1 \in G_1$ and $0 \neq w_1 \in W_1$ such that $\mu_1(g_1)(-n_1)w_1 = w_1$, and let $0 \neq w_2 \in W_2$. In both cases, $0 \neq w_1 \otimes w_2$ is a fixed vector of $\rho(g_1, g_2)n = \mu_1(g_1)n_1 \otimes \mu_2(g_2)n_2 = (-\mu_1(g_1)n_1) \otimes (-\mu_2(g_2)n_2)$. □

Corollary 3.2.2. *Let H be a finite non-abelian simple group with the $E1$-property. Then every finite direct product $G = H \times \cdots \times H$ has the $E1$-property.*

Proof. Suppose that G is a direct product of r copies of H. Let V be a non-trivial irreducible $\mathbb{R}G$-module of odd dimension. Then V is absolutely irreducible by Lemma 2.5.1. Hence there are irreducible $\mathbb{C}H$-modules V_i', $1 \leq i \leq r$ such that $\mathbb{C} \otimes V \cong V_1' \otimes \cdots \otimes V_r'$. For

$1 \leq i \leq r$, the isomorphism type of V_i' is uniquely determined by the isomorphism type of V, and thus the characters of the V_i' are real valued. As $\dim(V_i')$ is odd for $1 \leq i \leq r$, Lemma 2.5.1 implies the existence of $\mathbb{R}H$-modules V_i such that $\mathbb{C} \otimes V_i \cong V_i'$. Hence $V \cong V_1 \otimes \cdots \otimes V_r$ as $\mathbb{R}G$-modules. As V is non-trivial, at least one of the V_i is non-trivial and thus (H, V_i) has the $E1$-property by our assumption on H. It follows from Proposition 3.2.1 that V has the $E1$-property. $\square$

Corollary 3.2.3. *A minimal counterexample to* Theorem 1.1.5 *is a non-abelian simple group.*

Proof. Let G be a group of minimal order without the $E1$-property. Let V be a non-trivial irreducible $\mathbb{R}G$-module of odd dimension such that (G, V) does not have the $E1$-property, and let ρ denote the representation of G afforded by V. Then ρ is faithful.

Let $H \trianglelefteq G$ denote a non-trivial characteristic subgroup of G. If $H \lneq G$, then H has the $E1$-property by assumption. Moreover, $\rho(H)$ is non-trivial and characteristic in $\rho(G)$, as ρ is faithful. But then (G, V) has the $E1$-property by Corollary 3.1.4, contradicting our assumption.

Thus G is characteristically simple. In this case, Corollary 3.2.2 implies that G is simple. Lemma 2.5.2 and Example 1.1.4(b) imply that G is non-abelian. $\square$

Corollary 3.2.4. *A solvable group has the $E1$-property.* $\square$

4. The $E1$-property for the simple sporadic and alternating groups

The aim of this section is to prove the $E1$-property for the simple sporadic groups and the simple alternating groups. On the way to this, we establish further reductions.

4.1. **General notation.** We fix some pieces of notation that will be used throughout the remainder of this article.

Notation 4.1.1. Let G be a non-abelian finite simple group. Let V denote a non-trivial irreducible $\mathbb{R}G$-module of odd dimension, and let ρ be the representation of G afforded by V. Then ρ is faithful as G is simple. Moreover, we let $n \in \mathrm{GL}(V)$ denote an element of finite order normalizing $\rho(G)$. Finally, ν denotes the automorphism of G induced by n, i.e. $\nu(g) = \rho^{-1}({}^n\rho(g))$ for $g \in G$. $\square$

Notice that if χ denotes the character of G afforded by V, then χ is ν-invariant.

4.2. **On the structure of the problem.** Assume Notation 4.1.1. In this subsection we will identify G with its image $\rho(G) \leq \mathrm{GL}(V)$. Thus $G \leq \mathrm{GL}(V)$ is a non-abelian simple group which acts absolutely irreducibly on V. In particular, $C_{\mathrm{GL}(V)}(G) = \{x \cdot \mathrm{id}_V \mid 0 \neq x \in \mathbb{R}\}$. Also, $\nu = \mathrm{ad}_n$, and if $n' \in \mathrm{GL}(V)$ is of finite order normalizing G and inducing the automorphism ν of G, then $n' = \pm n$, since $n^{-1}n' \in C_{\mathrm{GL}(V)}(G)$. As G is perfect, we have $G \leq \mathrm{SL}(V)$. Notice that $N_{\mathrm{SL}(V)}(G)$ embeds into $\mathrm{Aut}(G)$, and thus every element of $N_{\mathrm{SL}(V)}(G)$ has finite order. As every element of finite order of $\mathrm{GL}(V)$ has determinant 1 or -1, it follows that the set of elements of finite order normalizing G is equal to the finite subgroup $N_{\mathrm{SL}(V)}(G) \times \langle -\mathrm{id}_V \rangle$ of $\mathrm{GL}(V)$.

Definition 4.2.1. Under the identification of G and $\rho(G)$, set $A := \langle G, n \rangle$ and $A_1 := A \cap \mathrm{SL}(V)$. Thus $A_1 \leq A \leq \mathrm{GL}(V)$. □

As A is a finite group, we have $C_A(G) \leq \langle -\mathrm{id}_V \rangle$. We now distinguish two cases.

Case 1. Suppose that $-\mathrm{id}_V \notin A$, so that, in particular, $-\mathrm{id}_V \notin \langle n \rangle$. Then $C_A(G) = \{\mathrm{id}_V\}$ and A embeds into $\mathrm{Aut}(G)$. We get a chain of groups

$$G \leq A_1 \leq A \leq N_{\mathrm{SL}(V)}(G) \times \langle -\mathrm{id}_V \rangle.$$

This case occurs, e.g., for $G = A_5$ and $A \cong S_5$ when $\dim(V) = 5$.

Case 2. Suppose that $-\mathrm{id}_V \in A$, so that $C_A(G) = \langle -\mathrm{id}_V \rangle$. Since $-\mathrm{id}_V \notin A_1$, we obtain

$$A = A_1 \times \langle -\mathrm{id}_V \rangle \leq N_{\mathrm{SL}(V)}(G) \times \langle -\mathrm{id}_V \rangle.$$

Thus $n = -n_1$ for some $n_1 \in A_1$. This case occurs, e.g., for $G = \mathrm{SL}_2(8)$ and $\dim(V) = 7$, where there exist an element $n \in \mathrm{GL}(V)$ of order 6 normalizing G such that $\langle G, n \rangle = N_{\mathrm{SL}(V)}(G) \times \langle -\mathrm{id}_V \rangle$.

We record a simple consequence.

Lemma 4.2.2. *Assume Notation* 4.1.1 *and suppose we are in* Case 2. *Then $A_1 = \langle G, n_1 \rangle$ and $|A_1/G|$ is odd. In particular, there is $g \in G$ such that $|gn_1|$ is odd.*

Proof. We have $A_1 = \langle G, n^2 \rangle = \langle G, n_1^2 \rangle \leq \langle G, n_1 \rangle \leq A_1$, and thus $\langle n_1 G \rangle = A_1/G = \langle n_1^2 G \rangle$, which implies that $|A_1/G|$ is odd. The last statement is clear, as the 2-part of n_1 lies in G. □

It is also worthwhile to take a more abstract point of view.

Definition 4.2.3. Set $A' := A$, respectively $A' := A_1$ if (G, V, n) is as in Case 1, respectively Case 2.

Set $G' := \langle \mathrm{Inn}(G), \nu \rangle \leq \mathrm{Aut}(G)$. □

Lemma 4.2.4. *There is a surjective homomorphism*

$$\rho' \colon A \to G'$$

with

$$gn^i \mapsto \mathrm{ad}_g \circ \nu^i \text{ for } g \in G \text{ and } i \in \mathbb{Z}. \tag{4.1}$$

Moreover, ρ' restricts to an isomorphism $A' \to G'$.

Proof. Let l denote the smallest positive integer such that $n^l \in G$. Then $\nu^l \in \mathrm{Inn}(G)$. Every element of A has a unique expression as gn^i for some $g \in G$ and some integer i with $0 \leq i < l$. We can thus define a surjective map $\rho' \colon A \to G'$ by (4.1). Clearly, ρ' is a homomorphism with kernel $C_A(G)$. This proves our assertions. □

Remark 4.2.5. Let $\chi \in \mathrm{Irr}(G)$ and $\chi' \in \mathrm{Irr}(A)$ denote the irreducible characters of G, respectively A, afforded by V. We also write χ' for the restriction of χ' to A'. Thus $\chi' \in \mathrm{Irr}(A')$ is an extension of χ.

The isomorphism $(\rho'|_{A'})^{-1} \colon G' \to A'$ from Lemma 4.2.4 makes V into an $\mathbb{R}G'$-module, and, by a slight abuse of notation, we also let χ' denote the character of G' afforded by V. Thus $\chi'(\rho'(a')) = \chi'(a')$ for all $a' \in A'$. □

4.3. **The large degree method.** The following criterion is often helpful in small situations. Notice that $G' = \langle \mathrm{Inn}(G), \mathrm{ad}_g \circ \nu \rangle$ for every $g \in G$.

Lemma 4.3.1. *Suppose that there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \nu$ has even order, and that, with the above notation,*

$$\mathrm{Res}^{G'}_{\langle \alpha \rangle}(\chi')$$

contains each of the real, irreducible characters of $\langle \alpha \rangle$ with positive multiplicity. Then (G, V, n) has the $E1$-property.

Proof. Let $\rho' \colon A \to G'$ be the homomorphism from Lemma 4.2.4. Then $\rho'(gn) = \alpha$. In Case 1, our hypothesis shows that $\mathrm{Res}^{A'}_{\langle gn \rangle}(\chi')$ contains the trivial character of $\langle gn \rangle$, and thus gn has eigenvalue 1.

Suppose that we are in Case 2 and put $n_1 = -n \in A'$. Since $n^{-1}n_1 = -\mathrm{id}_V$, we have $\rho'(n_1) = \nu$ and thus $\rho'(gn_1) = \alpha$. By hypothesis, $\mathrm{Res}^{A'}_{\langle gn_1 \rangle}(\chi')$ contains the non-trivial real irreducible character of $\langle gn_1 \rangle$, and thus gn_1 has eigenvalue -1. Hence $gn = -gn_1$ has eigenvalue 1. □

Corollary 4.3.2. *Suppose that there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \nu$ is an involution. Then (G, V, n) has the $E1$-property.*

Proof. As G' does not have a non-trivial abelian normal subgroup, α neither is in the kernel nor in the center of χ'. Hence he hypothesis of Lemma 4.3.1 is satisfied. $\square$

We close by showing that if $\dim(V)$ is large relative to certain subgroups of G, then (G, V, n) has the $E1$-property. We will use the following notation. If $\alpha \in \mathrm{Aut}(G)$, and if p is a prime dividing $|\alpha|$, we write $\alpha_{(p)}$ for an element of order p in $\langle\alpha\rangle$.

Lemma 4.3.3. *Assume* Notation 4.1.1. *Suppose that there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \nu$ has even order and that*

$$\dim(V) > (|\alpha| - 1)|C_G(\alpha_{(p)})|^{1/2},$$

for all primes p with $p \mid |\alpha|$.

Then (G, V, n) has the $E1$-property.

Proof. We will make use of Lemma 2.5.5 for the inclusion $\mathrm{Inn}(G) \trianglelefteq G'$. Notice that $|C_{\mathrm{Inn}(G)}(\beta)| = |C_G(\beta)|$ for every $\beta \in G'$. For every $1 \neq \beta \in \langle\alpha\rangle$, there is a prime p dividing $|\alpha|$ such that $C_G(\beta) \leq C_G(\alpha_{(p)})$. By hypothesis,

$$\chi'(1) > (|\alpha| - 1)|C_G(\beta)|^{1/2}$$

for all $1 \neq \beta \in \langle\alpha\rangle$. Hence $\mathrm{Res}^{G'}_{\langle\alpha\rangle}(\chi')$ contains all irreducible characters of $\langle\alpha\rangle$ with positive multiplicity by Lemma 2.5.5. The assertion follows from Lemma 4.3.1. $\square$

4.4. **Some special cases.** We prove the $E1$-property for non-abelian simple groups with special automorphism groups.

Lemma 4.4.1. *If $\nu \in \mathrm{Inn}(G)$, then (G, V, n) has the $E1$-property. In particular, G has the $E1$-property if* $\mathrm{Aut}(G) = \mathrm{Inn}(G)$.

Proof. If $\nu \in \mathrm{Inn}(G)$, there is $g \in G$ such that $\alpha = \mathrm{ad}_g \circ \nu$ is an involution. The claim follows from Corollary 4.3.2. $\square$

Lemma 4.4.2. *If* $\mathrm{Aut}(G)$ *is a split extension of* $\mathrm{Inn}(G)$ *with a group of order* 2*, then G has the $E1$-property.*

Proof. By hypothesis, there is $g \in G$ such that $\alpha = \mathrm{ad}_g \circ \nu$ is an involution. We are done by Corollary 4.3.2. $\square$

Corollary 4.4.3. *If $G = A_n$, the alternating group on n-letters with $n \neq 6$, or if G is a sporadic simple group, then G has the $E1$-property.*

Proof. It is well known that $\mathrm{Aut}(A_n) = S_n$, the symmetric group on n letters, unless $n = 6$, so the result follows from Lemma 4.4.2 in these cases.

If G is a sporadic simple group, then either $\mathrm{Aut}(G) \cong G$ or $\mathrm{Aut}(G)$ is a split extension of $\mathrm{Inn}(G)$ with a group of order 2; see [4]. We are done with Lemmas 4.4.1 and 4.4.2 $\square$

The group $G = A_6$ excluded in Corollary 4.4.3 will be treated as $G = \mathrm{PSL}_2(9)$.

5. Simple groups of Lie type

Here we introduce the simple groups of Lie type and some of their properties relevant to our investigations.

5.1. **The groups.** Let G be a finite simple group of Lie type defined over a field of characteristic r. For a concise introduction to these groups see [9, Section 2.2], for a thorough treatment refer to [2]. In the finitely many cases where r is not uniquely determined by G (cf. [9, Theorem 2.2.10]), we choose r to be odd. Every finite simple group of Lie type is isomorphic to exactly one of the groups listed in Table 5.1. In this table, q can be any power of r, subject to the conditions given in the last column of the table. The groups in Lines 1–6 are the classical groups, and we give their classical names as well as their "Lie type" names. For simplicity, we call the other groups exceptional groups of Lie type. The groups in Line 14 are the Suzuki groups, and the groups in Lines 15, 16 the Ree groups. Finally the group in Line 17 is known as the Tits group. The groups in Lines 2, 6 and 12–17 are called twisted groups, the others are the untwisted groups. Finally, in Lines 1–13, we let the positive integer f be such that $q = r^f$. Our notation for the Suzuki and Ree groups differs from the one used in [9]. What we write as ${}^2B_2(q)$, ${}^2G_2(q)$ and ${}^2F_4(q)$, is written as ${}^2B_2(\sqrt{q})$, ${}^2G_2(\sqrt{q})$, respectively ${}^2F_4(\sqrt{q})$, with $q = 2^{2m+1}$, 3^{2m+1}, respectively 2^{2m+1}; see [9, Definition 2.2.4].

5.2. **A σ-setup for G.** It is convenient to introduce a σ-setup $(\overline{G}, \sigma)$ for G; see [9, Definition 2.2.1]. We choose $\overline{G}$ as a simple, adjoint algebraic group over the algebraic closure $\mathbb{F}$ of the field with r elements, and σ as a suitable Steinberg morphism of $\overline{G}$, such that $G = O^{r'}(\overline{G}^{\sigma})$. If necessary, we will specify the choice of $\overline{G}$. Contrary to the usage in [9], we will in general write $\overline{H}^{\sigma}$ rather than $C_{\overline{H}}(\sigma)$ for the set of σ-fixed points of a σ-stable subgroup $\overline{H}$ of $\overline{G}$.

5.3. **The BN-pair and the Weyl group.** Here, we mainly follow [9, Chapters 1, 2]. See also [2, Subsections 8.5 and 13.5] and [3, Subsections 2.5 and 2.6].

Table 5.1. The simple groups of Lie type

Row	Names	Rank	Conditions
1	$A_{d-1}(q), \mathrm{PSL}_d(q)$	$d \geq 2$	$(d,q) \neq (2,2),(2,3),(2,4),(3,2)$
2	${}^2A_{d-1}(q), \mathrm{PSU}_d(q)$	$d \geq 3$	$(d,q) \neq (3,2),(4,2)$
3	$B_d(q), \mathrm{P}\Omega_{2d+1}(q)$	$d \geq 2$	$(d,q) \neq (2,2)$
4	$C_d(q), \mathrm{PSp}_{2d}(q)$	$d \geq 3$	q odd
5	$D_d(q), \mathrm{P}\Omega^+_{2d}(q)$	$d \geq 4$	
6	${}^2D_d(q), \mathrm{P}\Omega^-_{2d}(q)$	$d \geq 4$	
7	$G_2(q)$		$q \geq 3$
8	$F_4(q)$		
9	$E_6(q)$		
10	$E_7(q)$		
11	$E_8(q)$		
12	${}^3D_4(q)$		
13	${}^2E_6(q)$		
14	${}^2B_2(q)$		$q = 2^{2m+1} > 2$
15	${}^2G_2(q)$		$q = 3^{2m+1} > 3$
16	${}^2F_4(q)$		$q = 2^{2m+1} > 2$
17	${}^2F_4(q)'$		$q = 2$

The algebraic group $\overline{G}$ has a BN-pair $(\overline{B}, \overline{N})$, where $\overline{B}$ is a σ-stable Borel subgroup of $\overline{G}$ whose unipotent radical $\overline{U}$ is also σ-stable. Moreover, $\overline{B}$ contains a σ-stable maximal torus $\overline{T}$ such that $\overline{B} = \overline{T}\,\overline{U}$. Finally, $\overline{N} = N_{\overline{G}}(\overline{T})$. We will fix such a BN-pair and call $\overline{T}$ the standard (maximal) torus of $\overline{G}$. The pair $(\overline{T}, \overline{B})$ gives rise to the root system $\Sigma = \Sigma(\overline{G})$ of $\overline{G}$, the set $\Pi \subseteq \Sigma$ of fundamental roots and the Dynkin diagram of Σ. The root system Σ is a subset of $X(\overline{T}) = \mathrm{Hom}(\overline{T}, \mathbb{F}^*)$. For the possible Dynkin diagrams and their automorphisms see [3, Subsection 1.19]. The Weyl group of $\overline{G}$ with respect to $\overline{T}$ is denoted by $W(\overline{G})$. By definition, $W(\overline{G}) = N_{\overline{G}}(\overline{T})/\overline{T}$. Also, $W(\overline{G})$ can be identified with the Weyl group of Σ in a natural way.

For every subset $I \leq \Pi$, there is a Levi subgroup $\overline{L}_I$ of $\overline{G}$ contained in a parabolic subgroup $\overline{P}_I$, whose unipotent radical is denoted by $\overline{U}_I$; see [9, Theorem 1.13.2]. In fact, $\overline{P}_I = \overline{U}_I\overline{L}_I$, a semidirect product. The Levi subgroups $\overline{L}_I$, where I runs through the subsets of Π, are called the standard Levi subgroups of $\overline{G}$. A Levi subgroup of $\overline{G}$ is a subgroup of $\overline{G}$ which is conjugate in $\overline{N}$ to a standard Levi subgroup. The Weyl group of a standard Levi subgroup $\overline{L}_I$ is denoted by $W(\overline{L}_I)$. This is the parabolic subgroup $W(\overline{G})_I$ of $W(\overline{G})$ generated by the simple reflections corresponding to the roots in I.

We will assume that σ is in standard form relative to $\overline{B}$ and $\overline{T}$, i.e. that σ satisfies the conditions of [9, Theorem 2.2.3].

The BN-pair of $\overline{G}$ gives rise to a split BN-pair of G of characteristic r; see [3, Subsection 2.5] for the definition. Put $B = \overline{B} \cap G$, $N = N_{\overline{G}}(\overline{T}) \cap G$ and $T = \overline{T} \cap G$; see [9, Section 2.3]. We call T the standard (maximal) torus of G. We have $B = UT$ with $U := \overline{U}^{\sigma}$, the standard (maximal) unipotent subgroup of G. We also have $\overline{G}^{\sigma} = G\overline{T}^{\sigma}$ (see [9, Theorem 2.2.6(g)]), and $\overline{T}^{\sigma} \cap G = T$. In particular, $\overline{G}^{\sigma}/G \cong \overline{T}^{\sigma}/T$. Clearly, $W(\overline{G})$ is σ-stable, and we write $W(\overline{G})^{\sigma}$ for its subgroup of σ-fixpoints. Then $W(\overline{G})^{\sigma}$ is the Weyl group of the BN-pair of G, i.e. $W(\overline{G})^{\sigma} = N/T$; see [9, Theorem 2.3.4]. Since $\overline{T}$ is connected, we also have $N/T = W(\overline{G})^{\sigma} \cong N_{\overline{G}^{\sigma}}(\overline{T})/\overline{T}^{\sigma}$. Since $\overline{T}$ and $\overline{B}$ are σ-stable, the morphism σ acts on the root system Σ and fixes Π. In particular, σ determines a symmetry ι of the Dynkin diagram of Π.

A standard Levi subgroup $\overline{L}_I$ for $I \leq \Pi$ is σ-stable, if and only if I is fixed by ι. The standard Levi subgroups of G are the subgroups of the form $\overline{L}_I \cap G$, where I runs through the ι-stable subsets of Π. Let $I \subseteq \Pi$ be ι-stable. We then put $L_I := \overline{L}_I \cap G$. The Weyl group of L_I equals $W(\overline{G})_I^{\sigma}$; this is a parabolic subgroup of the Coxeter group $W(\overline{G})^{\sigma}$. The standard parabolic subgroup $\overline{P}_I$ of $\overline{G}$ is also σ-stable, and we put $P_I := \overline{P}_I \cap G$. The unipotent radical of P_I equals $U_I := \overline{U}_I \cap G = \overline{U}_I^{\sigma}$. By definition, a Levi subgroup of G is a subgroup of G which is conjugate in N to a standard Levi subgroup. This is sometimes also called a split Levi subgroup to distinguish it from the subgroups of the form $\overline{L}^{\sigma}$, where $\overline{L}$ is a σ-stable Levi subgroup of $\overline{G}$.

5.4. **Harish-Chandra theory.** We will apply Harish-Chandra theory to G; see [3, Chapters 9–11] or [6, Chapter 5]. If $P \leq G$ is a parabolic subgroup of G with Levi decomposition $P = UL$, where U denotes the unipotent radical of P, we write R_L^G and ${}^*R_L^G$ for Harish-Chandra induction from L, respectively Harish-Chandra restriction to L. It is known that these operations on $\mathbb{C}G$-mod, respectively $\mathbb{C}L$-mod, are independent of the chosen parabolic subgroup of G containing L as a Levi complement. Moreover, these operations are adjoint with respect to the standard scalar product on class function of G, respectively L.

By definition, a principal series module of G is a composition factor of $R_T^G(\mathbb{C})$, where $\mathbb{C}$ denotes the trivial $\mathbb{C}T$-module. (We use the term principal series in the narrow sense to mean the Harish-Chandra series corresponding to the cuspidal pair consisting of T and the trivial $\mathbb{C}T$-module.) The principal series modules are labeled by the irreducible characters of $W(\overline{G})^\sigma$, and we usually denote a principal series module of G by the corresponding irreducible character of $W(\overline{G})^\sigma$ or the label of this character. An irreducible $\mathbb{R}G$-module V' of odd dimension occurs in $R_T^G(\mathbb{R})$ as a composition factor, if and only if $\mathbb{C} \otimes V'$ is a principal series module; in this situation, we also call V' a principal series module.

Principal series modules are preserved by Harish-Chandra induction and restriction, in the sense that all composition factors of a Harish-Chandra induced principal series module are principal series modules; the analogous statement holds for Harish-Chandra restriction. For these statements see [6, Proposition 5.3.9].

A standard Levi subgroup L_I determines a parabolic subgroup $W(\overline{G})_I^\sigma$ of $W(\overline{G})^\sigma$. By the Howlett-Lehrer comparison theorem [11, Theorem 5.9], Harish-Chandra induction $R_{L_I}^G$ and restriction ${}^*R_{L_I}^G$ correspond to $\mathrm{Ind}_{W(\overline{G})_I^\sigma}^{W(\overline{G})^\sigma}$, respectively $\mathrm{Res}_{W(\overline{G})_I^\sigma}^{W(\overline{G})^\sigma}$. Taking $I = \emptyset$, we have $P_I = B$ and $L_I = T$. Thus, as already noted above, the composition factors of $R_T^G(\mathbb{C})$ correspond to the irreducible constituents of $\mathrm{Ind}_{\{1\}}^{W(\overline{G})^\sigma}(1)$, i.e. to the irreducible characters of $W(\overline{G})^\sigma$, and if the principal series module V' corresponds to $\vartheta \in \mathrm{Irr}(W(\overline{G})^\sigma)$, then the multiplicity of V' as a composition factor of $R_T^G(\mathbb{C})$ equals $\vartheta(1)$.

The following lemma provides the base for the applications of Harish-Chandra theory to our problem.

Lemma 5.4.1. *Let G be as in* Subsection 5.1. *Let (V, n, ν) be as in* Notation 4.1.1 *and let χ denote the character of V.*

Suppose that there is a ν-stable, proper parabolic subgroup P with a ν-stable Levi complement L and there is $\psi \in \mathrm{Irr}(L)$ real, of odd degree, ν-invariant and non-trivial, such that χ occurs with odd multiplicity in $R_L^G(\psi)$. Then if L has the $E1$-property, so does (G, V, n).

Proof. As in Subsection 4.2, we will identify G with is image in $\mathrm{GL}(V)$. Then P and L are n-invariant by assumption.

By hypothesis, $\langle R_L^G(\psi), \chi\rangle = \langle \psi, {}^*R_L^G(\chi)\rangle$ is odd. As ψ is real and of odd degree, Lemma 2.5.1 implies the existence of an irreducible $\mathbb{R}P$-submodule S of $\mathrm{Res}_P^G(V)$ such that the unipotent radical U of P acts trivially on S, and the character of S, viewed as an $\mathbb{R}L$-module, equals ψ. Moreover, the S-homogeneous component V_1 of $\mathrm{Res}_P^G(V)$ has odd dimension.

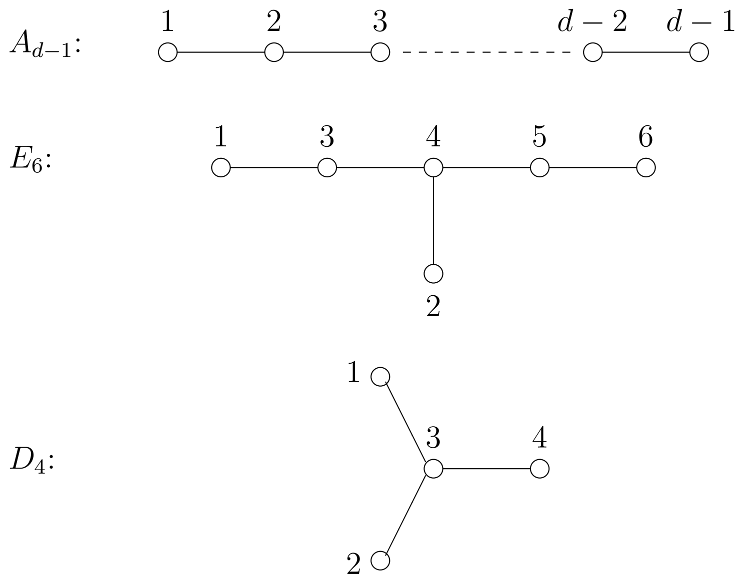
Since ψ is ν-invariant, nS is isomorphic to S as $\mathbb{R}L$-module. As U is n-invariant, U acts trivially on nS. Thus $nS \cong S$ as $\mathbb{R}P$-modules and hence $nS \leq V_1$. It follows that V_1 is n-invariant. As S is non-trivial and L has the $E1$-property, $(L, \mathrm{Res}_L^P(V_1))$ has the $E1$-property by Lemma 3.1.2. But then (P, V_1) also has the $E1$-property. Lemma 3.1.3 completes our proof. □

5.5. **Automorphisms.** The automorphisms of G are described in [9, Section 2.5]. According to [9, Theorem 2.5.12], we have $\mathrm{Aut}(G) = \mathrm{Inndiag}(G) \rtimes (\Gamma_G \Phi_G)$, where $\mathrm{Inndiag}(G)$ consists of the automorphisms of G induced by conjugation with elements of $\overline{G}^\sigma$, so that $\mathrm{Inndiag}(G) \cong \mathrm{Inn}(\overline{G}^\sigma) \cong \overline{G}^\sigma$.

For some small cases when r is odd, but more notably when $r = 2$, we will need a more precise description of $\mathrm{Aut}(G)$. For simplicity, we only consider the groups of Lines 1–12 of Table 5.1, and if G is one of $B_2(q)$, $G_2(q)$ or $F_4(q)$, we assume that $r \neq 2, 3, 2$ in the respective cases. The cases not treated here will be discussed when they occur. With these restrictions, we have $\Gamma_G \Phi_G = \Gamma_G \times \Phi_G$, and Γ_G is isomorphic to the group of symmetries of the Dynkin diagram of $\overline{G}$, if G is as in one of the Lines 1, 5 or 9 of Table 5.1, i.e. if G is one of the groups $\mathrm{PSL}_d(q)$, $\mathrm{P}\Omega_{2d}^+(q)$ or $E_6(q)$. Moreover, $|\Gamma_G| = 2$ in these cases, except for $G = \mathrm{P}\Omega_4^+(q)$, in which case Γ_G is isomorphic to the symmetric group on three letters; see the corresponding Dynkin diagrams displayed in Figure 5.1. In all other cases, Γ_G is trivial (under the restrictions on G imposed at the beginning of this paragraph). Also, Φ_G is cyclic, and we have $|\Phi_G| = f$, if G is untwisted, and $|\Phi_G| = 2f$ if G is twisted, unless $G = {}^3D_4(q)$, in which case $|\Phi_G| = 3f$.

Let us recall further terminology concerning automorphisms of G, following [9, Definition 2.5.13]. First, we choose a particular generator of Φ_G and a particular element of Γ_G. Let $\varphi := \varphi_r$ denote the standard Frobenius endomorphism of $\overline{G}$ introduced in [9, Theorem 1.15.4(a)], and put $\Phi_{\overline{G}} := \langle \varphi \rangle$ as in [9, Definition 1.15.5(a)]. The restriction of φ to G, also denoted by φ, is our preferred generator of Φ_G. Suppose that ι is a non-trivial symmetry of the Dynkin diagram of $\overline{G}$. We then denote, by the same letter, the element of $\Gamma_{\overline{G}}$ introduced in [9, Theorem 1.15.2(a), Definition 1.15.5(e)], as well as the restriction of the latter to G. Such a ι will be called a standard graph automorphism of $\overline{G}$, respectively G. Notice that the groups $\overline{B}^\sigma$, $\overline{T}^\sigma$ and $\overline{N}^\sigma$, and thus also, B, T and N are fixed, up to inner automorphisms of $\overline{G}^\sigma$, by $\mathrm{Aut}(G)$. Finally, we may choose notation so that $\sigma = \varphi^f$, if G is untwisted, and $\sigma = \iota \circ \varphi^f$ for some $1 \neq \iota \in \Gamma_{\overline{G}}$, otherwise.

FIGURE 5.1. The Dynkin diagrams of A_{d-1}, E_6 and D_4



Let $\alpha \in \mathrm{Aut}(G)$. Then α is of the form

$$\alpha = \mathrm{ad}_g \circ \iota \circ \varphi^b$$

with $g \in \overline{G}^\sigma$, $\iota \in \Gamma_G$ and $0 \leq b < |\Phi_G|$. Then α is an *inner-diagonal automorphism* of G if and only if $\iota = 1$ and $b = 0$. Suppose now that Γ_G is non-trivial. If $g = 1$ and $\iota = 1$, any $\mathrm{Aut}(G)$-conjugate of α is a *field automorphism* of G. If $\iota \neq 1$ and $b = 0$, any $\mathrm{Aut}(G)$-conjugate of α is a *graph automorphism* of G. Suppose now that Γ_G is trivial. If $g = 1$, any $\mathrm{Aut}(G)$-conjugate of α is a *field automorphism* of G, provided $|\alpha|$ is odd if G is twisted and $G \neq {}^3D_4(q)$, respectively $3 \nmid |\alpha|$ if $G = {}^3D_4(q)$. Finally, if G is twisted, α is a *graph automorphism* of G, if $|\varphi^b|$ is even, respectively divisible by 3 in case $G = {}^3D_4(q)$.

6. Simple groups of Lie type of odd characteristic

In this section we let G be one of the simple groups of Lie type listed in Table 5.1, where we assume that $q = r^f$ is odd. We use the terminology and the concepts introduced in Section 5. We also assume Notation 4.1.1. In particular, V is a non-trivial irreducible $\mathbb{R}G$-module of odd dimension, ρ is the corresponding homomorphism $G \to \mathrm{GL}(V)$, and n is an element of $\mathrm{GL}(V)$ of finite order normalizing $\rho(G)$. Moreover, ν denotes the automorphism of G induced by n. We say that V

is the Steinberg module of G, if $\mathbb{C} \otimes V$ affords the Steinberg representation of G. The main goal in this section is to show that the groups considered here are not minimal counterexamples to Theorem 1.1.5.

6.1. **Some special cases.** We first deal with the Steinberg modules in some groups of small rank.

Lemma 6.1.1. *Let*

$$G \in \{\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), G_2(3^f), {}^2G_2(3^{2m+1})\}$$

and suppose that V is the Steinberg module of G. Then (G, V) has the $E1$-property.

Proof. For some of the proofs below we rely on Lemma 4.3.3. For this, we need to estimate the order of subgroups $C_G(\beta)$ for certain automorphisms $\beta \in \mathrm{Aut}(G)$, for which we cite [9, Propositions 4.9.1, 4.9.2]. However, [9, Propositions 4.9.1] only gives $O^{r'}(C_G(\beta))$. Nevertheless, in combination with the tables of maximal subgroups determined in [1], we obtain the desired bounds.

Let us begin with $G = \mathrm{PSL}_2(q)$, with $q = r^f > 3$ odd. The elements of G are written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where the square brackets indicate the image in $\mathrm{PSL}_2(q)$ of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(q).$$

Let us define two automorphisms of G. First, choose $\zeta \in \mathbb{F}_q$ of order $(q-1)_2$. Let δ denote conjugation by the diagonal matrix

$$\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix},$$

i.e.

$$\delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & \zeta b \\ \zeta^{-1} c & d \end{bmatrix}.$$

Next, $\varphi := \varphi_r$ denotes the standard Frobenius morphism of G of order f, i.e.

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^r & b^r \\ c^r & d^r \end{bmatrix}.$$

Then $\varphi \circ \delta = \delta^r \circ \varphi$. By the results reported in Subsection 5.5, every automorphism of G is of the form $\mathrm{ad}_g \circ \delta^k \circ \varphi^l$, for some $g \in G$ and some integers k and l.

Let B and T denote the images of the subgroups of upper triangular matrices, respectively diagonal matrices, of $\mathrm{SL}_2(q)$ in $\mathrm{PSL}_2(q) = G$. We take B and T as our standard Borel subgroup, respectively standard maximal torus. Then B and T are invariant under δ and φ. A set of representatives for the left cosets of B in G is given by $\mathcal{R} = \mathcal{R}_q \cup \mathcal{R}_\infty$ with

$$\mathcal{R}_q := \left\{ \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \mid y \in \mathbb{F}_q \right\}$$

and

$$\mathcal{R}_\infty := \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

(These elements may be identified with the projective line over $\mathbb{F}_q$.)

By replacing n with a suitable element of its coset Gn, we may assume that $\nu = \delta^k \circ \varphi^l$ for some integers k and l. Then ν stabilizes B and permutes the elements of $\mathcal{R}_q$. Now identify G with the image of ρ in $\mathrm{GL}(V)$ and adopt the notation introduced in Subsection 4.2. In particular, $A = \langle G, n\rangle$; see Definition 4.2.1.

Suppose now that we are in Case 1 of Subsection 4.2. By Lemma 4.2.4, there is an isomorphism $A \to G' = \langle \mathrm{Inn}(G), \nu\rangle$, sending n to ν. In particular, $\langle n\rangle \cong \langle \nu\rangle$. As B is ν-invariant, $\langle B, n\rangle$ is an ad_n-invariant subgroup of A. Since B is a maximal subgroup of G, we have $\langle B, n\rangle \cap G = B$. Thus $\mathcal{R}$ is a set of representatives for the left cosets of $\langle B, n\rangle$ in A. Notice that

$$nx\langle B, n\rangle = nx\langle B, n\rangle n^{-1} = nxn^{-1}\langle B, n\rangle$$

for $x \in \mathcal{R}$. As $nxn^{-1} = \nu(x)$ for $x \in G$, left multiplication by n fixes the coset with representative in $\mathcal{R}_\infty$, and the action of $\langle n\rangle$ by left multiplication on the set of $\langle B, n\rangle$-cosets corresponding to $\mathcal{R}_q$ is equivalent to the action of $\langle \nu\rangle$ on $\mathcal{R}_q$. Thus, left multiplication by n also fixes the coset corresponding to the trivial element in $\mathcal{R}_q$.

Consider the character $\psi := \mathrm{Ind}_{\langle B,n\rangle}^{A}(1_{\langle B,n\rangle})$. Thus ψ is the permutation character of A acting by left multiplication on the cosets of $\langle B, n\rangle$. By Mackey's theorem, $\mathrm{Res}_G^A(\psi) = \mathrm{Ind}_B^G(1_B) = 1_G + \mathrm{St}_G$, where St_G denotes the character of the Steinberg module of G. Hence $\psi = 1_A + \mathrm{St}_A$, where St_A is a rational valued extension of St_G to A. As left multiplication by n fixes two $\langle B, n\rangle$-cosets, $\psi(n) \geq 2$ and so $\mathrm{St}_A(n)$ is a positive integer. The other extensions of St_G to A are of the form $\tilde{\lambda} \cdot \mathrm{St}_A$, where

$\tilde{\lambda}$ is the inflation of some $\lambda \in \mathrm{Irr}(A/G)$ to A. As A/G is cyclic, the only real extensions of St_G to A are of the form $\tilde{\lambda} \cdot \mathrm{St}_A$, where $\lambda \in \mathrm{Irr}(A/G)$ with $\lambda^2 = 1$. Viewing $\tilde{\lambda}$ as a character of $\langle B, n\rangle$ by restriction, we get $\mathrm{Ind}_{\langle B,n\rangle}^A(\tilde{\lambda}) = \tilde{\lambda} + \tilde{\lambda} \cdot \mathrm{St}_A$.

Let $\lambda \in \mathrm{Irr}(A/G)$ with $\lambda^2 = 1$ such that $\chi' := \tilde{\lambda} \cdot \mathrm{St}_A$ is the character of A afforded by V, where, again, $\tilde{\lambda}$ denotes the inflation of λ to a character of A. To show that n is represented by a matrix with eigenvalue 1, we have to show that $\mathrm{Res}_{\langle n\rangle}^A(\chi')$ contains a trivial constituent. Now, once more by Mackey's theorem, we have

$$\mathrm{Res}_{\langle n\rangle}^A(\chi') = -\mathrm{Res}_{\langle n\rangle}^A(\tilde{\lambda}) + \sum_{z\in\mathcal{R}'} \mathrm{Ind}_{\langle n\rangle_z}^{\langle n\rangle}(\tilde{\lambda}_z), \tag{6.1}$$

where $\mathcal{R}' \subseteq \mathcal{R}$ is a set of representatives for the $\langle n\rangle$-$\langle B, n\rangle$-double cosets of A, $\langle n\rangle_z := {}^z\langle B, n\rangle \cap \langle n\rangle$, and $\tilde{\lambda}_z := \mathrm{Res}_{\langle n\rangle_z}^{{}^z\langle B,n\rangle}({}^z\tilde{\lambda})$. If $\tilde{\lambda}$ is the trivial character, every summand $\mathrm{Ind}_{\langle n\rangle_z}^{\langle n\rangle}(\tilde{\lambda}_z)$ of (6.1) contains a trivial constituent. In this case $\mathrm{Res}_{\langle n\rangle}^A(\chi')$ contains the trivial character, as $\langle n\rangle\langle B, n\rangle = \langle B, n\rangle \lneq A$, and thus $|\mathcal{R}'| \geq 2$. Suppose that $\tilde{\lambda} \neq 1_A$. Then $\mathrm{Res}_{\langle n\rangle}^A(\tilde{\lambda}) \neq 1_{\langle n\rangle}$, as $\tilde{\lambda}(n) = -1$. Thus, in order to show that $\mathrm{Res}_{\langle n\rangle}^A(\chi')$ contains a trivial character, it suffices to show that there is $z \in \mathcal{R}'$ such that $\langle n\rangle_z$ is the trivial group.

Observe that $\langle n\rangle_z$ is the stabilizer in $\langle n\rangle$ of the coset $z\langle B, n\rangle$. Thus $\langle n\rangle_z$ is trivial, if and only if $z\langle B, n\rangle$ lies in a regular $\langle n\rangle$-orbit. This can only be the case if $z \in \mathcal{R}_q$, and we have to show that $\mathcal{R}_q$ contains a regular $\langle \nu\rangle$-orbit. The non-trivial elements of $\langle \nu\rangle$ are of the form $\delta^s \circ \varphi^t$ for integers s and t such that $\delta^s \neq 1$ or $\varphi^t \neq 1$. For $y \in \mathbb{F}_q$, let

$$z(y) := \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$

Then $\delta^s \circ \varphi^t(z(y)) = z(y')$ with $y' = \zeta^{-s}y^{r^t}$. Thus $\delta^s \circ \varphi^t$ fixes $z(y)$ if and only if $y^{r^t-1} = \zeta^s$. If r is not a Mersenne prime or if $f > 2$, choose a primitive prime divisor ℓ of $q-1 = r^f-1$, i.e. $\ell \mid r^f-1$ and $\ell \nmid r^e-1$ for all $1 \leq e < f$; see [12, Theorem IX.8.3]. Let $y \in \mathbb{F}_q^*$ denote an element of order ℓ. Then $y^{r^t-1} \neq 1$, unless $t = f$. As ζ has even order, it follows that $z(y)$ is only fixed by the trivial element of $\langle \nu\rangle$.

It remains to consider the case when r is a Mersenne prime and $f \leq 2$. Then $\mathrm{Out}(G)$ is an elementary abelian 2-group of order 4, respectively 2. In particular, $\nu^2 \in \mathrm{Inn}(G)$. As we are in Case 1, this implies that $n^2 \in G$. As G is in the kernel of $\tilde{\lambda}$ by definition, n^2 is in the kernel of ${}^z\tilde{\lambda}$ for all $z \in \mathcal{R}'$. Since $\langle n\rangle \cong \langle \nu\rangle$ is a cyclic 2-group, every proper subgroup of $\langle n \rangle$ is contained in $\langle n^2 \rangle$. Thus if $\langle n \rangle_z \lneq \langle n \rangle$ for some $z \in \mathcal{R}'$, the corresponding summand of (6.1) contains a trivial constituent. Since $\langle n \rangle$ is not a normal subgroup of A, it does not act trivially on the set of left cosets of $\langle B, n \rangle$ in A. Thus there is $z \in \mathcal{R}$ such that $z\langle B, n \rangle$ is not fixed by $\langle n \rangle$. We choose $\mathcal{R}'$ such that $z \in \mathcal{R}'$. Then $\langle n \rangle_z \lneq \langle n \rangle$ and we are done.

Now assume that we are in Case 2 of Subsection 4.2. With the notation introduced there, we have $n = -n_1$ for some $n_1 \in A_1$. By Lemma 4.2.2 we may assume that $|n_1|$ is odd. Lemma 4.2.4 implies that $|\nu|$ is odd, and so ν is a field automorphism of G. More precisely, consider $K := \langle \delta, \varphi \rangle \leq \mathrm{Aut}(G)$. This is a semidirect product $\langle \delta \rangle \rtimes \Phi_G$ with $\Phi_G = \langle \varphi \rangle$, and Φ_G contains a $2'$-Hall subgroup of K. Hence every element of odd order of K is conjugate in K to an element of Φ_G. In particular, $C_G(\nu')$ for $\nu' \in \langle \nu \rangle$ is isomorphic to $C_G(\varphi')$ for some $\varphi' \in \Phi_G$. Thus $C_G(\nu)$ contains a subgroup isomorphic to $\mathrm{PSL}_2(p)$.

Let g be an involution in $C_G(\nu)$, and put $\alpha := \mathrm{ad}_g \circ \nu$. Now make use of Lemma 4.3.3. If p is an odd prime diving $|\alpha|$, then $C_G(\alpha_{(p)})$ is isomorphic to $\mathrm{PSL}_2(q_0)$, where $q_0^p = q$; see [9, Proposition 4.9.1] and [1, Table 8.1]. In particular, $|C_G(\alpha_{(p)})| \leq q_0(q_0^2 - 1) \leq q$, as p is odd. If $p = 2$, then $|C_G(\alpha_{(p)})| = |C_G(g)| \leq q + 1$. If $f = 1$, then ν is trivial and we are done with Corollary 4.3.2. We will thus assume that $f > 1$, in which case $f \geq 3$, as $|\nu| = |n_1|$ is odd. This implies that $\dim(V) = q > (2f - 1)(q + 1)^{1/2}$, hence (G, V, n) has the $E1$-property by Lemma 4.3.3, as $|\alpha| \leq 2f$.

Let us now consider the case of $G = \mathrm{PSL}_3(q)$. The two standard Levi subgroups of G of type A_1 are conjugate in G. (This is not true for the corresponding standard parabolic subgroups.) Let L be one of these. As $\nu(L)$ is G-conjugate to a standard Levi subgroup, there is $g \in G$ such that $\mathrm{ad}_g \circ \nu$ fixes L. Again, identify G with its image $\rho(G)$ in $\mathrm{GL}(V)$. Replacing n by gn, we may assume that n normalizes L. Let St_G and St_L denote the Steinberg characters of G and L, respectively. Since St_L is invariant under every automorphism of L by [16, Theorem 2.5], the homogeneous component V_1 of $\mathrm{Res}^G_L(V)$ corresponding to St_L is n-invariant. To show that (G, V) has the $E1$-property, it suffices to show that (L, V_1) has this property; see Lemma 3.1.3. Using [19] and [20] or [3, Theorem 6.5.9], one checks that

$$\langle \mathrm{Res}^G_L(\mathrm{St}_G), \mathrm{St}_L \rangle = \begin{cases} 3, & \text{if } 3 \nmid q - 1 \\ 5, & \text{if } 3 \mid q - 1 \end{cases}$$

By Lemma 3.1.2, it suffices to show that a module affording St_L has the $E1$-property. Now St_L restricts to the Steinberg character $\mathrm{St}_{L'}$ of

$L' := [L, L] \cong \mathrm{SL}_2(q)$. By what we have already shown, a module affording $\mathrm{St}_{L'}$ has the $E1$-property. Thus a module affording St_L has the $E1$-property by Corollary 3.1.4. This completes the proof in case $G = \mathrm{PSL}_3(q)$.

We now consider the case $G = \mathrm{PSU}_3(q)$, where, once more, we are going to apply Lemma 4.3.3. Here, $\mathrm{Aut}(G) = \mathrm{Inndiag}(G) \rtimes \Phi_G$ with $\mathrm{Inndiag}(G) \cong \mathrm{PGU}_3(q)$ and Φ_G cyclic of order $2f$; see Subsection 5.5. We assume that a generator φ of Φ_G is the image of the standard Frobenius morphism of $\mathrm{SU}_3(q)$ which raises every entry of an element of $\mathrm{SU}_3(q)$ to its rth power. By pre-multiplying n with a suitable element of G, we may assume that $\nu = \mathrm{ad}_t \circ \mu$ with $\mu \in \Phi_G$, and where $t \in \mathrm{PGU}_3(q)$ is represented by the matrix $\hat{t} = \mathrm{diag}(1, \zeta, 1) \in \mathrm{GU}_3(q)$ with $\zeta \in \mathbb{F}_{q^2}^*$ of order dividing $(q+1)_3$. Notice that t is inverted by φ^f. If ν has even order, put $g := 1$. If ν has odd order, let $g \in G$ denote the image of $\mathrm{diag}(-1, 1, -1) \in \mathrm{SU}_3(q)$ in G. Let $\alpha := \mathrm{ad}_g \circ \nu$. Then α has even order.

Suppose first that $f \geq 2$. It follows from [9, Propositions 4.9.1, 4.9.2] in conjunction with [1, Tables 8.5, 8.6] that $|C_G(\alpha_p)| \leq |\mathrm{GU}_2(q)| \leq q^4 + q^3$ for all primes p dividing $|\alpha|$. Also, $|\alpha| \leq 2f(q+1)_3$, as ν centralizes g and $\langle \mu \rangle$ normalizes $\langle t \rangle$. As r is odd and $f \geq 2$, there exists a primitive prime divisor ℓ of $r^{2f} - 1$; see [12, Theorem IX.8.3]. That is, ℓ is a prime with $\ell \mid r^{2f} - 1$ but $\ell \nmid r^j - 1$ for all $1 \leq j < 2f$. In particular, $\ell \mid q+1$, and $2f \mid \ell - 1$. The latter implies that $\ell > 3$. As q is odd, $(q+1)_3 \mid (q+1)/(2\ell)$, and so $|\alpha| \leq 2f(q+1)_3 < (q+1)/2$. We conclude that

$$(|\alpha| - 1)^2(q^4 + q^3) < \frac{(q-1)^2}{4}(q^4 + q^3) = \frac{1}{4}(q^6 - q^5 - q^4 + q^3) < q^6.$$

Taking square roots we obtain

$$\dim(V) = q^3 > (|\alpha| - 1)|C_G(\alpha_p)^{1/2}$$

for all primes p dividing $|\alpha|$. Lemma 4.3.3 implies that (G, V, n) has the $E1$-property. Suppose now that $f = 1$. Then $\Phi_G = \langle \varphi \rangle$ with $|\varphi| = 2$. If $t = 1$ or if $t \neq 1$ and $\nu = \mathrm{ad}_t \circ \varphi$, then $|\alpha| = 2$, hence (G, V, n) has the $E1$-property by Corollary 4.3.2. It remains to consider the case $t \neq 1$ and $\mu = \mathrm{id}_G$. Then $G' = \langle \mathrm{Inn}(G), \nu \rangle \cong \mathrm{PGU}_3(q)$, and $\alpha = \mathrm{ad}_{gt}$. Let χ' denote the character of G' afforded by V according to Remark 4.2.5. For simplicity, identify G' with $\mathrm{PGU}_3(q)$ and α with gt. Then χ' is the Steinberg character of $\mathrm{PGU}_3(q)$, as the latter is the only real extension of the Steinberg character of G. Observe that $C_{G'}(y)$ is isomorphic to $\mathrm{GU}_2(q)$ for every non-trivial $y \in \langle gt \rangle$. Hence $\chi'(y) = -q$ for all such y; see [3, Theorem 6.5.9]. As $\chi'(1) = q^3$, this easily implies that

$\mathrm{Res}^{G'}_{\langle gt\rangle}(\chi')$ contains each of the two real irreducible characters of $\langle gt\rangle$ as constituents. We conclude from Lemma 4.3.1 that (G, V, n) has the $E1$-property.

We now consider the case of $G = G_2(q)$ with q a power of 3. Here, $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \Gamma_G\Phi_G$, where $\Gamma_G\Phi_G$ is cyclic of order $2f$; see [9, Theorem 2.5.12(a),(e)]. We may assume that $\nu \in \Gamma_G\Phi_G$. In particular, ν is a field or a graph-field automorphism of G in the notation of [9, Definition 2.5.13]. Thus $C_G(\nu) \cong G_2(3^{f'})$ for some $f' \mid f$ or $C_G(\nu) \cong {}^2G_2(3^{2m+1})$ for some positive integer m. In particular, $C_G(\nu)$ contains an involution. If ν has even order, let $g := 1$, and if ν has odd order, let g denote an involution in $C_G(\nu)$. Put $\alpha := \mathrm{ad}_g \circ \nu$. Then $|\alpha|$ is even, and if p is a prime dividing $|\alpha|$, then $\alpha_{(p)} \in \Gamma_G\Phi_G$, or $|g| = 2 = p$ and $\alpha_{(p)} = \mathrm{ad}_g$. If $\alpha_{(p)} \in \Gamma_G\Phi_G$, then $|C_G(\alpha_{(p)})| \leq |G_2(q_0)|$ with $q_0^p = q$ or $|C_G(\alpha_{(p)})| \leq |{}^2G_2(q)|$; see [9, Proposition 4.9.1(a)] and [1, Table 8.42]. In this case, $|C_G(\alpha_{(p)})| \leq q^7$. If $g \neq 1$, then $|C_G(\alpha_{(2)})| \leq |\mathrm{SL}_2(q)|^2 \leq q^6$; see [14]. Hence

$$(|\alpha| - 1)|C_G(\alpha_{(p)})|^{1/2} < q^6 = \dim(V)$$

for all primes p dividing $|\alpha|$. Lemma 4.3.3 implies our claim.

Finally, we deal with the case $G = {}^2G_2(q)$ with $q = 3^{2m+1}$ for some $m \geq 1$. Here, $\mathrm{Aut}(G)$ is a split extension of G with the group Φ_G of field automorphisms, the latter being cyclic of order $2m + 1$; see [9, Theorem 2.5.12]. We may thus assume that $\nu \in \Phi_G$, so that $|\nu|$ is odd. Then $C_G(\nu)$ is isomorphic to ${}^2G_2(q_0)$ for some root q_0 of q; see [9, Proposition 4.9.1(a)] and [1, Table 8.43]. In particular, ν centralizes some involution $g \in G$. Put $\alpha := \mathrm{ad}_g \circ \nu \in \Phi_G$. Then α has even order dividing $2(2m + 1)$. Let p be an odd prime dividing $|\alpha|$. Then $C_G(\alpha_{(p)}) \cong {}^2G_2(q_0)$ with $q = q_0^p$. Also, $|C_G(\alpha_{(2)})| = q(q^2 - 1)$; see [23, p. 62, 63]. Hence $|C_G(\alpha_{(2)})| = q_0^p(q_0^{2p} - 1) \geq q_0^3(q_0^6 - 1) \geq q_0^3(q_0 - 1)(q_0^3 + 1) = |C_G(\alpha_{(p)})|$. Now $\dim(V) = q^3 > q[q(q^2 - 1)]^{1/2} > (4m + 1)[q(q^2 - 1)]^{1/2} \geq (|\alpha| - 1)|C_G(\alpha_{(p)})|^{1/2}$ for all primes p dividing $|\alpha|$. The claim follows from Lemma 4.3.3. □

6.2. **Reductions.** We now work towards the main reductions in the present case.

Lemma 6.2.1. *Let $P \leq G$ be a parabolic subgroup of G with Levi decomposition $P = UL$, where U denotes the unipotent radical of P. Let S be an irreducible constituent of $\mathrm{Res}^G_P(V)$ of odd dimension. Then U acts trivially on S, i.e. S is a constituent of ${}^*R^G_L(V)$.*

Proof. As S is absolutely irreducible by Lemma 2.5.1, the claim follows from Lemma 2.5.2. □

Proposition 6.2.2. *Let $B = UT$ be the Borel subgroup of G. Then there is an irreducible $\mathbb{R}T$-module S of dimension 1 such that V occurs in $R_T^G(S)$ with odd multiplicity. If λ denotes the character of S, then $\lambda^2 = 1_T$ in the character group of T.*

Proof. Let V_1 denote a homogeneous component of $\mathrm{Res}_B^G(V)$ of odd dimension. Then U acts trivially on V_1 by Lemma 6.2.1, and thus V_1 is a homogeneous component of ${}^*R_T^G(V)$. Let $S \leq V_1$ denote an irreducible $\mathbb{R}B$-submodule. Then S is also irreducible as $\mathbb{R}T$-module. By adjointness of Harish-Chandra induction and Harish-Chandra restriction, V is a constituent of $R_T^G(S) = \mathrm{Ind}_B^G(S)$ with odd multiplicity.

As $\dim_{\mathbb{R}}(S)$ is odd, S is an absolutely irreducible $\mathbb{R}T$-module by Lemma 2.5.1, so that $\dim_{\mathbb{R}}(S) = \lambda(1) = 1$. The fact that λ is real implies that $\lambda^2 = 1_T$. This completes the proof. □

If $\lambda = 1_T$ in the notation of Proposition 6.2.2, then V is a principal series $\mathbb{R}G$-module. In this case we get a further reduction.

Lemma 6.2.3. *Suppose that V is a principal series $\mathbb{R}G$-module. Let P denote a parabolic subgroup of G with Levi decomposition $P = UL$. Suppose that there is $x \in G$ such that $\nu(U) = x^{-1}Ux$ and $\nu(L) = x^{-1}Lx$. Suppose also that the automorphisms of L fix the principal series characters of L and that L has the $E1$-property.*

Suppose that the multiplicity of V as a direct summand in $R_L^G(\mathbb{R})$ is even (including multiplicity 0), where $\mathbb{R}$ is the trivial $\mathbb{R}L$-module. Then (G, V, n) has the $E1$-property.

Proof. By replacing n with $\rho(x)n$, we may assume that ν fixes U and L.

Let V_1 denote a homogeneous component of $\mathrm{Res}_P^G(V)$ of odd dimension. Then U acts trivially on V_1 by Lemma 6.2.1. Let $S \leq V_1$ be an irreducible $\mathbb{R}P$-submodule of V_1. Then S is also irreducible as $\mathbb{R}L$-module. By adjointness of Harish-Chandra induction and Harish-Chandra restriction, V is a constituent of $R_L^G(S) = \mathrm{Ind}_P^G(S)$ with odd multiplicity. Let χ and ψ denote the characters of V, respectively S, the latter viewed as an $\mathbb{R}L$-module. Then ψ is real and $\langle R_L^G(\psi), \chi\rangle$ is odd.

By hypothesis, ψ is not the trivial character of L. As χ is a principal series character by hypothesis, so is ψ by the remarks in Subsection 5.4. In particular, ψ is ν-invariant by assumption. The assertion follows from Lemma 5.4.1. □

Corollary 6.2.4. *Suppose that V is a principal series module which corresponds to an irreducible character of $W(\overline{G}^{\sigma})$ of even degree. Then (G, V, n) has the $E1$-property.*

Proof. If V corresponds to $\vartheta \in \mathrm{Irr}(W(\overline{G}^{\sigma}))$, then the multiplicity of V as a direct summand in $R_T^G(\mathbb{R})$ equals $\vartheta(1)$; see Subsection 5.4. The claim follows from Lemma 6.2.3 applied to $P = B$ and $L = T$. □

6.3. **Non-principal series representations.** Here, we consider the case that V is not in the principal series. In the notation of Proposition 6.2.2, which we keep throughout this subsection, this means that $\lambda \neq 1_T$. Recall that $\mathrm{Aut}(G) = \mathrm{Inndiag}(G) \rtimes (\Gamma_G\Phi_G)$, with $\mathrm{Inndiag}(G) = \{\mathrm{ad}_h \mid h \in \overline{G}^{\sigma}\}$; see Subsection 5.5. Thus $\nu = \mathrm{ad}_h \circ \mu$ for some $h \in \overline{G}^{\sigma}$ and some $\mu \in \Gamma_G\Phi_G$. Since $\overline{G}^{\sigma} = G\overline{T}^{\sigma}$ (see Subsection 5.3), there is $g \in G$ such that $t := gh \in \overline{T}^{\sigma}$. By replacing n with $\rho(g)n$, we may and will thus assume that $\nu = \mathrm{ad}_t \circ \mu$. In particular, ad_t centralizes T, and B and T are ν-invariant.

Lemma 6.3.1. *Let $\alpha \in \mathrm{Aut}(G)$ fix T and act on T by powering its elements (i.e. there is an integer m such that $\alpha(s) = s^m$ for $s \in T$). Then λ is α-invariant.*

Proof. Notice that λ is uniquely determined by $\mathrm{Ker}(\lambda)$, as $\lambda^2 = 1_T$. By hypothesis, α fixes every subgroup of T and thus α fixes λ. □

Lemma 6.3.2. *The triple (G, V, n) has the $E1$-property under any of the following conditions.*

(a) *Some N-conjugate of λ is ν-invariant.*
(b) *The torus T is cyclic.*
(c) *The automorphism μ acts on T by powering its elements.*

Proof. (a) Observe that λ is cuspidal. If λ' is an N-conjugate of λ, then $R_T^G(\lambda) = R_T^G(\lambda')$; see [3, Proposition 8.2.7(ii)]. If, in addition, λ' is ν-invariant, (G, V, n) has the $E1$-property by Lemma 5.4.1, as T has the $E1$-property by Corollary 3.2.4.

(b) Since T is cyclic, $\lambda \in \mathrm{Irr}(T)$ is uniquely determined by $\lambda^2 = 1_T \neq \lambda$. Thus λ is ν-invariant and the claim follows from (a).

(c) This follows from Lemma 6.3.1 and (a). □

Corollary 6.3.3. *Suppose that G is not one of the groups in rows* 1, 5 *or* 9 *of* Table 5.1. *Then (G, V, n) has the $E1$-property.*

Proof. If $G = {}^2G_2(3^{2m+1})$ for some $m \geq 1$, then T is cyclic. In the other cases, $\mathrm{Aut}(G) = \mathrm{Inndiag}(G) \rtimes \Phi_G$, and the elements of Φ_G act on T by powering its elements. Our assertion thus follows from Lemma 6.3.2(b),(c). □

In the proofs of Propositions 6.3.4–6.3.6 below, we let $\kappa\colon \mathbb{F}_q^* \to \mathbb{C}^*$ denote the unique irreducible character of order 2.

Proposition 6.3.4. *Suppose that $G = \mathrm{PSL}_d(q)$ for some $d \geq 2$ and some prime power q. Then (G, V, n) has the $E1$-property.*

Proof. Let us work with $\tilde{G} := \mathrm{SL}_d(q)$, and view the characters of G as characters of $\tilde{G}$ via inflation. Let $\tilde{T}$ and $\tilde{N}$ denote the inverse images of T, respectively N, under the canonical epimorphism $\tilde{G} \to G$. Then $\tilde{T}$ is the standard torus of $\tilde{G}$, consisting of the diagonal matrices of determinant 1. For $1 \leq i \leq d$, let $\lambda_i \colon \tilde{T} \to \mathbb{C}^*, \mathrm{diag}(t_1, \ldots, t_d) \mapsto \kappa(t_i)$. Then every irreducible character of $\tilde{T}$ is a product of some λ_is. For $I \subset \{1, \ldots, d\}$ write $\lambda_I := \prod_{i \in I} \lambda_i$. Notice that $\lambda_I = \lambda_{I'}$ with $I' = \{1, \ldots, d\} \setminus I$, and that $\lambda_{\{1,\ldots,d\}} = 1_{\tilde{T}}$. Notice also that for $I, J \subseteq \{1, \ldots, d\}$, the characters λ_I and λ_J are conjugate by an element of $\tilde{N}$, if and only if $|I| = |J|$ or $|I| = d - |J|$. Finally, if ι denotes the standard graph automorphism of $\tilde{G}$, then ${}^{\iota}\lambda_i = \lambda_{d-i+1}$, for $1 \leq i \leq d$.

Let $I \leq \{1, \ldots, d\}$ be such that $\lambda = \lambda_I$. By the remarks in the previous paragraph, the $\tilde{N}$-orbit of λ contains a ι-stable element, if either d is odd, or d and $|I|$ are even. Then the N-orbit of λ contains a ν-stable element, and we are done by Lemma 6.3.2(a).

Suppose then that d is even and $|I|$ is odd. If $d = 2$, then T is cyclic and our claim follows from Lemma 6.3.2(b). Suppose then that $d \geq 4$. Let $\tilde{L}$ denote the ι-invariant Levi subgroup of type A_1 corresponding to the central node of the Dynkin diagram of $\tilde{G}$; see Figure 5.1. Assume, without loss of generality, that $|I \cap \{d/2, d/2 + 1\}| = 1$, and that $I \setminus \{d/2, d/2 + 1\}$ is invariant under reversing the elements. Then the orbit of λ under $\langle \iota \rangle$ equals the orbit of λ under $N_{\tilde{L}}(\tilde{T})$. This shows that $\psi := R_{\tilde{T}}^{\tilde{L}}(\lambda)$ is irreducible and fixed by ι. We may thus assume that ψ is ν-stable. As the character χ of V occurs in $R_{\tilde{L}}^{\tilde{G}}(\psi) = R_{\tilde{L}}^{\tilde{G}}(R_{\tilde{T}}^{\tilde{L}}(\lambda)) = R_{\tilde{T}}^{\tilde{G}}(\lambda)$ with odd multiplicity, we are done with Lemma 5.4.1. $\square$

Proposition 6.3.5. *If $G = E_6(q)$, then (G, V, n) has the $E1$-property.*

Proof. Write $\mu = \iota \circ \mu'$ for some $\iota \in \Gamma_G$ and some $\mu' \in \Phi_G$. As μ' acts on T by powering its elements, it suffices to show that some N-conjugate of λ contains a ι-stable element; see Lemma 6.3.1 and Lemma 6.3.2(a). This is trivial if $\iota = \mathrm{id}$. We may thus assume that ι equals the non-trivial graph automorphism of G. By its very definition, ι extends to the graph automorphism of $\overline{G}$ defined in [9, Theorem 1.15.2]; see [9, Definition 2.5.10(b)]. This extension, as well as its restriction to $\overline{G}^{\sigma}$, are also denoted by ι.

Since $[\overline{G}^{\sigma} : G] = [\overline{T}^{\sigma} : T]$ is odd, restriction of characters yields a bijection between $\{\tau \in \mathrm{Irr}(\overline{T}^{\sigma}) \mid \tau^2 = 1_{\overline{T}^{\sigma}}\}$ and $\{\tau \in \mathrm{Irr}(T) \mid \tau^2 = 1_T\}$. The mapping $\psi \mapsto \kappa \circ \psi|_{\overline{T}^{\sigma}}$ for $\psi \in X(\overline{T})$, yields a $\langle \iota \rangle$-equivariant isomorphism

$$X(\overline{T})/2X(\overline{T}) \to \{\tau \in \mathrm{Irr}(\overline{T}^{\sigma}) \mid \tau^2 = 1_{\overline{T}^{\sigma}}\}.$$

As $\overline{G}$ is of adjoint type, $X(\overline{T})$ has a basis consisting of the set Π of simple roots. Using this, it is easy to check with Chevie [7] that the Weyl group $N_{\overline{G}^{\sigma}}(\overline{T})/\overline{T}^{\sigma} = N/T$ of $\overline{G}^{\sigma}$ has exactly three orbits on $X(\overline{T})/2X(\overline{T})$, each of which contains a ι-stable element.

Hence some N-conjugate of λ is ν-stable and we are done by Lemma 6.3.2(a). □

Proposition 6.3.6. *If $G = \mathrm{P}\Omega_{2d}^{+}(q)$ for some $d \geq 4$, then (G, V, n) has the $E1$-property.*

Proof. Write $\mu = \iota \circ \mu'$ for some $\iota \in \Gamma_G$ and some $\mu' \in \Phi_G$. As in the proof of Proposition 6.3.5, it suffices to show that some N-conjugate of λ contains a ι-stable element, and we may thus assume that ι is non-trivial.

We begin with the case $|\iota| = 2$. Then ι is Γ_G-conjugate to the standard graph automorphism of G, and we will assume that ι is equal to the latter. We claim that ι stabilizes λ. To prove this, consider the group $\tilde{G} := \mathrm{SO}_{2d}^{+}(q)$, which contains G as a composition factor. Indeed, the commutator subgroup of $\tilde{G}$ equals $\Omega_{2d}^{+}(q)$, and G is the quotient of the latter by its center; see [22, Section 11].

To realize $\tilde{G}$ as a matrix group, equip the standard vector space $\mathbb{F}_q^{2d}$ with a non-degenerate symmetric bilinear form, and choose a basis $e_1, \ldots, e_d, e_d', \ldots, e_1'$ such that (e_i, e_i') is a hyperbolic pair for all $1 \leq i \leq d$. Then, with respect to this basis, $\tilde{G} = \{x \in \mathrm{SL}_{2d}(q) \mid x^t J x = J\}$, where J is the matrix with 1's along the antidiagonal, and 0's, elsewhere. The standard torus $\tilde{T}$ of $\tilde{G}$ is given by

$$\tilde{T} = \{\mathrm{diag}(\zeta_1, \ldots, \zeta_d, \zeta_d^{-1}, \ldots, \zeta_1^{-1}) \mid \zeta_1, \ldots, \zeta_d \in \mathbb{F}_q^*\}.$$

Then T is a quotient of $\tilde{T} \cap \Omega_{2d}^{+}(q)$, and we may view λ as an irreducible character of $\tilde{T} \cap \Omega_{2d}^{+}(q)$ via inflation. Now ι is induced by the standard graph automorphism of $\tilde{G}$ of order 2. The latter, denoted by $\tilde{\iota}$, is realized by conjugating $\tilde{G}$ with the element of $\mathrm{GL}_{2d}(q)$ which swaps the basis elements e_d and e_d', and fixes all the others.

Observe that λ, viewed as an irreducible character of $\tilde{T} \cap \Omega_{2d}^{+}(q)$, extends to some $\tilde{\lambda} \in \mathrm{Irr}(\tilde{T})$ satisfying $\tilde{\lambda}^2 = 1_{\tilde{T}}$, and it suffices to show that $\tilde{\iota}$ fixes $\tilde{\lambda}$. Let $\tilde{\lambda}_i \colon \tilde{T} \to \mathbb{C}^*$, $\mathrm{diag}(\zeta_1, \ldots, \zeta_d, \zeta_d^{-1}, \ldots, \zeta_1^{-1}) \mapsto \kappa(\zeta_i)$ for $1 \leq i \leq d$. Then $\tilde{\lambda} \in \langle \tilde{\lambda}_1, \ldots, \tilde{\lambda}_d \rangle$. Clearly, $\tilde{\lambda}_i$ is $\tilde{\iota}$-invariant for all $1 \leq i \leq d$, which proves our claim. This gives our assertion in case $|\iota| = 2$.

We are left with the case that $d = 4$ and $|\iota| = 3$. Recall that $\overline{T}^\sigma/T \cong \overline{G}^\sigma/G$. It thus follows from [9, Table 6.1.2] and [8, Theorem 2.5.20] that $\overline{T}^\sigma/T$ is elementary abelian of order 4. In particular, λ extends to four distinct characters $\bar{\lambda} \in \mathrm{Irr}(\overline{T}^\sigma)$ with $\bar{\lambda}^2 = 1_{\overline{T}^\sigma}$. We now argue as in the proof of Proposition 6.3.5. Using Chevie, we find that the Weyl group $N_{\overline{G}^\sigma}(\overline{T})/\overline{T}^\sigma$ of $\overline{G}^\sigma$ has exactly five orbits on $X(\overline{T})/2X(\overline{T})$, four of length 1 and one of length 12. The latter orbit contains the image of a root, as well as a ι-stable element, namely the image of the highest root. If $\bar{\lambda}' \in \mathrm{Irr}(\overline{T}^\sigma)$ corresponds to an image of a root, then $\mathrm{Res}_T^{\overline{T}^\sigma}(\bar{\lambda}')$ is non-trivial. Thus this orbit of length 12 accounts for the three irreducible characters of T of order 2. In particular, some N-conjugate of λ is ι-stable. This completes the proof. □

6.4. **Principal series representations.** Here, we consider the case that $\lambda = 1_T$, i.e. that V is a principal series module. As explained in the introduction to Subsection 6.3, we may assume that $\nu = \mathrm{ad}_t \circ \mu$ for some $t \in \overline{T}^\sigma$ and some $\mu \in \Gamma_G \Phi_G$. In order to apply Lemma 6.2.3, we choose P and L as a standard parabolic subgroup and a standard Levi subgroup, respectively. Then P and L are ad_t-invariant. With a suitable choice, we can also achieve that P and L are μ-invariant. Working inductively, we may assume that L has the $E1$-property. If the remaining hypothesis of this lemma is satisfied, only the constituents of $R_L^G(\mathbb{R})$ occurring with odd multiplicity have to be considered. If L is a large Levi subgroup, the number of such constituents is small, which restricts the possible $\mathbb{R}G$-modules V to be investigated.

We begin with the exceptional groups.

Proposition 6.4.1. *Let G be an exceptional group of Lie type, such that every proper subgroup of G has the $E1$-property. Then (G, V, n) has the $E1$-property.*

Proof. By Table 5.1, the group G is one of $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, ${}^2E_6(q)$ or ${}^3D_4(q)$ with q odd, or a Ree group $G = {}^2G_2(3^{2m+1})$ for some positive inter m.

The only non-trivial principal series module of a Ree group is its Steinberg module, so that the claim for these groups follows from Lemma 6.1.1. Thus, let us assume that G is not one of the groups ${}^2G_2(3^{2m+1})$ in the following.

The table below specifies, for each G, two or three standard Levi subgroups of G as follows. The second column gives the Weyl group $W(\overline{G})^\sigma$ of G, denoted by its Dynkin type. The third column lists subgraphs of the Dynkin diagram of $W(\overline{G})^\sigma$; these determine parabolic subgroups of $W(\overline{G})^\sigma$ and also standard parabolic subgroups and standard Levi subgroups of G. In case when $W(\overline{G})^\sigma$ is of type G_2, the two subgraphs A_1 and $\tilde{A}_1$ correspond to the long and to the short root of the fundamental system of this type, respectively.

G	$W(\overline{G})^\sigma$	L, L_1	Const.	Mult.
$G_2(q), {}^3D_4(q)$	G_2	A_1	${\phi_{1,3}}''$	1
		$\tilde{A}_1$		0
$F_4(q), {}^2E_6(q)$	F_4	B_3	$\phi_{9,2}$	1
		A_1		6
$E_6(q)$	E_6	A_5	$\phi_{15,4}$	1
		A_1		10
$E_7(q)$	E_7	E_6	$\phi_{7,1}, \phi_{21,3}, \phi_{27,2}$	$1, 1, 1$
		$D_5 \times A_1$		$1, 1, 2$
		A_1		$6, 16, 21$
$E_8(q)$	E_8	E_7	$\phi_{35,2}$	1
		A_1		28

Let P denote the standard parabolic subgroup of G specified in the first row of this table corresponding to G, and let P_1 denote the one specified in the other row, respectively, in case of $G = E_7(q)$, in one of the two other rows. We have Levi decompositions $P = UL$ and $P_1 = U_1L_1$ with the standard Levi subgroups L and L_1, and the unipotent radicals U and U_1 of P and P_1, respectively. The column of the table headed "Const." gives all non-trivial constituents of $R_L^G(\mathbb{R})$ of odd dimension, denoted by their labels as in [2, Subsection 13.9]. The last column contains the multiplicities of these constituents in $R_L^G(\mathbb{R})$ and in $R_{L_1}^G(\mathbb{R})$, respectively. By the Howlett-Lehrer comparison theorem [11, Theorem 5.9], these multiplicities can be computed by inducing the trivial characters of the corresponding parabolic subgroups of $W(\overline{G})^\sigma$ to $W(\overline{G})^\sigma$; see also Subsection 5.4. These computations are easily performed with Chevie [7].

Recall that $\nu = \mathrm{ad}_t \circ \mu$ for some $t \in \overline{T}^\sigma$ and some $\mu \in \Gamma_G\Phi_G$. If $G = G_2(q)$ with $q = 3^f$, then $\Gamma_G\Phi_G$ is cyclic of order $2f$; see [9, Theorem 2.5.12(d),(e)]. In this case, a generator ψ of $\Gamma_G\Phi_G$ swaps the two standard Levi subgroups of types A_1 and $\tilde{A}_1$. In all the other cases, μ, and hence ν, stabilizes the groups P, L, P_1 and L_1, and thus also U and U_1. Moreover, every automorphism of L and L_1 fixes their principal series characters; see [16, Theorem 2.5(a)]. By assumption, L and L_1

have the $E1$-property. If V is a constituent with even multiplicity (including multiplicity 0) in one of $R_L^G(\mathbb{R})$ or $R_{L_1}^G(\mathbb{R})$, then (G, V, n) has the $E1$-property by Lemma 6.2.3.

Thus (G, V, n) has the $E1$-property, except, possibly, if $G = G_2(q)$ with $q = 3^f$ and μ is an odd power of ψ. In this case, the non-trivial principal series characters of G of odd dimension are ${\phi_{1,3}}'$, ${\phi_{1,3}}''$ and $\phi_{1,6}$, the latter being the Steinberg character. As ν swaps the two representations ${\phi_{1,3}}'$ and ${\phi_{1,3}}''$ by [16, Theorem 2.5(d)], we are left with the case that V is the Steinberg module of G. This case is settled in Lemma 6.1.1. □

We finally deal with the classical groups.

Proposition 6.4.2. *Let G be a classical group, such that every proper subgroup of G has the $E1$-property. Then (G, V, n) has the $E1$-property.*

Proof. By Table 5.1, the group G is one of $\mathrm{PSL}_d(q)$, $d \geq 2$, $\mathrm{PSU}_d(q)$, $d \geq 3$, $\mathrm{P\Omega}_d(q)$, $d \geq 5$ odd, $\mathrm{PSp}_d(q)$, $d \geq 6$ even, or $\mathrm{P\Omega}_d^{\pm}(q)$, $d \geq 8$ even.

By hypothesis, V is a principal series $\mathbb{R}G$-module. Let χ denote the character of V.

We write $W := W(\overline{G})^\sigma$ for the Weyl group of G. Then W is a Coxeter group of type A_{d-1}, of type B_d or D_d. In the respective cases, the irreducible characters of W are labelled by partitions of d, by bipartitions of d, and by signed, unordered bipartitions of d; see, e.g. [3, Section 13.2]. Following the usage announced in Subsection 5.4, we will write χ^π for the character of the principal series module of G corresponding, via Harish-Chandra theory, to the irreducible character of W labelled by π; the latter will be denoted by ζ^π. Recall that the computation of Harish-Chandra induced trivial modules is reduced to the induction of the trivial character from parabolic subgroups of W. This is usually done with the branching rules.

Let us begin with the case $G = \mathrm{PSL}_d(q)$, $d \geq 2$. If $d < 4$, the only non-trivial principal series module of G of odd dimension is the Steinberg module. For this, our assertion has been settled in Lemma 6.1.1. So let us assume that $d \geq 4$ in the following. Here, W is a Coxeter group of type A_{d-1}, i.e. $W \cong S_d$, the symmetric group on d letters. First, consider the standard Levi subgroup L_I for $I \subseteq \Pi$ of type A_{d-3}, invariant under the graph automorphism; i.e. I is obtained from deleting the first and the last node of the Dynkin diagram of type A_{d-1}; see Figure 5.1. Then P_I and L_I are ν-invariant. By the branching rule for S_d, the non-trivial constituents of $R_{L_I}^G(\mathbb{R})$ have characters χ^π with $\pi \in \{(d-1, 1), (d-2, 2), (d-2, 1^2)\}$, where $\chi^{(d-1,1)}$ occurs twice. By

Lemma 6.2.3, we may assume that $\chi = \chi^\pi$ with π one of $(d-2,2)$ or $(d-2,1^2)$.

Now consider the standard Levi subgroup $L_{I'}$ of type $A_1 \times A_1$ corresponding to the two outer nodes of the Dynkin diagram of type A_{d-1}. Again, $P_{I'}$ and $L_{I'}$ are ν-invariant. Let ψ denote the Steinberg character of $L_{I'}$. Clearly, ψ is ν-invariant, real and of odd degree. By Lemma 5.4.1, it suffices to show that $\langle R^G_{L_{I'}}(\psi), \chi\rangle$ is odd. We claim that, in fact, $\langle R^G_{L_{I'}}(\psi), \chi\rangle = 1$. To prove this claim, observe that ψ corresponds to the sign-character ξ of the parabolic subgroup $W(\overline{G})^\sigma_{I'} \cong S_2 \times S_2$ of $W \cong S_d$. An application of the branching rule shows that

$$\langle \mathrm{Ind}^{S_d}_{S_2 \times S_2}(\xi), \zeta^\pi\rangle = 1$$

for $\pi \in \{(d-2,2),(d-2,1^2)\}$. The Howlett-Lehrer comparison theorem [11, Theorem 5.9] proves our claim. Thus (G, V, n) has the $E1$-property if $G = \mathrm{PSL}_d(q)$.

Next, assume that G is one of the groups $\mathrm{PSU}_{2d}(q)$, $\mathrm{PSU}_{2d+1}(q)$, $\mathrm{PSp}_{2d}(q)$, $\mathrm{P}\Omega_{2d+1}(q)$ or $\mathrm{P}\Omega^-_{2(d+1)}(q)$. In this case, all standard parabolic subgroups and standard Levi subgroups are μ-invariant and hence ν-invariant. The Weyl W of G is of type B_d; see, e.g. [9, Proposition 2.3.2]. By Lemma 6.1.1 we may assume that $d \geq 2$ if $G = \mathrm{PSU}_{2d+1}(q)$, as the only non-trivial principal series module of $\mathrm{PSU}_3(q)$ is the Steinberg module, which has been dealt with in Lemma 6.1.1. Hence $d \geq 2$ in all cases. Applying Lemma 6.2.3 to the standard parabolic subgroup of type B_{d-1}, we may assume that $\chi = \chi^\pi$ with $\pi \in \{(d-1,1),-),((d-1),(1))\}$. Using Corollary 6.2.4 and, once more, Lemma 6.2.3, the following table, which is proved by the branching rules, establishes our claim. In this table, the subgroup S_2 of W corresponds to the node of valency one at the end of the Dynkin diagram of B_d if $d > 2$, and if $d = 2$, to one of the two nodes of the Dynkin diagram of B_2.

π	$\zeta^\pi(1)$	$\langle \zeta^\pi, \mathrm{Ind}^W_{S_2}(1_{S_2})\rangle$
$((d-1,1),-)$	$d-1$	$d-2$
$((d-1),(1))$	d	$d-1$

We are left with the case $G = \mathrm{P}\Omega^+_{2d}(q)$ with $d \geq 4$. In this case, the Weyl group W of G is a Coxeter group of type D_d. We first deal with the case $d = 4$, where there is an exceptional graph automorphism of order 3 of the Dynkin diagram of D_4; see Figure 5.1. Let $I' \subseteq \Pi$ be of type A_1, invariant under all graph automorphisms of the Dynkin diagram, i.e. I' corresponds to node 3 in Figure 5.1. Then $P_{I'}$ and $L_{I'}$ are ν-invariant. Applying Corollary 6.2.4 and Lemma 6.2.3, we may assume that $\chi = \chi^\pi$ with π one of $\{(1^2),(1^2)\}^+$, $\{(1^2),(1^2)\}^-$ or $\{-,(2,1^2)\}$. By [16, Theorem 2.5(b)], the elements of order 3 in Γ_G permute the three characters with labels $\{(1^2),(1^2)\}^+$, $\{(1^2),(1^2)\}^-$ and $\{-,(2,1^2)\}$ transitively. We have $\nu = \mathrm{ad}_t \circ \iota \circ \varphi'$ with $\iota \in \Gamma_G$ and $\varphi' \in \Phi_G$. By, [16, Theorem 2.5], the unipotent characters of G are fixed by ad_t and φ'. Since χ is ν-invariant and Γ_G is isomorphic to S_3, we conclude that $|\iota| \leq 2$. This implies that there is a ι-invariant 3-element subset $I \subseteq \Pi$ containing the central node of the Dynkin diagram. Then P_I and L_I are ν-invariant. The non-trivial constituents of $R_{L_I}^G(\mathbb{R})$ are labeled by the unordered bipartitions $\{(1),(3)\}$ and $\{-,(3,1)\}$. Hence V does not occur in $R_{L_I}^G(\mathbb{R})$, and we conclude that V has the $E1$-property from Lemma 6.2.3.

Finally, assume that $d > 4$. Here, we take I of type D_{d-1} and I' of type A_1, the latter corresponding to the leaf of the Dynkin diagram whose removal gives I. Then I and I' are invariant under the symmetries of the Dynkin diagram, and thus P_I, L_I, $P_{I'}$ and $L_{I'}$ are ν-invariant. The non-trivial constituents of $R_{L_I}(\mathbb{R})$ are labeled by the unordered bipartitions $\{(1),(d-1)\}$ and $\{-,(d-1,1)\}$. Then the above table for the Coxeter group of type B_d also works for W of type D_d, and we are done by Corollary 6.2.4 and Lemma 6.2.3. $\square$

6.5. **Summary.** We summarize our results for the finite simple groups of Lie type of odd characteristic.

Theorem 6.5.1. *Let G be a finite group of Lie type of odd characteristic. Then G is not a minimal counterexample to* Theorem 1.1.5.

Proof. This follows from Proposition 6.2.2, Corollary 6.3.3, Propositions 6.3.4–6.3.6, Proposition 6.4.1 and Proposition 6.4.2. $\square$

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