THE EIGENVALUE ONE PROPERTY OF FINITE GROUPS, II

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Abstract. We prove a conjecture of Dekimpe, De Rock and Penninckx concerning the existence of eigenvalues one in certain elements of finite groups acting irreducibly on a real vector space of odd dimension. This yields a sufficient condition for a closed flat manifold to be an R_∞-manifold.

1. Introduction

The purpose of this article is to complete the proof of [14, Theorem 1.1.5], thereby confirming a conjecture of Dekimpe, De Rock and Penninckx [6, Conjecture 4.7]. More precisely, we establish the following result.

Theorem. *Let G be a finite group of Lie type of even characteristic. Then G is not a minimal counterexample to* [14, Theorem 1.1.5].

The proof of this theorem, which follows from Propositions 3.5, 5.6, 6.17, 6.27, and 7.13 below, is by far the most difficult part of our work. Let G be a finite simple group of Lie type of even characteristic. One reason for the complexity of our task is the fact that almost every irreducible character of G has odd degree. For example, if $G = E_6(q)$ with q even, then G has $q^6 + q^5 + x$ with $x < q^5$ irreducible characters, of which $q^6 + 8q^2$ have odd degree. Of course, not all of these are real, but the task is to identify the real characters among these. It is here, where we have to use the full power of Deligne-Lusztig theory, in particular Lusztig's generalized Jordan decomposition of characters. The book [11] by Geck and Malle is of enormous help in this respect. A rough sketch of the proof is given in [13].

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The general strategy is the same as in the first part of our work. The large degree method (see [14, Subsection 3.1]) rules out most of the instances to be considered. For the characters of smaller degrees we apply the restriction method (see [14, Subsection 4.3]). This general approach fails for numerous particular cases, be it for characters of very small degrees or for very small values of q or the Lie rank of G. These extra cases require lengthy and tedious calculations. The Tables [20, 21] by Frank Lübeck and extensive computations with GAP [9] and Chevie [10, 26] provide substantial contributions to our proofs.

Let us now briefly comment on the content of the individual sections. Section 2 collects a few preliminaries of general nature. In Section 3 we introduce the groups under examination, and prove our result for those with a trivial Schur multiplier and no graph automorphism of order 3. We are then left with G equal to one of $\mathrm{SL}_d(q)$ with $\gcd(d, q-1) > 1$, $\mathrm{SU}_d(q)$ with $\gcd(d, q+1) > 1$, $E_6(q)$ with $3 \mid q-1$, ${}^2E_6(q)$ with $3 \mid q+1$, or $\mathrm{P}\Omega_8^+(q)$. In Section 4 we provide the relevant theoretical background for these groups, including the specific σ-setups, duality, automorphisms and semisimple characters. In Section 5 we establish bounds on the orders of centralizers of particular automorphisms and elements in order to apply the large degree method. The results are sufficient to rule out $\mathrm{P}\Omega_8^+(q)$ as a minimal counterexample. Section 6 is devoted to the special linear and unitary groups. After lengthy preparations, and a special treatment for the groups of degree 3, we can finally show that these groups do not yield minimal counterexamples. The analogous result is then proved in Section 7 for the simple groups of Lie type E_6, twisted and untwisted.

In this second part of our series of two papers we will use the notation and preliminary results of the first part. Recall in particular [14, Notation 4.1.1]: The symbol G denotes a non-abelian finite simple group, which here is a group of Lie type of even characteristic. Next, V denotes a non-trivial irreducible $\mathbb{R}G$-module of odd dimension, and ρ the representation of G afforded by V. Moreover, $n \in \mathrm{GL}(V)$ is an element of finite order normalizing $\rho(G)$. Finally, ν denotes the automorphism of G induced by n.

2. General preliminaries

We begin with estimates for bounding character degrees and centralizer orders in linear and unitary groups.

Lemma 2.1. *Let $a \in \mathbb{R}$ and $m \in \mathbb{Z}$ with $a, m \geq 2$. Then the following inequalities hold.*

(a) *We have*

$$a^{m(m+1)/2}(1-a^{-1}-a^{-2}) \leq \prod_{i=1}^{m}(a^i-1) \leq a^{m(m+1)/2}.$$

(b) *We have*

$$a^{m(m+1)/2}(1-a^{-1}-a^{-2}) \leq \prod_{i=1}^{m}(a^i-(-1)^i) \leq a^{m(m+1)/2}(1-a^{-1}-a^{-2})^{-1}.$$

Proof. (a) The second estimate is trivial. According to [18, Proof of Lemma 4.1], we have

$$\prod_{i=1}^{\infty}(1-a^{-i}) \geq 1-a^{-1}-a^{-2}.$$

Thus $\prod_{i=1}^{m}(1-a^{-i}) \geq 1-a^{-1}-a^{-2}$. Multiplying this inequality with $a^{m(m+1)/2}$, we obtain $\prod_{i=1}^{m}(a^i-1) \geq a^{m(m+1)/2}(1-a^{-1}-a^{-2})$.

(b) To prove the first estimate, observe that $\prod_{i=1}^{m}(a^i-(-1)^i) \geq \prod_{i=1}^{m}(a^i-1)$, and use the first estimate of (a). To prove the second estimate, observe that

$$\begin{aligned}\prod_{i=1}^{m}(a^i-(-1)^i) &= \frac{\prod_{i=1}^{\lceil m/2\rceil}(a^{2i-1}-1)(a^{2i-1}+1)\prod_{i=1}^{\lfloor m/2\rfloor}(a^{2i}-1)^2}{\prod_{i=1}^{m}(a^i-1)} \\ &\leq \frac{\prod_{i=1}^{m}a^{2i}}{\prod_{i=1}^{m}(a^i-1)},\end{aligned}$$

and use the first estimate of (a). □

We will also need the following lemma on conjugation in algebraic groups.

Lemma 2.2. *Let H be an algebraic group (which could be finite), and let $M \leq H$ be a closed subgroup. Let $t, t' \in Z(M)$ with $C_H(t) = M$. Suppose that t and t' are conjugate in H.*

Then t and t' are conjugate in $N_H(M)$. If $N_H(M)$ is a product $N_H(M) = MS$ for some $S \leq N_H(M)$, then t and t' are conjugate in S. Moreover, if $M \cap S = \{1\}$, then S acts regularly on the set of elements of $Z(M)$ that are H-conjugate to t.

Proof. Let $h \in H$ with ${}^h t = t'$. Then

$$M \leq C_H(t') = {}^h C_H(t) = {}^h M.$$

This implies that $M = {}^h M$, as M and ${}^h M$ are closed subgroups of H of the same dimension. Hence $h \in N_H(M)$. The second statement is clear, since $t \in Z(M)$. If $M \cap S = \{1\}$, the centralizer of t in S is trivial. This implies the final statement. □

Let us introduce one further piece of notation. Let H be a group, $\beta \in \mathrm{Aut}(H)$ and l a positive integer. We then write

$$N_{\beta,l}(h) := \prod_{i=0}^{l-1} \beta^i(h)$$

for $h \in H$. With this notation, we have

$$(\mathrm{ad}_h \circ \beta)^l = \mathrm{ad}_{N_{\beta,l}(h)} \circ \beta^l \tag{1}$$

for every $\beta \in \mathrm{Aut}(H)$, every $h \in H$ and every positive integer l.

3. The elementary cases

In this section we introduce the groups to be investigated and prove our result for a large subclass of them by an elementary argument.

3.1. **The groups.** If $G = \mathrm{P}\Omega_{2d+1}(2)$ for $d \geq 3$ or if $G = {}^2F_4(2)'$, then G has the $E1$-property by [14, Lemma 4.4.1]. If $G = F_4(2)$, then G has the $E1$-property by [14, Lemma 4.4.2]. In order to prove the $E1$-property for simple groups of Lie type of even characteristic, we may therefore assume that G is one of the groups specified in part (a) of the following hypothesis. Part (b) lists a subset of these groups which require a deeper analysis.

Hypothesis 3.2. (a) Let G be one of the groups in Lines 1–3, 5–14 or 16 of [14, Table 1], where q is even. If G is as in Line 3 or 8 of this table, assume that $q > 2$. For the convenience of the reader, we reproduce the restricted list in Table 1.

(b) Let G be in one of the following subsets of the groups of Table 1.

(i) $G = \mathrm{PSL}_d(q)$ with $d \geq 3$ and $\gcd(d, q-1) > 1$,
(ii) $G = \mathrm{PSU}_d(q)$ with $d \geq 3$ and $\gcd(d, q+1) > 1$,
(iii) $G = E_6(q)$ and $3 \mid q-1$,
(iv) $G = {}^2E_6(q)$ and $3 \mid q+1$,
(v) $G = \mathrm{P}\Omega_8^+(q)$.

□

3.3. **Restricting to a maximal unipotent subgroup.** For G as in Hypothesis 3.2, let $B = UT$ denote the standard Borel subgroup of G, with standard unipotent subgroup U and standard torus T; see [14, Subsection 5.3]. Here we will restrict the character of V to U to deal with many of the cases listed in Hypothesis 3.2. In the following

TABLE 1. The relevant simple groups of Lie type of even characteristic

Row	Names	Rank	Conditions
1	$A_{d-1}(q), \mathrm{PSL}_d(q)$	$d \geq 2$	q even, $(d,q) \neq (2,2), (2,4), (3,2)$
2	${}^2A_{d-1}(q), \mathrm{PSU}_d(q)$	$d \geq 3$	q even, $(d,q) \neq (3,2), (4,2)$
3	$B_d(q), \mathrm{P}\Omega_{2d+1}(q)$	$d \geq 2$	$q > 2$ even
5	$D_d(q), \mathrm{P}\Omega^+_{2d}(q)$	$d \geq 4$	q even
6	${}^2D_d(q), \mathrm{P}\Omega^-_{2d}(q)$	$d \geq 4$	q even
7	$G_2(q)$		$q > 2$ even
8	$F_4(q)$		$q > 2$ even
9	$E_6(q)$		q even
10	$E_7(q)$		q even
11	$E_8(q)$		q even
12	${}^3D_4(q)$		q even
13	${}^2E_6(q)$		q even
14	${}^2B_2(q)$		$q = 2^{2m+1} > 2$
16	${}^2F_4(q)$		$q = 2^{2m+1} > 2$

proposition we use [14, Notation 4.1.1]; see the last paragraph of the introduction.

Proposition 3.4. *Let G be as in* Hypothesis 3.2(a). *Assume that U and T are stable under ν. Let χ denote the character of V. Then (G, V, n) has the $E1$-property, if any one of the two conditions below is satisfied.*

(a) *There is T-orbit $\mathcal{O}$ of linear characters of U fixed by ν such that $\langle \mathrm{Res}^G_U(\chi), \lambda \rangle$ is odd for every $\lambda \in \mathcal{O}$.*

(b) *Every T-orbit of linear characters of U is fixed by ν^2.*

Proof. (a) Let $\lambda \in \mathcal{O}$. If λ is the trivial character, $\langle \mathrm{Res}^G_U(\chi), \lambda \rangle$ equals the degree of ${}^*R^G_T(\chi)$, which then must be odd. Let $V_1 \leq V$ denote the subspace of U-fixed points of V. This is an $\mathbb{R}T$-module with character ${}^*R^G_T(\chi)$. Thus there is an irreducible $\mathbb{R}T$-submodule S of V_1 of odd dimension. As $|T|$ is odd, S is the trivial $\mathbb{R}T$-module. It follows that $\mathrm{Res}^G_B(V)$ contains a trivial constituent. Hence χ is a principal series character, and in particular unipotent. The only unipotent character of G of odd degree is the trivial character; see [24, Theorem 6.8], taking into account our restrictions on G. This is excluded by hypothesis. Thus λ is not the trivial character.

Since U is generated by involutions, U/U' is an elementary abelian 2-group, so that λ is realizable over $\mathbb{R}$. Let V_1 denote the $\mathbb{R}U$-submodule of $\mathrm{Res}^G_U(V)$ with character $\langle \mathrm{Res}^G_U(\chi), \lambda\rangle\lambda$, i.e. V_1 is the λ-homogeneous component of $\mathrm{Res}^G_U(V)$. As ν normalizes U and thus also U', the submodule nV_1 of $\mathrm{Res}^G_U(V)$ is the ${}^{\nu}\lambda$-homogeneous component of $\mathrm{Res}^G_U(V)$. Since ${}^{\nu}\lambda \in \mathcal{O}$, there is $t \in T$ such that $\rho(t)nV_1 = V_1$. Replacing n by $\rho(t)n$, we may assume that n stabilizes V_1. As U is a 2-group, it has the $E1$-property by [14, Corollary 3.2.4]. In particular, (U, V_1) has the E_1-property by [14, Lemma 3.1.2]. In turn (G, V, n) has the $E1$-property by [14, Lemma 3.1.3].

(b) Let $\mathcal{O}$ be a T-orbit of linear characters of U and let $\lambda \in \mathcal{O}$. Clearly, $\langle \mathrm{Res}^G_U(\chi), \lambda\rangle = \langle \mathrm{Res}^G_U(\chi), \lambda'\rangle$ for every other $\lambda' \in \mathcal{O}$, so $\mathcal{O}$ contributes $|\mathcal{O}|\langle \mathrm{Res}^G_U(\chi), \lambda\rangle$ to the degree of χ. If ${}^{\nu}\mathcal{O} \neq \mathcal{O}$, we obtain an even contribution to the degree of χ, as $\langle \mathrm{Res}^G_U(\chi), {}^{\nu}\lambda\rangle = \langle \mathrm{Res}^G_U({}^{\nu}\chi), {}^{\nu}\lambda\rangle = \langle {}^{\nu}\mathrm{Res}^G_U(\chi), {}^{\nu}\lambda\rangle = \langle \mathrm{Res}^G_U(\chi), \lambda\rangle$ and ${}^{\nu^2}\mathcal{O} = \mathcal{O}$. Thus there is a ν-invariant orbit $\mathcal{O}$ with $|\mathcal{O}|\langle \mathrm{Res}^G_U(\chi), \lambda\rangle$ odd for all $\lambda \in \mathcal{O}$. Hence the hypothesis of (a) is satisfied. □

We can now prove the $E1$-property for a large subclass of the simple groups of Lie type of even characteristic. Our argument involves the notion of regular characters of U, for which we follow [1, 14.B]. Notice that regular characters are called non-degenerate in [4, p. 253]. Recall the σ-setup for G introduced in [14, Subsection 5.2]. Recall in particular that $\overline{G}$ is a simple adjoint algebraic group so that $Z(\overline{G})$ is trivial. Under the hypothesis of the following proposition, we have $G = \overline{G}^{\sigma}$. We will also use the notation introduced in [14, Subsection 5.3] related to the BN-pair of G.

Proposition 3.5. *Let G be one of the groups of* Hypothesis 3.2(a) *not listed in* Hypothesis 3.2(b)(i)–(iv). *If $G = \mathrm{P}\Omega_8^+(q)$, assume that ν^2 is a product of an inner automorphism of G with a field automorphism. Then (G, V, n) has the $E1$-property.*

Proof. We have $\nu = \mathrm{ad}_g \circ \mu$ for some $g \in \overline{G}^{\sigma}$ and some $\mu \in \Gamma_G \times \Phi_G$; see [14, Subsection 5.5]. Since $G = \overline{G}^{\sigma}$, we may assume that $\nu \in \Gamma_G \times \Phi_G$. In particular, ν fixes U and T. Every element of Γ_G, except in the case where $G = \mathrm{P}\Omega_8^+(q)$, has order at most 2. Thanks to our assumption in the latter case, we have $\nu^2 \in \Phi_G$, and thus ν^2 stabilizes every standard Levi subgroup of G and the unipotent radical of every standard parabolic subgroup of G.

Let λ denote a linear character of U. By [1, 14.B], there is a standard parabolic subgroup $P_I = L_I U_I$ such that U_I is in the kernel of λ. Choose U_I maximal with this property. Then λ, viewed as a character of U/U_I, is regular. Indeed, by [4, Theorem 2.8.7], the unipotent radicals of parabolic subgroups of L_I are of the form U_J/U_I for subsets $J \subseteq I$. Now $L_I = \overline{L}_I^\sigma$ for the standard Levi subgroup $\overline{L}_I$ of $\overline{G}$. As $\overline{L}_I$ has connected center (see [4, Proposition 8.1.4]), the argument given in the proof of [4, Theorem 8.1.2(ii)] shows that then L_I has a unique T-orbit of regular characters. As ν^2 stabilizes U and U_I and sends λ to a regular character of U/U_I, it follows that ν^2 stabilizes the T-orbit containing λ. Our assertion follows from Proposition 3.4. $\square$

If $G = \mathrm{P}\Omega_8^+(q)$ and ν does not satisfy the hypothesis of Proposition 3.5, then $\nu = \mathrm{ad}_h \circ \iota \circ \mu$ for some $h \in G$, some $\iota \in \Gamma_G$ of order 3, and some $\mu \in \Phi_G$. This case will be treated in Proposition 5.6 below.

4. Preliminaries for the remaining groups

Here, we collect the properties relevant to our proofs for the remaining group to be considered.

4.1. **The σ-setup for the remaining groups.** Let G be one of the groups listed in Hypothesis 3.2(b). We now specify the choice of $\overline{G}$ in the σ-setup for G and recall some notation concerning automorphisms. In the present context, $\mathbb{F}$ is an algebraic closure of the field with two elements. In cases (i) and (ii) for G we take $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$, in cases (iii) and (iv) we take $\overline{G} = E_6(\mathbb{F})_{\mathrm{ad}}$, and in case (v) we let $\overline{G} = \mathrm{PCSO}_8(\mathbb{F})$, the simple algebraic group over $\mathbb{F}$ of adjoint type D_4. The standard torus $\overline{T}$ contained in the standard Borel subgroup $\overline{B}$ gives rise to the root system of $\overline{G}$, which is of type A_{d-1}, E_6 or D_4, respectively. We write Π for the set of simple roots determined by $\overline{T}$ and $\overline{B}$. The corresponding Dynkin diagrams are displayed in [14, Figure 1].

Recall the notation introduced in [14, Subsection 5.5]. In particular, $\varphi = \varphi_2$ is the standard Frobenius morphism of $\overline{G}$ defined in [12, Theorem 1.15.4(a)], and $\Phi_{\overline{G}} = \langle \varphi \rangle$. If ι denotes a symmetry of the Dynkin diagram of $\overline{G}$, then ι also denotes the corresponding standard graph automorphism of $\overline{G}$. Let f be a positive integer and $q = 2^f$. Taking $\sigma := \varphi^f$, we obtain the untwisted groups $\overline{G}^\sigma = \mathrm{PGL}_d(q)$, respectively $E_6(q)_{\mathrm{ad}}$, respectively $\mathrm{P}\Omega_8^+(q)$. The corresponding simple groups are $G = \mathrm{PSL}_d(q)$, respectively $E_6(q)$, respectively $\mathrm{P}\Omega_8^+(q)$. If $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$ or $\overline{G} = E_6(\mathbb{F})_{\mathrm{ad}}$, the group of symmetries of the Dynkin diagram of $\overline{G}$ has order 2, so that $\Gamma_{\overline{G}} = \langle \iota \rangle$ for the non-trivial such symmetry ι. Taking $\sigma := \iota \circ \varphi^f$, we obtain the twisted groups $\overline{G}^\sigma =$

$\mathrm{PGU}_d(q)$, respectively ${}^2E_6(q)_{\mathrm{ad}}$. The corresponding simple groups are $G = \mathrm{PSU}_d(q)$, respectively ${}^2E_6(q)$. Recall that $\overline{G}^\sigma = G\overline{T}^\sigma$; see [12, Theorem 2.2.6(g)].

For the sake of a uniform treatment, we will use the common ε-convention to distinguish between the twisted and the untwisted groups in cases (i)–(iv). Fix $\varepsilon \in \{-1, 1\}$. If $\varepsilon = 1$, let $\sigma = \varphi^f$, and if $\varepsilon = -1$, let $\sigma = \iota \circ \varphi^f$, where, as above, ι is the non-trivial symmetry of the Dynkin diagram of $\overline{G}$. We will thus understand $\mathrm{PSL}^\varepsilon(q)$ to be $\mathrm{PSL}_d(q)$ if $\varepsilon = 1$, and $\mathrm{PSU}_d(q)$, if $\varepsilon = -1$. Likewise, $E_6^\varepsilon(q)$ denotes the simple group $E_6(q)$ if $\varepsilon = 1$, and ${}^2E_6(q)$, if $\varepsilon = -1$. This ε-convention is also used for the groups $\overline{G}^\sigma$.

Clearly, $\overline{G}^\sigma$ and thus also G, are invariant under φ and ι for every symmetry ι of the Dynkin diagram of $\overline{G}$. The restrictions to $\overline{G}^\sigma$ or G of φ and any such ι are denoted by the same letters. In particular, $\Phi_G = \langle\varphi\rangle$. If $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$ or $\overline{G} = E_6(\mathbb{F})_{\mathrm{ad}}$, we have $\Gamma_{\overline{G}} = \langle\iota\rangle$ for a non-trivial symmetry ι of the Dynkin diagram of $\overline{G}$. If the corresponding G is untwisted, i.e. $G = \mathrm{PSL}_d(q)$ or $G = E_6(q)$, then $\Gamma_G = \langle\iota\rangle$ has order 2. If G is twisted, then $\iota = \varphi^f$, viewed as elements of $\mathrm{Aut}(G)$, so that $\iota \in \Phi_G$. Thus according to the conventions in [12, Definitions 2.2.4, 2.5.10], we have $\Gamma_G = \{1\}$, although $\Gamma_{\overline{G}}$ is non-trivial. If G is as in case (v), $\Gamma_{\overline{G}} \cong \Gamma_G$ is isomorphic to a symmetric group on three letters. In this case, Γ_G has elements of order 2 and 3; the latter are called triality automorphisms of G.

4.2. **The dual groups.** At some stage we will also need to consider the groups dual to $\overline{G}$. Namely, to describe the irreducible characters of G it is more convenient to realize G as a central quotient of the group $\overline{G}^{*\sigma^*}$, where $\overline{G}^*$ is a group dual to $\overline{G}$ and σ^* is a Steinberg morphism of $\overline{G}^*$ dual to σ; see [11, Definition 1.5.17]. For a reason that will become apparent in Subsection 4.3 below, we will henceforth usually suppress the asterisk in the notation of σ^*, and just write σ instead. Only in some proofs we stick, for reasons of clarity, to the more precise notation.

We take $\overline{G}^*$ as the simply connected version of the simple algebraic group of type A_{d-1}, E_6, respectively D_4. To be specific, if $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$, $E_6(\mathbb{F})_{\mathrm{ad}}$ or $\mathrm{PCSO}_8(\mathbb{F})$, we let $\overline{G}^* = \mathrm{SL}_d(\mathbb{F})$, $E_6(\mathbb{F})_{\mathrm{sc}}$ and $\mathrm{Spin}_8(\mathbb{F})$, respectively. There is an isogeny, i.e. a surjective homomorphism of algebraic groups with finite kernel,

$$\overline{G}^* \to \overline{G}. \tag{2}$$

If $\overline{G}^* = \mathrm{SL}_d(\mathbb{F})$, we have $\overline{G}^{*\sigma} = \mathrm{SL}_d(q)$, respectively $\mathrm{SU}_d(q)$. If $\overline{G}^* = E_6(\mathbb{F})_{\mathrm{sc}}$, the group $\overline{G}^{*\sigma}$ is the universal covering group of $G = E_6(q)$, respectively ${}^2E_6(q)$. In all cases, the center of $\overline{G}^{*\sigma}$ is cyclic and of odd order. The groups $\overline{G}^{\sigma}$ are almost simple, whereas the groups $\overline{G}^{*\sigma}$ are quasisimple.

4.3. **Automorphisms, I.** Assume that G is one of the groups of Hypothesis 3.2(b)(i)–(iv), so that $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$ or $\overline{G} = E_6(\mathbb{F})_{\mathrm{ad}}$. Here, we shortly comment on the connections between the automorphism groups of G, $\overline{G}^{\sigma}$, $\overline{G}$, and their duals. Let us recall the notation introduced in [12, Definition 1.15.1]. Thus $\mathrm{Aut}_1(\overline{G})$ is the set of all automorphisms α of $\overline{G}$ (as an abstract group), such that α or α^{-1} is an endomorphism of $\overline{G}$ as an algebraic group. Notice that $\sigma \in \mathrm{Aut}_1(\overline{G})$. Also, $\mathrm{Aut}_1(\overline{G})$ is a group; see [12, Theorem 1.15.7(a)]. Analogous notations will be used for $\overline{G}^*$.

The isogeny (2) gives rise to a natural map

$$\mathrm{Aut}_1(\overline{G}^*) \to \mathrm{Aut}_1(\overline{G}), \tag{3}$$

which is in fact an isomorphism; see [12, Theorems 1.15.6(c)]. By [12, Theorems 1.15.6(b)], the isomorphism (3) preserves isogenies.

Lemma 4.4. *Let $t \in \overline{T}$ and let $\mu \in \Gamma_{\overline{G}} \times \Phi_{\overline{G}}$ be an isogeny. Then $\alpha := \mathrm{ad}_t \circ \mu$ is an isogeny of $\overline{G}$. Let $\alpha^* \in \mathrm{Aut}_1(\overline{G}^*)$ be the inverse image of α under* (3)*. Then α^* is an isogeny of $\overline{G}^*$, which is dual to α in the sense of* [7, Proposition 11.1.11].

Proof. Clearly, α is an isogeny. As (3) preserves isogenies, α^* is an isogeny as well. To prove the second claim, we may assume that $t = 1$. We may also assume that $\alpha = \iota$, or $\alpha = \varphi$, the generators, introduced in Subsection 4.1, of $\Gamma_{\overline{G}}$ and $\Phi_{\overline{G}}$, respectively. The claim is then obvious. □

Now identify $\mathrm{Aut}_1(\overline{G}^*)$ and $\mathrm{Aut}_1(\overline{G})$ via (3). As $\sigma \in \mathrm{Aut}_1(\overline{G})$ is a Steinberg morphism, its inverse image in $\mathrm{Aut}_1(\overline{G}^*)$ is a Steinberg morphism of $\overline{G}^*$ dual to σ. This justifies our simplified notation introduced in Subsection 4.1.

Let $\alpha \in \mathrm{Aut}(G)$. Then α is the restriction to G of an element of $\mathrm{Aut}_1(\overline{G})$ commuting with σ; see [12, Theorem 2.5.4]. In fact, this element giving rise to α may be chosen to be an isogeny. Indeed, as α has finite order, $\alpha^{-1} = \alpha^l$ for some non-negative integer l. The claim follows from this, as either β or β^{-1} is an isogeny for every $\beta \in \mathrm{Aut}_1(\overline{G})$. Also, α extends to a unique automorphism of $\overline{G}^{\sigma}$ arising from the elements of $\mathrm{Aut}_1(\overline{G})$ commuting with σ and restricting to α, and we will tacitly view α as an element of $\mathrm{Aut}(\overline{G}^{\sigma})$ this way.

4.5. **Semisimple characters.** As already mentioned in Subsection 4.2, we are going to describe the irreducible characters of G through the epimorphism $\overline{G}^{*\sigma} \to G$ via inflation. This does not introduce new characters to be investigated, as $Z(\overline{G}^{*\sigma})$, the kernel of this epimorphism, is cyclic of odd order, so that every real irreducible character of $\overline{G}^{*\sigma}$ has $Z(\overline{G}^{*\sigma})$ in its kernel. (Contrary to a more common approach, we interchange the roles of the group and its dual here, as our focus is on the irreducible characters of the group $\overline{G}^{*\sigma}$.)

Viewed as element of $\mathrm{Irr}(\overline{G}^{*\sigma})$, the character of V is what is called a semisimple character. For this notion, we follow [11, Definition 2.6.9]. By [11, Corollary 2.6.18(a)] and the results in [1, Section 15], in particular [1, Théorème 15.10, Corollaire 15.11, Proposition 15.13 and Corollaire 15.14], this matches with [1, Définition 15.A].

The set $\mathrm{Irr}(\overline{G}^{*\sigma})$ is partitioned into Lusztig series $\mathcal{E}(\overline{G}^{*\sigma}, s)$, where s runs through the $\overline{G}^{\sigma}$-conjugacy classes of semisimple elements of $\overline{G}^{\sigma}$; see [11, Definition 2.6.1]. The following lemma collects the properties of semisimple characters relevant to our further investigation.

Lemma 4.6. *Let $\chi \in \mathrm{Irr}(\overline{G}^{*\sigma})$ and let $s \in \overline{G}^{\sigma}$ be semisimple such that $\chi \in \mathcal{E}(\overline{G}^{*\sigma}, s)$. Let $\alpha^* \in \mathrm{Aut}_1(\overline{G}^{*})$ be an isogeny commuting with σ. Let $\alpha \in \mathrm{Aut}_1(\overline{G})$ be an isogeny commuting with σ, dual to α^*. (Such an isogeny exists; see, e.g.* [11, 1.7.11].*) Then the following statements hold.*

(a) *The degree of χ is odd, if and only if χ is semisimple in the sense of* [11, Definition 2.6.9]. *If $\chi(1)$ is odd, then*

$$\chi(1) = [\overline{G}^{\sigma} : C_{\overline{G}^{\sigma}}(s)]_{2'}.$$

(b) *Assume that χ has odd degree. Then the following assertions are true.*

- (i) *The semisimple characters in $\mathcal{E}(\overline{G}^{*\sigma}, s)$ all have the same degree, and they correspond, via Lusztig's Jordan decomposition of characters, to the irreducible characters of $(C_{\overline{G}}(s)/C^{\circ}_{\overline{G}}(s))^{\sigma}$. In particular, if $C_{\overline{G}}(s)$ is connected, then χ is the unique semisimple character in $\mathcal{E}(\overline{G}^{*\sigma}, s)$.*
- (ii) *If χ is real, then s is real in $\overline{G}^{\sigma}$. If s is real in $\overline{G}^{\sigma}$ and $C_{\overline{G}}(s)$ is connected, then χ is real.*
- (iii) *If χ is α^*-invariant, then the $\overline{G}^{\sigma}$-conjugacy class of s is invariant under α. If the $\overline{G}^{\sigma}$-conjugacy class of s is α-invariant and $C_{\overline{G}^*}(s)$ is connected, then χ is α^*-invariant.*

(iv) *Let $\overline{L} \leq \overline{G}$ be a standard Levi subgroup of $\overline{G}$ with $s \in \overline{L}$. Let $\overline{L}^*$ denote the standard Levi subgroup of $\overline{G}^*$ dual to $\overline{L}$. Suppose that s is real in $\overline{L}^\sigma$ and that $C_{\overline{L}^*}(s)$ is connected.*

Then $\mathcal{E}(\overline{L}^{*\sigma}, s)$ contains a real, semisimple character ψ of odd degree, such that $\langle R_{\overline{L}^{*\sigma}}^{\overline{G}^{*\sigma}}(\psi), \chi\rangle = 1$.

Proof. (a) and (b)(i). Use the notion of unipotent characters of $C_{\overline{G}}(s)^\sigma$ as introduced in [22, 12]. Then, via Lusztig's generalized Jordan decomposition of characters, χ corresponds to a unipotent character λ of $C_{\overline{G}}(s)^\sigma$ such that $\chi(1) = \lambda(1)[\overline{G}^\sigma : C_{\overline{G}^\sigma}(s)]_{2'}$; see [22, Proposition 5.1].

Suppose that $\chi(1)$ is odd. Thus $\lambda(1)$ is odd, and hence λ lies over an odd degree unipotent character λ' of $C^\circ_{\overline{G}^\sigma}(s)$. Then [24, Theorem 6.8] implies that λ' is the trivial character. It follows that χ is semisimple by [11, Theorem 2.6.11(c) and Corollary 2.6.18(a)] and the definition of the generalized Jordan decomposition from the usual Jordan decomposition through a regular embedding of $\overline{G}^*$. This also implies that the semisimple characters in $\mathcal{E}(\overline{G}^{*\sigma}, s)$ are exactly those that correspond, via the generalized Jordan decomposition of characters, to the extensions of λ' to $C_{\overline{G}^\sigma}(s)$. Since $C_{\overline{G}^\sigma}(s)/C^\circ_{\overline{G}^\sigma}(s) = (C_{\overline{G}}(s)/C^\circ_{\overline{G}}(s))^\sigma$ is abelian by [1, Lemma 8.3], this implies (b)(i), one direction of (a), as well as the degree formula.

Suppose now that χ is semisimple. Then $\chi(1)$ is odd by [11, Theorem 2.6.11(b) and Corollary 2.6.18(a)]. This gives the other direction of (a).

(b)(ii) If $\overline{S}$ is a σ-stable maximal torus of $\overline{G}$ containing s, then $R_{\overline{S}}^{\overline{G}}(s)$ is complex conjugate to $R_{\overline{S}}^{\overline{G}}(s^{-1})$ by the character formula; see [11, Theorem 2.2.16]. The definition of the set $\mathcal{E}(\overline{G}^{*\sigma}, s)$ (see [11, Definition 2.6.1]) implies that the character complex conjugate to χ lies in $\mathcal{E}(\overline{G}^{*\sigma}, s^{-1})$. This proves the first assertion. If $C_{\overline{G}}(s)$ is connected, χ is the unique semisimple character in $\mathcal{E}(\overline{G}^{*\sigma}, s)$ by (i). Since s is real, the character complex conjugate to χ lies in $\mathcal{E}(\overline{G}^{*\sigma}, s)$, and is semisimple by (a). Hence χ is real.

(b)(iii) As α and α^* are dual isogenies, we have

$$^{\alpha^*}\mathcal{E}(\overline{G}^{*\sigma}, s) = \mathcal{E}(\overline{G}^{*\sigma}, \alpha^{-1}(s));$$

see [29, Proposition 7.2]. The proof now proceeds as in (ii).

(b)(iv) The first claim follows from (i), (ii), and (a), applied to $\overline{L}$. Further, under any Jordan decomposition of characters, ψ corresponds to the trivial character; see [11, Theorem 2.6.11(c)]. As χ is semisimple by (a), it corresponds, by (i), to an irreducible character λ of $C_{\overline{G}^\sigma}(s)$ which has $C^\circ_{\overline{G}^\sigma}(s)$ in its kernel. As $C_{\overline{L}}(s)$ is connected, we have $C_{\overline{L}^\sigma}(s) \leq$

$C^{\circ}_{\overline{G}^{\sigma}}(s)$. As $\overline{L}$ is 1-split, $C_{\overline{L}^{\sigma}}(s)$ is a 1-split Levi subgroup of $C^{\circ}_{\overline{G}}(s)$, and thus Lusztig induction from $C_{\overline{L}^{\sigma}}(s)$ to $C^{\circ}_{\overline{G}^{\sigma}}(s)$ is just usual Harish-Chandra induction. Thus the trivial character of $C_{\overline{L}^{\sigma}}(s)$, Harish-Chandra induced to $C_{\overline{G}^{\sigma}}(s)$ in the sense of [11, Definition 4.8.8], contains λ with multiplicity 1; see [11, Proposition 4.8.10]. The second claim now follows from [11, Theorem 4.8.24]. □

The next lemma paves the way for our applications.

Lemma 4.7. *Let G be as in* Hypothesis 3.2(b)(i)–(iv) *and suppose that every proper subgroup of G has the $E1$-property. Let ι denote the standard graph automorphism of $\overline{G}$ of order* 2. *Let (V, n, ν) be as in* [14, Notation 4.1.1]*, and let χ denote the character of V. View V as an $\mathbb{R}\overline{G}^{*\sigma}$-module and χ as a character of $\overline{G}^{*\sigma}$ via inflation. Let $s \in \overline{G}^{\sigma}$ be such that $\chi \in \mathcal{E}(\overline{G}^{*\sigma}, s)$. Then (G, V, n) has the $E1$-property under the following hypotheses.*

There is a ι-stable, proper standard Levi subgroup $\overline{L}$ of $\overline{G}$ and a $\overline{G}^{\sigma}$-conjugate $s' \in \overline{L}^{\sigma}$, such that the conditions (i),(ii),(iii) *hold.*

(i) *The element s' is real in $\overline{L}^{\sigma}$.*
(ii) *The centralizer $C_{\overline{L}}(s')$ is connected.*
(iii) *For every $\alpha \in \mathrm{Aut}(G)$ stabilizing $\overline{L}^{\sigma}$, the following holds: If $\alpha(s)$ and s are conjugate in $\overline{G}^{\sigma}$, then $\alpha(s')$ and s' are conjugate in $\overline{L}^{\sigma}$.*

Proof. Let the notation be as in Lemma 4.6. As $\overline{L}$ is ι-stable and a standard Levi subgroup of $\overline{G}$, it is σ-stable. As ι commutes with σ, the finite group $\overline{L}^{\sigma}$ is ι-stable as well. We may assume that $s = s' \in \overline{L}^{\sigma}$. Let $\overline{L}^{*}$ denote the standard Levi subgroup of $\overline{G}^{*}$ dual to $\overline{L}$. Then $\overline{L}^{*}$ is σ-stable. Let $\psi \in \mathrm{Irr}(\overline{L}^{*\sigma})$ denote the unique semisimple character in $\mathcal{E}(\overline{L}^{*\sigma}, s)$; see Lemma 4.6(b)(i). Then ψ is real, of odd degree, and χ occurs with multiplicity 1 in $R^{\overline{G}^{*}}_{\overline{L}^{*}}(\psi)$; see Lemma 4.6(b)(iv). Since χ is not the trivial character, $s \neq 1$, an thus ψ is not the trivial character of $\overline{L}^{*\sigma}$.

Write $\nu = \mathrm{ad}_h \circ \mu$ with $h \in \overline{G}^{\sigma}$ and $\mu \in \Gamma_G \times \Phi_G$; see [14, Subsection 5.5]. As $\overline{L}$ is ι-stable, μ stabilizes $\overline{L}^{\sigma}$ and the corresponding standard parabolic subgroup $\overline{P}^{\sigma}$. Since $\overline{G}^{\sigma} = G\overline{T}^{\sigma}$ (see Subsection 4.1), there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \nu = \mathrm{ad}_t \circ \mu$ with $t \in \overline{T}^{\sigma}$. In particular, α stabilizes $\overline{L}^{\sigma}$ and $\overline{P}^{\sigma}$. Also, α is the restriction to G of an isogeny $\alpha \in \mathrm{Aut}_1(\overline{G})$ commuting with σ, which fixes $\overline{T}$; see the penultimate paragraph in Subsection 4.3.

Let $\alpha^* \in \mathrm{Aut}_1(\overline{G}^{*})^{\sigma}$ denote the inverse image of α under the isomorphism (2) in Subsection 4.3. Then α and α^* are dual isogenies by

Lemma 4.4. Now α^* restricts to an automorphism of $\overline{G}^{*\sigma}$, which stabilizes $\overline{L}^{*\sigma}$ and the corresponding standard parabolic subgroup of $\overline{G}^{*\sigma}$. The automorphism induced by this α^* on $G \cong \overline{G}^{*\sigma}/Z(\overline{G}^{*\sigma})$ is the original α we started with. As such, α^* fixes χ, and so α^* also stabilizes the $\overline{G}^{\sigma}$-conjugacy class of s by Lemma 4.6(b)(iii). By Hypothesis (iii), the $\overline{L}^{\sigma}$-conjugacy class of s is α-stable as well. It follows that ψ is fixed by α^*, once more by Lemma 4.6(b)(iii). The assertion now follows from [14, Lemma 5.4.1], applied to the group $\overline{G}^{*\sigma}$ and the triple (V, gn, α). □

5. Bounds on orders of centralizers and elements

In this section we establish bounds on the orders of certain centralizers and elements. The results are enough to rule out $G = \mathrm{P}\Omega_8^+(q)$ as a minimal counterexample.

5.1. **Bounds on orders of centralizers and elements.** We need to estimate the sizes of the centralizers of certain automorphisms of the groups G listed in Hypothesis 3.2(b). We will use the notations introduced and summarized in Subsections 4.1. Recall in particular the σ-setup for G, the notion $q = 2^f$, and the significance of the symbols φ and ι, the latter denoting a non-trivial element of $\Gamma_{\overline{G}}$, as well as its restriction to G. Also recall the ε-convention for the linear and unitary groups as well as for the groups of type E_6. We also define the parameter $\delta \in \{1, 2\}$ by $\delta := 1$, if $\varepsilon = 1$, and $\delta := 2$, if $\varepsilon = -1$.

We begin with a definition.

Definition 5.2. With G as in Hypothesis 3.2(b), define the positive integer M_G as follows:

(a) If $G = E_6^{\varepsilon}(q)$, then $M_G := q^{48}$.

(b) If $G = \mathrm{PSL}_d^{\varepsilon}(q)$ with $d \geq 5$, then $M_G := q^{d(d+1)/2}$.

(c) If $G = \mathrm{PSL}_3^{\varepsilon}(q)$, then $M_G := \begin{cases} q^4, & \text{if } \varepsilon = 1 \text{ and } q \neq 16, \\ 62\,401, & \text{if } \varepsilon = 1 \text{ and } q = 16, \\ q^4 + q^3, & \text{if } \varepsilon = -1. \end{cases}$

(d) If $G = \mathrm{P}\Omega_8^+(q)$, then $M_G := q^{14} + q^{12}$.

Lemma 5.3. *Let p be a prime and let $\alpha \in \mathrm{Aut}(G) \setminus \mathrm{Inndiag}(G)$ with $|\alpha| = p$. If $G = \mathrm{P}\Omega_8^+(q)$ assume that $q \neq 2$. Then $|C_G(\alpha)| < M_G$, unless $G = E_6^{\varepsilon}(q)$ or $G = \mathrm{P}\Omega_8^+(q)$ and α is a graph automorphism of G of order 2. In the former case, $|C_G(\alpha)| < q^{52}$, in the latter case, $|C_G(\alpha)| < q^{21}$.*

Proof. Since $|\alpha| = p$ and $\alpha \notin \mathrm{Inndiag}(G)$, the order of α modulo $\mathrm{Inndiag}(G)$ also equals p. It follows that $\alpha = \mathrm{ad}_g \circ \mu$ for some $g \in \overline{G}^{\sigma}$

and some $\mu \in \Gamma_G \times \Phi_G$ of order p. Recall that $\Gamma_G = \{1\}$ if G is twisted. Write $\mu = \iota' \circ \varphi'$ with $\iota' \in \Gamma_G$ and $\varphi' \in \Phi_G$. As $|\mu| = p$ is a prime, we must have $|\varphi'| = p$ or $|\iota'| = p$. In the latter case, G is untwisted and $p = 2$ or $p = 3$, where $p = 3$ only occurs for $G = \mathrm{P}\Omega_8^+(q)$.

Suppose first that α is a graph automorphism of G. If G is untwisted, this means that $\varphi' = 1$, and either $p = 2$, or $p = 3$ and $G = \mathrm{P}\Omega_8^+(q)$. If G is twisted, this means that $\iota' = 1$ and $p = 2$. If $p = 2 = r$, then $C_G(\alpha) \cong F_4(q), \mathrm{Sp}_{2\lfloor d/2 \rfloor}(q), \mathrm{Sp}_2(q), \mathrm{SO}_7(q)$ in the respective cases of Definition 5.2; see [12, Proposition 4.2.9]. If $p = 3$, we have $C_G(\alpha) \cong G_2(q)$ by [12, Table 4.7.3A]. Notice that the latter reference only gives $O^{2'}(C_G(\alpha))$, but according to [2, Table 8.50], the only maximal subgroups of G containing a subgroup isomorphic to $G_2(q)$ are $G_2(q)$ itself or $\mathrm{Sp}_6(q)$. However, $G_2(q)$ is the only maximal subgroup of $\mathrm{Sp}_6(q)$ containing $G_2(q)$; see [2, Tables 8.28, 8.29]. Thus $O^{2'}(C_G(\alpha)) = C_G(\alpha)$. In any case, the given upper bound for $|C_G(\alpha)|$ is satisfied.

Now suppose that α is not a graph automorphism. Then $\varphi' \neq 1$ so that $|\varphi'| = p$. Recall that Φ_G is cyclic of order f if G is untwisted, and of order $2f$, if G is twisted. As p is odd in the twisted case, we have $p \mid f$ in any case. If G is twisted, ι is an involution in Φ_G. Then $\varphi^{f/p}$ in the untwisted case, and $\iota \circ \varphi^{f/p}$ in the twisted case, are elements of order p in Φ_G. Clearly, $C_G(\alpha) = C_G(\alpha^l)$ for all integers l prime to p. By Equation (1), we may thus assume that $\varphi' = \varphi^{f/p}$, respectively $\varphi' = \iota \circ \varphi^{f/p}$. Recall that φ and ι arise from restrictions of corresponding elements of $\mathrm{Aut}_1(\overline{G})$, denoted by the same letters. Viewed as such, $\mu = \iota' \circ \varphi'$ and $\alpha = \mathrm{ad}_g \circ \mu$ are Steinberg morphisms of $\overline{G}$. By our choice of φ', we have $\sigma = \varphi'^p$. Hence $\mu^p = \sigma$, as $|\iota'| \in \{1, p\}$. It follows that $C_{\overline{G}^\sigma}(\mu) = C_{\overline{G}}(\mu)$. Now $C_G(\alpha) \leq C_{\overline{G}}(\alpha) \cong C_{\overline{G}}(\mu) = C_{\overline{G}^\sigma}(\mu)$; for the isomorphism in the latter chain see [11, Lemma 1.4.14]. In particular, $|C_G(\alpha)| \leq |C_{\overline{G}}(\mu)| = |\overline{G}^\mu|$. The groups of maximal order among the $\overline{G}^\mu$ occur for G untwisted and $p = 2$. These are, in the respective cases, ${}^2E_6(q_0)$, $\mathrm{PGU}_d(q_0)$, $\mathrm{PGU}_3(q_0)$ and $\mathrm{P}\Omega_8^-(q_0)$, where $q = q_0^2$. It is easy to check that the orders of these groups satisfy the asserted bounds. Notice that $|\mathrm{PGU}_3(4)| = 62\,400$. □

We will also need the following technical result on the orders of some elements of $\mathrm{Aut}(G)$.

Lemma 5.4. *Suppose that G is as in* Hypothesis 3.2(b)(i)–(iv). *Let $\beta = \mathrm{ad}_t \circ \mu \in \mathrm{Aut}(G)$ with $t \in \overline{T}^\sigma$ and $\mu \in \Gamma_G \times \Phi_G$. Suppose that $3 \mid q - \varepsilon$. Then the following hold.*

(a) *If $t = 1$, then $|\beta|$ divides δf.*

(b) *If t is a 3-element, then $|\beta|$ divides $3\delta f$.*
(c) *If $\varepsilon = -1$, $|\mu|$ is even and $t^{q+1} = 1$, then $|\beta|$ divides $2f$.*

Proof. (a) This is clear, as $\Gamma_G \times \Phi_G$ has exponent δf.

(b) Put $c := \delta f/f_3$. If $\delta = 1$, then f is even as then $3 \mid 2^f - 1$. Thus c is even. Also, $|\mu^c|$ divides f_3. Hence $\mu^c = \varphi^{c'}$ for some even integer c'. By Equation (1), we have $\beta^c = \mathrm{ad}_x \circ \mu^c$ with a 3-element $x \in \overline{T}^\sigma$. To prove our claim, it suffices to show that $\beta^{3cf_3} = 1$.

Suppose that $f_3 := 3^b$ for some non-negative integer b. Then $(q - \varepsilon)_3 = 3^{b+1}$ by [16, Lemma IX.8.1(e)]. Observe that φ acts on $\overline{T}$ by squaring the elements; see [12, Theorem 1.12.1]. Put $q' := 2^{c'}$. Once more by Equation (1), we find that $\beta^{3c} = \mathrm{ad}_{x'} \circ \mu^{3c}$ with $x' = x^{1+q'+q'^2}$. As q' is an even power of 2, we have $3 \mid 1 + q' + q'^2$. Hence $|x'| = |x|/3$, unless $|x| = 1$, and $|\mu^{3c}| = |\mu^c|/3$, unless $|\mu^c|_3 = 1$. Iterating this argument, we find that $\beta^{cf_3} = \mathrm{ad}_y \circ \mathrm{id}$ with $y \in \overline{T}^\sigma$ of order 1 or 3. This completes the proof of (a).

(c) Since $\varepsilon = -1$, we have $\Gamma_G = \{\mathrm{id}\}$, and Φ_G is cyclic of order $2f$. Moreover, f is odd, as $3 \mid 2^f + 1$. As $|\mu|$ is even, Equation (1) implies that $\beta^f = \mathrm{ad}_x \circ \varphi^f$ for some $x \in \overline{T}^\sigma$ with $x^{q+1} = 1$. From Equation (1) we get $\beta^{2f} = \mathrm{ad}_{x^{1+q}} \circ \mathrm{id} = \mathrm{id}$. This yields our assertion. □

5.5. **The proof for the eight-dimensional orthogonal group.** As an illustration of Lemma 5.3, we complete the proof that the groups $\mathrm{P}\Omega_8^+(q)$ have the $E1$-property.

Proposition 5.6. *Let $G = \mathrm{P}\Omega_8^+(q)$. Assume that $\nu = \mathrm{ad}_h \circ \iota \circ \mu$, where $h \in G$, $\iota \in \Gamma_G$ of order 3 and $\mu \in \Phi_G$. Then (G, V, n) has the $E1$-property.*

Proof. By pre-multiplying n with h^{-1}, we may assume that $\nu = \iota \circ \mu$. If $|\nu|$ is even, we put $g := 1$. If $|\nu|$ is odd, we let $g \in \mathrm{P}\Omega_8^+(2) \leq G$ be a ι-stable involution, whose centralizer in $\mathrm{P}\Omega_8^+(2)$ has order $2^{10}3$. Using the fact that the centralizer of ι in $\mathrm{P}\Omega_8^+(2)$ is isomorphic to $G_2(2)$, the corresponding class fusion (computed with GAP) shows the existence of such a g. By construction, ν fixes g. The table of unipotent characters of G in Chevie [10] contains, in particular, the orders of the centralizers of the unipotent conjugacy classes of G. An inspection of these entries shows that $|C_G(g)| = q^{10}(q^2 - 1)$. Put $\alpha := \mathrm{ad}_g \circ \nu$. Assume that $q \neq 2$. Then $|C_G(\alpha_{(p)})| \leq q^{14} + q^{12}$ for every prime p dividing $|\alpha|$. For odd p or even $|\nu|$ this follows from Lemma 5.3, as α is not a graph automorphism of order 2. For odd $|\nu|$ we have $C_G(\alpha_{(2)}) = C_G(g)$.

By [14, Lemma 4.3.3] the triple (G, V, n) has the $E1$-property, if

$$\chi(1) > (|\alpha| - 1)(q^{14} + q^{12})^{1/2}. \tag{4}$$

Suppose first that $\chi(1) > q^9/8$. If $f \geq 5$, then $|\alpha|-1 \leq 6f-1 \leq 2^f = q$. For $f = 4$ we have $|\alpha| \leq 12 \leq 2^4 = q$. These bounds clearly imply that (4) is satisfied for $f \geq 4$. For $f = 2$ or 3, we have $|\alpha| = 6$. Using this and the exact values for $\chi(1)$ given in [21], we see that (4) also holds for $f = 2, 3$. We are left with the case $q = 2$, i.e. $G = \mathrm{P}\Omega_8^+(2)$. Here, Φ_G is trivial and thus $\alpha = \mathrm{ad}_g \circ \iota$, where g is a ι-stable involution with centralizer of order $2^{10}3$. Let $G^\diamond := \langle \mathrm{Inn}(G), \alpha \rangle = \langle \mathrm{Inn}(G), \nu \rangle$ and let $\chi^\diamond$ denote the character of $G^\diamond$ afforded by V; see [14, Remark 4.2.5]. Now $G^\diamond$ is a group of shape $G.3$. The character table of $G.3$, and thus of $G^\diamond$ is contained in the Atlas [5] and in Gap [9]. It is easy to locate the conjugacy classes of $G^\diamond$ containing the elements of $\langle \alpha \rangle$. Assume that χ does not satisfy (4). Using Gap, we find that then either $\chi(1) = 175$ or $\chi(1) = 525$. In either case, we check that $\mathrm{Res}^{G^\diamond}_{\langle \alpha \rangle}(\chi^\diamond)$ contains each of the real irreducible characters of $\langle \alpha \rangle$ as constituents. Hence (G, V, n) has the $E1$-property by [14, Lemma 4.3.1]. This completes the proof in case $\chi(1) > q^9/8$.

Now assume that $\chi(1) \leq q^9/8$. Using [21], we find that χ lies in one of two series of irreducible characters of G of degrees $(q-1)^2(q^2+q+1)(q^2+1)$ and $(q+1)^2(q^2+1)(q^2-q+1)$. We claim that χ is not invariant under ν. This would contradict the fact that χ extends to $G^\diamond = \langle \mathrm{Inn}(G), \nu \rangle$; see [14, Remark 4.2.5]. Using [20], it is easy to see that $\chi \in \mathcal{E}(G, s)$, where s is a semisimple element of G^* with $C_{G^*}(s) \cong \mathrm{GU}_4(q)$ or $C_{G^*}(s) \cong \mathrm{GL}_4(q)$. In fact, we may assume that $C_{\overline{G}^*}(s)$ is a standard Levi subgroup of $\overline{G}^*$ of type A_3. There are three such Levi subgroups, which are mutually non-conjugate in $\overline{G}^*$. Put $\nu^* = \iota^* \circ \mu^*$, where ι^* and μ^* denote the standard graph automorphism of $\overline{G}^*$ of order 3, dual to ι, and the standard field automorphism of $\overline{G}^*$ dual to μ, respectively. Then ν^* is an isogeny of $\overline{G}^*$ dual to ν. Notice that ι^* permutes the three standard Levi subgroups of $\overline{G}^*$ of type A_3 cyclically. On the other hand, each standard Levi subgroup is fixed by φ^*. Hence $C_{\overline{G}^*}(\nu^*(s))$ is not conjugate in $\overline{G}^*$ to $C_{\overline{G}^*}(s)$. It follows that $\nu^*(s)$ and s are not G^*-conjugate. As ${}^\nu\chi \in \mathcal{E}(G, \nu^{*-1}(s))$ by [29, Proposition 7.2], it follows that ${}^\nu\chi \neq \chi$, as claimed. □

6. The linear and unitary groups

In this section, let G be one of the groups of Hypothesis 3.2(b)(i),(ii). Thus $G = \mathrm{PSL}_d^\varepsilon(q)$ with $d \geq 3$ and $q = 2^f$, where $q \neq 2$ if $d = 3$ and $\varepsilon = -1$. Put $e := \gcd(d, q - \varepsilon)$ and recall that $e > 1$ by hypothesis. Also recall the definition of the parameter δ from the introduction to Subsection 5.1. We will further use the notations and results introduced in Subsections 4.1–4.3.

6.1. **On the natural representation of the linear and unitary groups.** We extend our ε-convention to the general linear and unitary groups in the obvious way. We will need some results on semisimple elements of $\mathrm{GL}_d^{\varepsilon}(q)$ acting on its natural vector space. In view of Lusztig's Jordan decomposition of characters, the description of real semisimple elements is relevant for the classification of real irreducible characters.

Let $E := {\mathbb{F}_{q^\delta}}^d$ denote the natural column vector space for $\mathrm{GL}_d^{\varepsilon}(q)$. If $\varepsilon = -1$, the group $\mathrm{GU}_d(q) \leq \mathrm{GL}_d(q^2)$ is the stabilizer of the Hermitian form

$$((x_1, \ldots, x_d)^t, (x_1, \ldots, x_d)^t) \mapsto \sum_{i=1}^{d} x_i y_{d-i+1}^q \tag{5}$$

on E. A subspace E_1 is totally isotropic, if the hermitian form (5) vanishes on $E_1 \times E_1$. A pair $(E_1, E_1^\dagger)$ of totally isotropic subspaces of E is called a complementary pair, if $E_1 \oplus E_1^\dagger$ is non-degenerate (which implies that $\dim(E_1) = \dim(E_1^\dagger)$).

Let Δ be a monic irreducible polynomial over $\mathbb{F}_{q^\delta}$. We write Δ^* for the monic polynomial whose roots in $\mathbb{F}$ are the inverses of the roots of Δ. Then Δ^* is an irreducible polynomial over $\mathbb{F}_{q^\delta}$. If $\delta = 2$, we write $\Delta^\dagger$ for the monic polynomial whose roots are the $-q$ths powers of the roots of Δ. Then $\Delta^\dagger$ is an irreducible polynomial over $\mathbb{F}_{q^2}$. The notions Δ^* and $\Delta^\dagger$ are extended to all monic polynomials over $\mathbb{F}_{q^\delta}$ by multiplicativity. We collect a few formal properties of these notions.

Lemma 6.2. *Let Δ be a monic irreducible polynomial over $\mathbb{F}_{q^\delta}$ of degree k. Then the following hold.*

(a) *If $\Delta = \Delta^*$, then k is even, unless $k = 1$ and 1 is the root of Δ.*

(b) *If $\delta = 2$ and $\Delta = \Delta^\dagger$, then k is odd.*

Proof. (a) Let $\zeta \in \mathbb{F}$ be a root of Δ. As q is even, $\zeta \neq \zeta^{-1}$, unless $\zeta = 1$.

(b) Suppose that $\Delta = \Delta^\dagger$ and let $\zeta \in \mathbb{F}$ be a root of Δ. As Δ is irreducible over $\mathbb{F}_{q^2}$, the roots of Δ are $\zeta, \zeta^{q^2}, \ldots, \zeta^{q^{2(k-1)}}$, and k is the smallest positive integer such that $\zeta^{q^{2k}} = \zeta$.

As $\Delta = \Delta^\dagger$, there is $0 \leq i \leq k-1$ such that $\zeta^{-q} = \zeta^{q^{2i}}$. Hence $\zeta^{q(q^{2i-1}+1)} = 1$. This implies $\zeta^{q^{2i-1}+1} = 1$ and thus $\zeta^{q^{2(2i-1)}-1} = 1$. It follows that $\zeta^{q^{2(2i-1)}} = \zeta$ which implies that k is odd. □

Lemma 6.3. *Let $\hat{s}, \hat{s}' \in \mathrm{GL}_d(q^\delta)$ be semisimple with characteristic polynomial Ξ and Ξ', respectively.*

(a) *The elements $\hat{s}$ and $\hat{s}'$ are conjugate in $\mathrm{GL}_d(q^\delta)$, if and only if $\Xi = \Xi'$.*

(b) *The element* $\hat{s}$ *is real, if and only if* $\Xi = \Xi^*$.

(c) *Suppose that* $\delta = 2$ *and that* $\hat{s} \in \mathrm{GU}_d(q)$. *Then* $\Xi = \Xi^\dagger$.

(d) *Suppose that* $\delta = 2$ *and that* $\hat{s}, \hat{s}' \in \mathrm{GU}_d(q)$. *Then* $\hat{s}$ *and* $\hat{s}'$ *are conjugate in* $\mathrm{GU}_d(q)$, *if and only if they are conjugate in* $\mathrm{GL}_d(q^2)$.

Proof. These assertions are well known. □

Lemma 6.4. *Let* $\hat{s} \in \mathrm{GL}_d(q^\delta)$ *be semisimple and real and let* d_1 *denote the dimension of the eigenspace of* $\hat{s}$ *for the eigenvalue* 1. *Then* $d \equiv d_1 \pmod 2$.

Proof. The eigenvalues of $\hat{s}$ different from 1 come in pairs of mutually inverse elements. □

Definition 6.5. Let $\hat{s} \in \mathrm{GL}_d^\varepsilon(q)$ be semisimple, and let Θ be a monic factor of the minimal polynomial of $\hat{s}$. We then put $E_\Theta(\hat{s}) := \ker(\Theta(\hat{s}))$.

Lemma 6.6. *Put* $\hat{G} := \mathrm{GL}_d^\varepsilon(q)$. *Let* $\hat{s} \in \hat{G}$ *be semisimple with characteristic polynomial* Ξ, *and let* Δ *be a monic irreducible factor of* Ξ *of multiplicity* m *and degree* k.

(a) *The centralizer* $C_{\hat{G}}(\hat{s})$ *stabilizes* $E_\Theta(\hat{s})$ *for every monic factor* Θ *of the minimal polynomial of* $\hat{s}$. *In particular,* $C_{\hat{G}}(\hat{s})$ *stabilizes* $E_\Delta(\hat{s})$. *If* $\delta = 1$, *then* $C_{\hat{G}}(\hat{s})$ *induces* $\mathrm{GL}_m(q^k)$ *on* $E_\Delta(\hat{s})$.

(b) *Suppose that* $\delta = 2$ *and let* Δ' *be a monic irreducible factor of* Ξ *with* $\Delta' \neq \Delta^\dagger$. *Then* $E_{\Delta'}(\hat{s})$ *and* $E_\Delta(\hat{s})$ *are orthogonal.*

(c) *Suppose that* $\delta = 2$ *and that* $\Delta = \Delta^\dagger$. *Then* $E_\Delta(\hat{s})$ *is non-degenerate and* $C_{\mathrm{GU}_d(q)}(\hat{s})$ *induces* $\mathrm{GU}_m(q^k)$ *on* $E_\Delta(\hat{s})$.

(d) *Suppose that* $\delta = 2$ *and that* $\Delta \neq \Delta^\dagger$. *Then* $E_\Delta(\hat{s})$ *is totally isotropic. In this case,* $E_\Delta(\hat{s}) \oplus E_{\Delta^\dagger}(\hat{s})$ *is non-degenerate and* $C_{\mathrm{GU}_d(q)}(\hat{s})$ *induces* $\mathrm{GL}_m(q^{2k})$ *on* $E_\Delta(\hat{s}) \oplus E_{\Delta^\dagger}(\hat{s})$.

Proof. These assertions are well known and easily proved; see e.g. [8, Proposition (1A)]. □

Lemma 6.7. *Let* $\hat{G} = \mathrm{GU}_d(q)$ *for some* $d \geq 2$. *Let* $1 \neq \hat{s} \in \hat{G}$ *be semisimple and real with characteristic polynomial* Ξ. *Let* Δ_1 *denote the monic polynomial of degree* 1 *with root* 1, *and let* d_1 *be the multiplicity of* Δ_1 *in* Ξ. *Let* $\Delta \neq \Delta_1$ *denote a monic irreducible factor of* Ξ *of degree* k *and multiplicity* m. *Then the following hold.*

(a) *If* $\Delta = \Delta^\dagger$, *then* $\Delta \neq \Delta^*$, *and* Δ^* *occurs with multiplicity* m *in* Ξ.

(b) *If* $\Delta \neq \Delta^\dagger$, *then* $\Delta^\dagger$ *occurs with multiplicity* m *in* Ξ.

(c) *We have* $d \geq d_1 + 2mk$.

(d) *We have* $|C_{\mathrm{GU}_d(q)}(\hat{s})|_{2'} \leq |\mathrm{GU}_m(q^k)|_{2'}^2 |\mathrm{GU}_{d-2mk}(q)|_{2'}$.

(e) *Let d' denote the greatest multiplicity of an irreducible factor of Ξ. If $d' = 1$, then $|C_{\mathrm{GU}_d(q)}(\hat{s})|_{2'} \leq (q+1)^d$. If $d' = 2$, then $|C_{\mathrm{GU}_d(q)}(\hat{s})|_{2'} \leq |\mathrm{GU}_2(q)|_{2'}^{\lfloor d/2 \rfloor}(q+1)^{\overline{d}}$ with $\overline{d} = d - 2\lfloor d/2 \rfloor$.*

Proof. The assertion in (c) follows from those in (a) and (b). To prove (a), observe that k is odd if $\Delta = \Delta^\dagger$; see Lemma 6.2(b). Lemma 6.2(a) then gives $\Delta \neq \Delta^*$. The remaining assertion follows from Lemma 6.3(b). The proof for (b) is analogous.

Let us now prove (d) and (e). If $\Delta = \Delta^\dagger$, then $\Delta \neq \Delta^*$ by (b) and we put $\Delta^\circ := \Delta^*$. If $\Delta \neq \Delta^\dagger$, we put $\Delta^\circ := \Delta^\dagger$. Then $E_1 := E_\Delta(\hat{s}) \oplus E_{\Delta^\circ}(\hat{s})$ is non-degenerate by Lemma 6.6(c)(d), and $C_{\mathrm{GU}_d(q)}(\hat{s})$ induces $\mathrm{GU}_m(q^k) \times \mathrm{GU}_m(q^k)$, respectively $\mathrm{GL}_m(q^{2k})$ on E_1. Let E_2 denote the orthogonal complement of E_1. Then $C_{\mathrm{GU}_d(q)}(\hat{s})$ fixes E_2 by Lemma 6.6(a)(b). Write $\hat{s} = \hat{s}_1 + \hat{s}_2$, where $\hat{s}_j$ is the projection of $\hat{s}$ to E_j, $j = 1, 2$. We then have, with a slight abuse of notation, $C_{\mathrm{GU}_d(q)}(\hat{s}) = C_{\mathrm{GU}_{2mk}(q)}(\hat{s}_1) \times C_{\mathrm{GU}_{d-2mk}(q)}(\hat{s}_2)$. The claim in (d) follows from $|\mathrm{GL}_m(q^{2k})|_{2'} < |\mathrm{GU}_m(q^k)|_{2'}^2$, which is easily proved by induction on m. To see (e), first assume that $d' = 1$. Clearly, $|\mathrm{GU}_1(q^k)|_{2'} = q^k + 1 \leq (q+1)^k$, and so the claim follows by induction on d. If $d' = 2$, suppose first that $d' = d_1$ and that all irreducible factors of Ξ different from Δ_1 occur with multiplicity 1. Applying the first part of the statement to the orthogonal complement of $E_{\Delta_1}(\hat{s})$, we obtain $|C_{\mathrm{GU}_d(q)}(\hat{s})|_{2'} \leq |\mathrm{GU}_2(q)|_{2'}(q+1)^{d-2}$, and thus, as $(q+1)^2 \leq |\mathrm{GU}_2(q)|_{2'}$, our assertion. If some irreducible factor of Ξ different from Δ_1 occurs with multiplicity 2, we can choose Δ such that $m = 2$. Now $|\mathrm{GU}_2(q^k)|_{2'} = (q^k+1)(q^{2k}-1) \leq (q+1)^k(q^2-1)^k = |\mathrm{GU}_2(q)|_{2'}^k$, where the middle inequality follows by induction on k. We are done by induction on d. ∎

Let $g \in \mathrm{PGL}_d^\varepsilon(q)$. An inverse image $\hat{g} \in \mathrm{GL}_d^\varepsilon(q)$ under the canonical epimorphism $\mathrm{GL}_d^\varepsilon(q) \to \mathrm{PGL}_d^\varepsilon(q)$ will be called a lift of g.

Lemma 6.8. *Let $s \in \mathrm{PGL}_d^\varepsilon(q)$ be semisimple and real. Then there exists a real lift $\hat{s} \in \mathrm{GL}_d^\varepsilon(q)$ of s.*

Proof. Let $y \in \mathrm{PGL}_d^\varepsilon(q)$ with $ysy^{-1} = s^{-1}$, and let $\hat{s}, \hat{y} \in \mathrm{GL}_d^\varepsilon(q)$ denote lifts of s and y, respectively. Then there is $\zeta \in \mathbb{F}_{q^\delta}^*$ such that $\hat{y}\hat{s}\hat{y}^{-1} = \zeta\hat{s}^{-1}$. Moreover, $\zeta^{q+1} = 1$ if $\delta = 2$. Let $\xi \in \mathbb{F}_{q^\delta}^*$ with $\xi^{-2} = \zeta$. (Recall that q is even.) Thus $\xi\hat{s} \in \mathrm{GL}_d^\varepsilon(q)$ is a lift of s and $\hat{y}(\xi\hat{s})\hat{y}^{-1} = (\xi\hat{s})^{-1}$. ∎

6.9. **Automorphisms, II.** Here, we extend the notations introduced in Subsections 4.3 in case of $\overline{G} = \mathrm{PGL}_d(\mathbb{F})$. Writing $\overline{\hat{G}} := \mathrm{GL}_d(\mathbb{F})$, we get $\overline{G} = \overline{\hat{G}}/Z(\overline{\hat{G}})$. This allows a convenient descriptions of $\mathrm{Aut}_1(\overline{G})$ using the natural matrix representations of $\overline{\hat{G}}$. Let $\hat{\varphi}$ denote the Steinberg morphism of $\overline{\hat{G}}$ which squares every matrix entry, and let $\hat{\iota}$ denote the automorphism of $\overline{\hat{G}}$, defined as the inverse-transpose automorphism, followed by conjugation with the matrix with 1s along the anti-diagonal, and 0s, elsewhere. Then $\hat{\iota}$ is an involution that commutes with $\hat{\varphi}$. Let $\Gamma_{\overline{\hat{G}}} := \langle \hat{\iota} \rangle \leq \mathrm{Aut}(\overline{\hat{G}})$ and $\Phi_{\overline{\hat{G}}} := \langle \hat{\varphi} \rangle \leq \mathrm{Aut}(\overline{\hat{G}})$, where $\mathrm{Aut}(\overline{\hat{G}})$ denotes the automorphism group of $\overline{\hat{G}}$ as an abstract group. Then $\Phi_{\overline{\hat{G}}} \times \Gamma_{\overline{\hat{G}}} \leq \mathrm{Aut}(\overline{\hat{G}})$ and $(\Gamma_{\overline{\hat{G}}} \times \Phi_{\overline{\hat{G}}}) \cap \mathrm{Inn}(\overline{\hat{G}})$ is trivial. We put

$$\mathrm{Aut}'(\overline{\hat{G}}) := \mathrm{Inn}(\overline{\hat{G}}) \rtimes (\Gamma_{\overline{\hat{G}}}\Phi_{\overline{\hat{G}}}).$$

Write ι and φ for the automorphisms of $\overline{G}$ induced by $\hat{\iota}$, respectively $\hat{\varphi}$. Then ι and φ are the standard graph automorphism, respectively the standard Frobenius morphism of $\overline{G}$ as in Subsection 4.1.

The natural homomorphism

$$\mathrm{Aut}'(\overline{\hat{G}}) \to \mathrm{Aut}_1(\overline{G}) \tag{6}$$

is in fact and isomorphism. For $\mu \in \mathrm{Aut}_1(\overline{G})$ we write $\hat{\mu}$ for its inverse image in $\mathrm{Aut}'(\overline{\hat{G}})$ under (6).

6.10. **Further preliminary results.** We collect more preliminary results.

Lemma 6.11. (a) *Some odd prime divides d, and if $\varepsilon = -1$, then some odd prime different from 7 divides d.*

(b) *If $q \geq 8$, then $q^3 \geq 2f(q+1)^2$ and $1 - q^{-1} - q^{-2} \geq q^{-1/4}$.*

(c) *If $q = 4$, we have $q^4 > 2f(q+1)^2$ and $1 - q^{-1} - q^{-2} \geq q^{-1/2}$.*

(d) *If $q = 2$, we have $q^5 > 2f(q+1)^2$ and $1 - q^{-1} - q^{-2} \geq q^{-2}$.*

(e) *We have $|\mathrm{GL}_d(q)| \leq q^{d^2}$ and $|\mathrm{GU}_d(q)| \leq (1 - q^{-1} - q^{-2})^{-1} q^{d^2}$. If $q \geq 4$, then $|\mathrm{GU}_d(q)| \leq q^{d^2+1/2}$, and if $q = 2$, then $|\mathrm{GU}_d(q)| \leq q^{d^2+2}$.*

Proof. (a) The first assertion is clear since $\gcd(d, q - \varepsilon) > 1$ and $q - \varepsilon$ is odd. Also, $7 \nmid q + 1$, as $2^f \bmod 7 \in \{1, 2, 4\}$. This yields the second assertion.

(c)–(d) These assertions are trivially verified.

(e) This follows from the known order formula for $\mathrm{GL}_d^{\varepsilon}(q)$, together with Lemma 2.1 and the estimates in (b)–(d). □

Lemma 6.12. *Let $t \in \overline{G}^{\sigma} = \mathrm{PGL}_d^{\varepsilon}(q)$ be semisimple and let $\hat{t} \in \mathrm{GL}_d^{\varepsilon}(q)$ denote a lift of t. Then $|C_{\overline{G}^{\sigma}}(t)| \leq |C_{\mathrm{GL}_d^{\varepsilon}(q)}(\hat{t})|$.*

Proof. Put $C := C_{\mathrm{GL}_d^\varepsilon(q)}(\hat{t})$ and let $\tilde{C}$ denote the inverse image of $C_{\overline{G}^\sigma}(t)$ in $\mathrm{GL}_d^\varepsilon(q)$. Then $C \leq \tilde{C}$ and $[\tilde{C}\colon C] \leq q - \varepsilon$. Indeed, the map $\tilde{C} \to Z(\mathrm{GL}_d^\varepsilon(q))$, $g \mapsto [g, t]$ is a homomorphism with kernel C. As $|C_{\overline{G}^\sigma}(t)| = |\tilde{C}|/(q - \varepsilon)$, our assertion follows. □

Let $u \in \mathrm{GL}_d(q^\delta)$ be an involution. We say that u consists of l Jordan blocks of size 2 if the Jordan normal form of u has l Jordan blocks of size (2×2) and $d - 2l$ Jordan blocks of size (1×1).

Lemma 6.13. *Let $u \in \mathrm{GL}_d^\varepsilon(q)$ be an involution consisting of l Jordan blocks of size 2 and let $C := C_{\mathrm{GL}_d^\varepsilon(q)}(u)$. Then C is a semidirect product $C = UL$ with a unipotent radical U of order q^{2ld-3l^2} and a complement $L \cong \mathrm{GL}_l^\varepsilon(q) \times \mathrm{GL}_{d-2l}^\varepsilon(q)$.*

In particular, $|C| \leq q^{2l^2-2ld+d^2}$ if $\varepsilon = 1$, and $|C| \leq q^{2l^2-2ld+d^2+4}/(q^2 - q - 1)^2$ if $\varepsilon = -1$.

Proof. It is clear that u is conjugate in $\mathrm{GL}_d(q^\delta)$ to the matrix

$$u' := \begin{bmatrix} \mathrm{Id}_l & 0 & \mathrm{Id}_l \\ 0 & \mathrm{Id}_{d-2l} & 0 \\ 0 & 0 & \mathrm{Id}_l \end{bmatrix}$$

Observe that $u' \in \mathrm{GU}_d(q)$. Thus u is conjugate to u' in $\mathrm{GU}_d(q)$ if $\varepsilon = -1$. We may thus replace u by u'. A routine matrix calculation then yields our assertion on the structure of C.

The estimates for $|C|$ follow from Lemma 2.1. □

6.14. **Modification of ν.** We now prove a crucial result on centralizers of certain automorphisms of G. The aim is the modification of the automorphism ν by an inner automorphism in order to minimize the centralizer orders of its powers. A priory, ν can be any element of $\mathrm{Aut}(G)$. Recall Definition 5.2 for the quantity M_G.

Proposition 6.15. *Let $\beta \in \mathrm{Aut}(G)$. Then there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \beta$ has even order and the following statements hold.*

(a) *We have $|C_G(\alpha_{(p)})| < M_G$ for every prime p dividing $|\alpha|$ and every element $\alpha_{(p)}$ of $\langle \alpha \rangle$ of order p.*

(b) *If $G = \mathrm{PSL}_d^\varepsilon(q)$ with $d \geq 5$, then $|\alpha|$ divides $\delta f(q - \varepsilon)$.*

(c) *If $G = \mathrm{PSL}_3^\varepsilon(q)$, then $|\alpha|$ divides $3\delta f$.*

(d) *For the specified values of ε, d and q, the order of α and the structure of $G^\diamond := \langle \mathrm{Inn}(G), \beta\rangle$ are as given in the following table.*

ε	d	q	$\|\alpha\|$	$G^\diamond$
1	3	4	6	$G.3 \cong \mathrm{PGL}_3(4)$
1	3	4	6	$G.6$
-1	3	8	6	$G.3$
-1	3	8	6	$G.6$
-1	3	8	18	$G.3$
-1	3	32	30	$\mathrm{PGU}_3(32) \cong \langle \mathrm{Inn}(G), \alpha^{10}\rangle$
-1	≥ 5	2	$2, 6$	
-1	≥ 5	4	$2, 4, 10$	
-1	6	2	6	$G.3 \cong \mathrm{PGU}_6(2)$
-1	6	8	$2, 6, 18$	

Proof. By Lemma 6.11(a), we have $d \notin \{4, 8, 16\}$, and if $\varepsilon = -1$, we also have $d \notin \{7, 14\}$. We represent the elements of $\overline{G}^\sigma = \mathrm{PGL}_d^\varepsilon(q)$ by elements in $\mathrm{GL}_d^\varepsilon(q)$ and make use of the notation introduced in Subsection 6.9. Let $\hat{t} \in \mathrm{GL}_d^\varepsilon(q)$ be a diagonal matrix. Then $\hat{\varphi}(\hat{t}) = \hat{t}^2$, and $\hat{\iota}(\hat{t})$ is obtained from $\hat{t}$ by reversing the order of its diagonal entries and inverting them. In particular, if $\hat{t}$ is palindromic, i.e. invariant under reversing its diagonal entries, then $\hat{\iota}(\hat{t}) = \hat{t}^{-1}$.

Recall that β has a factorization as $\beta = \mathrm{ad}_h \circ \mu$ with $h \in \mathrm{PGL}_d^\varepsilon(q)$ and $\mu \in \Gamma_G \times \Phi_G$, and that h and μ are uniquely determined by β. If $d = 3$, we may assume that $|h|$ is a 3-power by replacing h with $g'h$ for a suitable $g' \in G$. Let $\hat{h} \in \mathrm{GL}_d^\varepsilon(q)$ represent h. If $d = 3$, assume that $|\hat{h}|$ is a 3-power. Notice that, in any case, $\det(\hat{h})$ lies in the subgroup of $\mathbb{F}_{q^\delta}^*$ of order $q - \varepsilon$. We now choose a particular diagonal palindromic element $\hat{t} \in \mathrm{GL}_d^\varepsilon(q)$ with $\det(\hat{t}) = \det(\hat{h})$ as follows. For $3 \leq d \leq 6$, choose

$$\hat{t} := \begin{cases} \mathrm{diag}(\zeta, 1, \zeta), & \text{if } d = 3, \\ \mathrm{diag}(\zeta, 1, 1, 1, \zeta), & \text{if } d = 5, \\ \mathrm{diag}(\zeta, \zeta, 1, 1, \zeta, \zeta), & \text{if } d = 6, \\ \mathrm{diag}(\zeta, \zeta, 1, 1, 1, \zeta, \zeta), & \text{if } d = 7 \text{ (and } \varepsilon = 1), \end{cases}$$

where $\zeta \in \mathbb{F}_{q^\delta}^*$ is such that $\det(\hat{t}) = \det(\hat{h})$. This is possible since q is even. Notice that $\zeta^{q-\varepsilon} = 1$, so that $\hat{t}$ indeed lies in $\mathrm{GL}_d^\varepsilon(q)$. Notice also that $\hat{t} = 1$ if $h \in G$. Now suppose that $d \geq 9$. Define $\bar{d} \in \mathbb{Z}$ by $\bar{d} := 1$, if d is odd, and $\bar{d} := 2$, if d is even. Put $d' := \lfloor (d - \bar{d})/4 \rfloor$, so that

$d - \bar{d} = 4d'$ if $d - \bar{d}$ is divisible by 4, and $d - \bar{d} = 4d' + 2$, otherwise. Choose $\zeta \in \mathbb{F}_{q^\delta}^*$ of order $q - \varepsilon$. Put

$$\hat{t} := \begin{cases} \mathrm{diag}(\zeta \mathrm{Id}_{d'}, \mathrm{Id}_{d'}, \xi \mathrm{Id}_{\bar{d}}, \mathrm{Id}_{d'}, \zeta \mathrm{Id}_{d'}), & \text{if } d - \bar{d} = 4d', \\ \mathrm{diag}(\zeta \mathrm{Id}_{d'}, \mathrm{Id}_{d'}, \zeta^{-1}, \xi \mathrm{Id}_{\bar{d}}, \zeta^{-1}, \mathrm{Id}_{d'}, \zeta \mathrm{Id}_{d'}), & \text{otherwise.} \end{cases}$$

Here, ξ is chosen as a power of ζ in such a way that $\det(\hat{t}) = \det(\hat{h})$. Once more this is possible since q is even. As $\hat{t}$ is palindromic, $\hat{\mu}(\hat{t})$ is a power of $\hat{t}$ by what we have said above. In turn, $N_{\hat{\mu},l}(\hat{t})$ is a power of $\hat{t}$ for all integers l. Finally, $\hat{t} \in \mathrm{GL}_d^\varepsilon(q)$, as $\zeta^{q-\varepsilon} = 1$.

As $\det(\hat{t}) = \det(\hat{h})$, there is $\hat{y} \in \mathrm{SL}_d^\varepsilon(q)$ such that $\hat{y}\hat{h} = \hat{t}$. Let t and y denote the image of $\hat{t}$, respectively $\hat{y}$ in $\mathrm{PGL}_d^\varepsilon(q)$ and notice that $y \in G$. Put $\beta' := \mathrm{ad}_y \circ \beta = \mathrm{ad}_t \circ \mu$. As $\langle t \rangle$ is μ-invariant, Equation (1) shows that $|\beta'|$ divides $|t||\mu|$, and that $|\mu|$ divides $|\beta'|$. Since $|t|$ is odd, this implies that $|\beta'|$ is even if and only if $|\mu|$ is even. We now distinguish two cases. Suppose first that $|\mu|$ is even. Then $|\beta'|$ is even and we put $\alpha := \beta'$. Now suppose that $|\mu|$ is odd so that $|\beta'|$ is odd. Then μ is a power of the field automorphism φ. Choose an involution $\hat{z} \in \mathrm{SL}_d^\varepsilon(2) \leq \mathrm{GL}_d^\varepsilon(q)$ which commutes with $\hat{t}$ and whose number of Jordan blocks of size 2 is as large as possible. From the structure of $C_{\mathrm{GL}_d^\varepsilon(q)}(\hat{t})$ (see Lemma 6.6(c)), we conclude that we can choose $\hat{z}$ to have $\lfloor d/2 \rfloor$ Jordan blocks of size 2. Let $z \in G$ denote the image of $\hat{z}$, and put $\alpha := \mathrm{ad}_z \circ \beta' = \mathrm{ad}_{zt} \circ \mu$. Now ad_z commutes with β', as $\varphi(z) = z$ and z commutes with t. In particular, $|\alpha| = 2|\beta'|$ and $\alpha_{(p)} = \beta'_{(p)}$ for all odd primes p dividing $|\alpha|$. Clearly, $\alpha_{(2)} = \mathrm{ad}_z$.

(a) Let p denote a prime dividing $|\beta'|$. We claim that $|C_G(\beta'_{(p)})| \leq M_G$. If $\beta'_{(p)}$ is not an inner-diagonal automorphism of G, the claim follows from Lemma 5.3. Suppose then that $\beta'_{(p)}$ is an inner-diagonal automorphism. As $\beta'_{(p)}$ is a power of β', Equation (1) yields $\beta'_{(p)} = \mathrm{ad}_{N_{\mu,l}(t)}$ for some positive inter l. In particular, p is odd and $|N_{\mu,l}(t)| = p$. Also, $C_G(\beta'_{(p)}) = C_G(N_{\mu,l}(t))$. As $N_{\hat{\mu},l}(\hat{t})$ is a lift of $N_{\mu,l}(t))$, we obtain

$$|C_G(N_{\mu,l}(t))| \leq |C_{\mathrm{GL}_d^\varepsilon(q)}(N_{\hat{\mu},l}(\hat{t}))|$$

by Lemma 6.12. This bound is sufficient for our purpose, except if $d = 3$. Recall that $N_{\hat{\mu},l}(\hat{t}) = \hat{t}^k$ for some integer k. Since $N_{\hat{\mu},l}(\hat{t}) \neq 1$, and the eigenvalues of $\hat{t}$ lie in $\langle \zeta \rangle$, we have $\zeta^k \neq 1$. Also, $\zeta^k \neq \zeta^{-k}$ as q is even. However, we may have $\xi^k = \zeta^k$ or $\xi^k = 1$. This allows to estimate the dimensions of the eigenspaces of $N_{\hat{\mu},l}(\hat{t})$. Notice that the eigenvalues ζ' of $N_{\hat{\mu},l}(\hat{t})$ satisfy $\zeta'^{q-\varepsilon} = 1$. Lemma 6.6(c) implies that $C_{\mathrm{GL}_d^\varepsilon(q)}(N_{\hat{\mu},l}(\hat{t}))$ is a direct product of groups $\mathrm{GL}_{d_i}^\varepsilon(q)$, where d_i is the dimension of an eigenspace of $N_{\hat{\mu},l}(\hat{t})$. If $d = 3$, we have $N_{\hat{\mu},l}(\hat{t}) = \mathrm{diag}(\zeta', 1, \zeta')$ for some ζ' of order p so that $C_{\mathrm{GL}_d^\varepsilon(q)}(N_{\hat{\mu},l}(\hat{t})) \cong \mathrm{GL}_1^\varepsilon(q) \times \mathrm{GL}_2^\varepsilon(q)$. Clearly, every element of $\mathrm{GL}_3^\varepsilon(q)$, which commutes with $\mathrm{diag}(\zeta', 1, \zeta')$ up to a scalar, in fact commutes with $\mathrm{diag}(\zeta', 1, \zeta')$. Thus $C_{\overline{G}^\sigma}(N_{\mu,l}(t))$ is the image of $C_{\mathrm{GL}_3^\varepsilon(q)}(N_{\hat{\mu},l}(\hat{t}))$ under the canonical epimorphism. It follows that $|C_G(N_{\mu,l}(t))| \leq |C_{\overline{G}^\sigma}(N_{\mu,l}(t))| = |\mathrm{GL}_2^\varepsilon(q)| = q(q-\varepsilon)(q^2 - 1) < M_G$, as claimed in (a). Suppose now that $d > 3$. For $d = 5, 6$, we obtain $C_{\mathrm{GL}_d^\varepsilon(q)}(N_{\hat{\mu},l}(\hat{t})) \cong \mathrm{GL}_2^\varepsilon(q) \times \mathrm{GL}_3^\varepsilon(q)$, respectively $\mathrm{GL}_2^\varepsilon(q) \times \mathrm{GL}_4^\varepsilon(q)$. Using the exact values of the orders of theses groups, we obtain our claim in (a). For $d = 7$ we have $\varepsilon = 1$ and $C_{\mathrm{GL}_d(q)}(N_{\hat{\mu},l}(\hat{t})) \cong \mathrm{GL}_3(q) \times \mathrm{GL}_4(q)$. In this case, the claim follows from Lemma 6.11(e). Now suppose that $d \geq 9$. If $d - \bar{d} = 4d'$, then $C_{\mathrm{GL}_d^\varepsilon(q)}(N_{\mu,l}(\hat{t}))$ is isomorphic to one of $\mathrm{GL}_{2d'}^\varepsilon(q) \times \mathrm{GL}_{2d'+\bar{d}}^\varepsilon(q)$ or $\mathrm{GL}_{2d'}^\varepsilon(q) \times \mathrm{GL}_{2d'}^\varepsilon(q) \times \mathrm{GL}_{\bar{d}}^\varepsilon(q)$. If $d - \bar{d} = 4d' + 2$, then $C_{\mathrm{GL}_d^\varepsilon(q)}(N_{\hat{\mu},l}(\hat{t}))$ is isomorphic to one of $\mathrm{GL}_2^\varepsilon(q) \times \mathrm{GL}_{2d'}^\varepsilon(q) \times \mathrm{GL}_{2d'+\bar{d}}^\varepsilon(q)$ or $\mathrm{GL}_2^\varepsilon(q) \times \mathrm{GL}_{2d'}^\varepsilon(q) \times \mathrm{GL}_{2d'}^\varepsilon(q) \times \mathrm{GL}_{\bar{d}}^\varepsilon(q)$. In each case, the first named possibility for the centralizer has a larger order than the second. For $d > 10$, the estimate $|\mathrm{GL}_l^\varepsilon(q)| \leq q^{l^2+2}$ from Lemma 6.11(e) yields our bounds. For $d = 9, 10$ and $q \geq 4$, we use $|\mathrm{GL}_l^\varepsilon(q)| \leq q^{l^2+1/2}$, and for $d = 9, 10$ and $q = 2$, we use the exact values for $|\mathrm{GL}_l^\varepsilon(q)|$ to obtain our claim. This proves (a) in case $|\mu|$ is even.

Now suppose that $|\mu|$ is odd so that $\alpha = \mathrm{ad}_z \circ \beta'$. We have to show that $|C_G(\alpha_{(2)})| = |C_G(z)| \leq M_G$. If $d = 3$, then $|C_{\mathrm{GL}_3^\varepsilon(q)}(\hat{z})| = q^3(q-\varepsilon)^2$ by Lemma 6.13. The same lemma implies that $|C_{\mathrm{GL}_d^\varepsilon(q)}(\hat{z})| \leq q^{(d^2+1)/2}$ if $\varepsilon = 1$, and $|C_{\mathrm{GL}_d^\varepsilon(q)}(\hat{z})| \leq q^{(d^2+9)/2}/(q^2-q-1)^2$, if $\varepsilon = -1$. This implies that $|C_{\mathrm{GL}_d^\varepsilon(q)}(\hat{z})| \leq q^{d(d+1)/2}$ in all cases, except for $\varepsilon = -1$, $q = 2$ and $d \leq 8$. If $q = 2$, we have $3 \mid d$ so that we are left with $d = 6$ by Lemma 6.11(a) (recall that $(d, q) = (3, 2)$ is excluded by hypothesis, as $\mathrm{PSU}_3(2)$ is not a simple group). If $d = 6$, the involution $\hat{z}$ has 3 Jordan blocks of size 2, and hence $|C_{\mathrm{GL}_6^\varepsilon(q)}(\hat{z})| = q^9|\mathrm{GU}_3(q)|$ by Lemma 6.13. Inserting the exact values for $q = 2$, we obtain $|C_{\mathrm{GL}_6^\varepsilon(q)}(\hat{z})| \leq q^{d(d+1)/2}$ for $q = 2$ as well. Clearly, $C_{\overline{G}^\sigma}(z)$ is the image of $C_{\mathrm{GL}_d^\varepsilon(q)}(\hat{z})$ in $\overline{G}^\sigma$. It follows that $|C_G(\alpha_{(2)})| = |C_G(z)| \leq M_G$, thus completing the proof of (a).

(b) Suppose first that $|\mu|$ is even, so that $|\alpha| = |\beta'|$. Since $|\beta'|$ divides $|t||\mu|$ and $|t|$ and $|\mu|$ divide $q - \varepsilon$ and δf, respectively, we obtain (b). Now suppose that $|\mu|$ is odd, so that $|\alpha| = 2|\beta'|$. Since f is even if $\varepsilon = 1$, this implies that $|\mu|$ divides $\delta f/2$, so that $|\beta'|$ divides $\delta f(q-\varepsilon)/2$, which yields (b).

(c) By Lemma 5.4(b), the order of β' divides $3\delta f$, and $3\delta f/2$ if $|\beta'|$ is odd. This yields the assertion for $|\alpha|$.

(d) If $d = 3$, $\varepsilon = 1$ and $q = 4$, then $\Gamma_G \times \Phi_G$ is elementary abelian of order 4. In addition, $|\alpha| \in \{2, 6\}$ by (c). Suppose that $|\alpha| = 6$. If $|\beta'|$ is odd, then $\beta' = \mathrm{ad}_t$ and $G^\diamond \cong \mathrm{PGL}_3(4)$. If $|\beta'|$ is even, then $\alpha = \mathrm{ad}_t \circ \mu$ with $|t| = 3$ and μ an involution fixing t. As ι and φ invert t, we must have $\mu = \iota \circ \varphi$. This yields the second option for $G^\diamond$.

Suppose that $G = \mathrm{PSU}_3(8)$. Then $|\alpha| \in \{2, 6, 18\}$ by (c). We have $\beta' = \mathrm{ad}_t \circ \mu$ for some $t \in \mathrm{PGU}_3(8)$ of 3-power order and some $\mu \in \Phi_G$. Recall that $t = 1$ if $t \in G$. Hence either $t = 1$ or $|t| = 9$. If $|\beta'| = 3$ or $|\beta'| = 9$, then $|\mu| \in \{1, 3\}$. In either case, $\alpha \notin \mathrm{Inn}(G)$ but $\alpha^3 \in \mathrm{Inn}(G)$. Thus $G^\diamond \cong G.3$. If $|\alpha| = 18$, then $|\beta'| = 9$ by Lemma 5.4(a)(c). We are left with the case that $|\alpha| = |\beta'| = 6$. Then $|\mu| = 6$ and $\alpha = \beta'$. Hence $G^\diamond \cong G.6$.

Suppose that $G = \mathrm{PSU}_3(32)$ and $|\alpha| = 30$. Then $|\beta'| = 15$ by Lemma 5.4(c). It follows that $\alpha^{10} = \beta'^5 = \mathrm{ad}_{t'}$ for an element $t' \in \mathrm{PGU}_3(32) \setminus G$. This proves our assertion.

Suppose that $G = \mathrm{PSU}_d(4)$ for some $d \geq 5$. Then $|\beta'| \in \{1, 2, 4, 5\}$ by Lemma 5.4(c), which yields the claim for $|\alpha|$. The analogous proof works for $G = \mathrm{PSU}_d(2)$ for some $d \geq 5$ and $G = \mathrm{PSU}_6(8)$. If $G = \mathrm{PSU}_6(2)$ and $|\alpha| = 6$, then $|\beta'| = 3$ by Lemma 5.4(c), and thus $\beta' \in \mathrm{PGU}_6(2)$. □

6.16. **The linear groups of degree** 3**.** We can now prove that the linear groups of degree 3 have the $E1$-property.

Proposition 6.17. *Let* $G = \mathrm{PSL}_3^\varepsilon(q)$ *and let* (V, n, ν) *be as in* [14, Notation 4.1.1]*. Then* (G, V, n) *has the* $E1$*-property.*

Proof. Let χ denote the character of V. The character table of G is available in [28]. The irreducible characters of G of degree $q^2 + \varepsilon q + 1$ are not real. Thus χ is one of the remaining irreducible character of odd degree, and hence

$$\chi(1) > (q^3 - 2q^2)/3$$

by [28]. For these characters we are going to apply [14, Lemma 4.3.3], with α constructed from $\beta := \nu$ according to Proposition 6.15. Thus α has even order and $|C_G(\alpha_{(p)})| \leq M_G$ for all primes p dividing $|\alpha|$, where M_G is as in Definition 5.2(c). Moreover, $|\alpha|$ divides $3\delta f$ by Proposition 6.15(c). We also put $G^\diamond := \langle \mathrm{Inn}(G), \alpha \rangle = \langle \mathrm{Inn}(G), \nu \rangle$, and let $\chi^\diamond$ denote the extension of χ to $G^\diamond$; [14, Remark 4.2.5].

Suppose first that $\varepsilon = 1$. We aim to show that

$$(q^3 - 2q^2)/3 > (|\alpha| - 1)M_G^{1/2}. \quad (7)$$

Squaring, we see that (7) will follow from

$$q^6 - 4q^5 + 4q^4 > 9(|\alpha| - 1)^2 M_G.$$

If $q > 64$, we have $|\alpha| - 1 < 3f < q/4$ and $M_G = q^4$, and it is enough to show that

$$16(q^6 - 4q^5 + 4q^4) > 9q^6.$$

This is equivalent to

$$q^5(7q - 64) + 64q^4 > 0,$$

which is obviously true for $q > 64$. It remains to consider the cases $q \in \{4, 16, 64\}$. For $q = 64$, we have $M_G = q^4$ and $|\alpha| \leq 18$. With this, (7) holds. If $q = 16$, we have $|\alpha| \leq 6$ and $M_G = 62\,401$. Using the exact values for $\chi(1)$ from [28], we obtain $\chi(1) > 5 \cdot 62\,401^{1/2}$, which gives our result. Suppose that $q = 4$. Then $\chi(1) \in \{35, 63\}$ by the Atlas [5], and $|\alpha| \in \{2, 6\}$. If $|\alpha| = 2$, then (7) holds. Suppose then that $|\alpha| = 6$. By Proposition 6.15(d), either $G^\diamond \cong \mathrm{PGL}_3(4)$ or $G^\diamond \cong G.6$. According to the Atlas [5], no group of the latter shape has a character of degree 35 or 63. If $G^\diamond \cong \mathrm{PGL}_3(4)$, then $\chi(1) = 63$ and $\mathrm{Res}^{G^\diamond}_{\langle\alpha\rangle}(\chi^\diamond)$ contains each of the two irreducible real characters of $\langle\alpha\rangle$ with positive multiplicity; see the Atlas [5]. Hence (G, V, n) has the $E1$-property by [14, Lemma 4.3.1].

Suppose now that $\varepsilon = -1$. Then $M_G = q^4 + q^3$ and $|\alpha|$ divides $6f$. Suppose first that $f \geq 21$. Then $(6f - 1)^3 < 2^f$ and so $(|\alpha| - 1) < q^{1/3}$. By [14, Lemma 4.3.3], we have to show that

$$(q^3 - 2q^2)/3 > q^{1/3}(q^4 + q^3)^{1/2}, \tag{8}$$

as then

$$\dim(V) \geq (q^3 - 2q^2)/3 > q^{1/3}(q^4 + q^3)^{1/2} > (|\alpha| - 1)(q^4 + q^3)^{1/2}.$$

Squaring (8), it suffices to show that

$$q^6 - 4q^5 + 4q^4 > 9q^{2/3}(q^4 + q^3).$$

Noting that $4q^4 > 0$ and dividing by $q^{2/3}q^3$, it suffices to show that

$$q^{4/3}(q - 4) > 9(q + 1).$$

This is the case, as $q \geq 2^{21}$. For $q = 2^f$ with $7 \leq f \leq 19$, a GAP computation shows that

$$(q^3 - 2q^2)^2 > 9(6f - 1)^2(q^4 + q^3)$$

which proves our assertion, except for $q = 32$ or $q = 8$.

Suppose first that $q = 32$. Then $|\alpha| \leq 10$ or $|\alpha| = 30$. Now $(q^3 - 2q^2)^2 > 9(10 - 1)^2(q^4 + q^3)$, so that (G, V, n) has the $E1$-property if $|\alpha| \leq 10$. Suppose that $|\alpha| = 30$. Then $\mathrm{PGU}_3(q) \cong \langle G, \alpha^{10}\rangle \leq G^\diamond$

by Proposition 6.15(d). In particular, χ extends to $\mathrm{PGU}_3(q)$, so that $\chi(1) > q^3 - 2q^2$; see the character tables in [28]. As $(q^3 - 2q^2)^2 > 29^2(q^4 + q^3)$, this case also has the $E1$-property.

Suppose finally that $q = 8$. Then $|\alpha| \in \{2, 6, 18\}$ by Proposition 6.15(c). Also, $\chi(1) \in \{133, 399, 513\}$; see the Alas [5]. We have $\chi(1)^2 > (|\alpha| - 1)^2(q^4 + q^3)$, unless $|\alpha| = 18$ or $|\alpha| = 6$ and $\chi(1) = 133$. If $|\alpha| = 18$, then $G^\diamond \cong G.3$, by Proposition 6.15(d). There are three isomorphism types of groups $G.3$; see [5]. Only two of them have elements of order 18. We use Gap to compute $\mathrm{Res}^{G^\diamond}_{\langle\alpha\rangle}(\chi^\diamond)$, where $\chi^\diamond$ is the character of $G^\diamond$ on V; see [14, Remark 4.2.5]. It turns out that this restriction contains every irreducible character of $\langle\alpha\rangle$ with positive multiplicity. Thus (G, V, n) has the $E1$-property by [14, Lemma 4.3.1]. Now suppose that $|\alpha| = 6$ and $\chi(1) = 133$. By Proposition 6.15(d), either $G^\diamond \cong G.3$ or $G^\diamond \cong G.6$. A GAP computation as in the case of $|\alpha| = 18$ shows that (G, V, n) has the $E1$-property. □

6.18. **The characters of large degree.** From now on we can assume that $d \geq 5$. In the next two lemmas we identify the irreducible characters of G whose degree is large enough to satisfy the condition of [14, Lemma 4.3.3].

Lemma 6.19. *Let $\hat{G} = \mathrm{GL}^\varepsilon_d(q)$ for some $d \geq 5$. Let $1 \neq \hat{s} \in \hat{G}$ be semisimple and real. Let Ξ denote the characteristic polynomial of $\hat{s}$. Let Δ_1 denote the monic polynomial of degree 1 with root 1, and let d_1 be the multiplicity of Δ_1 in Ξ. Let $\Delta_2, \ldots, \Delta_l$ denote the distinct irreducible monic factors of Ξ different from Δ_1. Let d_j and k_j denote, respectively, the multiplicity of Δ_j in Ξ and the degree of Δ_j, $j = 2, \ldots, l$. Then the following hold.*

(a) *We have*

$$d = \sum_{j=1}^{l} d_j k_j, \tag{9}$$

with $k_1 = 1$.

(b) *Let $d' := \max\{d_j \mid 1 \leq j \leq l\}$. Put $\hat{C} := C_{\hat{G}}(\hat{s})$ and*

$$D := [\hat{G} \colon \hat{C}]_{2'} \cdot q^{-d(d+1)/4}.$$

Then

$$D \geq (1 - q^{-1} - q^{-2})^{l'+1} q^{d(d-2d'-1)/4}, \tag{10}$$

with $l' = 0$ if $\varepsilon = 1$ and $l' = l$ if $\varepsilon = -1$.

(c) *If $d_1 < d/3$ and $d' \geq \lfloor d/2 \rfloor$, then $d' = \lfloor d/2 \rfloor$ and $\Xi = \Delta_1^{d_1}\Delta^{\lfloor d/2 \rfloor}$, where Δ is a monic polynomial of degree 2 with $\Delta = \Delta^*$.*

Proof. (a) Equation (9) is clear.

(b) Suppose that $\varepsilon = 1$. For $1 \leq j \leq l$, each factor$\Delta_j^{d_j}$ of Ξ contributes the direct factor $\mathrm{GL}_{d_j}(q^{k_j})$ to $\hat{C}$; see Lemma 6.6(a). Suppose that $\varepsilon = -1$. For $1 \leq j \leq l$, the factor $\Delta_j^{d_j}$ of Ξ with $\Delta_j = \Delta_j^\dagger$ contributes the direct factor $\mathrm{GU}_{d_j}(q^{k_j})$ to $\hat{C}$, and each factor $(\Delta_j\Delta_j^\dagger)^{d_j}$ with $\Delta_j \neq \Delta_j^\dagger$ the direct factor $\mathrm{GL}_{d_j}(q^{2k_j})$; see Lemma 6.6(c),(d). Lemma 6.11(e) yields, for either case of ε,

$$\begin{aligned}(1-q^{-1}-q^{-2})^{l'}|\hat{C}|_{2'} &\leq \prod_{j=1}^{l} q^{k_j d_j(d_j+1)/2} \\ &\leq \prod_{j=1}^{l} q^{k_j d_j(d'+1)/2} \\ &= q^{(d'+1)/2\sum_{j=1}^{l} k_j d_j} \\ &= q^{d(d'+1)/2}.\end{aligned}$$

Also,

$$|\hat{G}|_{2'} \geq q^{d(d+1)/2}(1-q^{-1}-q^{-2}),$$

by Lemma 2.1, which gives (10).

(c) Let k' denote the maximum degree of the irreducible factors of Ξ occurring with multiplicity d'. Let Δ' be one of the $\Delta_1, \ldots, \Delta_l$, for which these values are obtained. By hypothesis, $d' \geq \lfloor d/2 \rfloor \geq d/3 > d_1$, and thus $\Delta' \neq \Delta_1$. If $d' > \lfloor d/2 \rfloor$, Equation (9) implies that $k' = 1$; in turn, $\Delta' \neq \Delta'^*$ by Lemma 6.2(a). As Δ'^* occurs with multiplicity k' in Ξ by Lemma 6.3(b), this contradicts Equation (9). Suppose now that $d' = \lfloor d/2 \rfloor$; then $k' \leq 2$, and Ξ is as claimed. □

Lemma 6.20. *Let $\hat{G} = \mathrm{GL}_d^\varepsilon(q)$ for some $d \geq 5$ such that $e = \gcd(d, q-\varepsilon) > 1$. Let $1 \neq \hat{s} \in \hat{G}$ be semisimple and real with characteristic polynomial Ξ. Let Δ_1 denote the monic polynomial of degree 1 with root 1, and let d_1 be the multiplicity of Δ_1 in Ξ. Put $\hat{C} := C_{\hat{G}}(\hat{s})$. Then one of the following holds.*

(a) *We have $d_1 \geq d/3$.*

(b) *We have $\Xi = \Delta_1^{d_1}\Delta^{\lfloor d/2 \rfloor}$, where Δ is a monic polynomial of degree 2 with $\Delta = \Delta^*$.*

(c) *We have $[\hat{G}\colon\hat{C}]_{2'} \geq \delta e f(q-\varepsilon) \cdot q^{d(d+1)/4}$.*

(d) *We have $\varepsilon = -1$, $q = 4$ and $[\hat{G}\colon\hat{C}]_{2'} \geq 45 \cdot q^{d(d+1)/4}$.*

(e) *We have $\varepsilon = -1$, $q = 2$ and $[\hat{G}\colon\hat{C}]_{2'} \geq 15 \cdot q^{d(d+1)/4}$.*

(f) *We have $\varepsilon = -1$, $(d, q) = (5, 4)$, and $[\hat{G}\colon\hat{C}]_{2'} > 12 \cdot q^{d(d+1)/4}$.*

(g) *We have $\varepsilon = -1$, $(d, q) = (6, 8)$ and $[\hat{G}\colon\hat{C}]_{2'} > 51 \cdot q^{d(d+1)/4}$.*

(h) *We have* $\varepsilon = -1$ *and* $(d, q) = (6, 2)$.

Proof. Adopt the notation of Lemma 6.19. Suppose that (a) and (b) do not hold. Then $d_1 < d/3$ and $d' \leq \lfloor d/2 \rfloor - 1$ by Lemma 6.19(c).

First assume that $\varepsilon = 1$. Lemma 6.19(b) gives

$$D \geq (1 - q^{-1} - q^{-2}) q^{d(d-2d'-1)/4}. \tag{11}$$

As $d' \leq \lfloor d/2 \rfloor - 1$, we have $d - 2d' - 1 \geq 1$, if d is even, and $d - 2d' - 1 \geq 2$, if d is odd. Since $\gcd(d, q-1) > 1$, we have $q \geq 4$. Hence $1 - q^{-1} - q^{-2} \geq 1 - 1/4 - 1/16 = 11/16$.

From now on we distinguish the cases d even and d odd. Suppose first that d is even, so that $d - 2d' - 1 \geq 1$. If $d > 12$, then

$$D \geq 11/16 \cdot q^{d/4} > 1/2 \cdot q^{7/2} > q^3 \geq ef(q-1),$$

where the first estimate arises from (11). We are thus in case (c). This leaves the cases $d = 6, 10, 12$. (The cases $d = 4, 8$ do not occur thanks to Lemma 6.11(a).)

For $d = 10$, we have $e = 5$ and $q \geq 16$, and we obtain

$$D \geq 11/16 \cdot q^{5/2} \geq 11/16 \cdot q^{3/2}(q-1) \geq 5f(q-1),$$

so that we are in case (c). For $d = 6$ or $d = 12$ we have $e = 3$. If $d = 12$, we obtain

$$D \geq 11/16 \cdot q^3 \geq 3f(q-1)$$

for all $q \geq 4$, so that we are in case (c). Suppose finally that $d = 6$. By checking all the possibilities for Ξ, using $d' \leq 2$ and $d_1 = 0$, we find that the largest possible value for $|\hat{C}|_{2'}$ is assumed for

$$|\hat{C}|_{2'} = (q^2 - 1)(q^2 - 1)(q^4 - 1).$$

Inserting the exact values for $|\hat{G}|_{2'}$ and $|\hat{C}|_{2'}$, we obtain

$$D \geq q^{-21/2}(q-1)(q^3-1)(q^5-1)(q^4+q^2+1).$$

Now $(q-1)(q^3-1)(q^5-1)(q^4+q^2+1) \geq q^{13}$, and so

$$D \geq q^{5/2}.$$

It is easily checked that

$$q^{5/2} \geq 3f(q-1)$$

for $q \geq 4$, so that we are again in case (c).

Assume now that d is odd, so that $d - 2d' - 1 \geq 2$. If $d \geq 7$, we obtain

$$D \geq 11/16 q^{d/2} > 1/2 \cdot q^{7/2} > q^3 \geq ef(q-1),$$

where the first estimate arises from (11). We are thus in case (c). If $d = 5$, we have $e = 5$ and $q \geq 16$. Moreover, $d' = 1$. Then $|\hat{C}|_{2'} \leq q^5 - 1$, and hence $[\hat{G} : \hat{C}]_{2'} \geq (q-1)(q^2-1)(q^3-1)(q^4-1)$. Now

$$(q-1)(q^2-1)(q^3-1)(q^4-1) > q^{15/2} \cdot 5f(q-1)$$

for all $q \geq 16$, so that we are in case (c). This completes the proof in case $\varepsilon = 1$.

Assume now that $\varepsilon = -1$. If $d' = d_1$, and each irreducible factor $\Delta \neq \Delta_1$ of Ξ occurs with multiplicity strictly smaller than d_1, put $\Delta' := \Delta_1$. Otherwise, choose a monic irreducible factor $\Delta' \neq \Delta_1$ of Ξ occurring with multiplicity d', and such that $\deg(\Delta')$ is maximal among the degrees of the irreducible factors of Ξ with multiplicity d'. Lemma 6.19(b) gives

$$D \geq (1 - q^{-1} - q^{-2})^{l+1} q^{d(d-2d'-1)/4}. \tag{12}$$

We now complete the proof with several claims. The end of the proof of each claim is indicated by the symbol $\diamond$.

Claim 1: If $d \leq 6$, we are in one of the Cases (c)–(h).

If $d = 5$ we have $d' = 1$ from $d' \leq \lfloor d/2 \rfloor - 1$. Hence $|\hat{C}|_{2'} \leq (q+1)^5$ by Lemma 6.7(d), and so

$$D \geq q^{-15/2}(q-1)^2(q^2-q+1)(q^2+1)(q^4-q^3+q^2-q+1).$$

As $d = 5$ we have $e = 5$ and $q \geq 4$. It is easy to check that

$$q^{-15/2}(q-1)^2(q^2-q+1)(q^2+1)(q^4-q^3+q^2-q+1) \geq 10 \cdot f(q+1)$$

for all $q > 4$ with $5 \mid q+1$, so that we are in Case (c) for $q > 4$. For $q = 4$ we are in Case (f), as can be checked directly.

Suppose now that $d = 6$. Then $e = 3$, and thus $3 \mid q+1$. Moreover, $d' \leq 2$, as $d' \leq \lfloor d/2 \rfloor - 1$. Also, $d_1 = 0$, as $d_1 < d/3$. We also assume that $q > 2$, as the case $q = 2$ is listed in (h). Suppose first that $d' = 2$. Then $k' = 1$ by Lemma 6.7(c). It follows that $\Xi = (\Delta_2\Delta_3)^2\Delta_4\Delta_5$, with polynomials $\Delta_2, \ldots, \Delta_5$ of degree 1, and $(\Delta_2\Delta_3)^* = \Delta_2\Delta_3 = (\Delta_2\Delta_3)^\dagger$. There are four such possibilities and Lemma 6.6 yields $|\hat{C}|_{2'} \leq |\mathrm{GU}_2(q)|_{2'}^2(q+1)^2 = (q+1)^4(q^2-1)^2$. Thus

$$D \geq q^{-21/2}(q-1)(q^2+q+1)(q^2-q+1)^2(q^2+1)(q^4-q^3+q^2-q+1).$$

It is easy to check that $(q-1)(q^2+q+1)(q^2-q+1)^2(q^2+1)(q^4 - q^3+q^2-q+1) \geq q^{21/2} \cdot 6f(q+1)$ for all $q > 8$ with $3 \mid q+1$, so that we are in Case (c) for $q > 8$. The case $q = 8$ gives the bound asserted in (g). If $d' = 1$, we have $k' \leq 3$ by Lemma 6.7(c), and $|\hat{C}|_{2'} \leq (q+1)^6$ by Lemma 6.7(e). In particular,

$$D \geq q^{-21/2}(q-1)^3(q^2+q+1)(q^2-q+1)^2(q^2+1)(q^4-q^3+q^2-q+1).$$

It is easy to check that $(q-1)^3(q^2+q+1)(q^2-q+1)^2(q^2+1)(q^4-q^3+q^2-q+1) \geq q^{21/2} \cdot 6f(q+1)$ for all $q > 2$ with $3 \mid q+1$, so that we are in Case (c). $\diamond$

Assume henceforth that $d > 6$. Then $d \geq 9$ by Lemma 6.11(a). Define the positive integer m by

$$d' = \lfloor d/2 \rfloor - m, \qquad m = 1, 2, \ldots$$

Claim 2: Suppose that $d \geq 6m'$ and $d \neq 6m'+1$ for some positive integer m'. If $d' = d_1$, then $m > m'$. In particular, if $d' = d_1$, then $m > 1$, as $d \geq 9$.

This follows from $\lfloor d/2 \rfloor - m = d' = d_1 < d/3$, and $\lfloor d/2 \rfloor - d/3 \geq m'$. $\diamond$.

Claim 3: If $m = 1$, then $\Delta' \neq \Delta_1$ and $k' = 1$.

By Claim 2 we have $d' > d_1$ and thus $\Delta' \neq \Delta_1$. Lemma 6.7(c) yields $d \geq 2k'(\lfloor d/2 \rfloor - 1)$, which gives $k' = 1$ as $d \geq 9$. $\diamond$

Claim 4: Suppose that $m = 1$. Then

$$D \geq (1 - q^{-1} - q^{-2})^5 q^{(3d-12)/4}. \tag{13}$$

if d is even, and

$$D \geq (1 - q^{-1} - q^{-2})^6 q^{(5d-27)/4}. \tag{14}$$

if d is odd.

By Claim 3 we have $\Delta' \neq \Delta_1$ and $k' = 1$. Moreover, $\Delta' \neq \Delta'^*$ by Lemma 6.2(a). It follows that $\Xi = \Delta_1^{d_1}(\Delta'\Delta'^*)^{\lfloor d/2 \rfloor - 1}$ or $\Xi = \Delta_1^{d_1}(\Delta'\Delta'^*)^{\lfloor d/2 \rfloor - 1}\Delta$ with Δ of degree 2. In the latter case, Δ is reducible, as otherwise $\Delta \neq \Delta^\dagger$ and $\Delta^\dagger$ would be a divisor of Ξ; see Lemmas 6.2(b), and 6.7(b). From Lemma 6.6 we can determine the structure of $\hat{C}$ in each case. Lemma 2.1 then gives the asserted estimates. $\diamond$

Claim 5: If $\Delta' \neq \Delta_1$, then $l \leq d - 2d' + 3$. In any case, $l \leq d - d' + 1$.

Suppose first that $\Delta' \neq \Delta_1$. In view of (9), we get $d \geq d_1 + 2d' + l - 3$ by Lemma 6.7(c) and its proof. The first assertion follows from this. If $\Delta' = \Delta_1$, the claim follows directly from (9). $\diamond$

Claim 6: We have $d - 2d' - 1 \geq 2m - 1$.

This follows from $d' = \lfloor d/2 \rfloor - m \leq d/2 - m$. $\diamond$.

Claim 7: We have

$$l + 1 \leq (d+5)/2 + m,$$

and

$$l + 1 \leq 2m + 5, \qquad \text{if } \Delta' \neq \Delta_1.$$

This follows from Claim 5 and $d' \geq (d-1)/2 - m$. $\diamond$.

Claim 8: If $q \geq 8$, we are in Case (c).

Suppose that $q \geq 8$. By Lemma 6.11(b) we have $1-q^{-1}-q^{-2} \geq q^{-1/4}$, and it suffices to show that $D > q^3$. Thus

$$D \geq q^{-(d+5)/8-m/4+d(2m-1)/4}$$

by Claims 6 and 7 and (12). We have $-(d+5)/8-m/4+d(2m-1)/4 \geq 3$, if and only if $d(4m-3)-2m \geq 29$. The latter holds if $m \geq 2$.

Suppose then that $m = 1$ and that d is even. Then

$$D \geq q^{-5/4+(3d-12)/4}$$

by (13). Hence $D \geq q^3$ for all $d \geq 10$. Suppose now that $m = 1$ and d is odd. Then

$$D \geq q^{-6/4+(5d-27)/4}$$

by (14). Hence $D \geq q^3$ for all $d \geq 9$. ◇

Claim 9: If $q = 4$, we are in Case (d).

Suppose that $q = 4$. Then $q + 1 = e = 5$, so that $5 \mid d$. Since $4^{11/4} > 45$, it suffices to show that $D \geq q^{11/4}$. By Lemma 6.11(c) we have $1 - q^{-1} - q^{-2} \geq q^{-1/2}$, and thus

$$D \geq q^{-(d+5)/4-m/2+d(2m-1)/4}$$

by Claims 6 and 7 and (12). We have $-(d+5)/4-m/2+d(2m-1)/4 \geq 11/4$, if and only if $d(m-2)+m(d-2) \geq 16$. If $m \geq 2$, this is the case for all $d \geq 10$. Suppose now that $m = 1$ and that d is even. Then

$$D \geq q^{-5/2+(3d-12)/4}$$

by (13). Hence $D \geq q^{11/4}$ for all even $d > 10$ with $5 \mid d$. Suppose that $d = 10$. Then $d' = 4$ and we obtain $k' = 1$ from Lemma 6.7(c). Lemma 6.7(d) yields $|\hat{C}|_{2'} \leq |\mathrm{GU}_4(q)|_{2'}^2 |\mathrm{GU}_2(q)|_{2'}$. Using the exact values for $|\mathrm{GU}_4(4)|_{2'}$ and $|\mathrm{GU}_{10}(4)|_{2'}$, one checks that the bound on $[\hat{G} : \hat{C}|_{2'}$ in (d) is satisfied. If $m = 1$ and d is odd, we have

$$D \geq q^{-6/2+(5d-27)/4}$$

by (14). Hence $D \geq q^{11/4}$ for all odd $d > 5$ with $5 \mid d$. ◇

Claim 10: If $q = 2$, we are in Case (e).

Suppose that $q = 2$. Then $q + 1 = e = 3$, so that $3 \mid d$. It suffices to show that $D \geq q^4$. By Lemma 6.11(c) we have $1 - q^{-1} - q^{-2} \geq q^{-2}$. Suppose first that $\Delta' = \Delta_1$. Then $d_1 = d' > 1$ by our choice of Δ', and

$$D \geq q^{-d-5-2m+d(2m-1)/4}$$

by Claims (6) and (7) and (12). We have $-d-5-2m+d(2m-1)/4 \geq 4$, if and only if

$$d(m-5)+m(d-8) \geq 36. \tag{15}$$

By Claim 2 we have $m \geq 5$ if $d \geq 24$. In particular, (15) holds for $d \geq 24$. This leaves the cases $9 \leq d \leq 21$. Using $1 < d' = d_1 < d/3$ and $d \equiv d_1 \pmod 2$ to enumerate the remaining possibilities for d and $d' = \lfloor d/2 \rfloor - m$, we find that (15) is satisfied unless $(d, d') \in \{(15, 3), (12, 2)\}$. Applying Lemma 6.7(e) to the orthogonal complement of the fixed space of $\hat{s}$, we get $|\hat{C}|_{2'} \leq |\mathrm{GU}_3(2)|_{2'} |\mathrm{GU}_2(2)|_{2'}^6$ if $d = 15$ and $|\hat{C}|_{2'} \leq |\mathrm{GU}_2(2)|_{2'}(2+1)^{10}$ if $d = 12$. A computation yields

$$[\hat{G} : \hat{C}]_{2'} > 15 \cdot 2^{d(d+1)/4} \tag{16}$$

in these two cases.

Suppose now that $\Delta' \neq \Delta_1$. Then

$$D \geq q^{-4m-10+d(2m-1)/4}$$

by Claims (6), (7) and (12). We have $-4m - 10 + d(2m-1)/4 \geq 4$, if and only if $d(m-1) + m(d-16) \geq 56$. For $m \geq 2$, the latter holds for $d \geq 30$, and if $m \geq 4$, the condition holds for $d \geq 18$.

Suppose then that $m = 1$ and d is even. Then

$$D \geq q^{-10+(3d-12)/4}$$

by (13). Hence $D \geq q^4$ for all even $d \geq 18$ with $3 \mid d$. If $m = 1$ and d is odd, we have

$$D \geq q^{-12+(5d-27)/4}$$

by (14). Hence $D \geq q^4$ for all odd $d \geq 21$ with $3 \mid d$.

We finally investigate the cases which cannot be ruled out with the above arguments. These are included in the cases $9 \leq d \leq 15$ and all possible m such that $d_1 \leq d'$, and the cases $18 \leq d \leq 27$ and $1 \leq m \leq 3$. If $d' \leq 2$, we use the bounds on $|\hat{C}|_{2'}$ of Lemma 6.7(e). If $d' > 2$, we use the estimate $|\hat{C}|_{2'} \leq |\mathrm{GU}_{d'}(2)|_{2'}^2 |\mathrm{GU}_{d-2d'}(2)|_{2'}$ of Lemma 6.7(d). Inserting these upper bounds and the correct value for $|\mathrm{GU}_d(2)|_{2'}$ into the definition of D, we get (16) in all cases, except if $d = 9$ and $d' = 3$. Suppose that $d = 9$ and $d' = 3$. Since $d_1 < d/3 = 1$, we get that $\Xi = \Delta_1(\Delta'\Delta'^*)^3\Delta$, where Δ is a reducible polynomial of degree 2. Hence $|\hat{C}|_{2'} \leq |\mathrm{GU}_3(2)|_{2'}^2 (2+1)^3$, which is enough to get (16). ◇

This completes the proof. □

6.21. **The characters of small degree.** In the next two lemmas we show that the irreducible characters of G whose degrees are too small to apply [14, Lemma 4.3.3], satisfy the hypotheses of Lemma 4.7. Recall the notation $\mathrm{Aut}'(\mathrm{GL}_d(\mathbb{F}))$ introduced in Subsection 6.9.

Lemma 6.22. *Let $\hat{s} \in \mathrm{GL}_d(\mathbb{F})$ be semisimple and real with characteristic polynomial Ξ. Let Δ_1 denote the monic polynomial of degree 1 with root 1, and let d_1 be the multiplicity of Δ_1 in Ξ.*

(a) *Suppose that* $\Xi = \Delta_1^{d_1}\Delta^{d_2}$ *with* $\Delta = \Delta_2\Delta_2^*$*, where* $\Delta_2 \neq \Delta_1$ *is monic of degree* 1*. Suppose also that* $d_1 \neq d_2$*. If* $\kappa\hat{s}$ *is real for some* $\kappa \in \mathbb{F}^*$*, then* $\kappa = 1$*.*

(b) *Let* $\hat{\alpha} \in \mathrm{Aut}'(\mathrm{GL}_d(\mathbb{F}))$ *be such that* $\hat{\alpha}(\hat{s})$ *is conjugate to* $\kappa\hat{s}$ *for some* $\kappa \in \mathbb{F}^*$*. If* $d_1 \geq d/3$*, then* $\hat{\alpha}(\hat{s})$ *is conjugate to* $\hat{s}$*.*

Proof. (a) Let ζ denote the root of Δ_2. Then $\kappa\hat{s}$ has the eigenvalues κ, $\kappa\zeta$, $\kappa\zeta^{-1}$ with multiplicities d_1, d_2, d_2. Suppose that $\kappa\hat{s}$ is real. Then $\kappa^{-1} = \kappa$, as $d_1 \neq d_2$. This gives our claim.

(b) Notice that the elements of $\mathrm{Aut}'(\mathrm{GL}_d(\mathbb{F}))$ preserve dimensions of eigenspaces for the eigenvalue 1. Thus $\hat{\alpha}(\hat{s})$ has the eigenvalues 1 and κ, each with multiplicity d_1. As $\hat{\alpha}(\hat{s})$ is real, it also has the eigenvalue κ^{-1} with multiplicity d_1. Suppose that $\kappa \neq 1$. As $d_1 \geq d/3$, this implies $d_1 = d/3$, and 1, κ, κ^{-1} are the eigenvalues of $\hat{\alpha}(\hat{s})$, and hence of $\kappa\hat{s}$, each with multiplicity $d/3$. It follows that $\hat{s} = \kappa^{-1}(\kappa\hat{s})$ has the eigenvalues κ^{-1}, 1, κ^{-2}, each with multiplicity $d/3$. Since $\hat{s}$ is real, we must have $\kappa^{-2} = \kappa$. Hence $\hat{\alpha}(\hat{s})$ and $\hat{s}$ have the same multiset of eigenvalues, and so they are conjugate. ☐

Lemma 6.23. *Let* $\hat{G} = \mathrm{GL}_d^{\varepsilon}(q)$ *for some* $d \geq 5$ *such that* $e = \gcd(d, q-\varepsilon) > 1$*. If* $\varepsilon = -1$*, assume that* $(d, q) \neq (6, 2)$*. Let* $1 \neq \hat{s} \in \hat{G}$ *be semisimple and real and let* Ξ *denote the characteristic polynomial of* $\hat{s}$*. Let* Δ_1 *denote the monic polynomial of degree* 1 *with root* 1*, and let* d_1 *be the multiplicity of* Δ_1 *in* Ξ*.*

Suppose that Ξ *is as in* (a) *or* (b) *of* Lemma 6.20*. Then the following statements hold.*

(a) *There is a standard* $\hat{\iota}$*-stable Levi subgroup* $\hat{L}$ *of* $\hat{G}$*, and there is a* $\hat{G}$*-conjugate* $\hat{s}' \in \hat{L}$ *such* $\hat{s}'$ *is real in* $\hat{L}$*. Moreover, if* $\hat{\alpha} \in \mathrm{Aut}(\hat{G})$ *stabilizes* $\hat{L}$ *and the* $\hat{G}$*-conjugacy class of* $\hat{s}$*, then* $\hat{\alpha}(\hat{s}')$ *is conjugate to* $\hat{s}'$ *in* $\hat{L}$*.*

(b) *The image of* $\hat{L}$ *in* $\overline{G}^{\sigma} = \mathrm{PGL}_d^{\varepsilon}(q)$ *is of the form* $\overline{L}^{\sigma}$ *for a standard* ι*-stable Levi subgroup* $\overline{L}$ *of* $\overline{G}$*, and the image* s *of* $\hat{s}$ *in* $\overline{G}^{\sigma}$ *satisfies the hypotheses of* Lemma 4.7 *with respect to* $\overline{L}^{\sigma}$*.*

Proof. (a) Since $\hat{s}$ is real, there is $\hat{g} \in \hat{G}$ such that $\hat{s}^{-2} = [\hat{g}, \hat{s}] \in \mathrm{SL}_d^{\varepsilon}(q)$. As q is even, this implies that $\det(\hat{s}) = 1$.

Suppose first that Ξ is as in (a) of Lemma 6.20. Notice that $d - d_1 \geq 2$, as $\hat{s}$ is non-trivial and real. Put $d_1' := \lfloor d_1/2 \rfloor$, and let $\hat{L} := \hat{L}_I$ of $\hat{G}$, where I corresponds to the nodes $d_1' + 1, d_1' + 2, \ldots, d - d_1' - 1$ of the Dynkin diagram of $\mathrm{GL}_d(\mathbb{F})$. Thus $\hat{L} \cong \hat{S} \times \mathrm{GL}_{d-2d_1'}^{\varepsilon}(q)$, where $\hat{S}$ is the standard torus of $\mathrm{GL}_{2d_1'}(q)$ if $\varepsilon = 1$, and of $\mathrm{GL}_{d_1'}(q^2)$ if $\varepsilon = -1$. As $1 \leq d_1' \leq d - 2$ and I is $\hat{\iota}$-invariant, $\hat{L}$ is a proper, non-trivial (i.e. different from $\hat{T}$), standard, $\hat{\iota}$-invariant Levi subgroup of $\hat{G}$. By hypothesis, there is $\hat{x} \in \mathrm{GL}^{\varepsilon}_{d-2d'_1}(q)$, such that $\hat{s}$ is conjugate in $\hat{G}$ to

$$\hat{s}' = \mathrm{diag}(\mathrm{Id}_{d'_1}, \hat{x}, \mathrm{Id}_{d'_1}) \in \hat{L}.$$

The characteristic polynomial of $\hat{x}$ equals $\Xi/\Delta_1^{2d'_1}$. As $\hat{s}$ is real, we have $\Xi^* = \Xi$, and thus $(\Xi/\Delta_1^{2d'_1})^* = \Xi/\Delta_1^{2d'_1}$. It follows that $\hat{x}$ is conjugate to $\hat{x}^{-1}$ in $\mathrm{GL}^{\varepsilon}_{d-2d'_1}(q)$, and so $\hat{s}$ is real in $\hat{L}$. Now suppose that $\hat{\alpha} \in \mathrm{Aut}(\hat{G})$ stabilizes $\hat{L}$ and the $\hat{G}$-conjugacy class of $\hat{s}$. In particular, $\hat{\alpha}$ stabilizes

$$[\hat{L}, \hat{L}] = \{\mathrm{diag}(\mathrm{Id}_{d'_1}, \hat{z}, \mathrm{Id}_{d'_1}) \mid \hat{z} \in \mathrm{SL}^{\varepsilon}_{d-2d'_1}(q)\}.$$

Since $1 = \det(\hat{s}) = \det(\hat{s}') = \det(\hat{x})$, we have $\hat{s}' \in [\hat{L}, \hat{L}]$ and thus

$$\hat{\alpha}(\hat{s}') = \mathrm{diag}(\mathrm{Id}_{d'_1}, \hat{x}', \mathrm{Id}_{d'_1})$$

for some $\hat{x}' \in \mathrm{SL}^{\varepsilon}_{d-d_1}(q)$. As $\hat{\alpha}$ stabilizes the $\hat{G}$-conjugacy class of $\hat{s}'$, the characteristic polynomial of $\hat{\alpha}(\hat{s}')$ equals Ξ. Hence the characteristic polynomial of $\hat{x}'$ equals $\Xi/\Delta_1^{2d'_1}$, which is the characteristic polynomial of $\hat{x}$. Thus $\hat{x}$ is conjugate to $\hat{x}'$ in $\mathrm{GL}^{\varepsilon}_{d-2d'_1}(q)$ and so $\hat{s}'$ is conjugate to $\hat{\alpha}(\hat{s}')$ in $\hat{L}$.

Suppose now that $\hat{s}$ satisfies condition (b) of Lemma 6.20. Put $d' := d - 4$. Consider the standard Levi subgroup $\hat{L} := \hat{L}_I$, where I corresponds to all the nodes of the Dynkin diagram of $\mathrm{GL}_d(\mathbb{F})$ without the nodes 2 and $d-2$. Thus $\hat{L} \cong \mathrm{GL}_2(q) \times \mathrm{GL}_{d'}(q) \times \mathrm{GL}_2(q)$ if $\varepsilon = 1$, and $\hat{L} \cong \mathrm{GL}_2(q^2) \times \mathrm{GU}_{d'}(q)$ if $\varepsilon = -1$. If $\varepsilon = 1$, let $\hat{y} \in \mathrm{GL}_2(q)$ denote an element with characteristic polynomial Δ. Then $\det(\hat{y}) = 1$, as the two roots of Δ are mutually inverse. If $\varepsilon = -1$, then Δ is reducible, as otherwise $\Delta \neq \Delta^\dagger$ by Lemma 6.2(b), contradicting Lemma 6.7(b). Thus $\Delta = \Delta_2\Delta_2^*$ for a monic polynomial $\Delta_2 \neq \Delta_1$ of degree 1. Let $\zeta \in \mathbb{F}_{q^2}$ denote the root of Δ_2. In this case, put

$$\hat{y} := \mathrm{diag}(\zeta, \zeta^{-1}) \in \mathrm{GL}_2(q^2).$$

Then $\hat{y} = \hat{\iota}_2(\hat{y})$, where $\hat{\iota}_2$ denotes the standard graph automorphism of $\mathrm{GL}_2(q^2)$; see Subsection 6.9. In either case of ε, there is $\hat{x} \in \mathrm{GL}^{\varepsilon}_{d'}(q)$, such that $\hat{s}$ is conjugate in $\hat{G}$ to

$$\hat{s}' = \mathrm{diag}(\hat{y}, \hat{x}, \hat{y}) \in \hat{L}.$$

Notice that if $\varepsilon = -1$, the displayed element indeed lies in $\mathrm{GU}_d(q)$, since $\hat{y} = \hat{\iota}_2(\hat{y})$. If $d = 6$ and $\varepsilon = 1$, we take $\hat{x} = \hat{y}$. It follows that $\hat{s}'$ is real in $\hat{L}$, as $\hat{y}$ is real in $\mathrm{GL}_2(q^\delta)$ and $\hat{x}$ is real in $\mathrm{GL}^{\varepsilon}_{d'}(q)$, since its characteristic polynomial equals $\Delta_1^{d_1}\Delta^{\lfloor d/2 \rfloor - 2}$.

As $\det(\hat{y}) = \det(\hat{x}) = 1$, we have $\hat{s}' \in [\hat{L}, \hat{L}]$, where $[\hat{L}, \hat{L}] \cong \mathrm{SL}_2(q) \times \mathrm{SL}_{d'}(q) \times \mathrm{SL}_2(q)$, respectively $[\hat{L}, \hat{L}] \cong \mathrm{SL}_2(q^2) \times \mathrm{SU}_{d'}(q)$, with the obvious embeddings of the direct factors into $\hat{L}$. If $d = 5$, then $\mathrm{SL}_{d'}(q)$ is the trivial group. If $d = 6$, then $\mathrm{SL}^{\varepsilon}_{d'}(q) \cong \mathrm{SL}_2(q)$ is nonabelian simple, as $q > 2$ in this case (recall that $(d, q) \neq (6, 2)$ if $\varepsilon = -1$). The case $d = 7$ and $\varepsilon = -1$ does not occur thanks to Lemma 6.11(a). Hence for $d > 6$, the group $\mathrm{SL}^{\varepsilon}_{d'}(q)$ is non-abelian simple and not isomorphic to the group $\mathrm{SL}_2(q^{\delta})$.

Now let $\hat{\alpha} \in \mathrm{Aut}(\hat{G})$ normalize $\hat{L}$, and thus $[\hat{L}, \hat{L}]$. By the remarks in the previous paragraph, $\hat{\alpha}$ permutes the direct factors $\mathrm{SL}_2(q)$ of $[\hat{L}, \hat{L}]$ if $d = 6$ and $\varepsilon = 1$. In all other cases, $\hat{\alpha}$, normalizes the direct factor $\mathrm{SL}^{\varepsilon}_{d'}(q)$ of $[\hat{L}, \hat{L}]$, as well as the factor $\mathrm{SL}_2(q^2)$ if $\varepsilon = -1$, whereas it permutes the two factors $\mathrm{SL}_2(q)$ if $\varepsilon = 1$. Hence $\hat{\alpha}(\hat{s}')$ is of the form

$$\hat{\alpha}(\hat{s}') = \mathrm{diag}(\hat{\beta}(\hat{y}), \hat{\gamma}(\hat{x}), \hat{\beta}'(\hat{y})),$$

with automorphisms $\hat{\beta}, \hat{\beta}'$ of $\mathrm{SL}_2(q^{\delta})$ and $\hat{\gamma}$ of $\mathrm{SL}^{\varepsilon}_{d'}(q)$. If $\varepsilon = -1$, we have $\hat{\beta}' = \hat{\iota}_2 \circ \hat{\beta}$, and $\hat{\beta}$ and $\hat{\gamma}$ are the restrictions of $\hat{\alpha}$ to the respective subgroups of $[\hat{L}, \hat{L}]$. Since $\det(\hat{\beta}(\hat{y})) = 1 = \det(\hat{\beta}'(\hat{y}))$, the two eigenvalues of $\hat{\beta}(\hat{y})$, respectively $\hat{\beta}'(\hat{y})$, are mutually inverse. As $\hat{\alpha}$ stabilizes the $\hat{G}$-conjugacy class of $\hat{s}'$, the characteristic polynomial of $\hat{\alpha}(\hat{s}')$ equals Ξ. This implies that $\hat{\beta}(\hat{y})$, $\hat{\gamma}(\hat{x})$ and $\hat{\beta}'(\hat{y})$ have characteristic polynomial Δ, Ξ/Δ^2, respectively Δ. Hence $\hat{\beta}(\hat{y})$ and $\hat{\beta}'(\hat{y})$ are conjugate to $\hat{y}$ in $\mathrm{GL}_2(q^{\delta})$, and $\hat{\gamma}(\hat{x})$ is conjugate to $\hat{x}$ in $\mathrm{GL}^{\varepsilon}_{d'}(q)$, which implies the result.

(b) Let $\overline{\hat{L}}$ denote the standard Levi subgroup of $\overline{\hat{G}} := \mathrm{GL}_d(\mathbb{F})$ corresponding to the subset I specified above in the two cases, and let $\overline{L}$ denote the image of $\overline{\hat{L}}$ in $\overline{G}$ under the canonical epimorphism

$$\overline{\hat{G}} \to \overline{G}. \tag{17}$$

Then $\overline{L}$ is ι-stable, and the image of $\hat{L}$ in $\overline{G}^{\sigma}$ equals $\overline{L}^{\sigma}$. We may assume that $\hat{s} = \hat{s}' \in \hat{L}$. Then, in both cases, Condition (i) of Lemma 4.7 is satisfied.

To establish the second condition, suppose first that $\hat{s}$ is as in Case (a) of Lemma 6.20. The natural embedding $\mathrm{SL}_{d-2d_1'}(\mathbb{F}) \to [\overline{\hat{L}}, \overline{\hat{L}}]$ yields an isomorphism $\mathrm{SL}_{d-2d_1'}(\mathbb{F}) \to [\overline{L}, \overline{L}]$ of algebraic groups. In particular, $[\overline{L}, \overline{L}]$ is simply connected, and thus $C_{\overline{L}}(s)$ is connected by a theorem of Steinberg; see [4, Theorem 3.5.6]. Suppose now that $\hat{s}$ is as in Case (b) of Lemma 6.20. The inverse image of $C_{\overline{L}}(s)$ in $\overline{\hat{L}}$ under the map (17) equals

$$\{\hat{g} \in \overline{\hat{L}} \mid \hat{g}\hat{s}\hat{g}^{-1} = \kappa\hat{s} \text{ for some } \kappa \in \mathbb{F}^*\}. \tag{18}$$

Lemma 6.22(a) implies that the group (18) equals $C_{\overline{\hat{L}}}(\hat{s})$. As centralizers of semisimple elements in $\overline{\hat{L}}$ are connected by the theorem of Steinberg cited above, the image $C_{\overline{L}}(s)$ of (18) is connected as well. Thus Condition (ii) of Lemma 4.7 is satisfied in both cases.

Now let $\alpha \in \mathrm{Aut}(\overline{G}^\sigma)$ stabilize $\overline{L}^\sigma$ and the $\overline{G}^\sigma$-conjugacy class of s. By the discussion in Subsection 6.9, there is an an automorphism $\hat{\alpha} \in \mathrm{Aut}'(\overline{\hat{G}})$, which stabilizes $\hat{L}$ and such that $\hat{\alpha}(\hat{s})$ is $\hat{G}$-conjugate to $\kappa\hat{s}$ for some $\kappa \in \mathbb{F}_q$. Lemma 6.22(b) implies that $\hat{\alpha}(\hat{s})$ is conjugate to $\hat{s}$ in $\hat{G}$, and so Condition (iii) of Lemma 4.7 follows from (a). □

Remark 6.24. An analogous result as in Lemma 6.23(b) does not hold for $d = 3$. For example, suppose that $G = \mathrm{PSL}_3(4)$. Then $\overline{G}^\sigma = \mathrm{PGL}_3(4)$ and $\overline{G}^{*\sigma} = \mathrm{SL}_3(4)$. There are two conjugacy classes in $\overline{G}^\sigma$ of elements of order 5, whose centralizer is the cyclic maximal torus of order 15. Let s be an element in one of these classes. Then $\mathcal{E}(\overline{G}^{*\sigma}, s)$ contains a unique element, which has degree 63, is real and has $Z(\overline{G}^{*\sigma})$ in its kernel. Thus these elements are relevant for our investigation.

Let $\hat{s}$ be a real lift of s of order 5. Then the characteristic polynomial of $\hat{s}$ has the form $\Delta_1\Delta$, where Δ_1 is as in Lemma 6.23, and where Δ is irreducible of degree 2 with roots of order 5. Thus $\hat{s}$ is as in Lemma 6.20(a)(b).

The only ι-stable standard Levi subgroups of $\overline{G}^\sigma$ are $\overline{T}^\sigma$ and $\overline{G}^\sigma$ itself. No conjugate of s lies in $\overline{T}^\sigma$.

Of course, there is a ι-stable conjugate of a proper standard Levi subgroup of $\overline{G}^\sigma$ meeting the conjugacy class of s, but this does not lie in any ι-stable parabolic subgroups, so the argument arising from Proposition 3.5 cannot be applied. □

6.25. **The main result.** We are now ready to establish the main result of this section.

Lemma 6.26. *Let $\chi \in \mathrm{Irr}(\mathrm{SL}_d^\varepsilon(q))$, and let $\hat{\chi} \in \mathrm{Irr}(\mathrm{GL}_d^\varepsilon(q))$ lying above χ. Then there is a positive integer e' with $e' \mid e$ such that $\chi(1) = \hat{\chi}(1)/e'$.*

Proof. Clearly, $Z(\mathrm{GL}_d^\varepsilon(q))\mathrm{SL}_d^\varepsilon(q)$ stabilizes χ, and

$$[\mathrm{GL}_d^\varepsilon(q) \colon Z(\mathrm{GL}_d^\varepsilon(q))\mathrm{SL}_d^\varepsilon(q)] = e.$$

As $\mathrm{GL}_d^\varepsilon(q)/\mathrm{SL}_d^\varepsilon(q)$ is cyclic, the restriction of $\hat{\chi}$ to $\mathrm{SL}_d^\varepsilon(q)$ is a sum of e' distinct conjugates of χ, where $e' \mid e$. Thus $\chi(1) = \hat{\chi}(1)/e'$. □

Proposition 6.27. *Let $G = \mathrm{PSL}_d^\varepsilon(q)$ with q even, $d \geq 5$ and $e = \gcd(d, q-\varepsilon) > 1$. Then G is not a minimal counterexample to* [14, Theorem 1.1.5].

Proof. Suppose that G is a minimal counterexample to [14, Theorem 1.1.5]. Let (V, n, ν) be as in [14, Notation 4.1.1]. Let χ be the character of V, viewed as a character of $\mathrm{SL}_d^\varepsilon(q) = \overline{G}^{*\sigma}$. Let $s \in \overline{G}^{\sigma}$ be semisimple such that $\chi \in \mathcal{E}(\overline{G}^{*\sigma}, s)$. As χ is real, s is real by Lemma 4.6(c)(iii). By Lemma 6.8(a), there exists a real lift $\hat{s} \in \mathrm{GL}_d^\varepsilon(q)$ of s.

Suppose that $\hat{s}$ is as in Cases (a) or (b) of Lemma 6.20. Lemma 6.23 shows that s satisfies the hypotheses of Lemma 4.7. Hence (G, V, n) has the $E1$-property, as we have assumed that G is a minimal counterexample.

Assume now that $\hat{s}$ is as in one of the remaining cases of Lemma 6.20. Let α be constructed from $\beta := \nu$ as in Proposition 6.15. Then $|C_G(\alpha_{(p)})| < M_G$ for every prime p dividing $|\alpha|$ and every element $\alpha_{(p)}$ of $\langle\alpha\rangle$ of order p. Here, $M_G = q^{d(d+1)/2}$; see Definition 5.2. By [14, Lemma 4.3.3], it suffices to show that

$$\chi(1) \geq (|\alpha| - 1)M_G^{1/2} = (|\alpha| - 1)q^{d(d+1)/4}. \tag{19}$$

Bounds for $|\alpha|$ are given in Proposition 6.15(b)(d). We will use these bounds below without any further reference.

If $\hat{s}$ is as in (c) of Lemma 6.20, then (19) holds. Suppose that we are in Case (d) of Lemma 6.20. Then $\chi(1) \geq [\hat{G} : \hat{C}]_{2'}/5$ by Lemma 6.26, and $|\alpha| \leq 10$. It follows that (19) is satisfied. Analogous proofs work in Case (e) and (g) of Lemma 6.20.

Suppose now that we are in Case (f) of Lemma 6.20. Then $d = e = 5$ and $q = 4$. Also, $\chi(1) \geq 12 \cdot 2^{15}/5 > 78\,634$. On the other hand, (19) holds for $\chi(1) > 9 \cdot 2^{15} = 294\,912$. The character table of $G = \mathrm{PSU}_5(4)$ is available in GAP [9]. The two inequalities above imply that $\chi(1) = 81\,549$. Then, however, χ is not invariant in $\mathrm{PGU}_5(q)$. As χ is α-invariant and Φ_G has order 4, this implies that $\alpha \in \Phi_G$. If $|\alpha| = 2$, then G is not a minimal counterexample by [14, Corollary 4.3.2]. Hence $|\alpha| = 4$. As d is odd, the graph automorphism φ^2 of G has fixed subgroup isomorphic to $\mathrm{Sp}_4(4)$; see [12, Proposition 9.4.2(b)(2)]. We have $|\mathrm{Sp}_4(4)| = 979\,200$ and hence $|\mathrm{Sp}_4(4)|^{1/2} < 1\,000$. Then [14, Lemma 4.3.3] shows that (G, V, n) has the $E1$-property, as $\dim(V) = 81\,549 > 3 \cdot 1\,000$.

Suppose finally that we are in Case (g) of Lemma 6.20. Then $d = 6$ and $q = 2$. Moreover, $|\alpha| = 2$ or $|\alpha| = 6$. In the former case, (G, V, n) has the $E1$-property by [14, Corollary 4.3.2]. Assume then that $|\alpha| = 6$.

Then $G^\diamond := \langle \mathrm{Inn}(G), \alpha \rangle \cong \mathrm{PGU}_6(q)$ by Proposition 6.15(d). Also, χ extends to a character $\chi^\diamond$ of $G^\diamond$; see [14, Remark 4.2.5]. The character table of $\mathrm{PGU}_6(q)$ is available in the Atlas as well as in GAP. If $\chi(1) > 5 \cdot 2^{21/2}$, then (19) is satisfied. Using this, we are left with the two cases $\chi(1) \in \{231, 385\}$. With the notation of [14, Lemma 4.3.1], it suffices to show that $\mathrm{Res}^{G^\diamond}_{\langle\alpha\rangle}(\chi^\diamond)$ contains each of the two irreducible, real characters of $\langle\alpha\rangle$ with positive multiplicity. This is easily checked to be true. □

7. The groups of type E_6

In this section we complete the proof that no finite simple group of Lie type E_6 is a minimal counterexample to [14, Theorem 1.1.5].

7.1. **The Weyl group and the graph automorphism.** Let $G = E_6^\varepsilon(q)$. The strategy to deal with these groups is the same as the one employed for the linear and unitary groups in Subsection 6. Suppose that $\overline{G} = E_6(\mathbb{F})$. For the explicit computations in the Weyl group $W(\overline{G})$ and the root system $\Sigma = \Sigma(\overline{G})$ reported below, we use the GAP3 package Chevie [10] with its extensions by Jean Michel [26]. In particular, we follow the numbering of the roots used in [10]. Thus r_j denotes the root of position j, $1 \leq j \leq 36$, in the list of positive roots of Σ provided by [10]. The simple roots are $r_1, \ldots, r_6$, where the root r_j is attached to the node with number j, $j = 1, \ldots, 6$, in the Dynkin diagram of Σ displayed in [14, Figure 1]. Finally, $s_j \in W(\overline{G})$ denotes the reflection on the hyperplane perpendicular to r_j. The elements $n_j \in N_{\overline{G}}(\overline{T})$ introduced in [3, Lemma 6.4.2] are lifts of the reflections s_j; see [3, Theorem 7.2.2]. As $\mathbb{F}$ has characteristic 2, the standard torus $\overline{T}$ has a complement in $N_{\overline{G}}(\overline{T})$, namely the subgroup generated by the elements n_j, $j = 1, \ldots, 6$. This follows from the Steinberg relations as exhibited in [3, p. 192–193]. Although the Steinberg relations define the simply connected group $\overline{G}^*$ of type E_6, the claimed result for the adjoint group $\overline{G}$ follows from this, as $\overline{G}$ is a quotient of $\overline{G}^*$ with a kernel contained in its standard torus. In the following, we will thus identify $W(\overline{G})$ with a subgroup of $N_{\overline{G}}(\overline{T})$.

The graph automorphism $\iota \in \mathrm{Aut}(\overline{G})$ arises from the non-trivial symmetry, also denoted by ι, of the Dynkin diagram. The symmetry ι of the Dynkin diagram induces an automorphism of $W(\overline{G})$, which is, in fact, the restriction of $\iota \in \mathrm{Aut}(\overline{G})$ to $W(\overline{G})$. Once again, this automorphism of $W(\overline{G})$ will also be denoted by ι. If $\varepsilon = -1$, then ι equals the automorphism of $W(\overline{G})$ induced by σ; see [14, Subsection 5.3] and

Subsection 4.1. As φ acts trivially on $W(\overline{G})$, an element $\mu \in \Gamma_{\overline{G}} \times \Phi_{\overline{G}}$ fixes $w \in W(\overline{G})$, if and only if ι fixes w.

7.2. **Some Levi subgroups.** For $I \subseteq \Pi$, let $\overline{L}_I$ denote the corresponding standard Levi subgroup of $\overline{G}$; see [14, Subsection 5.3]. Notice that $\overline{L}_I$ is φ-invariant, and $\overline{L}_I$ is ι-invariant if and only if I is invariant under symmetry ι of the Dynkin diagram. More generally, if $J \subseteq \Sigma$, the subsystem subgroup of $\overline{G}$ corresponding to the closed subsystem of Σ generated by J, is denoted by $\overline{L}_J$; see [25, Section 13.1] for the definition and construction of subsystem subgroups.

For $I \subseteq \Pi$, the normalizer of the standard Levi subgroup $\overline{L}_I$ can be computed by an algorithm of Howlett [15]. In fact, $N_{\overline{G}}(\overline{L}_I)/\overline{L}_I \cong \mathrm{Stab}_{W(\overline{G})}(I)$, and $N_{\overline{G}}(\overline{L}_I)$ is generated by $\overline{L}_I$ together with elements in $N_{\overline{G}}(\overline{T})$, whose images in $W(\overline{G})$ generate $\mathrm{Stab}_{W(\overline{G})}(I)$. With our identification of $W(\overline{G})$ with a subgroup of $N_{\overline{G}}(\overline{T})$, we have $N_{\overline{G}}(\overline{L}_I) = \overline{L}_I\mathrm{Stab}_{W(\overline{G})}(I)$.

We record a result on twisting of Levi subgroups; for this concept see [11, 3.3.1].

Lemma 7.1. *Let $I \subseteq \Pi$ be σ-stable, so that $\overline{L}_I$ and $\mathrm{Stab}_{W(\overline{G})}(I)$ are σ-stable. Let $y \in \mathrm{Stab}_{W(\overline{G})^\sigma}(I)$ and choose $g \in \overline{G}$ with $g^{-1}\sigma(g) = y$. Put $\overline{M} := {}^g\overline{L}_I$, and $S := {}^g\mathrm{Stab}_{W(\overline{G})}(I)$.*

(a) *The groups $\overline{M}$ and S are σ-stable and $N_{\overline{G}}(\overline{M})$ is a semidirect product $N_{\overline{G}}(\overline{M}) = \overline{M}S$. Moreover, g^{-1} conjugates S^σ to the set*

$$C_{I,\sigma}(y) := \{u \in \mathrm{Stab}_{W(\overline{G})}(I) \mid uy\sigma(u)^{-1} = y\},$$

the σ-centralizer of y.

(b) *Let $t \in Z(\overline{M}^\sigma)$ with $C_{\overline{G}}(t) = \overline{M}$. Then S^σ acts regularly on the set of $\overline{G}^\sigma$-conjugates of t in $Z(\overline{M}^\sigma)$. In particular, $|S^\sigma|$ is even, if t is non-trivial and real in $\overline{G}^\sigma$.*

Proof. (a) This is well known. It can be proved by a direct calculation.

(b) By (a), we have $N_{\overline{G}}(\overline{M})^\sigma = \overline{M}^\sigma S^\sigma$, a semidirect product. Notice that $Z(\overline{M}^\sigma) = Z(\overline{M})^\sigma$ by [4, Proposition 3.6.8]. It $t' \in Z(\overline{M}^\sigma)$ is conjugate to t by an element $g \in \overline{G}^\sigma$, then $g \in N_{\overline{G}}(\overline{M})^\sigma$ by Lemma 2.2. It follows that t and t' are conjugate by an element of S^σ. Since the stabilizer of t in S^σ is trivial, we get our assertion. If t is real, t is conjugate to t^{-1} in S^σ. Since t is non-trivial and q is even, $t \neq t^{-1}$, and so a conjugating element must have even order. □

The following two cases are of particular relevance for our proof. For simplicity, the elements of $I \subseteq \Sigma$ are denoted by their Chevie numbers;

see [10]. The root r_{36} occurring below is the unique positive root of maximal height. The corresponding reflection s_{36} is ι- and hence σ-stable.

Lemma 7.2. (a) *Let* $I = \{1, 3, 4, 5, 6\}$. *Then* $\mathrm{Stab}_{W(\overline{G})}(I) = \langle s_{36} \rangle$. *Moreover,* s_{36} *is conjugate to* s_4 *by a* ι*-stable element* x *such that* $W(\overline{L}_I) \cap {}^{x^{-1}}W(\overline{L}_I) = W(\overline{L}_{\{3,4,5\}})$.

(b) *Let* $I = \{2, 3, 4, 5\}$. *Then* $\mathrm{Stab}_{W(\overline{G})}(I) = \langle v, w \rangle$ *is a non-abelian group of order* 6. *Here,* v *and* w *are distinct involutions, and* v *is the unique* ι*-stable involution in* $\mathrm{Stab}_{W(\overline{G})}(I)$. *Also,* v *acts like* ι *on the subdiagram of the Dynkin diagram of* E_6 *induced by the nodes* 2, 3, 4, 5, *and* vw *permutes its leaves cyclically.*

Moreover, there is a unique $x \in W(\overline{G})$ *such that the following hold.*

(i) *The element* x *is a* ι*-stable involution.*
(ii) *We have* ${}^{x}v = s_1 s_6$ *and* ${}^{x}w = s_1 s_3 s_1 s_5$.
(iii) *We have* $W(\overline{L}_I) \cap {}^{x^{-1}}W(\overline{L}_{\{1,3,4,5,6\}}) = W(\overline{L}_{\{2,3,5\}})$.

Proof. This can be proved with a Chevie [10] computation. □

We will need some properties of certain semisimple elements of $\overline{G}^{\sigma}$. The semisimple elements of $\overline{G}^{\sigma}$ are organized in class types. By definition, two semisimple elements $s, s' \in \overline{G}^{\sigma}$ belong to the same class type, if and only if $C_{\overline{G}}(s)$ and $C_{\overline{G}}(s')$ are conjugate in $\overline{G}^{\sigma}$. A list of these class types and the corresponding centralizers is provided by Frank Lübeck in [20]. In fact, there is one list for each value of ε. In Lemma 7.8 below, we will follow [20] in the labelling of the class types. Before this, we have to explore the structure of a particular Levi subgroup of $\overline{G}$.

Lemma 7.3. *Let* $\overline{L} := \overline{L}_{\{1,3,4,5,6\}}$ *denote the standard Levi subgroup of* $\overline{G}$ *of type* A_5*; see* [14, Figure 1]. *Then* $\overline{L} = [\overline{L}, \overline{L}] \times Z(\overline{L})$, *where* $Z(\overline{L})$ *is a split torus of rank* 1. *Moreover,* $N_{\overline{G}}(\overline{L}) = \langle \overline{L}, s_{36} \rangle$, *where* s_{36} *centralizes* $[\overline{L}, \overline{L}]$ *and inverts the elements of* $Z(\overline{L})$.

Also, $\overline{L}$ *is* σ*-stable and* $\overline{L}^{\sigma} = [\overline{L}, \overline{L}]^{\sigma} \times Z(\overline{L})^{\sigma}$, *where* $[\overline{L}, \overline{L}]^{\sigma} \cong \mathrm{PGL}_6^{\varepsilon}(q)$ *and* $Z(\overline{L})^{\sigma} = Z(\overline{L}^{\sigma})$ *is cyclic of order* $q - 1$. *Moreover,* $N_{\overline{G}^{\sigma}}(\overline{L}) = \langle \overline{L}^{\sigma}, s_{36} \rangle$ *and* $N_{\overline{G}^{\sigma}}(\overline{L}) = N_{\overline{G}^{\sigma}}(\overline{L}^{\sigma})$, *unless* $q = 2$, *in which case* $N_{\overline{G}^{\sigma}}(\overline{L}^{\sigma}) = [\overline{L}, \overline{L}]^{\sigma} \times H_2$, *with* $H_2 \cong \mathrm{SL}_2(2)$.

Proof. As is easily checked with Chevie [10], the root system Σ of $\overline{G}$ contains a closed subsystem of type $A_5 + A_1$, generated by the positive roots $r_1, r_3, r_4, r_5, r_6, r_{36}$. Let $\overline{H} := \overline{L}_{\{1,3,4,5,6,36\}}$ denote the corresponding subsystem subgroup of $\overline{G}$; see [25, Section 13.1].

Put $\overline{H}_1 := [\overline{L}, \overline{L}]$, and $\overline{H}_2 := [\overline{L}_{\{36\}}, \overline{L}_{\{36\}}]$. Then $\overline{H}_i$ is semisimple for $i = 1, 2$. It follows that $\overline{H}_1$ is generated by the root subgroups corresponding to the roots $\pm r_1, \pm r_3, \pm r_4, \pm r_5, \pm r_6$, and $\overline{H}_2$ is generated by the root subgroups corresponding to $\pm r_{36}$; see, e.g. [25, Theorem 8.21(a)]. Since its root system has rank 6, the group $\overline{H}$ is semisimple as well, and we obtain $\overline{H} = \langle \overline{H}_1, \overline{H}_2 \rangle$. As $r_j \pm r_{36}$ is not a root for all $j \in \{1, 3, 4, 5, 6\}$, the commutator relations show that $\overline{H}_1$ and $\overline{H}_2$ centralize each other. Since q is even, the center of $\overline{H}_2$ is trivial. Hence $\overline{H} = \overline{H}_1 \times \overline{H}_2$. Since $\overline{L} \leq \overline{H}$ and $\overline{H}_1 \leq \overline{L}$, we have $\overline{L} = [\overline{L}, \overline{L}] \times (\overline{L} \cap \overline{H}_2)$ with $\overline{L} \cap \overline{H}_2 = \overline{T} \cap \overline{H}_2 = Z(\overline{L})$. As $s_{36} \in \overline{H}_2$ it follows that s_{36} centralizes $[\overline{L}, \overline{L}]$. Clearly, s_{36} inverts the elements of the standard torus $\overline{T} \cap \overline{H}_2$ of $\overline{H}_2$. By Lemma 7.2(a), we have $\mathrm{Stab}_{W(\overline{G})}(\{1, 3, 4, 5, 6\}) = \langle s_{36} \rangle$. Hence $N_{\overline{G}}(\overline{L}) = \langle \overline{L}, s_{36} \rangle$.

As $\overline{L}$ and $\overline{H}_2$ are ι-stable, they are σ-stable as well. We obtain $\overline{H}^{\sigma} = [\overline{L}, \overline{L}]^{\sigma} \times \overline{H}_2^{\sigma}$ with $\overline{H}_2^{\sigma} \cong \mathrm{SL}_2(q)$. This yields $\overline{L}^{\sigma} = [\overline{L}, \overline{L}]^{\sigma} \times Z(\overline{L})^{\sigma} = [\overline{L}, \overline{L}]^{\sigma} \times Z(\overline{L}^{\sigma})$. Clearly, $Z(\overline{L}^{\sigma})$ is cyclic of order $q - 1$, as $\overline{L}$ is a standard Levi subgroup of $\overline{G}$ of semisimple rank 5. If $q = 2$, then $\varepsilon = -1$, and $\overline{L}^{\sigma} = [\overline{L}, \overline{L}]^{\sigma} \cong \mathrm{PGU}_6(2)$; see the Atlas [5, p. 191 and p. 115]. Suppose then that $q > 2$. Then, by the tables in [20], the group $\overline{L}^{\sigma}$ is the centralizer of a semisimple element of class type [8, 1, 1]. These tables show that the center of $[\overline{L}, \overline{L}]^{\sigma}$ is trivial. Hence $[\overline{L}^{\sigma}, \overline{L}^{\sigma}]$ is the simple group $\mathrm{PSL}_6^{\varepsilon}(q)$, which has index 3 in $[\overline{L}, \overline{L}]^{\sigma}$. Moreover, $[\overline{L}, \overline{L}]^{\sigma}$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{PSL}_6^{\varepsilon}(q))$. It follows from general principles about isogenies, that the standard torus of $[\overline{L}, \overline{L}]^{\sigma}$ has the same order as the standard torus of $\mathrm{PGL}_6^{\varepsilon}(q)$. This implies that $[\overline{L}, \overline{L}]^{\sigma} = \langle [\overline{L}^{\sigma}, \overline{L}^{\sigma}], t \rangle$, where t is an element of the standard torus of $[\overline{L}, \overline{L}]^{\sigma}$. In particular, t induces an inner diagonal automorphism on $[\overline{L}^{\sigma}, \overline{L}^{\sigma}]$, and so $[\overline{L}, \overline{L}]^{\sigma} \cong \mathrm{PGL}_6^{\varepsilon}(q)$ as claimed.

As s_{36} is σ-stable, we get $N_{\overline{G}}(\overline{L})^{\sigma} = \langle \overline{L}^{\sigma}, s_{36} \rangle$. To prove the final statement, suppose first that $q \neq 2$. Observe that $C_{\overline{G}}(Z(\overline{L}^{\sigma})) = \overline{L}$. Indeed, $\overline{L} \leq C_{\overline{G}}(Z(\overline{L}^{\sigma}))$, and by the tables in [20] we must have equality. Hence

$$N_{\overline{G}^{\sigma}}(\overline{L}^{\sigma}) \leq N_{\overline{G}^{\sigma}}(Z(\overline{L}^{\sigma})) \leq N_{\overline{G}^{\sigma}}(C_{\overline{G}}(Z(\overline{L}^{\sigma}))) = N_{\overline{G}^{\sigma}}(\overline{L}) \leq N_{\overline{G}^{\sigma}}(\overline{L}^{\sigma}).$$

Now suppose that $q = 2$. Then, $\overline{L}^{\sigma} = [\overline{L}, \overline{L}]^{\sigma}$, and thus $[\overline{L}, \overline{L}]^{\sigma} \times H_2 \leq N_{\overline{G}^{\sigma}}(\overline{L}^{\sigma})$. The list of maximal subgroups of $\overline{G}^{\sigma}$ given in the Atlas (see [5, p. 191] and [17, Appendix 2]) now proves our claim. □

Lemma 7.4. *Let the notation be as in* Lemma 7.3 *and its proof. Let* $g \in \overline{H}_2$ *be such that* $g^{-1}\sigma(g) = s_{36}$, *and put* $\overline{L}' := {}^{g}\overline{L}$. *Then* $\overline{L}'$ *is* σ*-stable and* $[\overline{L}', \overline{L}'] = [\overline{L}, \overline{L}]$. *Moreover,* $\overline{L}'^{\sigma} = [\overline{L}', \overline{L}']^{\sigma} \times Z(\overline{L}')^{\sigma}$, $Z(\overline{L}')^{\sigma}$ *is cyclic of order* $q + 1$.

Proof. This follows from Lemma 7.1 and the fact that g commutes with $[\overline{L}, \overline{L}]$. □

7.5. **Some class types and their centralizers.** For the concept of e-split Levi subgroups of $\overline{G}$ used below, see [11, 3.5.1].

Remark 7.6. The Levi subgroups $\overline{L}$ and $\overline{L}'$ described in Lemmas 7.3 and 7.4 are 1-split and 2-split Levi subgroups of $\overline{G}$ of type A_5, respectively. They are representatives, with respect to $\overline{G}^\sigma$-conjugation, for the centralizers of semisimple elements of class types $[8, 1, 1]$ and $[8, 1, 2]$ (in the notation of [20]), respectively. □

In the course of the proof, we will consider various other class types. The following lemma gives a construction of the corresponding centralizers.

Lemma 7.7. *Let $s \in \overline{G}^\sigma$ be a semisimple element in one of the class types $[8, 1, k]$ or $[14, 1, k]$ with $k = 1, 2$.*

Then $C_{\overline{G}}(s)$ is connected. In fact, $C_{\overline{G}}(s)$ is $\overline{G}$-conjugate to $\overline{L}_I$, where I is one of $\{1, 3, 4, 5, 6\}$ or $\{2, 3, 4, 5\}$, according as s is of class type $[8, 1, k]$ or $[14, 1, k]$, respectively, $k = 1, 2$.

An element s of class type $[14, 2, 1]$ has order 3 and lies in the center of the centralizer of an element of class type $[14, 1, 1]$. In this case, $C^\circ_{\overline{G}}(s)$ is $\overline{G}$-conjugate to $\overline{L}_{\{2,3,4,5\}}$, and $|C_{\overline{G}}(s)/C^\circ_{\overline{G}}(s)| = 3$.

Proof. This is implicit in [20]. □

The above lemma gives representatives, up to $\overline{G}^\sigma$-conjugation, of the centralizers of the class types in the split case, i.e. where $C^\circ_{\overline{G}}(s)$ is $\overline{G}^\sigma$-conjugate to a standard Levi subgroup of $\overline{G}$. Representatives for the 2-split cases will be constructed in the following lemma. Recall the significance of the symbol ι discussed in the introductory paragraphs of this subsection.

Lemma 7.8. *Let $s \in \overline{G}^\sigma$ be a semisimple element in one of the class types $[8, 1, k]$, $[14, 1, k]$ with $k = 1, 2$, or $[14, 2, 1]$. In the latter case assume that $q \neq 2$. Suppose that s is real in $\overline{G}^\sigma$. Then there is a σ-stable standard Levi subgroup $\overline{L}$ of $\overline{G}$, and there is $s' \in \overline{L}^\sigma$ such that s and s' are $\overline{G}^\sigma$-conjugate, and the following conditions hold.*

(i) *The element s' is real in $\overline{L}^\sigma$.*
(ii) *The centralizer $C_{\overline{L}}(s')$ is connected.*
(iii) *If $\alpha \in \mathrm{Aut}(G)$ stabilizes $\overline{L}^\sigma$ and the $\overline{G}^\sigma$-conjugacy class of s, then $\alpha(s')$ and s' are conjugate in $\overline{L}^\sigma$.*

Table 2. Twisting elements for some class types (explanations in the proof of Lemma 7.8)

class type	I	$\mathrm{Stab}_{W(\overline{G})}(I)$	ε	y	z	$\lvert C_{I,\sigma}(y)\rvert$
$[8,1,1]$	$\{1,3,4,5,6\}$	$\langle s_{36}\rangle$	± 1	1	1	2
$[8,1,2]$				s_{36}	s_4	2
$[14,1,1]$	$\{2,3,4,5\}$	$\langle v,w\rangle$	1	1	1	6
$[14,1,2]$				v	s_1s_6	2
$[14,1,2]$			-1	1	1	2
$[14,1,1]$				v	s_1s_6	6

Proof. By Lemma 7.7, the group $C^{\circ}_{\overline{G}}(s)$ is $\overline{G}$-conjugate to $\overline{L}_I$, where I is one of $\{1,3,4,5,6\}$ or $\{2,3,4,5\}$; notice that I is ι-stable and thus also σ-stable. Put $\overline{K} := \overline{L}_{\{1,3,4,5,6\}}$. The groups $\mathrm{Stab}_{W(\overline{G})}(I)$ have been determined in Lemma 7.2, whose notation is used in Column 3 of Table 2. Let x be a ι-stable element with the properties exhibited in Lemma 7.2. The elements $y \in \mathrm{Stab}_{W(\overline{G})}(I)$ given in the fifth column of Table 2 are σ-stable. Let us write $z := {}^{x}y$, so that $z \in W(\overline{K})^{\sigma} \leq \overline{K}^{\sigma}$, where the first inclusion follows from Lemma 7.2. The elements z are given in the sixth column of Table 2.

Fix I, ε and $k \in \{1,2\}$. We claim that twisting $\overline{L}_I$ with the element y associated to these data in Table 2 (see [11, 3.3.1]), yields representatives for the centralizers of the corresponding class types $[8,1,k]$ respectively $[14,1,k]$. To see this, first observe that the two y-elements associated to distinct ks lie in distinct σ-conjugacy classes of $\mathrm{Stab}_{W(\overline{G})}(I)$. This is clear for $I = \{1,3,4,5,6\}$, as then $\mathrm{Stab}_{W(\overline{G})}(I)$ has order 2. For $I = \{2,3,4,5\}$, the corresponding σ-centralizers $C_{I,\sigma}(y)$ of y (see Lemma 7.1(b)) have different orders. Next, choose $h \in \overline{K}$ with $h^{-1}\sigma(h) = z$; if $z = 1$, choose $h = 1$. Putting $g := hx$, we get $g^{-1}\sigma(g) = y$. Now let $\overline{M} := {}^{g}\overline{L}_I$. By Lemma 7.1(b), the number of σ-stable elements of $Z(\overline{M})$ with centralizer $\overline{M}$ equals $\lvert({}^{g}\mathrm{Stab}_{W(\overline{G})}(I))^{\sigma}\rvert$. By Lemma 7.1(a), the latter number equals $\lvert C_{I,\sigma}(y)\rvert$. On the other hand, the number in question can be read off the tables in [20] as the denominator of the leading coefficient in the polynomial giving the number of conjugacy classes in a given class type. This proves our claim.

We now prove our assertions if s has class type $[8,1,k]$ or $[14,1,k]$, $k = 1,2$, with $\overline{L} := \overline{K} = \overline{L}_{\{1,3,4,5,6\}}$. By the claim in the previous paragraph, $\overline{M} = {}^h({}^x\overline{L}_I)$ is the centralizer in $\overline{G}$ of an element of class type s, i.e. $C_{\overline{G}}(s)$ is $\overline{G}^\sigma$-conjugate to $\overline{M}$. In particular, s is $\overline{G}^\sigma$-conjugate to an element $s' \in Z(\overline{M})$ with $C_{\overline{G}}(s') = \overline{M}$.

(i) As $Z({}^x\overline{L}_I) \leq {}^x\overline{T} = \overline{T} \leq \overline{K}$ and $h \in \overline{K}$, we have $Z(\overline{M}) = Z({}^g\overline{L}_I) \leq \overline{K}$, and thus $s' \in Z(\overline{M})^\sigma \leq \overline{K}^\sigma$. Now s' is real in $\overline{G}^\sigma$ by hypothesis. Lemma 7.1(b) implies that s' is inverted by a σ-stable element of ${}^g\mathrm{Stab}_{W(\overline{G})}(I) = {}^h({}^x\mathrm{Stab}_{W(\overline{G})}(I))$. Hence s' is real in $\overline{K}^\sigma$, as ${}^x\mathrm{Stab}_{W(\overline{G})}(I) \leq \overline{K}$ and $h \in \overline{K}$.

(ii) By construction, the centralizer $C_{\overline{K}}(s')$ is $\overline{K}$-conjugate to ${}^x\overline{L}_I \cap \overline{K}$. The latter is the centralizer in $\overline{G}$ of the subtorus ${}^xZ(\overline{L}_I)Z(\overline{K})$ of $\overline{T}$; see [25, Propositions 12.6 and 3.9]. Hence ${}^x\overline{L}_I \cap \overline{K}$ is a Levi subgroup of $\overline{G}$ and thus connected. To be specific, Lemma 7.2 implies that ${}^x\overline{L}_I \cap \overline{K}$ is $\overline{G}$-conjugate to the standard Levi subgroup $\overline{L}_{\{3,4,5\}}$, respectively $\overline{L}_{\{2,3,5\}}$. Hence $C_{\overline{K}}(s')$ is connected.

(iii) We may assume that $s = s'$. We begin with a special case. Suppose first that $\alpha = \mu \in \Gamma_G \times \Phi_G$. Then $\alpha = \mu$ certainly stabilizes $\overline{K}^\sigma$, as $\overline{K}$ is σ-stable by Lemma 7.3, and μ commutes with σ. By the same argument, $\overline{L}_I$ and $\overline{L}_I^\sigma$ are μ-stable.

Suppose that s' is conjugate in $\overline{G}^\sigma$ to $\mu(s')$. Then $g^{-1}s'g$ is conjugate in $\overline{G}$ to $\mu(g)^{-1}\mu(s')\mu(g)$. Now $g^{-1}s'g$ and $\mu(g)^{-1}\mu(s')\mu(g)$ are contained in $Z(\overline{L}_I)$, as $\overline{L}_I$, and hence $Z(\overline{L}_I)$ are μ-stable. By Lemma 2.2, there is $u \in \mathrm{Stab}_{W(\overline{G})}(I)$ such that $ug^{-1}s'gu^{-1} = \mu(g)^{-1}\mu(s')\mu(g)$. As $g = hx$ and $\mu(x) = x$, we obtain

$$\mu(s') = [\mu(h)xux^{-1}h^{-1}]s'[\mu(h)xux^{-1}h^{-1}]^{-1}.$$

Now $h \in \overline{K}$ and $\overline{K}$ is μ-stable. Moreover, $xux^{-1} \in \overline{K}$ by Lemma 7.2. Thus $\mu(h)xux^{-1}h^{-1} \in \overline{K}$ and so s' and $\mu(s')$ are conjugate in $\overline{K}$. As $C_{\overline{K}}(s')$ is connected, s' and $\mu(s')$ are conjugate in $\overline{K}^\sigma$; see [11, Example 1.4.10].

Now let $\alpha \in \mathrm{Aut}(G)$ stabilize $\overline{K}^\sigma$ and the $\overline{G}^\sigma$-conjugacy class of s'. Then $\alpha = \mathrm{ad}_c \circ \mu$ with $c \in \overline{G}^\sigma$ and $\mu \in \Gamma_G \times \Phi_G$. As μ stabilizes $\overline{K}^\sigma$, we must have $c \in N_{\overline{G}^\sigma}(\overline{K}^\sigma)$. Now $\alpha(s') = c\mu(s')c^{-1}$ is conjugate in $\overline{G}^\sigma$ to s'. Hence $\mu(s')$ is conjugate to s' in $\overline{G}^\sigma$. By what we have already proved, there is $c' \in \overline{K}^\sigma$ such that $\mu(s') = c's'c'^{-1}$. Thus $\alpha(s') = c\mu(s')c^{-1} = (cc')s'(cc')^{-1}$ and so $\alpha(s')$ and s' are conjugate in $N_{\overline{G}^\sigma}(\overline{K}^\sigma)$.

As s' is real in $\overline{K}^\sigma$ and $\overline{K}^\sigma = [\overline{K}, \overline{K}]^\sigma \times Z(\overline{K})^\sigma$ by Lemma 7.3, we must have $s' \in [\overline{K}, \overline{K}]^\sigma$. By the structure of $N_{\overline{G}^\sigma}(\overline{K}^\sigma)$ given in Lemma 7.3, the elements $\alpha(s')$ and s' are conjugate in $\overline{K}^\sigma$. This completes our proof if s is of class type $[8, 1, k]$ or $[14, 1, k]$, $k = 1, 2$.

We finally prove our assertions for elements of class type $[14, 2, 1]$. Suppose we are in the situation where $\overline{M}$ is the centralizer of an element of class type $[14, 1, 1]$, i.e. $I = \{2, 3, 4, 5\}$. Thus there $t \in Z(\overline{M}^{\sigma})$ of order 3 such that $C^{\circ}_{\overline{G}}(t) = \overline{M}$ and $|(C_{\overline{G}}(t)/C^{\circ}_{\overline{G}}(t))^{\sigma}| = 3$; see Lemma 7.7. Now $C_{\overline{G}}(t) \leq N_{\overline{G}}(C^{\circ}_{\overline{G}}(t)) = N_{\overline{G}}(\overline{M}) = \overline{M}{}^g\mathrm{Stab}_{W(\overline{G})}(I)$. Thus t is centralized by the elements of order 3 in ${}^g\mathrm{Stab}_{W(\overline{G})}(I)$. As t is real, it is inverted by the involutions in ${}^g\mathrm{Stab}_{W(\overline{G})}(I)$. Now every element of ${}^g\mathrm{Stab}_{W(\overline{G})}(I)$ is σ-stable, and so t is real in $\overline{K}^{\sigma}$. Also, $C_{\overline{L}}(t)$ is not connected, as the Sylow 3-subgroup of ${}^g\mathrm{Stab}_{W(\overline{G})}(I)$ centralizes t and normalizes $\overline{M} \cap \overline{K} = C^{\circ}_{\overline{K}}(t)$. By the proof of (ii) above, the group $\overline{M} \cap \overline{K}$ is $\overline{G}$-conjugate to $\overline{L}_{\{2,3,5\}}$. A Chevie computation shows that the latter is $\overline{G}^{\sigma}$-conjugate to $\overline{L}_{\{1,4,6\}} \leq \overline{K}$.

Now $\overline{K}^{\sigma} = [\overline{K}, \overline{K}]^{\sigma} \times Z(\overline{K})^{\sigma}$, where $Z(\overline{K})^{\sigma}$ is a cyclic group of order $q - 1$ and $[\overline{K}, \overline{K}]^{\sigma} \cong \mathrm{PGL}_6^{\varepsilon}(q)$; see Lemma 7.3. As t is real in $\overline{K}^{\sigma}$, we must have $t \in [\overline{K}, \overline{K}]^{\sigma}$. Let $\hat{G} = \mathrm{GL}_6^{\varepsilon}(q)$ and consider the natural map $\hat{G} \to [\overline{L}, \overline{L}]^{\sigma}$. Let $\hat{t}$ be a real lift of t in $\hat{G}$. Then $\hat{t}$ has order 3, since $\hat{t}^3$ is a real central element of $\hat{G}$. The structure of $C_{\overline{K}}(t)$ exhibited above, together with Lemmas 6.3 and 6.6, imply that $\hat{t}$ has eigenvalues $1, \zeta, \zeta^{-1}$, each with multiplicity 2, where $\zeta \in \mathbb{F}_{q^{\delta}}$ has order 3. In particular, the minimal polynomial of $\hat{t}$ is as in (a) of Lemma 6.20.

Let $\overline{L} := \overline{L}_{\{3,4,5\}} \leq \overline{K} = \overline{L}_{\{1,3,4,5,6\}}$. Then $\overline{L}$ is a σ-stable standard Levi subgroup of $\overline{G}$. Lemma 6.23, applied to $\hat{t}$ and $[\overline{K}, \overline{K}]^{\sigma}$, shows that there is an element $t' \in \overline{L}^{\sigma}$ conjugate to t in $[\overline{K}, \overline{K}]^{\sigma}$, such that t satisfies (i) and (iii). Moreover, $C_{\overline{L}}(t')$ is connected, since the standard Levi subgroup $\overline{L}^* \leq \overline{G}^*$ dual to $\overline{L}$ has connected center; see [20]. This completes our proof for the case that s has class type $[14, 2, 1]$. □

Lemma 7.9. *The elements of the class types* $[3, 2, 3]$, $[4, 1, 1]$, $[5, 1, 1]$, $[9, 1, 2]$, $[9, 1, 1]$, $[14, 2, 2]$ *and* $[14, 1, 3]$ *are not real.*

Proof. If s is of class type $[14, 2, 2]$ or $[3, 2, 3]$, then the center of $C_{\overline{G}^{\sigma}}(s)$ has order 3, and there are two conjugacy classes of this class type; thus s and s^{-1} lie in different $\overline{G}^{\sigma}$-conjugacy classes. For the class types $[4, 1, 1]$, $[5, 1, 1]$ and $[14, 1, 3]$, the claim follows from the last statement of Lemma 7.1(b).

Suppose that the semisimple element $s \in \overline{G}^{\sigma}$ is of class type $[9, 1, 1]$ or $[9, 1, 2]$ and that s is real in $\overline{G}^{\sigma}$. Then s is real in $\overline{G}$. Let $J := \{1, 3, 4, 5\}$ and put $\overline{M} := \overline{L}_J$. As $C_{\overline{G}}(s)$ is $\overline{G}$-conjugate to $\overline{M}$ by [20], the center of $\overline{M}$ contains a real element t with centralizer $\overline{M}$. By Lemma 2.2, there is an element in $\mathrm{Stab}_{W(\overline{G})}(J)$ inverting t.

Now $\overline{M} = \overline{L}_J \leq \overline{L}_I$ with $I = \{1, 3, 4, 5, 6\}$. By Table 2, we have $\mathrm{Stab}_{W(\overline{G})}(J) = \mathrm{Stab}_{W(\overline{G})}(I) = \langle s_{36} \rangle$. By Lemma 7.3, we may write $t = t_1 t_2$ with $t_1 \in [\overline{L}_I, \overline{L}_I]$ and $t_2 \in Z(\overline{L}_I)$. Moreover, t_1 is centralized by s_{36}. Now $t_1 \neq t_1^{-1}$ as $C_{\overline{G}}(t) = \overline{L}_J$ and q is even. Hence t is not inverted by s_{36}, a contradiction. □

7.10. **Modification of ν.** The following is the analogue of Proposition 6.15.

Proposition 7.11. *Let G be as in* Hypothesis 3.2(b)(iii),(iv)*, and let $\beta \in \mathrm{Aut}(G)$. Write $\beta = \mathrm{ad}_h \circ \mu$ for some $h \in \overline{G}^\sigma$ and some $\mu \in \Gamma_G \times \Phi_G$. Then there is $g \in G$ such that $\alpha := \mathrm{ad}_g \circ \beta$ has even order and the following statements hold.*

(a) *We have $|C_G(\alpha_{(p)})| < q^{48}$ for every prime p dividing $|\alpha|$ and every element $\alpha_{(p)}$ of $\langle \alpha \rangle$ of order p, unless $p = 2$ and $\alpha_{(2)}$ is a graph automorphism of G. In the latter case, $|C_G(\alpha_{(p)})| < q^{52}$.*

(b) *The order of α divides $3\delta f$.*

(c) *If $h \in G$, then $|\alpha|$ divides δf.*

(d) *If $\varepsilon = -1$ and $|\mu|$ is even, then $|\alpha|$ divides $2f$.*

Proof. Let $\overline{L} := \overline{L}_{\{2,3,4,5\}}$. Then $[\overline{L}, \overline{L}]$ is a simple group of type D_4, and thus $[\overline{L}, \overline{L}]^\sigma \cong \mathrm{P}\Omega_8^\varepsilon(q)$, as q is even. This is a non-abelian simple group, so that $[\overline{L}, \overline{L}]^\sigma = [\overline{L}^\sigma, \overline{L}^\sigma]$. It follows that $\overline{L}^\sigma = [\overline{L}^\sigma, \overline{L}^\sigma] \times Z$ with Z isomorphic to $[q-1] \times [q-1]$, respectively to $[q^2-1]$; see [20]. Write Z_3 for the Sylow 3-subgroup of Z. Then Z_3 is invariant under $\Gamma_G \times \Phi_G$. Recall from Subsection 4.1 that $\overline{G}^\sigma = G\overline{T}^\sigma$, and thus $\overline{G}^\sigma = G\overline{L}^\sigma$. Now $[\overline{L}^\sigma, \overline{L}^\sigma] \leq [\overline{G}^\sigma, \overline{G}^\sigma] = G$. Hence $Z \not\leq G$ and so $\overline{G}^\sigma = GZ_3$.

Thus there is $y \in G$ such that $\beta' := \mathrm{ad}_y \circ \beta$ is of the form

$$\beta' = \mathrm{ad}_t \circ \mu, \tag{20}$$

with $t \in Z_3$. We choose $t = 1$ if $h \in G$. Equation (1) and the fact that Z_3 is invariant in $\Gamma_G \times \Phi_G$, imply that

$$\beta'^l = \mathrm{ad}_{t'} \circ \mu^l \tag{21}$$

with $t' \in Z_3$ for every integer l.

(a) Let p be a prime dividing $|\beta'|$. The claim that $|C_G(\beta'_{(p)})| < q^{48}$, unless $p = 2$ and $\beta'_{(2)}$ is a graph automorphism of G, in which case $|C_G(\beta'_{(p)})| < q^{52}$. This claim follows from Lemma 5.3, except if $\beta'_{(p)}$ is an inner-diagonal automorphism. In the latter case, $\beta'_{(p)} = \mathrm{ad}_{t'}$ for some $t' \in Z_3$ by Equation (21). Since t' is an element of order 3 in the center of $\overline{L}^\sigma$, the tables in [20] imply that $|C_{\overline{G}^\sigma}(\beta'_{(3)})| = |C_{\overline{G}^\sigma}(t')|$ is bounded above by $(q - \varepsilon)|\mathrm{P}\Omega_{10}^\varepsilon(q)|$. A rough upper bound for the latter order is q^{48}, proving our claim.

Once more with Equation (21), we see that $|\beta'|$ is even, if and only if $|\mu|$ is even. In this case we put $\alpha := \beta'$. Then α satisfies the assertions. Suppose now that $|\mu|$ is odd. Then $\mu \in \Phi_G$. In particular, μ is a power of φ. Let $g \in [\overline{L}, \overline{L}]$ be the product of a ι-stable set of three pairwise commuting φ-stable root elements. Then g is a ι-stable involution in G. Moreover, g belongs to the class $3A1$ of unipotent elements in the notation of [19]. Put $\alpha := \mathrm{ad}_g \circ \beta'$. As μ is a power of φ and thus fixes g, and as g commutes with t, we have $\alpha = \beta' \circ \mathrm{ad}_g$. Also, $C_G(\alpha_{(p)}) = C_G(\beta'_{(p)})$ if p is odd, and $|C_{\overline{G}^\sigma}(\alpha_{(2)})| = |C_{\overline{G}^\sigma}(g)| = q^{31}(q-\varepsilon)^3(q+\varepsilon)^2(q^2+\varepsilon q+1) \leq q^{48}$. In case of $\varepsilon = 1$, the order of $C_{\overline{G}^\sigma}(g)$ has been computed by Mizuno (see [27, Lemma 4.2]), in case $\varepsilon = -1$ by Gunter Malle (see [23, Proposition 3]). This completes the proof of (a).

The statements (b)–(d) follow from Lemma 5.4. □

7.12. **The main result.** We are now ready to finish the proof for the groups $E_6^\varepsilon(q)$.

Proposition 7.13. *Suppose that* $G = E_6^\varepsilon(q)$ *with* q *even and* $3 \mid q - \varepsilon$. *Then* G *is not a minimal counterexample to* [14, Theorem 1.1.5].

Proof. Suppose that G is a minimal counterexample to [14, Theorem 1.1.5], let (V, n) be a pair without the $E1$-property and let χ denote the character of V. Let α be constructed as in Proposition 7.11 from $\beta := \nu$. By [14, Lemma 4.3.3] and Proposition 7.11 we have

$$\chi(1) \leq (|\alpha| - 1)q^{26}. \tag{22}$$

By Proposition 7.11(b), we have $|\alpha| \leq 3\delta f$. Hence $|\alpha| \leq q$ if $f > 3$. For $f = 1, 2, 3$ we have $3\delta f = 6, 6, 18$.

Now consider the lists of character degrees and their multiplicities of $\overline{G}^{*\sigma}$ provided by Frank Lübeck in [21]. In fact there are two such lists, one for each $\varepsilon \in \{-1, 1\}$; all the considerations below are true for each of these lists. It turns out that exactly 116 of the character degrees of $\overline{G}^{*\sigma}$ are odd. Let (d_i, m_i), $i = 1, \ldots, 116$ denote the entries of Lübeck's list corresponding to the odd degree characters, where d_i denotes the degree of a character and m_i its multiplicity, $i = 1, \ldots, 116$. We choose the notation that that $i < j$ if and only if (d_i, m_i) comes before (d_j, m_j) in the full list [21]. The quantities d_i, m_i, $i = 1, \ldots, 116$ are polynomials in q with rational coefficients. Using GAP [9] and some obvious estimates, one checks that $d_i > q^{27}$ for $q > 8$ and $i > 14$. We also have $d_i > 17 \cdot 8^{26}$ if $q = 8$ and $i > 14$, as well as $d_i > 5 \cdot 4^{26}$ if $q = 4$ and $i > 14$. Finally, if $q = 2$, we have $d_i > 5 \cdot 2^{26}$ for all $i > 20$.

However, $m_i = 0$ for $q = 2$ and $15 \leq i \leq 19$. By (22) we may thus assume that $\chi(1) = d_i$ for some $1 \leq i \leq 14$.

The semisimple elements of $\overline{G}^{\sigma}$ and their centralizers are enumerated and described in the tables of [20]. Again, there are two such tables, but the statements below hold for each of these tables. As already mentioned in the paragraph preceding Lemma 7.8, the semisimple conjugacy classes are combined to class types. Using Lemma 4.6(a), we find that the characters of degrees d_i, $i = 1, \ldots, 14$ correspond, via Lusztig's generalized Jordan decomposition of characters, to the following class types: $[1, 1, 1]$, $[4, 1, 1,]$, $[8, 1, 2]$, $[8, 1, 1]$, $[14, 2, 2]$, $[14, 2, 1]$, $[14, 1, 3]$, $[14, 1, 2]$, $[14, 1, 1]$, $[5, 1, 1]$, $[9, 1, 2]$, $[9, 1, 1]$, $[3, 2, 3]$, $[3, 2, 1]$, where the order of the class types in this list corresponds to the order $d_1, d_2, \ldots, d_{14}$ if $\varepsilon = 1$. If $\varepsilon = -1$, this order has to be changed according to the permutation $(5, 6)(7, 9)$. The trivial character of degree $d_1 = 1$ corresponds to the trivial element, i.e. to the class type $[1, 1, 1]$. By Lemma 7.9, the elements of the class types $[4, 1, 1]$, $[14, 2, 2]$, $[14, 1, 3]$, $[5, 1, 1]$, $[9, 1, 2]$, $[9, 1, 1]$ and $[3, 2, 3]$ are not real. In all these cases, χ is not real by Lemma 4.6(b)(ii).

In the following, we may assume that s belongs to one of the remaining 6 class types. We begin by ruling out the three characters corresponding to the class type $[3, 2, 1]$ (this class type consists of a unique conjugacy class of elements of order 3) as candidates for χ. Suppose that χ corresponds to s in class $[3, 2, 1]$. Using the exact value of $\chi(1)$, we find $\chi(1) > q^{27}/3$. Clearly, $3\delta f < q/3$ for $f \geq 6$. By Proposition 7.11(b) and (22), we are left with the cases $f \leq 5$. Observe that χ is not invariant in $\overline{G}^{\sigma}$. On the other hand, χ is invariant under ν, and thus also under β', with $\beta' = \mathrm{ad}_y \circ \nu = \mathrm{ad}_t \circ \mu$ as in (20). This implies that $t \in G$ if $3 \nmid f$. In the latter case, $|\alpha| \leq \delta f$ by Proposition 7.11(c). Since $\delta f - 1 < q/3$ for $f = 2, 4, 5$, we are left with the cases $f = 1$ and $f = 3$. If $f = 1$, then α is an involution, and we are done with [14, Corollary 4.3.2]. Suppose finally that $f = 3$. Assume first that $|\mu|$ is odd. Then $|\alpha|$ divides 18 by Proposition 7.11(b), and $|C_G(\alpha_{(p)})| < q^{48}$ for $p = 2, 3$ by Proposition 7.11(a). Since $q^{27}/3 > 17q^{24}$, this case is ruled out by [14, Lemma 4.3.3]. Suppose finally that $|\mu|$ is even. Then $|\alpha| \in \{2, 6\}$ by Proposition 7.11(d). Once more by [14, Corollary 4.3.2], we may assume that $|\alpha| = 6$. Put $G^{\diamond} = \langle \mathrm{Inn}(G), \nu \rangle$ as in [14, Definition 4.2.3] and let $\chi^{\diamond}$ denote the extension of χ to $G^{\diamond}$ as in [14, Remark 4.2.5]. The elements of order 6 in $\langle \alpha \rangle$ square to elements of order 3, and $\alpha_{(2)}$ is the unique element of order 2 in $\langle \alpha \rangle$. Thus $|C_G(\alpha')| < q^{48}$ for every element $\alpha' \in \langle \alpha \rangle$ of order 3 or 6, and $|C_G(\alpha_{(2)})| < q^{52}$; see Proposition 7.11(a). Now

$q^{27}/3 > q^{26} + 4q^{24}$. If follows from [14, Lemma 2.5.5] that the restriction of $\chi^\diamond$ to $\langle\alpha\rangle$ contains the two real irreducible characters of $\langle\alpha\rangle$ with positive multiplicity. Then [14, Lemma 4.3.1] implies that this case does not yield a counterexample.

We are thus left with the cases that s belongs to one of the class types $[8, 1, k]$ or $[14, 1, k]$, $k = 1, 2$, or $[14, 2, 1]$. As we have assumed that G is a minimal counterexample, these cases are ruled out by Lemmas 7.8(b) and 4.7. □

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