ON THE SOURCE ALGEBRA EQUIVALENCE CLASS OF BLOCKS WITH CYCLIC DEFECT GROUPS, II

GERHARD HISS AND CAROLINE LASSUEUR

Abstract. This series of papers is a contribution to the program of classifying p-blocks of finite groups up to source algebra equivalence, starting with the case of cyclic blocks. To any p-block $\mathbf{B}$ of a finite group with cyclic defect group D, Linckelmann associated an invariant $W(\mathbf{B})$, which is an indecomposable endo-permutation module over D, and which, together with the Brauer tree of $\mathbf{B}$, essentially determines its source algebra equivalence class.

In Parts II–IV of our series of papers, we classify, for odd p, those endo-permutation modules of cyclic p-groups arising from p-blocks of quasisimple groups.

In the present Part II, we reduce the desired classification for the quasisimple classical groups of Lie type B, C, and D to the corresponding objective for the general linear and unitary groups; the classification is completed for the latter groups.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. By a *p-block*, or simply a *block*, we mean an indecomposable direct algebra factor of the group algebra kG for some finite group G. A block is *cyclic*, if it has a cyclic defect group.

Our work falls into the general program of classifying blocks up to certain categorical equivalences, which, in turn, is motivated by the following basic question: Can we determine which k-algebras occur as blocks? The approach to this question is guided by two concepts of equivalence, the general notion of *Morita equivalence* for finite-dimensional k-algebras, and the stronger notion of *source-algebra equivalence*, which applies to blocks only. In this respect, two prominent conjectures, one by Donovan and one by Puig, predict that the number

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of Morita equivalence classes, respectively source-algebra equivalence classes, of blocks is finite, provided a defect group is fixed. We notice, however, that classifying blocks up to Morita equivalence, respectively source-algebra equivalence is a much more involved task than proving the finiteness of the set of equivalence classes.

Donovan's Conjecture is known to hold for a fairly good number of small defect groups. An excellent summary of recent works and advances is given by the fast-growing database of blocks maintained by Charles Eaton [Ea18]. On the side of source-algebra equivalences much less is known: Puig's conjecture is known to hold when the defect groups are cyclic for an arbitrary prime number p by [Li96], and for Klein-four defect groups when $p = 2$ by [CEKL11]. In the latter case, [CEKL11] also provides us with a classification of blocks up to source-algebra equivalence. In the cyclic case, though it may sound like everything is known, no such classification is available at present. In fact, we do not even have a classification of cyclic blocks up to Morita equivalence.

The Morita equivalence class of a cyclic block $\mathbf{B}$ is encoded by its embedded Brauer tree. The source-algebra equivalence class of $\mathbf{B}$ is encoded by its embedded Brauer tree together with a sign function on the vertices of this tree and an invariant $W(\mathbf{B})$ associated to $\mathbf{B}$ by Linckelmann; see [Li96]. This is a certain indecomposable endo-permutation kD-module, where D is a defect group of $\mathbf{B}$. In [HL24], which constitutes Part I of our series of papers, we investigated the modules $W(\mathbf{B})$ and started, for odd p, the classification of those $W(\mathbf{B})$ that arise from a cyclic block $\mathbf{B}$ of a quasisimple group G. We proved that $W(\mathbf{B})$ is trivial, in the sense that $W(\mathbf{B}) \cong k$, in a large number of cases. For an accurate account see [HL24, Section 6]. In particular, $W(\mathbf{B})$ can only be non-trivial if G is a finite group of Lie type of characteristic different from p, and if $G/Z(G)$ does not have an exceptional Schur multiplier; moreover, if G is an exceptional group of Lie type, then $p = 3$, or G is of type E_8 and $p = 3$ or 5.

In [HL24] we have announced one subsequent article containing the classification of the non-trivial invariants $W(\mathbf{B})$. It has turned out that it is more appropriate to further divide this material into three portions, making up Parts II – IV of our series of papers. This division follows three well-defined, methodologically disjoint, sections.

Let us now describe the content of the present Part II in more detail. In Section 2 we provide a large collection of preliminary results, which will also be used in Parts III and IV. Our classification program starts in Section 3. In order to describe the results, suppose henceforth, until otherwise said, that G is a quasisimple group of Lie type of characteristic different from p, and that $\mathbf{B}$ is a cyclic p-block of G with defect group D. We also choose a suitable algebraic group $\mathbf{G}$, such that G is the set of fixed points of some Steinberg morphism of $\mathbf{G}$. By the results summarized two paragraphs above, every group we are left to consider is of this form.

In [HL24, Section 3] we showed how to determine $W(\mathbf{B})$ from the character table of G, in particular from the sign sequence of a non-exceptional character of $\mathbf{B}$ on the powers of a generator of D. This result is one of our main tools. Namely, suppose that such a non-exceptional character is an irreducible Deligne–Lusztig character, arising from a linear character of p'-order of a maximal torus T of G containing D. Then the character formula for Deligne–Lusztig characters allows for a computation of this sign sequence; see Lemma 2.5.7. The relative ranks of the $\mathbf{G}$-centralizers of the elements of D play a crucial role here. This approach is particularly fruitful if the block $\mathbf{B}$ is regular in the sense of Broué; see [Br90, Théorème 3.1].

Let D_1 denote the unique subgroup of D of order p. By its very definition, $W(\mathbf{B}) \cong W(\mathbf{c})$, where $\mathbf{c}$ is a Brauer correspondent of $\mathbf{B}$ in $C_G(D_1)$. This observation is our second main tool.

Suppose that G is a quasisimple classical group different from $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$. Our main reduction, presented in Section 3, shows that there is a block $\mathbf{B}'$ of a general linear or unitary group G', such that $\mathbf{B}$ and $\mathbf{B}'$ have isomorphic defect groups, and that $W(\mathbf{B}) \cong W(\mathbf{B}')$ after an identification of the defect groups; see Theorem 3.6.4. So even though we are interested in quasisimple groups, we now have to investigate the general linear and unitary groups. This is achieved in Section 4.

It turns out a posteriori, that the general linear groups do not yield examples of blocks $\mathbf{B}$ with $W(\mathbf{B}) \ncong k$. Thus let $G = \mathrm{GU}_n(q)$ for some prime power q with $p \nmid q$ and some integer $n \geq 2$. The case $p \mid q+1$ is essential, in the sense that the general case can be reduced to this; see Theorem 4.2.1. Suppose that $p \mid q+1$. Then $p \mid |Z(G)|$, so that $G = C_G(D_1)$. The p-blocks of G have been determined by Fong and Srinivasan in [FoSr82]. In our case, we may assume that $\mathbf{B}$ is a regular block with respect to a cyclic torus T of G of order $q^n - (-1)^n$ containing a defect group of $\mathbf{B}$. Then the non-exceptional character of $\mathbf{B}$ is an irreducible Deligne–Lusztig character arising from a linear character of T of p'-order and in general position. As indicated above, this allows us to determine $W(\mathbf{B})$. We find that $W(\mathbf{B}) \cong k$, unless n is odd and $p \equiv -1 \pmod 4$; see Corollary 4.1.4. In this case, the module $W(\mathbf{B})$ is described in terms of the parameters p, q and n.

As a result of our investigations, we present an explicit example where the Bonnafé–Dat–Rouquier Morita equivalence [BDR17, Theorem 1.1] does not preserve the source algebras; see Example 4.2.2. This ends the description of the results of Part II.

In Part III we will determine the non-trivial invariants $W(\mathbf{B})$ arising from cyclic blocks of quasisimple groups G with $G/Z(G) \cong \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$. Finally, Part IV achieves the desired classification for the exceptional groups of Lie type.

2. Preliminaries

Before we return to the setup of [HL24] in Subsection 2.4 below, we introduce some specific notation to simplify the formulation of our results.

2.1. **Some general notation.** Here, we set up some notation that will be used in the formulation of our results.

Definition 2.1.1. Let X be a set and $\chi : X \to \mathbb{R}$ a map. Put $\sigma_\chi := \operatorname{sgn} \circ \chi$, where sgn is the sign function on real numbers, so that, for $x \in X$,

$$\sigma_\chi(x) := \begin{cases} +1, & \text{if } \chi(x) > 0, \\ 0, & \text{if } \chi(x) = 0, \\ -1, & \text{if } \chi(x) < 0. \end{cases}$$

□

Definition 2.1.2. Let H be a finite group, p a prime and $t \in H$ a p-element. Let X be a set and let $\rho : H \to X$ be a map. Define, for every positive integer m, an element $\rho^{[m]}(t) \in X^m$ by

$$\rho^{[m]}(t) := (\rho(t^{p^{m-1}}), \rho(t^{p^{m-2}}), \dots, \rho(t^p), \rho(t)).$$

Thus, $\rho^{[m]}(t)$ contains, in reverse order, the values of ρ at the elements $t, t^p, t^{p^2}, \dots, t^{p^{m-1}}$. □

In the notation of Definition 2.1.2, suppose that $|t| = p^l$. Then $|t^{p^{m-j}}| = p^{l-\min\{l,m-j\}}$ for $1 \leq j \leq m$. If $m \leq l$, then $|t^{p^{m-j}}| = p^j$ for $1 \leq j \leq m$; if $m > l$, then $|t^{p^{m-j}}| = 1$ for $1 \leq j \leq m-l$, and $|t^{p^{m-j}}| = p^{j-(m-l)}$ for $m-l < j \leq m$.

2.2. **Labels.** In the following, we let $\mathbb{F}_2 = \{0,1\}$ denote the field with 2 elements. Let l be a positive integer, and put $\Lambda := \{0, 1, \dots, l-1\}$. By $\mathcal{P}(\Lambda)$ we denote the power set of Λ. The symmetric difference of sets equips $\mathcal{P}(\Lambda)$ with the structure of an abelian group of exponent two, thus of an $\mathbb{F}_2$-vector space of dimension l.

We write the elements of $\mathbb{F}_2^\Lambda$ as tuples $(\alpha_0, \alpha_1, \dots, \alpha_{l-1})$. For $A \subseteq \Lambda$ we let $\mathbf{1}_A \in \mathbb{F}_2^\Lambda$ denote the characteristic function of A, i.e. $\mathbf{1}_A = (\alpha_0, \alpha_1, \dots, \alpha_{l-1})$ with $\alpha_j = 1$ if and only if $j \in A$. Then the map

$$\mathcal{P}(\Lambda) \to \mathbb{F}_2^\Lambda, \qquad A \mapsto \mathbf{1}_A \tag{1}$$

is an isomorphism of vector spaces.

The set $\{-1, 1\}^l \subseteq \mathbb{Z}^l$ is an $\mathbb{F}_2$-vector space with component-wise multiplication. We define

$$\omega_\Lambda : \mathbb{F}_2^\Lambda \to \{-1, 1\}^l \tag{2}$$

by

$$\omega_\Lambda(\alpha_0, \dots, \alpha_{l-1})_i = \begin{cases} +1, & \text{if } \sum_{j=0}^{i-1} \alpha_j = 0 \\ -1, & \text{if } \sum_{j=0}^{i-1} \alpha_j = 1 \end{cases} \qquad \text{for } 1 \le i \le l.$$

Then ω_Λ is an $\mathbb{F}_2$-vector space isomorphism.

A set $I \subseteq \Lambda$ is called an *interval*, if I is the intersection of an interval of real numbers with Λ. A non-empty interval I is written as $I = [a, b]$, if a, respectively b is the smallest, respectively the largest element of I. If $a, b \in \Lambda$ with $a > b$, then $[a, b]$ denotes the empty interval. The *distance* of two intervals is the Euclidean distance of subsets of the reals.

Lemma 2.2.1. *Let $I = [a, b] \subseteq \Lambda$ be an interval with $a \le b$. Then*

$$\omega_\Lambda(\mathbf{1}_I)_i = \begin{cases} 1, & 0 \le i \le a, \\ (-1)^{i-a}, & a+1 \le i \le b, \\ (-1)^{b-a+1}, & b+1 \le i \le l, \end{cases}$$

and $\omega_\Lambda(\mathbf{1}_\emptyset)_i = 1$ for all $i \in \Lambda$.

Proof. This is clear from the definition. □

2.3. **The Dade group and the invariant $W(\mathbf{B})$.** Let p be an odd prime and let k denote an algebraically closed field of characteristic p. Let D denote a cyclic group of order $p^l > 1$. Recall that for $1 \le j \le l$, the unique subgroup of D of order p^j is denoted by D_j. Let $\Lambda = \{0, 1, \dots, l-1\}$ as in Subsection 2.2. Recall from [HL24, Subsection 3.1] that the Dade group $\mathbf{D}_k(D)$ of D is isomorphic to $\mathbb{F}_2^l$ and consists of the isomorphism classes of the indecomposable capped endo-permutation kD-modules

$$W_D(\alpha_0, \dots, \alpha_{l-1}) := \Omega_{D/D_0}^{\alpha_0} \circ \Omega_{D/D_1}^{\alpha_1} \circ \cdots \circ \Omega_{D/D_{l-1}}^{\alpha_{l-1}}(k),$$

with $\alpha_j \in \mathbb{F}_2$ for $j \in \Lambda$, where Ω_{D/D_j} denotes the relative syzygy operator with respect to $D_j \leq D$. Moreover, the addition in $\mathbf{D}_k(D)$ is given by

$$W_D(\alpha_0, \dots, \alpha_{l-1}) + W_D(\alpha'_0, \dots, \alpha'_{l-1}) = W_D(\alpha_0 + \alpha'_0, \dots, \alpha_{l-1} + \alpha'_{l-1}),$$

so that the map $\mathbb{F}_2^\Lambda \to \mathbf{D}_k(D), (\alpha_0, \dots, \alpha_{l-1}) \mapsto W_D(\alpha_0, \dots, \alpha_{l-1})$ is an isomorphism.

For later purposes, it is convenient to introduce a slightly different notation for these modules. We use the bijection (1) to replace the label $(\alpha_0, \dots, \alpha_{l-1}) \in \mathbb{F}_2^\Lambda$ of an element of $\mathbf{D}_k(D)$ by a subset of Λ. Let us give a formal definition.

Definition 2.3.1. For $A \subseteq \Lambda$ we put $W_D(A) := W_D(\mathbf{1}_A)$. In particular, $W_D(\emptyset) \cong k$. □

Lemma 2.3.2. *Let $A \subseteq \Lambda$ and put $W := W_D(A)$. Let t denote a generator of D. Then, in the notation of* Definition 2.1.2*,*

$$\omega_W^{[l]}(t) = \omega_\Lambda(\mathbf{1}_A),$$

with $\omega_W := \sigma_{\rho_W}$, where ρ_W is the ordinary character of the lift of determinant 1 *of W; see* [HL24, Subsection 3.1]*. In particular, W is uniquely determined, up to isomorphism, by $\omega_W^{[l]}(t)$.*

Proof. The first statement follows from [HL24, Lemma 3.2] and (2), the second from the fact that ω_Λ is an isomorphism. □

Let G be a finite group and let $\mathbf{B}$ be a p-block of G with a non-trivial cyclic defect group D. We then write $W(\mathbf{B})$ for the indecomposable capped endo-permutation kD-module associated to $\mathbf{B}$ as in [HL24, Subsection 2.4]. The isomorphism class of $W(\mathbf{B})$ is an element of $\mathbf{D}_k(D)$. Recall that $\mathbf{B}$ is nilpotent, if and only if it has a unique irreducible Brauer character. This is the case if $D_1 \leq Z(G)$.

Remark 2.3.3. With the notation introduced above, let $|D| = p^l$. Suppose that $D_1 \leq Z(G)$ so that $\mathrm{Irr}(\mathbf{B})$ contains a unique non-exceptional character, χ, say. Put $W := W(\mathbf{B})$ and let $t \in D$ be a generator. Then, [HL24, Lemma 3.3], reformulated in the notation of Definition 2.1.2, implies that

$$\sigma_\chi^{[l]}(t) = \omega_W^{[l]}(t).$$

In particular, W is uniquely determined by $\sigma_\chi^{[l]}(t)$ up to isomorphism, and if $\sigma_\chi^{[l]}(t) = \omega_\Lambda(\mathbf{1}_A)$ for some $A \in \Lambda \setminus \{0\}$, then $W = W_D(A)$. □

In order to apply this remark, it is necessary to identify the non-exceptional character of a cyclic block under the given hypothesis. This can be achieved by looking at character values. Let $\mathcal{G}$ denote the subgroup of $\mathrm{Gal}(\mathbb{Q}(\sqrt[|G|]{1})/\mathbb{Q})$ which fixes the roots of unity of p'-order. Then $\mathcal{G}$ acts on the set of characters of G. A character of G is p-rational, if and only if it is fixed by $\mathcal{G}$.

Lemma 2.3.4. *Assume the notation and hypotheses of* Remark 2.3.3. *Then the non-exceptional character is the unique p-rational element of* $\mathrm{Irr}(\mathbf{B})$.

Proof. The non-exceptional character in $\mathrm{Irr}(\mathbf{B})$ is p-rational by [Dor72, Theorem 68.1(8)]. The same reference implies that the matrix of generalized decomposition numbers of $\mathbf{B}$, as defined in [Fei82, Section IV.6, P. 175], has a unique p-rational column. Then [Fei82, Lemma IV.6.10] shows that $\mathrm{Irr}(\mathbf{B})$ has a unique p-rational character. $\square$

2.4. **Preliminaries from representation theory.** We continue with the general notation of the first part [HL24] of this work. Thus G denotes a finite group, p is a prime and $(K, \mathcal{O}, k)$ is a splitting p-modular system for G, as introduced in [HL24, Subsection 2.1]. Except in Lemma 2.4.1 below, we assume that p is odd throughout this subsection.

We specify our usage of the term Brauer correspondent. Suppose $\mathbf{B}$ is a p-block of G with an abelian defect group D. If $D' \leq D$, any block $\mathbf{b}'$ of $C_G(D')$ such that $(D', \mathbf{b}') \leq (D, \mathbf{b}_0)$ for some maximal $\mathbf{B}$-Brauer pair $(D, \mathbf{b}_0)$, will be called *a Brauer correspondent* of $\mathbf{B}$ in $C_G(D')$.

We continue with a result on the compatibility of Brauer correspondence and domination of blocks. For the concept of domination of blocks see [NT89, Subsection 5.8.2].

Lemma 2.4.1. *Let $Y \trianglelefteq Z(G)$ and set $\bar{G} := G/Y$. Let $\bar{\mathbf{B}}$ be a block of $k\bar{G}$ and let $\mathbf{B}$ be the unique block of kG dominating $\bar{\mathbf{B}}$. Let D be a defect group of $\mathbf{B}$. Then $\bar{D} := DY/Y$ is a defect group of $\bar{\mathbf{B}}$.*

Let $H \leq G$ with $N_G(D) \leq H$, and put $\bar{H} := H/Y$. Then $N_{\bar{G}}(\bar{D}) \leq \bar{H}$. Let $\mathbf{b}$ be the Brauer correspondent of $\mathbf{B}$ in H and let $\bar{\mathbf{b}}$ be the Brauer correspondent of $\bar{\mathbf{B}}$ in $\bar{H}$. If D is abelian, then $\mathbf{b}$ is the unique block of kH dominating $\bar{\mathbf{b}}$.

Proof. Since $G/Y \cong (G/O_{p'}(Y)/(Y/O_{p'}(Y))$, the group $\bar{D}$ is a defect group of $\bar{\mathbf{B}}$ by [NT89, Theorems 5.8.8 and 5.8.10].

There is a unique block $\mathbf{b}_0$ of kH dominating $\bar{\mathbf{b}}$. To prove that $\mathbf{b}_0 = \mathbf{b}$, it suffices to show that $\mathbf{b}$ contains a module which is the inflation from $\bar{H}$ to H of an indecomposable $\bar{\mathbf{b}}$-module.

As $\mathbf{B}$ dominates $\bar{\mathbf{B}}$, there is a simple $\bar{\mathbf{B}}$-module $\bar{V}$, whose inflation $V := \mathrm{Inf}_{\bar{G}}^{G}(\bar{V})$ to G belongs to $\mathbf{B}$. The defect group D of $\mathbf{B}$ being abelian, it must be a vertex of V by Knörr's theorem (see [Kno79, Corollary 3.7(ii)]), and similarly $\bar{D}$ is a vertex of $\bar{V}$. Write $f(V)$, respectively $f(\bar{V})$, for the Green correspondent of V in H, respectively of $\bar{V}$ in $\bar{H}$. As the Green correspondence commutes with the Brauer correspondence, $f(V)$ belongs to $\mathbf{b}$ and $f(\bar{V})$ belongs to $\bar{\mathbf{b}}$. Finally, [Har08, Proposition 5] yields

$$f(V) \cong \mathrm{Inf}_{\bar{H}}^{H}(f(\bar{V})),$$

as $Y \leq N_G(D) \leq H$. This proves our claim. □

In the following lemma we investigate the behavior of $W(\mathbf{B})$ with respect to domination of blocks. Recall that if D is a non-trivial cyclic p-group, D_1 denotes its unique subgroup of order p.

Lemma 2.4.2. *Let $Y \leq Z(G)$ with $O_p(Y) \neq \{1\}$. Set $\bar{G} := G/Y$ and let a be the positive integer such that $|O_p(Y)| = p^a$.*

Let $\bar{\mathbf{B}}$ be a block of $\bar{G}$ and let $\mathbf{B}$ be the unique block of G dominating $\bar{\mathbf{B}}$. Suppose that a defect group D of $\mathbf{B}$ is cyclic of order p^l with $l \geq 1$.

Then the following assertions hold.

(a) *We have $G = N_G(D_1) = C_G(D_1)$, the blocks $\mathbf{B}$ and $\bar{\mathbf{B}}$ are nilpotent, and $\bar{D} := D/O_p(Y)$ is a defect group of $\bar{\mathbf{B}}$.*

(b) *We have $W(\mathbf{B}) \cong \mathrm{Inf}_{\bar{G}}^{G}(W(\bar{\mathbf{B}}))$ or $W(\mathbf{B}) \cong \mathrm{Inf}_{\bar{G}}^{G}(\Omega(W(\bar{\mathbf{B}})))$.*

(c) *Suppose that $W(\mathbf{B}) = W_D(\alpha_0, \ldots, \alpha_{l-1}) = W_D(\mathbf{1}_A)$ for some $A \subseteq \{0, \ldots, l-1\}$. Then $\alpha_0 = \cdots = \alpha_{a-1} = 0$, i.e. $A \subseteq \{a, \ldots, l-1\}$.*

Setting $\bar{A} = \{j - a \mid j \in A \setminus \{a\}\} \subseteq \{1, \ldots, l-a-1\}$, we have $W(\bar{\mathbf{B}}) = W_{\bar{D}}(\bar{A})$.

Proof. (a) As D is a defect group and $O_p(Y)$ is a normal p-subgroup of $Z(G)$, certainly $O_p(Y)$ is contained in $O_p(G) \leq D$. Thus $D_1 \leq O_p(Y) \leq D$ and so G centralizes D_1. Thus $\mathbf{B}$ is nilpotent and so must be $\bar{\mathbf{B}}$, proving the first assertion of (a). The second one follows from Lemma 2.4.1.

(b) By (a), up to isomorphism, there is a unique simple $\mathbf{B}$-module, say V, on which D_1 acts trivially. Thus, by definition, $W(\mathbf{B})$ is a kD-source of V. Similarly, there is, up to isomorphism, a unique simple $\bar{\mathbf{B}}$-module, say $\bar{V}$, and $\mathrm{Inf}_{\bar{G}}^{G}(\bar{V}) = V$ since $\mathbf{B}$ dominates $\bar{\mathbf{B}}$. In the quotient, we have two possibilities, namely either $W(\bar{\mathbf{B}})$ is a source of $\bar{V}$ or $\Omega(W(\bar{\mathbf{B}}))$ is a source of $\bar{V}$. This follows from the facts summarized in [HL20, Subsections 3.5 and 4.2]. However, if S denotes a $k\bar{D}$-source of $\bar{V}$, then by [Har08, Proposition 2] we have

$$W(\mathbf{B}) \cong \mathrm{Inf}_{\bar{D}}^{D}(S).$$

Assertion (b) follows.

(c) Suppose that

$$W_{\bar{D}}(\bar{\alpha}_0,\dots,\bar{\alpha}_{l-a-1}) = \Omega^{\bar{\alpha}_0}_{\bar{D}/\bar{D}_0} \circ \Omega^{\bar{\alpha}_1}_{\bar{D}/\bar{D}_1} \circ \cdots \circ \Omega^{\bar{\alpha}_{l-a-1}}_{\bar{D}/\bar{D}_{l-a-1}}(k)$$

represents the class of S in the Dade group of $\bar{D}$. Then $W(\mathbf{B}) = \mathrm{Inf}^D_{\bar{D}}(S) = \mathrm{Inf}^D_{D/D_a}(S)$ is represented by

$$W_D(\alpha_0,\dots,\alpha_{l-1}) = \Omega^{\alpha_0}_{D/D_0} \circ \Omega^{\alpha_1}_{D/D_1} \circ \cdots \circ \Omega^{\alpha_{l-1}}_{D/D_{l-1}}(k)$$

with $\alpha_0 = \cdots = \alpha_{a-1} = 0$ and $\alpha_{j+a} = \bar{\alpha}_j$ for $0 \le j \le l-a-1$. Notice that $S = W(\bar{\mathbf{B}})$, if and only if $\bar{\alpha}_0 = 0$. Otherwise, $\bar{\alpha}_0 = 1$ and $W(\bar{\mathbf{B}}) = \Omega(S)$. Now $\Omega(S)$ is represented by $W_{\bar{D}}(\bar{\alpha}'_0,\dots,\bar{\alpha}'_{l-a-1})$ with $\bar{\alpha}'_0 = \bar{\alpha}_0 + 1$, and $\bar{\alpha}'_j = \bar{\alpha}_j$ for all $1 \le j \le l-1-1$. This proves our assertions. □

2.5. **Preliminaries on finite reductive groups.** Let $\mathbb{F}$ denote an algebraic closure of a finite field of characteristic r. The prime p introduced in Subsection 2.4 is assumed to be distinct from r. Let $\mathbf{G}$ be a connected reductive linear algebraic group over $\mathbb{F}$ and let F denote a Steinberg morphism of $\mathbf{G}$. Recall that if $\mathbf{H}$ is a closed, F-stable subgroup of $\mathbf{G}$, we write $H := \mathbf{H}^F$ for the set of F-fixed points of $\mathbf{H}$, and $\mathbf{H}^\circ$ for the connected component of $\mathbf{H}$. In particular, the group G introduced in Subsection 2.4 now is of the form $G = \mathbf{G}^F$. To avoid cumbersome double superscripts, we write $Z^\circ(\mathbf{G}) := Z(\mathbf{G})^\circ$ and $Z^\circ(G) := Z^\circ(\mathbf{G})^F$. Also, if $s \in G$, we write $C^\circ_{\mathbf{G}}(s) := C_{\mathbf{G}}(s)^\circ$ and $C^\circ_G(s) := C^\circ_{\mathbf{G}}(s)^F$. Let $\mathbf{G}^*$ be a reductive group dual to $\mathbf{G}$, equipped with a dual Steinberg morphism, also denoted by F. If $\mathbf{T}$ and $\mathbf{T}^*$ are F-stable maximal tori of $\mathbf{G}$ and $\mathbf{G}^*$, respectively, and if $s \in T^*$ and $\theta \in \mathrm{Irr}(T)$ are such that the G^*-conjugacy class of $(\mathbf{T}^*, s)$ and the G-conjugacy class of $(\mathbf{T}, \theta)$ correspond under the bijection exhibited in [DiMi20, Proposition 11.1.6], we say that $(\mathbf{T}^*, s)$ and $(\mathbf{T}, \theta)$ *are in duality*. When we talk about duality, we always tacitly assume that the necessary choices have been made. If $s \in G^*$ is semisimple, we let $\mathcal{E}(G, s)$ denote the corresponding Lusztig series of characters; see [GeMa20, Definition 2.6.1]. The elements of $\mathcal{E}(G, 1)$ are the unipotent characters of G. By a regular subgroup of $\mathbf{G}$ we mean an F-stable Levi subgroup of $\mathbf{G}$.

2.5.1. *Component groups.* Let $s \in \mathbf{G}$ be a semisimple element. The finite group

$$A_{\mathbf{G}}(s) := C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$$

is called the *component group* of s. If $s \in G$, then $A_{\mathbf{G}}(s)$ is F-stable, and we write $A_{\mathbf{G}}(s)^F$ for the set of its F-fixed points. We have $A_{\mathbf{G}}(s)^F = C_G(s)/C^\circ_G(s)$ by the Lang–Steinberg theorem.

Let $s \in \mathbf{G}^*$ be semisimple. There is an F-equivariant embedding of $A_{\mathbf{G}^*}(s)$ into $Z(\mathbf{G})/Z^\circ(\mathbf{G})$; see [Bon06, (8.4) and Lemme 4.10]. In particular, $A_{\mathbf{G}^*}(s)$ is abelian, and $A_{\mathbf{G}^*}(s) = 1$ if $Z(\mathbf{G})$ is connected. Moreover, every prime divisor of $|A_{\mathbf{G}^*}(s)|$ divides $|s|$; see [Bon06, Lemma 8.3].

2.5.2. *Regular embeddings and conjugacy classes.* Consider a regular embedding $\mathbf{G} \to \tilde{\mathbf{G}}$; see [GeMa20, Definition 1.7.1]. Then there is a bijection $\tilde{\mathbf{L}} \mapsto \tilde{\mathbf{L}} \cap \mathbf{G}$ between the regular subgroups of $\tilde{\mathbf{G}}$ and those of $\mathbf{G}$. The inverse image of a regular subgroup $\mathbf{L}$ of $\mathbf{G}$ under this map equals $\mathbf{L}Z(\tilde{\mathbf{G}})$.

Lemma 2.5.3. *Let $\mathbf{G} \to \tilde{\mathbf{G}}$ be a regular embedding.*

(a) *Let $\tilde{\mathbf{L}}$ be a regular subgroup of $\tilde{\mathbf{G}}$. Then*

$$[\tilde{L}\colon L] = [\tilde{G}\colon G].$$

(b) *Let $s \in G$ be semisimple. Then $C_{\tilde{\mathbf{G}}}(s)$ is a regular subgroup of $\tilde{\mathbf{G}}$ if and only if $C_{\mathbf{G}}(s)$ is a regular subgroup of $\mathbf{G}$. If this condition is satisfied, the G-conjugacy class of s is a $\tilde{G}$-conjugacy class.*

Proof. (a) Since $\tilde{\mathbf{L}}$ is a regular subgroup of $\tilde{\mathbf{G}}$, we have $Z^\circ(\mathbf{G}) \leq \tilde{\mathbf{L}}$. Moreover, $\tilde{\mathbf{L}} = \mathbf{L}Z(\tilde{\mathbf{G}})$ and $\tilde{\mathbf{G}} = \mathbf{G}Z(\tilde{\mathbf{G}})$. The claim follows from [GeMa20, Lemma 1.7.8].

(b) The first assertion is clear. The second follows from (a), which implies

$$[\tilde{G}\colon C_{\tilde{G}}(s)] = [G\colon C_G(s)].$$

□

2.5.4. *A crucial invariant.* We now introduce a crucial invariant of a semisimple element of G, based on the notion of relative rank. The *relative F-rank* of $\mathbf{G}$, denoted by $r_F(\mathbf{G})$, is defined as in [DiMi20, Definition 7.1.5]. It can be computed from the order polynomial of the generic finite reductive group associated with the pair $(\mathbf{G}, F)$; see the discussion following [GeMa20, Definition 2.2.11].

Example 2.5.5. Let n be a positive integer and q a prime power.

(a) If $(\mathbf{G}, F)$ is such that $G \cong \mathrm{GL}_n(q)$, then $r_F(\mathbf{G}) = n$.

(b) If $(\mathbf{G}, F)$ is such that $G \cong \mathrm{GU}_n(q)$, then $r_F(\mathbf{G}) = \lfloor n/2 \rfloor$. □

As usual, we also write $\varepsilon_{\mathbf{G}} := (-1)^{r_F(\mathbf{G})}$. Let us now define the aforementioned invariant.

Definition 2.5.6. For a semisimple element $s \in G$ put

$$\omega_{\mathbf{G}}(s) := \varepsilon_{\mathbf{G}}\varepsilon_{C^\circ_{\mathbf{G}}(s)}.$$

□

Let $\mathbf{T}$ denote an F-stable maximal torus of $\mathbf{G}$. By $R_{\mathbf{T}}^{\mathbf{G}}$ we denote Deligne–Lusztig induction, which maps generalized characters of T to generalized characters of G; see [Ca85, Subsection 7.2]. Also, recall Definition 2.1.1.

Lemma 2.5.7. *Let $\mathbf{T}$ denote an F-stable maximal torus of $\mathbf{G}$. Let $\theta \in \mathrm{Irr}(T)$ and put $\chi := \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. Let $t \in T$ with $\theta(t) = 1$. Then $\chi(t)$ is a non-zero integer and $\sigma_\chi(t) = \omega_{\mathbf{G}}(t) \in \{-1, 1\}$.*

Proof. By the character formula given in [Ca85, Proposition 7.5.3], the value of $\chi(t)$ equals $\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{T}}\varepsilon_{C^\circ_{\mathbf{G}}(t)}$ times a positive rational number. Hence $\chi(t)$ is an integer, since χ is a generalized character. This gives our claims. ☐

We will also need a more specific version of Lemma 2.5.7.

Lemma 2.5.8. *Let $\mathbf{T}$ denote an F-stable maximal torus of $\mathbf{G}$ and let $\theta \in \mathrm{Irr}(T)$. Let $t \in T$ such that $\langle t \rangle$ is a Sylow p-subgroup of T. Put $\mathbf{H} := C_{\mathbf{G}}(t)$. Then*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(t') = \frac{\varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{H}^\circ}|N_G(\mathbf{H})|}{|T||\mathbf{H}^{\circ F}|_r}$$

for every $t' \in \langle t \rangle$ with $\theta(t') = 1$ and $C_{\mathbf{G}}(t') = \mathbf{H}$ (recall that r is the characteristic of $\mathbb{F}$).

Proof. Let t' be as in the assertion. Let $g \in G$. We claim that $g^{-1}t'g \in T$, if and only if $g \in N_G(\mathbf{H})$. Suppose first that $g \in N_G(\mathbf{H})$. Then $g^{-1}t'g \in Z(\mathbf{H})$. Since $\mathbf{T}$ is a maximal torus of $\mathbf{H}$, we have $Z(\mathbf{H}) \leq \mathbf{T}$, and thus $g^{-1}t'g \in T$. Suppose now that $t'' := g^{-1}t'g \in T$. By hypothesis, $t'' \in \langle t \rangle \leq Z(\mathbf{H})$. Hence $\mathbf{H} \leq C_{\mathbf{G}}(t'') = g^{-1}\mathbf{H}g$. It follows that $\mathbf{H} = g^{-1}\mathbf{H}g$, as g has finite order. This proves our claim. Our assertion now follows from [Ca85, Proposition 7.5.3]. ☐

2.5.9. *Isogenies.* An isogeny between algebraic groups is a surjective homomorphism with finite kernel. We record some properties which are transferred by isogenies.

Lemma 2.5.10. *Let $\mathbf{G}'$ be a connected reductive algebraic group over $\mathbb{F}$ and let F' be a Steinberg morphism of $\mathbf{G}'$. Let $\nu : \mathbf{G} \to \mathbf{G}'$ be an isogeny with $\nu \circ F = F' \circ \nu$. Then the following statements hold.*

(a) *Let $\mathbf{H} \leq \mathbf{G}$ be a closed subgroup. Then the restriction $\nu_{\mathbf{H}} : \mathbf{H} \to \nu(\mathbf{H})$ is an isogeny.*

(b) *Let e be a positive integer and let $\mathbf{S}$ denote a Φ_e-torus of $\mathbf{G}$. Then $\mathbf{S}' := \nu(\mathbf{S})$ is a Φ_e-torus of $\mathbf{G}'$. If $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$, then $\mathbf{L}' := \nu(\mathbf{L})$ is an e-split Levi subgroup of $\mathbf{G}'$. (For the notions of Φ_e-tori and e-split Levi subgroups see* [BrMa92]*).*

(c) *Let* $\mathbf{L}'$ *be an* F'*-stable Levi subgroup of* $\mathbf{G}'$*. Then there is an* F*-stable Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$ *with* $\nu(\mathbf{L}) = \mathbf{L}'$*.*

(d) *We have* $r_F(\mathbf{G}) = r_{F'}(\mathbf{G}')$*.*

(e) *We have* $\nu(Z^\circ(\mathbf{G})) = Z^\circ(\mathbf{G}')$ *and* $|Z^\circ(\mathbf{G}')^{F'}|$ *divides* $|Z(G)|$*.*

(f) *The map* $\chi' \mapsto \chi' \circ \nu|_G$ *is a bijection between the unipotent characters of* $G' = \mathbf{G}'^{F'}$ *and those of* $G = \mathbf{G}^F$*, which preserves the sets of* e*-cuspidal characters (see* [GeMa20, Definition 3.5.19]*) for every positive integer* e*.*

Proof. (a) This follows from the definition of an isogeny.

(b) Clearly, $\mathbf{S}'$ is a torus of $\mathbf{G}'$. Moreover, $|\mathbf{S}^{F^m}| = |\mathbf{S}'^{F'^m}|$ for all positive integers m (see [GeMa20, Proposition 1.4.13(c)]), implying the first claim. By definition, $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$ for some Φ_e-torus $\mathbf{S}$ of $\mathbf{G}$. As $\mathbf{L}' = \nu(\mathbf{L}) = C_{\mathbf{G}'}(\nu(\mathbf{S}))$ by [Bor91, Corollary 2 to Proposition 11.14], the second assertion follows from the first one.

(c) As $\mathbf{L}'$ is a Levi subgroup of $\mathbf{G}'$, we have $\mathbf{L}' = C_{\mathbf{G}'}(Z(\mathbf{L}')^\circ)$ by [Bor91, Corollary 14.19]. Put $\mathbf{L} := (\nu^{-1}(\mathbf{L}'))^\circ$. Then $\mathbf{L}$ is an F-stable closed, connected subgroup of $\mathbf{G}$ and $\nu(\mathbf{L}) = \mathbf{L}'$. It remains to show that $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$. It follows from [Bor91, 11.14(1)], that the unipotent radical of $\mathbf{L}$ is sent to the unipotent radical of $\mathbf{L}'$, which is trivial, as $\mathbf{L}'$ is reductive. This implies that the unipotent radical of $\mathbf{L}$ is trivial, since the kernel of ν is finite. Hence $\mathbf{L}$ is reductive. Now, put $\mathbf{Z} := Z(\mathbf{L})^\circ$. Then

$$\nu(\mathbf{Z}) = \nu(Z(\mathbf{L})^\circ) = (\nu(Z(\mathbf{L})))^\circ = Z(\mathbf{L}')^\circ,$$

where the second equality arises from [Hum75, Proposition 7.4.B(c)], and the third one from [GeMa20, 1.3.10(c)]. Hence

$$\nu(C_{\mathbf{G}}(\mathbf{Z})) = C_{\mathbf{G}'}(\nu(\mathbf{Z})) = \mathbf{L}',$$

by [Bor91, Corollary 2 to Proposition 11.14]. In particular, $C_{\mathbf{G}}(\mathbf{Z}) \trianglelefteq \nu^{-1}(\mathbf{L}')$. As $C_{\mathbf{G}}(\mathbf{Z})$ is connected, we obtain $C_{\mathbf{G}}(\mathbf{Z}) \leq \mathbf{L}$, and thus $\mathbf{L} = C_{\mathbf{G}}(\mathbf{Z})$. This implies that $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$.

(d) Let $\mathbf{T}$ denote a maximally split torus of $\mathbf{G}$. Then $\mathbf{T}' := \nu(\mathbf{T})$ is a maximally split torus of $\mathbf{G}'$. Moreover, ν induces an isomorphism $\Phi : \mathbb{R} \otimes_{\mathbb{Z}} X(\mathbf{T}') \to \mathbb{R} \otimes_{\mathbb{Z}} X(\mathbf{T})$ such that $F \circ \Phi = \Phi \circ F'$; see, e.g. [DiMi20, Theorem 2.4.8]. Thus Φ maps the F'-eigenspaces on $\mathbb{R} \otimes_{\mathbb{Z}} X(\mathbf{T}')$ to the F-eigenspaces on $\mathbb{R} \otimes_{\mathbb{Z}} X(\mathbf{T})$, which proves our claim.

(e) We have $\nu(Z(\mathbf{G})) = Z(\mathbf{G}')$ by [GeMa20, 1.3.10(c)]. In turn, $\nu(Z^\circ(\mathbf{G})) = Z^\circ(\mathbf{G}')$. Since $\nu_{Z^\circ(\mathbf{G})} : Z^\circ(\mathbf{G}) \to Z^\circ(\mathbf{G}')$ is an isogeny by (a), we get $|Z^\circ(\mathbf{G})^F| = |Z^\circ(\mathbf{G}')^{F'}|$ from [GeMa20, Proposition 1.4.13(c)]. As $Z^\circ(\mathbf{G})^F$ is a subgroup of $Z(\mathbf{G})^F$ and $Z(\mathbf{G})^F = Z(G)$ by [Ca85, Proposition 3.6.8], we obtain our assertion.

(f) For the first statement see [GeMa20, Proposition 2.3.15]. This bijection preserves the degrees of the characters and thus e-cuspidality by [BMM93, Proposition 2.9]. □

2.5.11. *Basic sets and p-rational characters.* Recall that p is a prime with $p \neq r$, the characteristic of $\mathbb{F}$. For a semisimple p'-element $s \in G^*$ we define $\mathcal{E}_p(G, s)$ as in [BrMi89, p. 57]. Then $\mathcal{E}_p(G, s)$ is the set of characters of a union of p-blocks of G; see [BrMi89, Théorème 2.2].

The following lemma on p-rational characters in a block will be useful. Recall the definition of the Galois group $\mathcal{G}$ introduced prior to Lemma 2.3.4. For the concept of a basic set see [GeHi91, Subsection 1.1].

Lemma 2.5.12. *Let $s \in G^*$ denote a semisimple p'-element.*

(a) *Suppose that the pairs $(\mathbf{T}^*, s)$ and $(\mathbf{T}, \theta)$ are in duality. Then the order of θ is coprime to p and $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is p-rational. Moreover, $\mathcal{G}$ stabilizes $\mathcal{E}(G, s)$ as a set.*

(b) *Let $\mathbf{B}$ be a p-block of G such that $\mathrm{Irr}(\mathbf{B}) \subseteq \mathcal{E}_p(G, s)$. If the restrictions of the characters of $\mathrm{Irr}(\mathbf{B}) \cap \mathcal{E}(G, s)$ to the p-regular classes form a basic set for $\mathbf{B}$, every character of $\mathrm{Irr}(\mathbf{B}) \cap \mathcal{E}(G, s)$ is p-rational.*

Proof. (a) The first assertion follows from the reasoning of [GeMa20, *Remark* 2.5.15], the second from the character formula [Ca85, Theorem 7.2.8], and the third is contained in [GeMa20, Proposition 3.3.15].

(b) Two elements of $\mathrm{Irr}(G)$ in a $\mathcal{G}$-orbit have the same restrictions to the p-regular classes of G. Hence $\mathcal{G}$ fixes $\mathrm{Irr}(\mathbf{B})$ and thus also $\mathrm{Irr}(\mathbf{B}) \cap \mathcal{E}(G, s)$ by (a). The claim then follows from our hypothesis. □

By [Ge93, Theorem A], the hypothesis of Lemma 2.5.12(b) is in particular satisfied if p is good for $\mathbf{G}$ and p does not divide $|Z(\mathbf{G})/Z^0(\mathbf{G})|$. The group $G = G_2(q)$ has unipotent characters, i.e. elements of $\mathcal{E}(G, 1)$, that are not 3-rational. Thus the conclusion of Lemma 2.5.12(b) does not hold in general.

2.5.13. *Regular blocks.* Let us keep the notation introduced in 2.5.11. In particular, p is a prime distinct from r, the characteristic of $\mathbb{F}$. Lemma 2.5.7 is especially useful in the context of regular blocks. Recall that a semisimple element $s \in \mathbf{G}$ is called regular, if $C^\circ_{\mathbf{G}}(s)$ is a maximal torus. Clearly, a regular semisimple element is non-trivial, unless $\mathbf{G}$ is a torus. If we want to emphasize the underlying group, we also say that s is regular in $\mathbf{G}$, or in G, if $s \in G$ and $\mathbf{G}$ is only given implicitly.

Definition 2.5.14. (a) Let $s \in G$ be semisimple. We call s *strictly regular*, if $C_G(s)$ is a maximal torus of G; in other words, if s is regular in $\mathbf{G}$ and $A_{\mathbf{G}}(s)^F = 1$ (for the notion $A_{\mathbf{G}}(s)$ see 2.5.1).

(b) Let $s \in G^*$ be a semisimple p'-element and let $\mathbf{B}$ be a p-block of G with $\mathrm{Irr}(\mathbf{B}) \subseteq \mathcal{E}_p(G,s)$. Suppose that $C^\circ_{\mathbf{G}^*}(s) = \mathbf{T}^*$ for some maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ dual to $\mathbf{T}^*$. We then call $\mathbf{B}$ *regular* with respect to T; if, in addition, $C_{G^*}(s) = T^*$, we call $\mathbf{B}$ *strictly regular* with respect to T. □

Notice that if centralizers of semisimple elements in $\mathbf{G}^*$ are connected, the notions of regular and strictly regular blocks coincide. Since, in the notation of Definition 2.5.14(b), the torus $\mathbf{T}^*$ is uniquely determined by s, the torus T is determined by s up to conjugation in G. The reason for the attribute "with respect to T" in the above definition will become clear in Lemma 2.5.16 and Corollary 2.5.17 below.

Before we prove these results, we investigate regular elements under regular embeddings. Let $i : \mathbf{G} \to \tilde{\mathbf{G}}$ denote a regular embedding. Then there is a reductive group $\tilde{\mathbf{G}}^*$, dual to $\mathbf{G}^*$, an epimorphism $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$, and a compatible Steinberg morphism of $\tilde{\mathbf{G}}^*$, also denoted by F; see, e.g. [Bon06, 2.B, 2.D]. Recall that if $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(T)$, then θ is *in general position*, if and only if the stabilizer $N_G(\mathbf{T},\theta)$ of θ in $N_G(\mathbf{T})$ equals T. This is the case if and only if $\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ is an irreducible character of G; [GeMa20, Corollary 2.2.9(a)].

Lemma 2.5.15. *Let $\mathbf{T}^*$ be an F-stable maximal torus of $\mathbf{G}^*$, and let $s \in T^*$ be a regular element, so that $C^\circ_{\mathbf{G}^*}(s) = \mathbf{T}^*$. Choose an element $\tilde{s} \in \tilde{G}^*$ with $i^*(\tilde{s}) = s$. Then $\tilde{s}$ is regular, with $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) = \tilde{\mathbf{T}}^*$, where $\tilde{\mathbf{T}}^* := (i^*)^{-1}(\mathbf{T}^*)$.*

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ and let $\theta \in \mathrm{Irr}(T)$ be such that the pairs $(\mathbf{T}^*, s)$ and $(\mathbf{T}, \theta)$ are in duality. Put $\tilde{\mathbf{T}} := \mathbf{T}Z(\tilde{\mathbf{G}})$. Then there is an extension $\tilde{\theta}$ of θ to $\tilde{T}$ such that $(\tilde{\mathbf{T}}^*, \tilde{s})$ and $(\tilde{\mathbf{T}}, \tilde{\theta})$ are in duality.

Put $\chi := \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ and $\tilde{\chi} := \varepsilon_{\tilde{\mathbf{G}}}\varepsilon_{\tilde{\mathbf{T}}}R^{\tilde{\mathbf{G}}}_{\tilde{\mathbf{T}}}(\tilde{\theta})$.

We have $\mathcal{E}(\tilde{G}, \tilde{s}) = \{\tilde{\chi}\}$ and

$$\chi = \mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi}) = \sum_{i=1}^{m} \chi_i$$

with pairwise distinct irreducible characters χ_i, $i = 1, \ldots, m$ of G. Moreover, $m = |A_{\mathbf{G}^*}(s)^F|$ and $\mathcal{E}(G,s) = \{\chi_1, \ldots, \chi_m\}$.

Finally, θ is in general position, if and only if s is strictly regular.

Proof. As $Z(\tilde{\mathbf{G}})$ is connected, $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ is connected by a theorem of Steinberg; see [Ca85, Theorem 4.5.9]. As $i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C^\circ_{\mathbf{G}^*}(s) = \mathbf{T}^*$ (see, e.g. [Bon06, p. 36]), we obtain $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) = (i^*)^{-1}(\mathbf{T}^*) = \tilde{\mathbf{T}}^*$.

For the second statement see [Bon06, Lemme 9.3(b)]. By Lusztig's Jordan decomposition of characters we get $\mathcal{E}(\tilde{G}, \tilde{s}) = \{\tilde{\chi}\}$. Now $\chi = \mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi})$ by [Bon06, Proposition 10.10]. By [Lu88, Section 10], the restriction $\mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi})$ is multiplicity free, and its constituents are in bijection with the unipotent characters of $C_{G^*}(s)$. As $A_{\mathbf{G}^*}(s)^F$ is abelian and $C^\circ_{\mathbf{G}^*}(s)$ is a torus, the number of unipotent characters of $C_{G^*}(s)$ equals $|A_{\mathbf{G}^*}(s)^F|$. The statement about $\mathcal{E}(G, s)$ follows from [Bon06, Proposition 11.7]. The last statement follows from the previous ones. □

We next investigate regular blocks and their defect groups.

Lemma 2.5.16. *Assume the notation and hypotheses of* Lemma 2.5.15. *Assume in addition that s is a p'-element, and choose an inverse image $\tilde{s}$ under i^* as a p'-element as well.*

Then $p \nmid m$ and $N_G(\mathbf{T}, \theta)/T \cong A_{\mathbf{G}^*}(s)^F$.

Let $\tilde{\mathbf{B}}$ be the p-block of $\tilde{G}$ containing $\tilde{\chi}$, and let $\mathbf{B}$ be a block of G covered by $\tilde{\mathbf{B}}$. Then $\mathrm{Irr}(\mathbf{B}) \subseteq \mathcal{E}_p(G, s)$, the Sylow p-subgroup $\tilde{D}$ of $\tilde{T}$ is a defect group of $\tilde{\mathbf{B}}$, and $D := \tilde{D} \cap G$ is a defect group of $\mathbf{B}$. Also, $\mathrm{Irr}(\mathbf{B}) \cap \mathcal{E}(G, s)$ is a non-empty subset of $\{\chi_i \mid 1 \leq i \leq m\}$.

Suppose that $t \in D$ is a p-element such that $C_{\mathbf{G}}(t)$ is a regular subgroup of $\mathbf{G}$. Then $\chi_i(t) = \chi_j(t)$ for all $1 \leq i, j \leq m$. In particular,

$$\chi_i(1) = \frac{1}{m}[G^* : T^*]_{r'}$$

for all $1 \leq i \leq m$.

Proof. As s has p'-order, $p \nmid |A_{\mathbf{G}^*}(s)^F|$; see 2.5.1. Clearly, $\mathrm{Irr}(\mathbf{B}) \subseteq \mathcal{E}_p(G, s)$, since the irreducible constituents of χ lie in $\mathcal{E}(G, s)$ by Lemma 2.5.15,

Let $\mathbf{e}^G_s$ and $\mathbf{e}^T_s$ denote the central idempotents of $\mathcal{O}G$, respectively $\mathcal{O}T$, corresponding to $\mathcal{E}_p(G, s)$, respectively $\mathcal{E}_p(T, s)$. Notice that θ is the unique element in $\mathcal{E}(T, s)$, and that $\mathcal{E}_p(T, s)$ is a single block of kT. The assertion $N_G(\mathbf{T}, \theta)/T \cong A_{\mathbf{G}^*}(s)^F$ thus follows from [BDR17, (7.1)].

The fact that $\tilde{D}$ is a defect group of $\tilde{\mathbf{B}}$ is contained in [Br90, Théorème 3.1]. As $p \nmid [N_G(\mathbf{T}, \theta) : T]$, the blocks of $N_G(\mathbf{T}, \theta)$ covering $\mathbf{e}^T_s \mathcal{O}T$ have the Sylow p-subgroup $\tilde{D} \cap G$ of T as defect group. The corresponding assertion for $\mathbf{e}^G_s \mathcal{O}G$ follows from [BDR17, Theorem 7.7].

Recall that $\mathcal{E}(G, s) = \{\chi_i \mid 1 \leq i \leq m\}$ by Lemma 2.5.15. Since $\mathrm{Irr}(\mathbf{B}) \subseteq \mathcal{E}_p(G, s)$ and s is a p'-element, we have $\mathrm{Irr}(\mathbf{B}) \cap \mathcal{E}(G, s) \neq \emptyset$ by [Hi90, Theorem 3.1].

As $C_{\mathbf{G}}(t)$ is a regular subgroup of $\mathbf{G}$ by hypothesis, the G-conjugacy class of t equals the $\tilde{G}$-conjugacy class of t; see Lemma 2.5.3(b). This implies the claim on the values of $\chi_i(t)$, since the χ_i, $1 \leq i \leq m$, are conjugate under the action of $\tilde{G}$. The final claim follows by choosing $t = 1$. □

By Lemma 2.5.15, every regular block arises as one of the blocks $\mathbf{B}$ considered in Lemma 2.5.16. Regular blocks have first been introduced and investigated by Broué in [Br90], under the more restrictive condition that $C_{\mathbf{G}^*}(s)$, in the notation of Lemma 2.5.16, is a torus. Under this condition we have $N_G(\mathbf{T}, \theta) = T$; in other words, $\theta \in \mathrm{Irr}(T)$ is in general position. Moreover, $m = 1$. As our applications are mostly concerned with the latter case, we formulate the relevant results in a corollary. In the following, we will identify the elements of $\mathrm{Irr}(T)$ of p-power order with their restrictions to the Sylow p-subgroup D of T.

Corollary 2.5.17. *Let the hypotheses and notation be as in* Lemma 2.5.16. *Assume in addition that $m = 1$, i.e. $A_{\mathbf{G}^*}(s)^F = 1$, so that* $\mathbf{B}$ *is strictly regular.*

Then $\theta \in \mathrm{Irr}(T)$ is in general position and $\chi \in \mathrm{Irr}(G)$, so that $\mathcal{E}(G, s) = \{\chi\}$. The Sylow p-subgroup D of T is a defect group of $\mathbf{B}$ *and* $\mathbf{B}$ *is the unique block of G covered by* $\tilde{\mathbf{B}}$.

Also, $\mathrm{Irr}(\mathbf{B}) = \{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\lambda\theta) \mid \lambda \in \mathrm{Irr}(D)\}$, *and* $|\mathrm{Irr}(\mathbf{B})| = |D|$. *In particular* $\mathbf{B}$ *has a unique irreducible Brauer character. If D is cyclic, χ is the non-exceptional character of* $\mathbf{B}$.

Proof. The statements, except those in the last paragraph, immediately follow from Lemma 2.5.16. Let us prove the final statements.

Clearly, $\lambda\theta$ is in general position for every $\lambda \in \mathrm{Irr}(D)$. We conclude that $|\{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\lambda\theta) \mid \lambda \in \mathrm{Irr}(T)\}| = |D|$. It follows from [DiMi20, Corollary 7.3.5], that restriction of class functions to the p-regular elements commutes with the Lusztig map. Hence the elements of the above set all have the same restriction to the p-regular elements, and all of them lie in $\mathrm{Irr}(\mathbf{B})$. Moreover, χ is p-rational by Lemma 2.5.12(b).

In the notation of Lemma 2.5.15, we have $C_{\mathbf{G}^*}(ts) = \mathbf{T}^*$ for every p-element $t \in T^*$. It follows that $|\mathcal{E}_p(G, s)| = |D|$. If D is cyclic, χ is the non-exceptional character of $\mathbf{B}$ by Lemma 2.3.4. We have proved all our assertions. □

The last statement of Corollary 2.5.17 can be generalized to the situation when $\mathbf{B}$ in Lemma 2.5.16 is cyclic and nilpotent. Under these hypotheses, the non-exceptional character in $\mathbf{B}$ is the unique character in $\mathrm{Irr}(\mathbf{B}) \cap \mathcal{E}(G, s)$. In view of Lemma 2.3.4, this is a consequence of the following, slightly more general, result. Due to its hypothesis, we cannot use Lemma 2.5.12(b) right away.

Lemma 2.5.18. *Keep the hypotheses and notation of* Lemma 2.5.16. *Suppose that* $\mathrm{Irr}(\mathbf{B})$ *contains a unique p-rational character* ψ_1*, and that* ψ_1° *is the unique irreducible Brauer character of* $\mathbf{B}$. *(Here,* ψ_1° *denotes the restriction of* ψ_1 *to the p-regular elements of* G*.)*

Then the elements $\chi_1, \ldots, \chi_m$ *of* $\mathcal{E}(G, s)$ *lie in* m *distinct p-blocks of* G*, and* χ_i *is the unique p-rational character in its block for* $1 \leq i \leq m$.

Proof. Recall that $\tilde{\mathbf{B}}$ is a strictly regular block of $\tilde{G}$ covering $\mathbf{B}$. By Corollary 2.5.17, applied with $G = \tilde{G}$ and $\mathbf{B} = \tilde{\mathbf{B}}$, all irreducible characters of $\tilde{\mathbf{B}}$ have the same degree. Let $\tilde{\psi} \in \mathrm{Irr}(\tilde{\mathbf{B}})$ such that $\psi := \mathrm{Res}_G^{\tilde{G}}(\tilde{\psi})$ contains ψ_1 as a constituent. Then $\psi = \sum_{i=1}^{l} \psi_i$ with pairwise distinct irreducible characters ψ_i, $1 \leq i \leq l$; see [Lu88, Section 10]. For $1 \leq i \leq l$, let $\mathbf{B}_i$ denote the block of G containing ψ_i. Then the blocks $\mathbf{B} = \mathbf{B}_1, \ldots, \mathbf{B}_l$ are pairwise distinct, since conjugation of characters preserves their p-rationality. Moreover, $\mathbf{B}_1, \ldots, \mathbf{B}_l$ are all the blocks of G covered by $\tilde{\mathbf{B}}$.

Choose the notation such that $\chi_1 \in \mathbf{B} = \mathbf{B}_1$. Then $\psi_1(1) = \psi_1^\circ(1) \leq \chi_1(1)$. Since every block covered by $\tilde{\mathbf{B}}$ contains at least one of the characters $\chi_1, \ldots, \chi_m$ by Lemma 2.5.16, we have $l \leq m$. It follows that $\psi(1) = l\psi_1(1) \leq m\chi_1(1) = \chi(1) = \psi(1)$, and so $l = m$. Thus each of the blocks $\mathbf{B}_1, \ldots, \mathbf{B}_m$ contains exactly one of the characters $\chi_1, \ldots, \chi_m$.

Since $\chi = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is p-rational by Lemma 2.5.12(a), the Galois group $\mathcal{G}$ permutes the constituents $\chi_1, \ldots, \chi_m$ of χ. As $\mathcal{G}$-conjugate characters lie in the same block, $\mathcal{G}$ fixes χ_1, which then is p-rational. This completes the proof. □

3. Reduction theorems for classical groups

The aim in this section is to reduce the computation of the source algebra equivalence classes of cyclic blocks of classical groups for the primes excluded in [HL24, Proposition 6.5(a)] to a corresponding problem for the general linear and unitary groups. Throughout this section, we fix a prime power q, an odd prime p not dividing q, and an integer $n \geq 1$. An algebraic closure of the finite field with q elements is denoted by $\mathbb{F}$.

3.1. **The relevant classical groups.** It is convenient to study the classical groups via their natural representations. For the notions and concepts with respect to classical groups, see [Tay92]. For the notions regarding Spin groups, see [Jac80, Gro02].

Let V be an n-dimensional vector space over the finite field $\mathbb{F}_{q^\delta}$, where $\delta \in \{1, 2\}$. We assume that V is equipped with a form κ of one of the following types: a non-degenerate hermitian form, a non-degenerate symplectic form, a non-degenerate quadratic form or the zero-form. If κ is hermitian, we let $\delta = 2$; in all other cases, $\delta = 1$. If κ is a quadratic form and $\dim(V)$ is odd, we assume that q is odd. By $I(V, \kappa)$ we denote the full isometry group of (V, κ), and by $\Omega(V, \kappa)$ the commutator subgroup of $I(V, \kappa)$. If κ is a quadratic form, we write $\mathrm{Spin}(V, \kappa)$ for the corresponding spin group. (Notice that $\mathrm{Spin}(V, \kappa) = \Omega(V, \kappa)$ in case q is even.) If κ is the zero form or hermitian, $I(V, \kappa) \cong \mathrm{GL}_n(q)$, respectively $\mathrm{GU}_n(q)$ and $\Omega(V, \kappa) \cong \mathrm{SL}_n(q)$, respectively $\mathrm{SU}_n(q)$, unless $(n, q) = (2, 2)$. If κ is a symplectic form, $I(V, \kappa) = \Omega(V, \kappa) \cong \mathrm{Sp}_n(q)$, unless (n, q) is one of $(2, 2), (2, 3), (4, 2)$. If κ is a quadratic form, the index of $\Omega(V, \kappa)$ in $I(V, \kappa)$ is a power of 2; see [Tay92, Section 11]. Finally, $G(V, \kappa)$ denotes one of the following groups:

$$G(V, \kappa) := \begin{cases} \mathrm{Spin}(V, \kappa), & \text{if } \kappa \text{ is a quadratic form,} \\ I(V, \kappa), & \text{otherwise.} \end{cases} \tag{3}$$

We also consider the homomorphism $\nu : G(V, \kappa) \to I(V, \kappa)$, defined as follows. If κ is a quadratic form, ν is the *vector representation of* $\mathrm{Spin}(V, \kappa)$, whose image equals $\Omega(V, \kappa)$. In the other cases, ν is the identity map. Put $Z := \ker(\nu)$. Then Z is trivial unless κ is a quadratic form and q is odd, in which case Z has order 2. This notation is used until the end of this section.

3.2. **Minimal polynomials.** We also need some notation concerning polynomials. Let Δ be a monic irreducible polynomial over $\mathbb{F}_{q^\delta}$. If $\delta = 1$, we write Δ^* for the monic, irreducible polynomial over $\mathbb{F}_q$ with the property: $\zeta \in \mathbb{F}$ is a root of Δ, if and only if ζ^{-1} is a root of Δ^*. If $\delta = 2$, we write $\Delta^\dagger$ for the monic, irreducible polynomial over $\mathbb{F}_{q^2}$ with the property: $\zeta \in \mathbb{F}$ is a root of Δ, if and only if ζ^{-q} is a root of $\Delta^\dagger$.

Lemma 3.2.1. *Let Δ be a monic irreducible polynomial over $\mathbb{F}_q$ of degree e, and let $\zeta \in \mathbb{F}$ be a root of Δ with $|\zeta| = p$.*

Then e is the order of q modulo p. If $e = 2d$ is even, then $p \mid q^d + 1$ and $\Delta = \Delta^*$. If $\Delta = \Delta^*$, then e is even.

Proof. As Δ is irreducible, we have $[\mathbb{F}_q[\zeta] : \mathbb{F}_q] = e$. Hence $p \mid q^e - 1$ as $|\zeta| = p$. Suppose that $p \mid q^j - 1$ for some positive integer j. Then $\mathbb{F}_{q^j}$ contains all elements of $\mathbb{F}^*$ of order p, and so $\zeta \in \mathbb{F}_{q^j}$. It follows that $\mathbb{F}_{q^e} = \mathbb{F}_q[\zeta] \subseteq \mathbb{F}_{q^j}$, and hence $e \leq j$.

Suppose that $e = 2d$ is even. Then $p \mid (q^d - 1)(q^d + 1)$. As p is odd and $2d$ is the order of q modulo p, we conclude that $p \mid q^d + 1$. As $|\zeta| = p$, we get $\zeta^{q^d+1} = 1$, and so $\zeta^{-1} = \zeta^{q^d}$. As ζ^{q^d} is a root of Δ, we get $\Delta = \Delta^*$. On the other hand, if $\Delta = \Delta^*$, then e is even. This concludes the proof. $\square$

Lemma 3.2.2. *Suppose that $p \mid q + 1$. Let Δ be a monic irreducible polynomial over $\mathbb{F}_{q^2}$ of odd degree e, and let $\zeta \in \mathbb{F}$ be a root of Δ of p-power order. Then $\Delta = \Delta^\dagger$.*

Proof. The claim is true if $\zeta = 1$. Thus assume that $\zeta \neq 1$. By hypothesis, $q \equiv -1 \pmod{p}$, and thus $q^e \equiv -1 \pmod{p}$. Now $\zeta^{q^{2e}-1} = 1$, and thus ζ^{q^e-1} or $\zeta^{q^e+1} = 1$, as p is odd. As $\zeta \neq 1$, the former case would imply $p \mid q^e - 1$, which is impossible. Hence $\zeta^{q^e} = \zeta^{-1}$. It follows that $\zeta^{-q} = \zeta^{q^{e+1}}$. As $e + 1$ is even, $\zeta^{q^{e+1}}$ is a root of Δ, which proves our claim. $\square$

The next lemma is used to investigate centralizers of possible defect groups.

Lemma 3.2.3. *Let $G := G(V, \kappa)$ with $G(V, \kappa)$ as in (3). Let $D = \langle t \rangle \leq G$ denote a non-trivial cyclic radical p-subgroup of G. Write $\bar{t} := \nu(t)$, and assume that $\bar{t}$ has no non-trivial fixed vector on V. Let $\bar{t}_1$ denote a power of $\bar{t}$ of order p. Let Γ denote the minimal polynomial of either $\bar{t}$ or of $\bar{t}_1$ on V.*

Then Γ has at most two irreducible factors. If κ is the zero form, Γ is irreducible. If κ is a hermitian form, either $\Gamma = \Gamma^\dagger$ is irreducible, or $\Gamma = \Delta\Delta^\dagger$ where Δ is a monic irreducible polynomial over $\mathbb{F}_{q^2}$ with $\Delta \neq \Delta^\dagger$. In the other cases, either $\Gamma = \Gamma^*$ is irreducible, or $\Gamma = \Delta\Delta^*$, where Δ is a monic irreducible polynomial over $\mathbb{F}_q$ with $\Delta \neq \Delta^*$. If Γ has two irreducible factors, each f them occurs with the same multiplicity in the characteristic polynomial of $\bar{t}$, respectively $\bar{t}_1$ on V.

Finally, $\bar{t}_1$ has non non-zero fixed vector on V.

Proof. Suppose first that Γ is the minimal polynomial of $\bar{t}$. Notice that $\nu(G)$ is a normal subgroup of $I(V, \kappa)$ of 2-power index. Thus $\bar{D} := \langle \bar{t} \rangle$ is a radical p-subgroup of $I(V, \kappa)$ by [HL24, Lemma 2.3(b)(c)], and hence $\bar{D} = O_p(C_{I(V,\kappa)}(\bar{D}))$ by [HL24, Lemma 2.2(b)(c)]. The primary decomposition of $\bar{t}$ implies our claims; see, e.g. [FoSr82, §1] and [FoSr89, (1.10), (1.13)] (although the latter reference assumes that q is odd, the results are also valid for even q).

Now suppose that Γ is the minimal polynomial of $\bar{t}_1$. If $\bar{t}_1 = \bar{t}^j$, the roots of Γ are the jth powers of the roots of the minimal polynomial of $\bar{t}$. Suppose first that κ is not a hermitian form so that the ground field is $\mathbb{F}_q$. Let φ denote the Frobenius morphism of $\mathbb{F}$ raising every element to its qth power. Let $\zeta \in \mathbb{F}$ denote a root of Γ. Then Γ is irreducible, if and only if every root of Γ is conjugate under $\langle \varphi \rangle$ to ζ.

The fact that $\Gamma = \Gamma^*$ is irreducible, or $\Gamma = \Delta\Delta^*$, where Δ is a monic irreducible polynomial over $\mathbb{F}_q$ with $\Delta \neq \Delta^*$, is equivalent to the fact that any other root of Γ is conjugate under $\langle\varphi\rangle$ to ζ or to ζ^{-1}. Thus the asserted properties of Γ follow from the corresponding properties of the minimal polynomial of $\bar{t}$. An analogous argument works in the case when κ is a hermitian form, replacing φ by φ^2 and ζ^{-1} by ζ^{-q}.

The claim about the multiplicity in the characteristic polynomial follows from $\bar{t}, \bar{t}_1 \in I(V, \kappa)$.

The last assertion follows from the previous ones, which imply that all non-trivial eigenvalues of $\bar{t}$ have the same order. □

3.3. **A reduction lemma.** Let $\mathbf{B}$ be a p-block of a group $G(V, \kappa)$ with a non-trivial cyclic defect group D. The following lemma reduces the computation of the invariant $W(\mathbf{B})$ to the case where the fixed space of $\nu(D)$ on V is trivial.

Lemma 3.3.1. *Let $G := G(V, \kappa)$ be one of the groups introduced in* (3) *and let $\mathbf{B}$ be a p-block of G with a non-trivial cyclic defect group D. Put $\bar{D} := \nu(D)$.*

Let V^0 be the fixed space of $\bar{D}$ on V, and let κ^0 denote the restriction of κ to V^0. Then (V^0, κ^0) is non-degenerate, unless κ is the zero form. Let V' denote the orthogonal complement of V^0 (or any complement in case κ is the zero form). Finally, let κ' denote the restriction of κ to V'.

Then $G(V', \kappa')$ embeds into G, and D is contained in the image G' of this embedding. Moreover, there is a block $\mathbf{B}'$ of G' with defect group D such that $W(\mathbf{B}) \cong W(\mathbf{B}')$.

Proof. Write $\bar{G} := I(V, \kappa)$. As D is a defect group of a p-block of G, it is a radical p-subgroup of G. Put $\bar{D}_1 := \nu(D_1)$, where D_1 denotes the unique subgroup of D of order p. Let t denote a generator of D and put $\bar{t} := \nu(t)$.

Suppose that κ is not the zero form. Then (V^0, κ^0) is non-degenerate, since the eigenspaces of $\bar{t}$ on V for eigenvalues unequal to 1 are orthogonal to V^0. Identify $I^0 := I(V^0, \kappa^0)$ and $I' := I(V', \kappa')$ with subgroups of $\bar{G}$, and $\bar{t}$ with an element of I' in the natural way.

If κ is a quadratic form and q is odd, let $G^0 := \nu^{-1}(\Omega(V^0, \kappa^0))$ and $G' := \nu^{-1}(\Omega(V', \kappa'))$. Then $G^0 \cong \mathrm{Spin}(V^0, \kappa^0)$ and $G' \cong \mathrm{Spin}(V', \kappa')$; see, e.g. [Art57, Chapter V, Section 4]. Otherwise, let $G^0 := G(V^0, \kappa^0)$ and $G' := G(V', \kappa')$.

We claim that $D \leq G'$ and that $N_{G'}(D_1)$ is a normal subgroup of $N_G(D_1)$. We only prove these claims in case κ is a quadratic form. The proof in the other cases is analogous but simpler. Assume that κ

Table 1. Some classical algebraic groups

Case	$\mathbf{G}$	Condition	F	$\mathbf{G}^F$	κ
1	$\mathrm{GL}_n(\mathbb{F})$	$n \geq 1$	F_{+1}	$\mathrm{GL}_n(q)$	zero
			F_{-1}	$\mathrm{GU}_n(q)$	hermitian
2	$\mathrm{Sp}_n(\mathbb{F})$	$n \geq 4$ even	F	$\mathrm{Sp}_n(q)$	symplectic
3	$\mathrm{Spin}_n(\mathbb{F})$	$n \geq 7$ odd, q odd	F	$\mathrm{Spin}_n(q)$	quadratic
4	$\mathrm{Spin}_n(\mathbb{F})$	$n \geq 8$ even	F_{+1}	$\mathrm{Spin}_n^+(q)$	quadratic
			F_{-1}	$\mathrm{Spin}_n^-(q)$	quadratic

is a quadratic form and write $\Omega^0 := \Omega(V^0, \kappa^0)$ and $\Omega' := \Omega(V', \kappa')$. Then $\bar{t} \in \Omega'$, as the index of Ω' in I' is a power of 2. Thus $t \in \nu^{-1}(\Omega') = G'$, which is our first claim. As V^0 is the fixed space of $\bar{D}_1$ by Lemma 3.2.3, we get $\Omega^0 \times N_{\Omega'}(\bar{D}_1) \leq N_{\nu(G)}(\bar{D}_1) \leq I^0 \times N_{I'}(\bar{D}_1)$. In particular, $N_{\Omega'}(\bar{D}_1) \trianglelefteq N_{\nu(G)}(\bar{D}_1)$. Now $\nu(N_G(D_1)) = N_{\nu(G)}(\bar{D}_1)$ by [HL24, Lemma 2.2(a)]. The kernel of ν is contained in $N_G(D_1)$, and hence $N_G(D_1) = \nu^{-1}(\nu(N_G(D_1))) = \nu^{-1}(N_{\nu(G)}(\bar{D}_1)$. Thus $\nu^{-1}(N_{\Omega'}(\bar{D}_1)) \trianglelefteq N_G(D_1)$. As $\nu^{-1}(N_{\Omega'}(\bar{D}_1)) = N_{G'}(D_1)$, once more by [HL24, Lemma 2.2(a)], we obtain our second claim.

Let $\mathbf{b}$ denote the Brauer correspondent of $\mathbf{B}$ in $N_G(D_1)$ and let $\mathbf{b}'$ be a block of $N_{G'}(D_1)$ covered by $\mathbf{b}$. As $D \leq N_{G'}(D_1) \trianglelefteq N_G(D_1)$, there is a defect group D' of $\mathbf{b}'$ conjugate to D in $N_G(D_1)$. We may thus assume that $D' = D$, and let $\mathbf{B}'$ denote the Brauer correspondent of $\mathbf{b}'$ in G'. Then $\mathbf{B}'$ has defect group D and we find $W(\mathbf{B}) \cong W(\mathbf{b}) \cong W(\mathbf{b}') \cong W(\mathbf{B}')$, where the middle isomorphism follows from [HL24, Corollary 4.4]. This completes our proof. □

3.4. **The underlying algebraic groups.** To continue, we also need to consider the algebraic groups and their Steinberg morphisms that give rise to the finite groups introduced in Subsection 3.1. At the same time we specify the degrees n relevant to our investigation.

Let $\mathbf{G}$ denote one of the following classical groups over $\mathbb{F}$, namely $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{Sp}_n(\mathbb{F})$ with n even, or $\mathrm{Spin}_n(\mathbb{F})$ with q odd if n is odd. Then there is an $\mathbb{F}_q$-rational structure on $\mathbf{G}$ and a corresponding Frobenius morphism F, such that $G = \mathbf{G}^F \cong G(V, \kappa)$ in the notation of Subsection 3.1. The various cases are displayed in Table 1.

Let us now be more specific. For a positive integer m, let J_m denote the $(m \times m)$-matrix with 1's along the anti-diagonal, and 0's elsewhere. Let $\mathbf{V} := \mathbb{F}^n$, the standard column space of dimension n over $\mathbb{F}$. Gram matrices of bilinear forms on $\mathbf{V}$ are taken with respect to the standard basis. In Case 1, we put $\mathbf{G} := \mathrm{GL}(\mathbf{V})$. In Case 2, we have $n = 2m$ even, and we let $\boldsymbol{\omega}$ denote the symplectic form on $\mathbf{V}$, whose Gram matrix equals $\begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$. In Case 3, we have q odd, and we let $\boldsymbol{\omega}$ denote the quadratic form on $\mathbf{V}$, whose polar form has Gram matrix J_n. In Case 4, we have $n = 2m$ even, and we let $\boldsymbol{\omega}$ denote the quadratic form on $\mathbf{V}$ defined by

$$\boldsymbol{\omega}((x_1, \dots, x_n)^t) = \sum_{i=1}^{m} x_i x_{n-i+1}$$

for $(x_1, \dots, x_n)^t \in \mathbf{V}$. Then the polar form of $\boldsymbol{\omega}$ has Gram matrix J_n. In Cases 2–4, the forms $\boldsymbol{\omega}$ are non-degenerate, and we write $I(\mathbf{V}, \boldsymbol{\omega})$ for their full isometry groups. In Case 2, we put $\mathbf{G} := I(\mathbf{V}, \boldsymbol{\omega})$.

In Cases 3 and 4, we let $\mathbf{G}$ denote the spin group of $\boldsymbol{\omega}$, and we have a homomorphism $\nu : \mathbf{G} \to I(\mathbf{V}, \boldsymbol{\omega})$ of algebraic groups, the vector representation of $\mathbf{G}$. Its kernel has order $\gcd(2, q-1)$, its image $\bar{\mathbf{G}} := \mathrm{SO}_n(\mathbb{F})$ is the connected component of $I(\mathbf{V}, \boldsymbol{\omega})$. If q is odd, $\mathrm{SO}_n(\mathbb{F}) = I(\mathbf{V}, \boldsymbol{\omega}) \cap \mathrm{SL}_n(\mathbb{F})$. As the kernel of ν consists of F-fixed points, there is a unique Steinberg morphism of $\bar{\mathbf{G}}$, also denoted by F, such that ν is F-equivariant. Then $\bar{G} = \bar{\mathbf{G}}^F = \nu(\mathbf{G})^F$ is a special orthogonal group of the appropriate type, and $\nu(G) = \nu(\mathbf{G}^F) \cong \Omega(V, \kappa)$. If n is even and q is odd, $[\bar{G} : \nu(G)] = 2$; otherwise, $\bar{G} = \nu(G)$. Finally, there is a Steinberg morphism F on $I(\mathbf{V}, \boldsymbol{\omega})$, which restricts to the given Steinberg morphism F on $\bar{\mathbf{G}}$ such that $I(\mathbf{V}, \boldsymbol{\omega})^F \cong I(V, \kappa)$.

Lemma 3.4.1. *Let $(\mathbf{G}, F)$ be as in one of the Cases* 3 *or* 4 *of* Table 1. *Write Z for the kernel of ν. Let $t \in G$ be a p-element and put $\bar{t} := \nu(t)$. Then there is a short exact sequence*

$$1 \longrightarrow Z \longrightarrow C_{\mathbf{G}}(t) \xrightarrow{\nu} C_{\bar{\mathbf{G}}}(\bar{t}) \longrightarrow 1. \tag{4}$$

In particular, $|C_G(t)| = |C_{\bar{G}}(\bar{t})|$.

Proof. As p is odd, $C_{\bar{\mathbf{G}}}(\bar{t})$ is connected by [GeHi91, Corollary 2.6]. The first claim thus follows from [GeMa20, 1.3.10(e)], and the second one from [GeMa20, Proposition 1.4.13(c)]. □

3.5. **The symplectic and orthogonal groups.** Here we investigate the centralizers of possible defect groups of cyclic p-blocks in the symplectic and orthogonal groups, under the restrictions suggested by Lemma 3.3.1. Let $(\mathbf{G}, F)$ be as in Cases 2–4 of Table 1, so that $\mathbf{G}^F = G \cong G(V, \kappa)$ with $G(V, \kappa)$ as in (3), where κ is a symplectic form if $\mathbf{G}$ is as in Case 2 of Table 1, and a quadratic form, otherwise. If κ is a symplectic form, put $\bar{\mathbf{G}} := \mathbf{G}$, and let $\nu : \mathbf{G} \to \bar{\mathbf{G}}$ denote the identity map. If κ is a quadratic form, let $\bar{\mathbf{G}} = \mathrm{SO}_n(\mathbb{F})$ denote the image of the vector representation ν of $\mathbf{G}$. Let Z denote the kernel of ν.

From now on, we will use the common ε-convention for the groups $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$. Let $\varepsilon \in \{1, -1\}$. Then $\mathrm{GL}_n^{\varepsilon}(q)$ denotes the general linear group $\mathrm{GL}_n(q)$, if $\varepsilon = 1$, and the general unitary group $\mathrm{GU}_n(q)$, if $\varepsilon = -1$. Analogous conventions are used for $\mathrm{SL}_n^{\varepsilon}(q)$ and $\mathrm{PSL}_n^{\varepsilon}(q)$.

Lemma 3.5.1. *Assume the notation and hypothesis introduced at the beginning of* Subsection 3.5*.*

Let $t \in G$ be a non-trivial p-element such that $\langle t \rangle$ is a radical p-subgroup of G. Let t_1 denote a power of t of order p.

Put $\bar{t}_1 := \nu(t_1)$. Assume that $\bar{t}_1$ does not fix any non-trivial vector of V, and let Γ denote the minimal polynomial of $\bar{t}_1$. By Lemmas 3.2.3 *and* 3.2.1*, the degree of Γ is even, $2d$, say, and Γ has at most two irreducible constituents. Put $\varepsilon := -1$, if Γ is irreducible, and $\varepsilon := 1$, otherwise.*

Then $n = 2md$ for some positive integer m,

$$C_{\bar{\mathbf{G}}}(\bar{t}_1) \cong \mathrm{GL}_m(\mathbb{F}) \times \cdots \times \mathrm{GL}_m(\mathbb{F}),$$

with d factors, and

$$C_{\bar{G}}(\bar{t}_1) \cong \mathrm{GL}_m^{\varepsilon}(q^d).$$

Proof. Clearly, $n = 2md$, where m denotes the common multiplicity of the irreducible factors of Γ in the characteristic polynomial of $\bar{t}_1$. In particular, $\mathbf{G}$ is not as in Case 3 of Table 1.

Let $\zeta \in \mathbb{F}$ denote a root of Γ, and write $\mathbf{V}_\zeta$ for the corresponding eigenspace of $\bar{t}_1$ on $\mathbf{V}$. Then $\mathbf{V}_\zeta$ is $C_{I(\mathbf{V},\boldsymbol{\omega})}(\bar{t}_1)$-invariant and $\dim(\mathbf{V}_\zeta) = m$. Let ζ' be a root of Γ with $\zeta' \neq \zeta^{-1}$. A straightforward calculation shows that $\mathbf{V}_\zeta$ and $\mathbf{V}_{\zeta'}$ are orthogonal with respect to $\boldsymbol{\omega}$ in Case 2, respectively to the polar form of $\boldsymbol{\omega}$ in Case 4. In particular, $\mathbf{V}_\zeta$ is totally isotropic since $\zeta \neq \pm 1$, and $\mathbf{V}_\zeta \oplus \mathbf{V}_{\zeta^{-1}}$ is non-degenerate, as $\boldsymbol{\omega}$ in Case 2, respectively the polar form of $\boldsymbol{\omega}$ in Case 4, are non-degenerate. The restriction of $C_{I(\mathbf{V},\boldsymbol{\omega})}(\bar{t}_1)$ to $\mathbf{V}_\zeta \oplus \mathbf{V}_{\zeta^{-1}}$ is isomorphic to $\mathrm{GL}_m(\mathbb{F})$. Hence $C_{I(\mathbf{V},\boldsymbol{\omega})}(\bar{t}_1) \cong \mathrm{GL}_m(\mathbb{F}) \times \cdots \times \mathrm{GL}_m(\mathbb{F})$ with d factors. In particular, $C_{I(\mathbf{V},\boldsymbol{\omega})}(\bar{t}_1)$ is connected.

It follows that $C_{I(\mathbf{V},\boldsymbol{\omega})}(\bar{t}_1) \leq \bar{\mathbf{G}}$ as $\bar{\mathbf{G}} = I(\mathbf{V}, \boldsymbol{\omega})^\circ$. Thus $C_{I(\mathbf{V},\boldsymbol{\omega})}(\bar{t}_1) = C_{\bar{\mathbf{G}}}(\bar{t}_1)$, proving our first claim. Now $C_{\bar{G}}(\bar{t}_1) = C_{I(V,\omega)}(\bar{t}_1)$, and by [FoSr89, (1.13)] (which is also valid for even q), we obtain $C_{I(V,\omega)}(\bar{t}_1) \cong \mathrm{GL}_m^{\varepsilon}(q^d))$. This is our second claim. □

Remark 3.5.2. Assume the notation and hypotheses of Lemma 3.5.1. If Γ is irreducible, then $2d$ is the order of q modulo p, and $p \mid q^d + 1$.

On the other hand, if $\Gamma = \Delta\Delta^*$ with $\Delta \neq \Delta^*$, then d is odd, and d is the order of q modulo p. Thus $p \mid q^d - \varepsilon$.

Suppose that κ is a quadratic form. If $\Gamma = \Delta\Delta^*$ with $\Delta \neq \Delta^*$, then $G = \mathrm{Spin}_n^+(q)$. On the other hand, if Γ is irreducible, then $G = \mathrm{Spin}_n^+(q)$ if m is even, and $G = \mathrm{Spin}_n^-(q)$ if m is odd. These results are proved in [HM19, Lemmas 4.1, 4.5]. □

3.6. **Reduction to the general linear and unitary groups.** We now put our focus on the centralizers of elements of order p, under the hypotheses of Lemma 3.5.1. The lemma below reduces the computation of the invariants $W(\mathbf{B})$ in such centralizers to the corresponding question in general linear and unitary groups. The groups $\mathbf{G}$ and $\bar{\mathbf{G}}$ in Lemma 3.6.1 below play the roles of the groups $C_{\mathbf{G}}(t_1)$, respectively $C_{\bar{\mathbf{G}}}(\bar{t}_1)$ of Lemma 3.5.1. Notice that in the situation of Lemma 3.5.1, the groups $C_G(t_1)$ and $C_{\bar{G}}(\bar{t}_1)$ need not be isomorphic, although they have the same order by Lemma 3.4.1. For example, suppose that $G = \mathrm{Spin}_n^+(q)$ with q odd and $4 \mid n$. Then $Z(G)$ is elementary abelian of order 4; see [GLS98, Table 6.1.2]. In this case, we cannot have $C_G(t_1) \cong C_{\bar{G}}(\bar{t}_1)$, as $C_{\bar{G}}(\bar{t}_1) \cong \mathrm{GL}_m^{\varepsilon}(q^d)$ has a cyclic center.

Fix $\varepsilon \in \{-1, 1\}$. In view of Remark 3.5.2, replacing q^d by q, the hypothesis $p \mid q - \varepsilon$ of Lemma 3.6.1 below is justified. This lemma is based mainly on [CaEn99]. For a concise and in parts more general account of these results see [KeMa15, Sections 2, 3].

Lemma 3.6.1. *Let $\mathbf{G}$ be a connected reductive algebraic group and let $\bar{\mathbf{G}}$ denote a direct product of groups isomorphic to $\mathrm{GL}_n(\mathbb{F})$. Let F and $\bar{F}$ be Frobenius morphisms of $\mathbf{G}$ and $\bar{\mathbf{G}}$, respectively, arising from $\mathbb{F}_q$-rational structures of $\mathbf{G}$ and $\bar{\mathbf{G}}$, respectively, such that $\bar{G} = \bar{\mathbf{G}}^{\bar{F}} \cong \mathrm{GL}_n^{\varepsilon}(q)$. Assume that $p \mid q - \varepsilon$.*

Suppose that there is an isogeny $\nu : \mathbf{G} \to \bar{\mathbf{G}}$ with $\nu \circ F = \bar{F} \circ \nu$ and $|\ker(\nu)| \leq 2$.

Let $\mathbf{B}$ be a p-block of G with a non-trivial cyclic defect group D. Then there is an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, such that $\mathbf{B}$ is regular with respect to T; see Definition 2.5.14(b). *By* Lemma 2.5.16, *we may thus assume that D is a Sylow p-subgroup of T.*

Put $\bar{\mathbf{T}} := \nu(\mathbf{T})$. Then $\bar{T}$ is cyclic of order $q^n - \varepsilon^n$.

Suppose that $\bar{\mathbf{B}}$ is a regular p-block of $\bar{G}$ with respect to $\bar{T}$. Then $W(\mathbf{B}) \cong W(\bar{\mathbf{B}})$, if D and $\bar{D}$ are identified.

Proof. We will make use of the main results of [CaEn99]. Put $\bar{\mathbf{G}}^* := \bar{\mathbf{G}}$, and let $\mathbf{G}^*$ be a reductive group dual to $\mathbf{G}$. Let F be a Frobenius morphism of $\mathbf{G}^*$ such that the pairs $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F)$ are in duality. There is a dual isogeny $\nu^* : \bar{\mathbf{G}}^* \to \mathbf{G}^*$ satisfying $F \circ \nu^* = \nu^* \circ \bar{F}$;

see [DiMi20, Proposition 11.1.11]. By Lemma 2.5.10(e) and [GeMa20, 1.3.10(c)] we have $\nu(Z^\circ(\mathbf{G})) = Z^\circ(\bar{\mathbf{G}}) = Z(\bar{\mathbf{G}}) = \nu(Z(\mathbf{G}))$. Hence $Z(\mathbf{G}) = Z^\circ(\mathbf{G})\cdot\ker(\nu)$, and thus $|Z(\mathbf{G})/Z^\circ(\mathbf{G})| \leq 2$. By the analogous argument, $Z(\mathbf{G}^*)$ is connected. Thus $3 \in \Gamma(\mathbf{G}, F) \cap \Gamma(\bar{\mathbf{G}}, \bar{F})$ in the notation of [CaEn99, Definition-Notation 4.3], so that the results of [CaEn99] apply to $(\mathbf{G}, F)$ and to $(\bar{\mathbf{G}}, \bar{F})$ also in case $p = 3$; see [CaEn99, Subsection 5.2]. Notice that p is good for $\mathbf{G}$, as $\mathbf{G}$ is of type A.

Let e denote the order of q modulo p. Thus $e = 1$ if $p \mid q-1$, i.e. if $\varepsilon = 1$, and $e = 2$, if $p \mid q+1$, i.e. if $\varepsilon = -1$. According to [CaEn99, Theorem], there is an e-cuspidal pair $(\mathbf{L}, \chi')$ associated to $\mathbf{G}$ which determines $\mathbf{B}$. This means that $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$, and χ' is an e-cuspidal irreducible character of L. Moreover, χ' lies in a Lusztig series of L determined by a p'-element. Let $\mathbf{M}$ and $Z = O_p(Z(M))$ be as in [CaEn99, Definition-Notation 4.3], respectively [CaEn99, Definition 4.6]. Then $\mathbf{L} = \mathbf{M}$ by [CaEn99, Proposition 3.2] and thus $Z = O_p(Z(L))$. By [CaEn99, Lemma 4.16], we may assume that $O_p(Z(L)) \leq D$. In particular, $O_p(Z(L))$ is cyclic.

Now $\bar{\mathbf{L}} := \nu(\mathbf{L})$ is an e-split Levi subgroup of $\bar{\mathbf{G}}$ by Lemma 2.5.10(b). Furthermore, $O_p(Z(L)) \cong O_p(Z(\bar{L}))$, as $\nu(Z(\mathbf{L})) = Z(\bar{\mathbf{L}})$; see [GeMa20, 1.3.10(c)]. We also have $\bar{L} \cong \mathrm{GL}^\varepsilon_{n_1}(q)\times\cdots\times\mathrm{GL}^\varepsilon_{n_c}(q)$ with $\sum_{j=1}^c n_j = n$. As $O_p(Z(\bar{L}))$ is cyclic, we get $c = 1$, thus $\bar{\mathbf{L}} = \bar{\mathbf{G}}$, and hence $\mathbf{L} = \mathbf{G}$. In particular, χ' is an e-cuspidal character of G, and $\chi' \in \mathrm{Irr}(\mathbf{B})$. We now apply [CaEn99, Theorem 4.2] which shows that χ' satisfies the Jordan criterion, i.e. Condition (J) of [CaEn99, Subsection 1.3].

Let $s \in G^*$ be a semisimple p'-element such that χ' lies in the Lusztig series $\mathcal{E}(G, s)$ and put $\mathbf{T}^* := C_{\mathbf{G}^*}(s)^\circ$. Then $\mathbf{T}^*$ is an F-stable Levi subgroup of $\mathbf{G}^*$, as the latter is a group of type A. By Lemma 2.5.10(c), there is an $\bar{F}$-stable Levi subgroup $\bar{\mathbf{T}}^*$ of $\bar{\mathbf{G}}^*$ with $\nu^*(\bar{\mathbf{T}}^*) = \mathbf{T}^*$.

Condition (J_2) of [CaEn99, Subsection 1.3] implies that $C^\circ_{G^*}(s)$ has an e-cuspidal unipotent character. Then $\bar{T}^*$ has an e-cuspidal unipotent character by Lemma 2.5.10(f). In turn, $\bar{\mathbf{T}}^*$ is a maximal torus by [BMM93, Proposition 2.9]. In particular, $\mathbf{T}^*$ is a maximal torus. Notice that ν^* maps the Sylow Φ_e-torus of $\bar{\mathbf{T}}^*$ to the Sylow Φ_e-torus of $\mathbf{T}^*$; see Lemma 2.5.10(b). Condition (J_1) of [CaEn99, Subsection 1.3] and the description of the maximal tori of $\mathrm{GL}^\varepsilon_n(q)$ now show that $\bar{T}^*$ is cyclic of order $q^n - \varepsilon^n$.

Let $\mathbf{T}$ denote an F-stable maximal torus of $\mathbf{G}$, dual to $\mathbf{T}^*$. Then $\mathbf{B}$ is regular with respect to $\mathbf{T}$ by definition. Also, $\bar{\mathbf{T}} = \nu(\mathbf{T})$ is dual to $\bar{\mathbf{T}}^*$. Hence $\mathbf{T}$ is cyclic of order $q^n - \varepsilon^n$. Let $\theta \in \mathrm{Irr}(T)$ such that the pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}^*, s)$ correspond via duality, and put $\chi := \varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{G}} R^{\mathbf{G}}_{\mathbf{T}}(\theta)$. Then χ' is an irreducible constituent of χ by Lemma 2.5.16. Since

$Z^\circ(\bar{\mathbf{G}})^{\bar{F}} = Z(\bar{\mathbf{G}})^{\bar{F}} = Z(\bar{G})$ has order $q - \varepsilon$, we conclude from Lemma 2.5.10(e) and our assumption that $p \mid |Z(G)|$. In particular, $\mathbf{B}$ is nilpotent, and so the hypotheses of Lemma 2.5.18 are satisfied. This then implies that χ', being an element of $\mathcal{E}(G, s)$, is the non-exceptional character in $\mathbf{B}$.

Let $\bar{\chi}$ denote the non-exceptional character of $\bar{\mathbf{B}}$. By Corollary 2.5.17 there is $\bar{\theta} \in \mathrm{Irr}(\bar{T})$ of p'-order, such that $\bar{\chi} = \varepsilon_{\bar{\mathbf{G}}}\varepsilon_{\bar{\mathbf{T}}} R_{\bar{\mathbf{T}}}^{\bar{\mathbf{G}}}(\bar{\theta})$. Suppose that $|D| = p^l$ and let t be a generator of D. Put $\bar{t} := \nu(t)$. As $Z(\mathbf{G}^*)$ and $Z(\bar{\mathbf{G}}^*)$ are connected, the centralizers in $\mathbf{G}$, respectively in $\bar{\mathbf{G}}$, of the powers of t, respectively of $\bar{t}$, are connected. By [GeMa20, Theorem 1.3.10(e)] and Lemma 2.5.10(d) we get $\omega_{\bar{\mathbf{G}}}^{[l]}(\bar{t}) = \omega_{\mathbf{G}}^{[l]}(t)$. Then $\sigma_{\bar{\chi}}^{[l]}(\bar{t}) = \sigma_{\chi}^{[l]}(t) = \sigma_{\chi'}^{[l]}(t)$, where the first equality arises from Lemma 2.5.7, and the second one from the last statement of Lemma 2.5.16. Using Remark 2.3.3, we conclude that $W(\mathbf{B}) \cong W(\bar{\mathbf{B}})$. □

Corollary 3.6.2. *Let* $(\mathbf{G}, F)$ *be as in* Case 1 *of* Table 1, *so that* $G = \mathrm{GL}_n^\varepsilon(q)$. *Suppose that* $p \mid q - \varepsilon$, *and let* $\mathbf{B}$ *be a* p*-block of* G *with a cyclic defect group* D.

Then $\mathbf{B}$ *is regular with respect to a cyclic maximal torus* T *of* G *of order* $q^n - \varepsilon^n$. *In particular, we may assume that* D *is a Sylow* p*-subgroup of* T.

Let a *be the positive integer such that* p^a *is the highest power of* p *dividing* $q - \varepsilon$, *and define the non-negative integer* a' *by* $n = mp^{a'}$ *with* $p \nmid m$. *Then* $|D| = p^{a+a'}$.

Proof. The first part follows from Lemma 3.6.1, if we let $\mathbf{G} = \bar{\mathbf{G}}$ and ν the identity map. The statement on $|D|$ follows from this. □

The statements in Corollary 3.6.2 were first proved by Fong and Srinivasan in [FoSr82]. Notice however, that their notation differs from the one used here. Most notably, the group we denote by $\mathrm{GU}_n(q)$ here, is denoted by $U(n, \mathbb{F}_{q^2})$ or $U(n, q^2)$ in [FoSr82].

To make use of the last part of Lemma 3.6.1, we need to investigate when $\mathrm{GL}_n^\varepsilon(q)$ has a regular block with respect to the cyclic maximal torus T of order $q^n - \varepsilon^n$. We may realize $G = \mathrm{GL}_n^\varepsilon(q)$ as $G = \mathbf{G}^F$ with $\mathbf{G} = \mathrm{GL}_n(\mathbb{F})$ and some suitable Steinberg morphism F of $\mathbf{G}$. As $\mathbf{G}$ has connected center, the notions of regular and strictly regular blocks for G coincide; see the remarks following Definition 2.5.14. We may also identify G with its dual group and T with its dual torus T^*. Then a regular block of G with respect to T exists, if and only if T contains a regular p'-element; see Lemma 2.5.15. The last part of the following lemma is only used at a later stage of our work.

Lemma 3.6.3. *Let* $\mathbf{G} := \mathrm{GL}_n(\mathbb{F})$ *for some* $n > 2$, *and let* F *be a Steinberg morphism of* $\mathbf{G}$ *such that* $G = \mathbf{G}^F = \mathrm{GL}_n^\varepsilon(q)$. *Assume that* $p \mid q - \varepsilon$ *and that* $(q, n) \neq (2, 3)$.

Then there is a prime f *with* $f \mid q^n - \varepsilon^n$, *but* $f \nmid q^j - \varepsilon^j$ *for all* $1 \leq j < n$.

Let T *be a cyclic maximal torus of* G *of order* $q^n - \varepsilon^n$, *and let* $s \in T$ *be of order* f. *Then* s *is regular in* G *and the image* s' *of* s *in* $\mathrm{PGL}_n^\varepsilon(q)$ *is strictly regular; i.e.* $C_{\mathrm{PGL}_n^\varepsilon(q)}(s')$ *is a maximal torus of* $\mathrm{PGL}_n^\varepsilon(q)$; *see* Definition 2.5.14(a).

Proof. Define the integer m by

$$m := \begin{cases} n, & \text{if } \varepsilon = 1, \\ 2n, & \text{if } \varepsilon = -1 \text{ and } n \text{ is odd}, \\ n, & \text{if } \varepsilon = -1 \text{ and } 4 \mid n, \\ n/2, & \text{if } \varepsilon = -1 \text{ and } n \equiv 2 \pmod 4. \end{cases}$$

Then $m \geq 3$. If $q = 2$, then $\varepsilon = -1$, as p is a prime dividing $q - \varepsilon$. Hence $(q, m) = (2, 6)$ corresponds to the case $(q, n) = (2, 3)$, which is excluded. It follows that there exists a primitive prime divisor f of $q^m - 1$, i.e. f is a prime with $f \mid q^m - 1$ but $f \nmid q^j - 1$ for $1 \leq j < m$; see [HuBl82, Theorem IX.8.3]. It is easy to check that $f \nmid q^j - \varepsilon^j$ for all $1 \leq j < n$.

Clearly s is regular in $\mathbf{G}$, as every maximal torus of G containing s is G-conjugate to T. As $|A_{\mathrm{PGL}_n(\mathbb{F})}(s')^F|$ divides $\gcd(q - \varepsilon, n)$, the statement about s' follows from 2.5.1. ∎

We can now formulate the main result of this section.

Theorem 3.6.4. *Let* $\mathbf{G}$ *and* F *be as in one of the Cases* 2–4 *of* Table 1. *Let* $\mathbf{B}$ *be a* p*-block of* G *with a cyclic defect group* D. *Assume that* $W(\mathbf{B}) \ncong k$.

Then there is $\varepsilon \in \{-1, 1\}$, *an integer* $n' \geq 3$, *a power* q' *of* q *with* $p \mid q' - \varepsilon$, *and a block* $\mathbf{B}'$ *of* $\mathrm{GL}_{n'}^\varepsilon(q')$, *regular with respect to the cyclic maximal torus* T' *of* $\mathrm{GL}_{n'}^\varepsilon(q')$ *of order* ${q'}^{n'} - \varepsilon^{n'}$, *such that* D *is isomorphic to a Sylow* p*-subgroup* D' *of* T', *and* $W(\mathbf{B}) \cong W(\mathbf{B}')$, *if* D *and* D' *are identified.*

Proof. Use the notation of the previous subsections. Let $t_1 \in D$ denote an element of order p and put $\bar{t}_1 = \nu(t_1)$. By Lemma 3.3.1, we may assume that $\bar{t}_1$ has no non-trivial fixed point on V. Let us assume this from now on. By Lemma 3.4.1 there is an isogeny $\nu : C_{\mathbf{G}}(t_1) \to C_{\bar{\mathbf{G}}}(\bar{t}_1)$ with kernel of order at most two. By Lemma 3.5.1, the group $C_{\bar{\mathbf{G}}}(\bar{t}_1)$ is isomorphic to a direct product of general linear groups of degree m, and

$C_{\bar{G}}(\bar{t}_1) \cong \mathrm{GL}_m^{\varepsilon}(q^d)$, for some $\varepsilon \in \{-1, 1\}$, where m, d satisfy $2md = n$. Put $n' := m$ and $q' := q^d$. Then $p \mid q' - \varepsilon$ by Remark 3.5.2. Thus the hypotheses of Lemma 3.6.1 hold for ν, n' and q'.

In order to find $W(\mathbf{B})$ we may replace G by $C_G(t_1)$ and $\mathbf{B}$ by a Brauer correspondent $\mathbf{c}$ of $\mathbf{B}$ in $C_G(t_1)$, but keeping D. Lemma 3.6.1 implies that D is a Sylow p-subgroup of a torus of $C_G(t_1)$ of order ${q'}^{n'} - \varepsilon^{n'}$. Recall that $q' - \varepsilon$ divides $Z(C_G(t_1))$ by Lemma 2.5.10(e). Thus $n' \leq 2$ would imply $D \leq Z(C_G(t_1))$, and then $W(\mathbf{c}) \cong k$ by [HL24, Lemma 3.6(b)], a case we have excluded. Hence $n' > 2$.

The claim then follows from Lemmas 3.6.3 and 3.6.1, unless $(q', n') = (2, 3)$. In the latter case, $G = \mathrm{Sp}_6(2)$ and $p = 3$. But the cyclic 3-blocks of $\mathrm{Sp}_6(2)$ have defect 1 or 0; see [GAP21]. Then $W(\mathbf{B}) \cong W(\mathbf{c}) \cong k$, once more by [HL24, Lemma 3.6(b)], contrary to our hypothesis. □

4. The general linear and unitary groups

As a next step we consider the general linear and unitary groups. Fix a sign $\varepsilon \in \{1, -1\}$ and let $\mathbf{G}$ and $F = F_\varepsilon$ be as in Case 1 of Table 1. Then $G = \mathbf{G}^F = \mathrm{GL}_n^{\varepsilon}(q)$. We may and will identify $\mathbf{G}$ with its dual group $\mathbf{G}^*$.

By Corollary 3.6.2, if p is an odd prime with $p \mid q - \varepsilon$, a cyclic p-block of G is regular with respect to the cyclic maximal torus T of G order $q^n - \varepsilon^n$. If $n = 1$, the principal block of G has this property. Suppose that $n = 2$ and $G \neq \mathrm{GU}_2(2)$. Then if $p \mid q - \varepsilon$, an element $\theta \in \mathrm{Irr}(T)$ of order $(q^2 - 1)_{p'}$ is in general position, so that regular blocks with respect to T also exist in this case. Moreover, if $n \geq 3$, such regular blocks exist, unless $G = \mathrm{GU}_3(2)$ and $p = 3$; see Lemma 3.6.3.

4.1. The crucial case.

Throughout this subsection we let p be an odd prime satisfying $p \mid q - \varepsilon$, and we denote by a the positive integer such that p^a is the highest power of p dividing $q - \varepsilon$. We will use the parameter δ in the meaning of Subsection 3.1. Thus, $\delta = 1$, if $\varepsilon = 1$, and $\delta = 2$, if $\varepsilon = -1$.

The following lemma on certain p-elements of G will be used several times. The terms minimal polynomial and eigenvalue refer to the action of G on its natural vector space $V = \mathbb{F}_{q^\delta}^n$.

Lemma 4.1.1. *Let $t \in G$ be semisimple and let Γ denote the minimal polynomial of t over $\mathbb{F}_{q^\delta}$. Let T denote a cyclic maximal torus of G of order $q^n - \varepsilon^n$.*

(a) *If Γ is irreducible, the minimal polynomial of t^j is irreducible for all integers j.*

(b) *Suppose that* $t \in T$. *Then* Γ *is irreducible, unless* $\varepsilon = -1$, n *is even and* $\Gamma = \Delta\Delta^\dagger$ *for a monic irreducible polynomial* Δ *with* $\Delta \neq \Delta^\dagger$.

(c) *Let* Δ *be a monic irreducible polynomial over* $\mathbb{F}_{q^\delta}$, *and let* $\zeta \in \mathbb{F}$ *be a root of* Δ. *If* ζ *is of* p*-power order, then* $\deg(\Delta) = p^b$ *with*

$$b = \begin{cases} 0, & \text{if } |\zeta| \leq p^a \\ a', & \text{if } |\zeta| = p^{a+a'} \text{ with } a' \geq 0. \end{cases}$$

(d) *If* $t \in T$ *is of* p*-power order, then* Γ *is irreducible.*

(e) *Suppose that* Γ *is irreducible and put* $d := \deg(\Gamma)$. *Thus* $n = n'd$ *for some positive integer* n'. *Let* $\zeta \in \mathbb{F}$ *be an eigenvalue of* t. *Then*

$$\det(t) = \zeta^{n'(q^{\delta d}-1)/(q^\delta-1)}.$$

Proof. (a) This is certainly well known. For convenience, we sketch a proof. As Γ is irreducible, the subalgebra $\mathbb{F}_{q^\delta}[t]$ of the full matrix algebra is a field. It is, in particular, a finite extension of $\mathbb{F}_{q^\delta}$. Thus t^j is algebraic over $\mathbb{F}_{q^\delta}$, and so $\mathbb{F}_{q^\delta}[t^j]$ is a field. This implies the claim.

(b) Suppose that $\varepsilon = 1$ or that n is odd. Then T lies in a Coxeter torus of $\mathrm{GL}_n(q^\delta)$. Hence t is a power of an element of $\mathrm{GL}_n(q^\delta)$ acting irreducibly on V. Thus Γ is irreducible by (a).

Now suppose that $\varepsilon = -1$ and that $n = 2m$ is even. Then T is a Coxeter torus of a Levi subgroup $L \cong \mathrm{GL}_m(q^2)$ of G. Now $V = V_1 \oplus V_2$ with L-invariant totally isotropic subspaces V_1, V_2 of V, such that L acts on V_1 in the natural way, i.e. as $\mathrm{GL}_m(q^2)$ acts on $\mathbb{F}_{q^2}^m$, whereas the action of L on V_2 is the natural action twisted by an automorphism of L. Thus, by the first part of the proof, if Γ is reducible, we have $\Gamma = \Delta\Delta'$ with monic irreducible polynomials $\Delta \neq \Delta'$. As t lies in the unitary group, we must have $\Delta' = \Delta^\dagger$.

(c) Put $d = \deg(\Delta)$. If $|\zeta| \leq p^a$, then $\zeta \in \mathbb{F}_{q^\delta}$, and thus $d = 1$. Suppose then that $|\zeta| = p^{a+a'}$ for some non-negative integer a'. Observe that the highest power of p dividing $q^\delta - 1$, respectively $q^{\delta p^{a'}} - 1$, equals p^a, respectively $p^{a+a'}$. Hence $d = [\mathbb{F}_{q^\delta}[\zeta] : \mathbb{F}_{q^\delta}] = p^{a'}$.

(d) Suppose that Γ is reducible, and let $\zeta \in \mathbb{F}$ denote a root of Δ, where Δ is as in (b). Then ζ is of p-power order, and thus $\deg(\Delta)$ is odd by (c). But then $\Delta = \Delta^\dagger$ by Lemma 3.2.2, a contradiction.

(e) As Γ is irreducible, the roots of Γ are

$$\zeta, \zeta^{q^\delta}, \zeta^{q^{\delta\cdot 2}}, \ldots, \zeta^{q^{\delta\cdot(d-1)}},$$

and the product of these roots equals

$$\zeta^{(q^{\delta d}-1)/(q^\delta-1)}.$$

This proves our claim. □

We will also need the following elementary result.

Lemma 4.1.2. *Let m, b be positive integers. Then the following hold.*
(a) *If m is even or if $p \equiv 1 \pmod 4$, then $\lfloor m/2 \rfloor + \lfloor mp^b/2 \rfloor$ is even.*
(b) *If m is odd and $p \equiv -1 \pmod 4$, then $\lfloor m/2 \rfloor + \lfloor mp^b/2 \rfloor \equiv b \pmod 2$.*

Proof. Suppose first that m is even. Then $\lfloor m/2 \rfloor + \lfloor mp^b/2 \rfloor = m(1 + p^b)/2$ is even as p is odd. This gives the first part of (a).

Suppose now that m is odd. Then $\lfloor m/2 \rfloor + \lfloor mp^b/2 \rfloor = m(1+p^b)/2-1$. This is even if and only if $(1 + p^b)/2$ is odd, i.e. if and only if $p^b \equiv 1 \pmod 4$. This proves the remaining results. □

Assume from now on that that $n \geq 2$. Define the integers a' and m by $n = p^{a'}m$ and $p \nmid m$. Then a' is non-negative and m is positive. Let $\mathbf{B}$ denote a p-block of G with a non-trivial cyclic defect group D. Let $\mathbf{T}$ denote an F-stable maximal torus of $\mathbf{G}$ such that T is cyclic of order $q^n - \varepsilon^n$. (Notice that $\mathbf{T}$ is a Coxeter torus of $\mathbf{G}$, unless $G = \mathrm{GU}_n(q)$ and n is even; in any case, $\mathbf{T}$ is uniquely determined in $\mathbf{G}$ up to conjugation in G.) By Corollary 3.6.2 we may assume that D is a Sylow p-subgroup of T, which implies $|D| = p^{a+a'}$. Also, there is $\theta \in \mathrm{Irr}(T)$ in general position and of p'-order, such that the unique non-exceptional character of $\mathrm{Irr}(\mathbf{B})$ is the character $\chi := \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R_{\mathbf{T}}^{\mathbf{G}}(\theta)$; see Corollary 2.5.17.

Lemma 4.1.3. *Assume the notation and assumptions introduced in the previous paragraph. Let $t \in T$ be a non-trivial p-element and let l be a positive integer such that $t^{p^{l-1}} \in Z(G)$. Put $\Lambda := \{0, 1, \ldots, l-1\}$.*

Then $\sigma_\chi^{[l]}(t) = \omega_\Lambda(\mathbf{1}_I)$, for an interval $I \subseteq \Lambda \setminus \{0\}$.

Moreover, $I = \emptyset$, except if $\varepsilon = -1$, n is odd, $p \equiv -1 \pmod 4$ and $|t| = p^{a+a''}$ for some $1 \leq a'' \leq a'$. In this case, $I = [l - a'', l-1]$.

Proof. By Lemma 2.5.7 we have $\sigma_\chi^{[l]}(t) = \omega_{\mathbf{G}}^{[l]}(t)$. Choose $j \in \Lambda \setminus \{0\}$ and let $u := t^{p^{l-j}}$. We have to compute $\omega_{\mathbf{G}}(u)$.

Clearly, $C_G(u) = G$ if $|u| \mid p^a$, in which case $\omega_{\mathbf{G}}(u) = 1$. Suppose that $|u| = p^{a+b}$ for some integer $b > 0$. As $|D| = p^{a+a'}$, we have $b \leq a'$. By Lemma 4.1.1, the minimal polynomial of u is irreducible, and the eigenvalues of u span a field extension of $\mathbb{F}_{q^\delta}$ of degree p^b. Thus $C_G(u) \cong \mathrm{GL}^\varepsilon_{mp^{a'-b}}(q^{p^b})$. By Example 2.5.5, we get

$$\omega_{\mathbf{G}}(u) = (-1)^{mp^{a'}+mp^{a'-b}}$$

if $\varepsilon = 1$, and

$$\omega_{\mathbf{G}}(u) = (-1)^{\lfloor mp^{a'}/2 \rfloor + \lfloor mp^{a'-b}/2 \rfloor}$$

if $\varepsilon = -1$.

Thus $\omega_{\mathbf{G}}(u) = 1$ if $\varepsilon = 1$. Suppose that $\varepsilon = -1$. Lemma 4.1.2 then implies that $\omega_{\mathbf{G}}(u) = 1$ unless m and b are odd and $p \equiv -1 \pmod 4$, in which case $\omega_{\mathbf{G}}(u) = -1$.

Suppose now that $\varepsilon = -1$, that m is odd, that $p \equiv -1 \pmod 4$ and that $|t| = p^{a+a''}$ for some positive integer a''. As $t^{p^{l-1}} \in Z(G)$ by hypothesis, we have $a'' < l$. By what we have proved so far, we get

$$\sigma_\chi^{[l]}(t) = (1, \ldots, 1, -1, 1, -1, \ldots, \pm 1),$$

where the first -1 appears at position $l - a'' + 1$. The claim now follows from Lemma 2.2.1. □

We are now able to determine $W(\mathbf{B})$. By Lemma 2.4.2(c), this is of the form $W_D(A)$ for some $A \subseteq \{a, \ldots, a + a' - 1\}$.

Corollary 4.1.4. *Assume the notation and assumptions introduced in the paragraph preceding* Lemma 4.1.3*.*

Then $W(\mathbf{B}) \cong k$*, unless* $\varepsilon = -1$*,* n *is odd and* $p \equiv -1 \pmod 4$*. In this case,* $W(\mathbf{B}) \cong W_D([a, a + a' - 1])$*.*

Proof. Let t denote a generator of D. Then $|t| = p^{a+a'}$. Put $l := a + a'$ and let $\Lambda := \{0, \ldots, l-1\}$. Lemma 4.1.3 yields $\sigma_\chi^{[l]}(t) = \omega_\Lambda(\mathbf{1}_I)$ with $I = \emptyset$ and $I = [a, a + a' - 1]$ in the respective cases.

Our assertion follows from Lemma 2.3.2 and [HL24, Lemma 3.3]. □

4.2. **The general case.** Here, we will show how to reduce the computation of $W(\mathbf{B})$ for an arbitrary cyclic block $\mathbf{B}$ of G to Corollary 4.1.4. We will also show, by way of an example, how to use Corollary 4.1.4 to construct a non-uniserial cyclic block $\mathbf{B}$ of a suitable group $\mathrm{GU}_n(q)$ with $W(\mathbf{B}) \not\cong k$. We will also discuss an important consequence of Corollary 4.1.4 to the special linear and unitary groups.

Keep the notation introduced at the beginning of Section 4. In addition, fix an odd prime p with $p \nmid q$. To allow for a uniform treatment, we denote by d the order of εq modulo p. Then $p \mid q^d + 1$ if $\varepsilon = -1$ and d is odd, and we say that p is a *unitary prime* for $\mathrm{GU}_n(q)$. If $\varepsilon = -1$ and d is even, then $p \mid q^d - 1$, and p is called a *linear prime* for $\mathrm{GU}_n(q)$.

Theorem 4.2.1. *Let d denote the order of εq modulo p. Assume that $d > 1$. Let $\mathbf{B}$ denote a p-block of G with a non-trivial cyclic defect group D. Assume that the fixed space of D on the natural vector space for G is trivial. Put $L := C_G(D_1)$, where D_1 denotes the unique subgroup of D of order p.*

Then there are non-negative integers m, a' *with* $p \nmid m$ *and* $n = mdp^{a'}$, *such that* $L \cong \mathrm{GL}^{\varepsilon(d)}_{mp^{a'}}(q^d)$, *where* $\varepsilon(d) = 1$ *if* $\varepsilon = 1$ *or if* $\varepsilon = -1$ *and* d *is even, and* $\varepsilon(d) = -1$, *otherwise.*

Let $\mathbf{c}$ *denote a Brauer correspondent of* $\mathbf{B}$ *in* L. *Then* $W(\mathbf{B}) \cong W(\mathbf{c})$ *and* $W(\mathbf{c})$ *can be computed from* Corollary 4.1.4.

Proof. Let $t \in D$ denote a generator of D and let t_1 be a power of t generating D_1. By assumption, t has no non-trivial fixed vector on the natural vector space for G. Then [FoSr82, Theorem (3C)] implies that the primary decomposition of t in the sense of [FoSr82, §1] has a unique term.

By [FoSr82, Proposition (4A))], we obtain $C_G(D) \cong \mathrm{GL}^{\varepsilon(d)}_m(q^{dp^{a'}})$. The structure of L follows from this, as t_1 is a power of t. The assertion about $W(\mathbf{c})$ is clear, since $p \mid q^d - \varepsilon(d)$. □

As a further application of Corollary 4.1.4, we provide an explicit example demonstrating that the Bonnafé–Dat–Rouquier Morita equivalence [BDR17] is not a source algebra equivalence in general. It also gives an example for a non-uniserial block $\mathbf{B}$ with $W(\mathbf{B}) \ncong k$.

Example 4.2.2. Assume that $\varepsilon = -1$ and that $p \equiv -1 \pmod 4$. Let d be an odd, positive integer with $d > 1$. Suppose that the order of $-q$ modulo p equals d (given p and d, this is a condition on q). Assume also that $n = pd$.

Let $\mathbf{T}$ denote a Coxeter torus of $\mathbf{G}$ and let D be a Sylow p-subgroup of T. Then $C_{\mathbf{G}}(D) = \mathbf{T}$, as a generator of D has $dp = n$ distinct eigenvalues. Write D_1 for the subgroup of D of order p and put $\mathbf{L} := C_{\mathbf{G}}(D_1)$. Then $L \cong \mathrm{GU}_p(q^d)$.

Now suppose that $s \in T$ is a p'-element with $C_G(s) \cong \mathrm{GU}_d(q^p)$ (the eigenvalues of s span an extension field of $\mathbb{F}_{q^2}$ of degree p). Identify $\mathbf{T}$ with its dual torus $\mathbf{T}^* \leq \mathbf{G}^* = \mathbf{G}$. Let $\theta \in \mathrm{Irr}(T)$ such that the pairs (T, s) and (T, θ) are in duality. Then $[N_G(\mathbf{T}, \theta)\colon T] = d$, and thus is prime to p. (Recall that $N_G(\mathbf{T}, \theta)$ denotes the stabilizer of θ in $N_G(\mathbf{T})$.) As $C_{\mathbf{G}}(D) = \mathbf{T}$, we have $N_G(\mathbf{T}) = N_G(T)$. Let $\mathbf{B}$ denote the p-block of G with defect group D corresponding to the p-block of T determined by θ. Now $s \in T$ is regular in $\mathbf{L}$, and thus there is a block $\mathbf{c}$ of L such that $\mathrm{Irr}(\mathbf{c})$ contains the irreducible Deligne–Lusztig character $\varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{L}}R^{\mathbf{L}}_{\mathbf{T}}(\theta)$; see Lemma 2.5.16. Then $\mathbf{c}$ is a Brauer correspondent of $\mathbf{B}$ and thus $W(\mathbf{B}) \cong W(\mathbf{c})$. By Corollary 4.1.4, where $(G, n, q, \mathbf{B})$ now take the values $(L, p, q^d, \mathbf{c})$, we have $W(\mathbf{c}) \ncong k$.

By the results [FoSr90] of Fong and Srinivasan, the Brauer tree of $\mathbf{B}$ has exactly d edges and is a straight line. In particular, $\mathbf{B}$ is not uniserial.

The Bonnafé–Dat–Rouquier reduction [BDR17] establishes a Morita equivalence between $\mathbf{B}$ and a unipotent block $\mathbf{b}$ of $C_G(s) \cong \mathrm{GU}_d(q^p)$. Once more by [FoSr90], the block $\mathbf{b}$ is the principal block of $C_G(s)$, as d is also equal to the order of q^p modulo p. Hence $W(\mathbf{b}) \cong k$. Thus the Bonnafé–Dat–Rouquier reduction does not preserve the source algebra equivalence class of blocks in general.

A specific instance for the parameters $(d, p, q, |s|))$ as above is given by $(3, 7, 5, 449)$. In this particular case, $W(\mathbf{c}) = W_D([1, 1]) = \Omega_{D/D_1}(k)$, where D has order 7^2. □

Let us discuss an important consequence of the above investigations to the special linear and unitary groups.

Remark 4.2.3. Let $G' = \mathrm{SL}_n^{\varepsilon}(q) \leq G = \mathrm{GL}_n^{\varepsilon}(q)$. Assume that $p \nmid q - \varepsilon$.

(a) Let $\mathbf{B}'$ denote a p-block of G' with a non-trivial cyclic defect group, and let $\mathbf{B}$ be a p-block of G covering $\mathbf{B}'$. As $p \nmid q - \varepsilon$, any defect group of $\mathbf{B}'$ is also a defect group of $\mathbf{B}$ and we have $W(\mathbf{B}') \cong W(\mathbf{B})$ by [HL24, Lemma 4.3]. By Theorem 4.2.1, the invariant $W(\mathbf{B})$ can be computed with Corollary 4.1.4.

(b) Let $\bar{\mathbf{B}}'$ be a cyclic p-block of a central quotient $\bar{G}'$ of G'. By [HL24, Lemma 4.1], there is a block $\mathbf{B}'$ of G' such that $W(\mathbf{B}') \cong W(\bar{\mathbf{B}}')$, and $W(\mathbf{B}')$ can be computed by (a). □

Thus the computation of $W(\bar{\mathbf{B}}')$ for a cylic p-block $\bar{\mathbf{B}}'$ of a central quotient of $\mathrm{SL}_n^{\varepsilon}(q)$ is reduced to the case $p \mid q - \varepsilon$. This will be settled in Part III.

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Gerhard Hiss, Lehrstuhl für Algebra und Zahlentheorie, RWTH Aachen, 52056 Aachen, Germany.
Email address: gerhard.hiss@math.rwth-aachen.de

Caroline Lassueur, Caroline Lassueur, RPTU Kaiserslautern–Landau, Fachbereich Mathematik, 67653 Kaiserslautern, Germany and Leibniz Universität Hannover, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Welfengarten 1, 30167 Hannover, Germany.
Email address: lassueur@mathematik.uni-kl.de