ON THE SOURCE ALGEBRA EQUIVALENCE CLASS OF BLOCKS WITH CYCLIC DEFECT GROUPS, III

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ABSTRACT. This series of papers is a contribution to the program of classifying p-blocks of finite groups up to source algebra equivalence, starting with the case of cyclic blocks. To any p-block $\mathbf{B}$ of a finite group with cyclic defect group D, Linckelmann associated an invariant $W(\mathbf{B})$, which is an indecomposable endo-permutation module over D, and which, together with the Brauer tree of $\mathbf{B}$, essentially determines its source algebra equivalence class.

In Part II of our series, assuming that p is an odd prime, we reduced the classification of the invariants $W(\mathbf{B})$ arising from cyclic p-blocks $\mathbf{B}$ of quasisimple classical groups to the classification for cyclic p-blocks of quasisimple quotients of special linear or unitary groups. This objective is achieved in the present Part III.

1. INTRODUCTION

Let p be an odd prime. The purpose of this article is to determine the invariants $W(\bar{\mathbf{B}})$ for all cyclic p-blocks $\bar{\mathbf{B}}$ of quasisimple groups with simple quotients $\mathrm{PSL}_n^\varepsilon(q)$. As usual, $\varepsilon \in \{-1, 1\}$ and $\mathrm{PSL}_n^\varepsilon(q) = \mathrm{PSL}_n(q)$ if $\varepsilon = 1$, and $\mathrm{PSL}_n^\varepsilon(q) = \mathrm{PSU}_n(q)$ if $\varepsilon = -1$. The analogous convention is used for the groups $\mathrm{SL}_n^\varepsilon(q)$ and $\mathrm{GL}_n^\varepsilon(q)$. We collect the notation used in this manuscript, and in Parts I and II of this series of articles, in form of a glossary in Section 5.

Here is the first main result of our analysis.

Theorem 1.1. *Let $n \geq 2$ be an integer, q a prime power and $G := \mathrm{SL}_n^\varepsilon(q)$. Let p be an odd prime with $p \mid q - \varepsilon$.*

Let $\bar{G}$ denote a central quotient of G and let $\bar{\mathbf{B}}$ be a p-block of $\bar{G}$ with a non-trivial cyclic defect group $\bar{D}$ of order p^l.

(a) *If $G = \mathrm{SL}_n(q)$, then $W(\bar{\mathbf{B}}) \cong k$.*

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(b) *If* $G = \mathrm{SU}_n(q)$ *and* $p \equiv -1 \pmod 4$, *then* $W(\bar{\mathbf{B}}) \cong W_{\bar{D}}(A)$ *for a subset* $A \subseteq \{1, \ldots, l-1\}$ *satisfying one of the following conditions.*

(i) A *is an interval (possibly the empty set);*

(ii) $A = [a, l-1] \setminus \{l-a\}$ *for some* $1 \leq a \leq l/2$*;*

(iii) $A = \{l-a\} \cup [a, l-1]$ *for an integer* $l/2 < a \leq l-1$.

(c) *If* $G = \mathrm{SU}_n(q)$ *and* $p \equiv 1 \pmod 4$, $W(\bar{\mathbf{B}}) \cong W_{\bar{D}}(A)$ *for a subset* $A \subseteq \{1, \ldots, l-1\}$ *with* $|A| \leq 1$. □

A proof of Theorem 1.1 is provided by Propositions 3.27, 3.29 and 3.31, where more precise information is given.

In the notation of the theorem, the cases $p \nmid q - \varepsilon$ have already been treated in [HL24, Proposition 6.3] if $p \mid q$, and in [HL25, Remark 4.2.3] if $p \nmid q$.

In order to prove this theorem, we assume that $\bar{G} = G/Y$ with a non-trivial subgroup $Y \leq Z(G)$, and we let $\bar{\mathbf{B}}$ be a cyclic p-block of $\bar{G}$. To determine $W(\bar{\mathbf{B}})$, we may assume that Y is a p-group. Indeed, $\hat{G} := G/O_p(Y)$ is a central extension of $\bar{G}$ by a p'-group, and if $\hat{\mathbf{B}}$ denotes the p-block of $\hat{G}$ dominating $\bar{\mathbf{B}}$, then $W(\bar{\mathbf{B}}) = W(\hat{\mathbf{B}})$ by [HL24, Lemma 4.1].

Let $\mathbf{B}$ denote the block of G dominating $\bar{\mathbf{B}}$. Then a defect group D of $\mathbf{B}$ is abelian with at most 2 generators, but not necessarily cyclic. Let $\mathbf{c}$ denote a Brauer correspondent of $\mathbf{B}$ in $C_G(D)$. Now embed G into $\tilde{G} = \mathrm{GL}_n^{\varepsilon}(q)$, and let $\tilde{\mathbf{c}}$ denote a block of $C_{\tilde{G}}(D)$ covering $\mathbf{c}$. Then a defect group $\tilde{D}$ of $\tilde{\mathbf{c}}$ is abelian with at most 3 generators. This situation is analyzed in detail in Section 3, which leads to a complete enumeration of the possibilities for $W(\bar{\mathbf{B}})$ arising from the blocks $\bar{\mathbf{B}}$.

In Section 4 we show that all such possibilities listed in Theorem 1.1 arise for suitable choices of q and n, leading to the following result.

Theorem 1.2. *Let* p *be an odd prime,* $l \geq 1$ *an integer and let* $\bar{D}$ *be a cyclic* p*-group of order* p^l.

Let A *be a subset of* $\{1, \ldots, l-1\}$. *If* $p \equiv -1 \pmod 4$, *assume that* A *satisfies one of the conditions* (i)–(iii) *of* Theorem 1.1(b). *If* $p \equiv 1 \pmod 4$, *assume that* $|A| = 1$.

Then there is an integer $n \geq 2$, *a prime power* q *such that* $p \mid q+1$, *a central quotient* $\bar{G}$ *of* $\mathrm{SU}_n(q)$ *and a* p*-block* $\bar{\mathbf{B}}$ *of* $\bar{G}$ *with cyclic defect group isomorphic to* $\bar{D}$, *such that* $W(\bar{\mathbf{B}}) \cong W_{\bar{D}}(A)$. □

A proof of Theorem 1.2 is provided in Propositions 4.2–4.5, which in fact give more precise information.

Remark 1.3. Let p be an odd prime, $l \geq 1$ an integer and let $\bar{D}$ be a cyclic p-group of order p^l.

If $l \geq 2$, the number of intervals in $[1, l-1]$, including the empty one, equals $l(l-1)/2+1$. The number of subsets of $[1, l-1]$ as in Theorem 1.1(b)(ii) or (iii) equals $l-1$, and of these, two are intervals, if $l \geq 3$.

Thus the number of isomorphism classes of endo-permutation $k\bar{D}$-modules of the form $W(\bar{\mathbf{B}})$ arising in Theorem 1.1 equals l, if $p \equiv 1 \pmod 4$, and $l(l+1)/2-2$ if $l \geq 3$ and $p \equiv -1 \pmod 4$.

On the other hand, the number of isomorphism classes of endo-permutation kD-modules equals 2^l since p is odd. □

2. Preliminaries

In this section, G denotes a finite group, and p a prime.

By $r(G)$ we denote the smallest size of a generating set of G. The following standard results on $r(G)$ are stated without proof.

Lemma 2.1. (a) *If $1 \to L \to G \to H \to 1$ is a short exact sequence of finite groups, then $r(G) \leq r(L) + r(H)$.*

(b) *If L and H are finite p-groups, then $r(L \times H) = r(L) + r(H)$.*

(c) *If G is abelian and $H \leq G$, then $r(H) \leq r(G)$.*

We will also need the following supplement to [HL24, Lemma 2.2]. Although this is well known, we include a proof for the convenience of the reader.

Lemma 2.2. *Let $Z \leq Z(G)$ be a p-group and let $t \in G$ be a p-element. Then the index of $C_G(t)/Z$ in $C_{G/Z}(tZ)$ is a p-power. In particular, if $N_G(C_G(t))/C_G(t)$ is a p'-group. Then*

$$C_{G/Z}(tZ) = C_G(t)/Z.$$

Proof. Let $C \leq G$ with $C/Z = C_{G/Z}(tZ)$. Then $C_G(t) \leq C \leq N_G(C_G(t))$, so that the second assertion follows from the first. The map

$$C \to Z, c \mapsto ctc^{-1}t^{-1}$$

is a group homomorphism with kernel $C_G(t)$. This proves our first assertion. □

Finally, we state a result on Brauer pairs needed later on.

Lemma 2.3. *Let $D \leq G$ be an abelian p-subgroup. Let $E \leq D$ and $H := C_G(E)$. Then $D \leq H$. Suppose that $\mathbf{c}$ is a p-block of H with defect group D. Let $\mathbf{B}$ denote the p-block of G such that $(E, \mathbf{c})$ is a $\mathbf{B}$-Brauer pair. Then $\mathbf{B}$ has defect group D.*

Proof. As $E \leq D$, we have $C_G(D) \leq H$ and thus $C_G(D) = C_H(D)$. Let $\mathbf{c}_0$ denote a Brauer correspondent of $\mathbf{c}$ in $C_H(D)$. Then $\mathbf{c}_0$ has defect group D by [Lin18, Corollary 6.3.12]. Hence $(D, \mathbf{c}_0)$ is a maximal $\mathbf{B}$-Brauer pair in G by [AlBr79, Theorem 3.10]. In turn, D is a defect group of $\mathbf{B}$ by [Lin18, Theorem 6.3.7]. □

We adopt a common and useful diction in a Clifford theory situation. Namely, if N is a normal subgroup of G and if χ and ψ are irreducible K-characters of G, respectively N, we say that ψ lies below χ or that χ lies above ψ, if ψ is a constituent of the restriction of χ to N.

3. Analysis

Here, we analyze the relevant configurations, leading to a proof of Theorem 1.1.

3.1. **The groups.** Let us begin by introducing the groups and some corresponding notation, used throughout this section.

In order not to overload the notation, we slightly change the global conventions used in [HL25]. As our focus is on the special linear and unitary groups, $\mathrm{SL}_n^\varepsilon(q)$ is denoted by G, and $\mathrm{GL}_n^\varepsilon(q)$ by $\tilde{G}$; see Notation 3.2 below. Also, D does not necessarily denote a cyclic group, and symbols such as D_1 attain a new significance.

Notation 3.2. (i) Let $\varepsilon \in \{-1, 1\}$ and $n \geq 2$ be integers, p an odd prime and q a power of a prime r with $p \mid q - \varepsilon$. Let $\mathbb{F}$ denote an algebraic closure of the finite field with r elements.

(ii) Put $\tilde{\mathbf{G}} := \mathrm{GL}_n(\mathbb{F})$ and $\mathbf{G} := \{g \in \tilde{\mathbf{G}} \mid \det(g) = 1\}$. Let $\mathbf{V} := \mathbb{F}^n$, the natural vector space for $\tilde{\mathbf{G}}$.

(iii) Let $F := F_\varepsilon$ denote a Steinberg morphism of $\tilde{\mathbf{G}}$ such that $\tilde{G} = \tilde{\mathbf{G}}^F = \mathrm{GL}_n^\varepsilon(q)$. Then $\mathbf{G}$ is F-stable and $G = \mathbf{G}^F = \mathrm{SL}_n^\varepsilon(q)$. Put $Z := Z(G)$. Notice that $|Z| = \gcd(q - \varepsilon, n)$.

(iv) Let $\delta = 1$ if $\varepsilon = 1$, and $\delta = 2$, if $\varepsilon = -1$, and put $V := \mathbb{F}_{q^\delta}^n \subseteq \mathbf{V}$. Then V is the natural vector space for $\tilde{G}$. (This is consistent with the notation introduced in [HL25, Subsection 3.1].)

(v) Let p^a and p^b denote the highest powers of p dividing $q - \varepsilon$, respectively n, and put $c := \min\{a, b\}$. Then p^c is the highest power of p dividing $|Z|$. □

3.3. **Preliminaries on $\tilde{G}$.** Before we continue, we record two results on $\tilde{G}$. The first of these is purely group theoretical in nature.

Lemma 3.4. (a) *Let* $\tilde{\mathbf{H}} \le \tilde{\mathbf{G}}$ *be a regular subgroup. Then* $\mathbf{H} := \mathbf{G} \cap \tilde{\mathbf{H}}$ *is a regular subgroup of* $\mathbf{G}$ *and*

$$[\tilde{H} : H] = q - \varepsilon.$$

(b) *Let* $s \in \tilde{G}$ *be semisimple. Then*

$$[C_{\tilde{G}}(s) : C_G(s)] = q - \varepsilon.$$

In particular, if $s \in G$*, the* G*-conjugacy class of* s *is a* $\tilde{G}$*-conjugacy class.*

Proof. (a) The inclusion $\mathbf{G} \to \tilde{\mathbf{G}}$ is a regular embedding; see [GeMa20, Definition 1.7.1]. Moreover, $C_{\tilde{\mathbf{G}}}(s)$ is a regular subgroup of $\tilde{\mathbf{G}}$. The claims follow from [HL25, Lemma 2.5.3]. □

For the following result recall the notion $r(H)$, introduced in Subsection 2 for a finite group H.

Lemma 3.5. *Let* $\tilde{D}$ *be an abelian defect group of some* p*-block of* $\tilde{G}$ *with* $r(\tilde{D}) = 2$*. Then* $p^{2a} \mid |\tilde{D}|$*.*

Proof. This follows from [FoSr82, Theorem (3C)]. □

3.6. **The blocks and their defect groups.** We introduce the principal object of our study and set up further notation.

Notation 3.7. (i) Let $Y \le Z$ be a p-group, put $\bar{G} := G/Y$, and write $\bar{\ } : G \to \bar{G}$ for the natural epimorphism.

(ii) Let $\bar{\mathbf{B}}$ denote a p-block of $\bar{G}$ with a non-trivial cyclic defect group.

(iii) Let $\mathbf{B}$ denote the p-block of G dominating $\bar{\mathbf{B}}$ and let D be a defect group of $\mathbf{B}$. (Then $\bar{D} = D/Y$ is a defect group of $\bar{\mathbf{B}}$ by [HL25, Lemma 2.4.1].)

(iv) Choose $t \in D$ with $\bar{D} = \langle \bar{t} \rangle$.

(v) Define the non-negative integer c' by $p^{c'} = |\langle t \rangle \cap Y|$. □

Notice that, as Y is a p-group, for any block $\mathbf{B}$ of G there is a unique block $\bar{\mathbf{B}}$ dominated by $\mathbf{B}$; see [NT89, Theorem 5.8.11]. Let us record some easy observations.

Lemma 3.8. (a) *We have* $Y \le Z(\tilde{G})$*.*

(b) *The kernel of the natural epimorphism* $\langle t \rangle \to \bar{D}$ *equals* $\langle t \rangle \cap Y$*, so that* $|\bar{D}| = |t|/p^{c'}$*.*

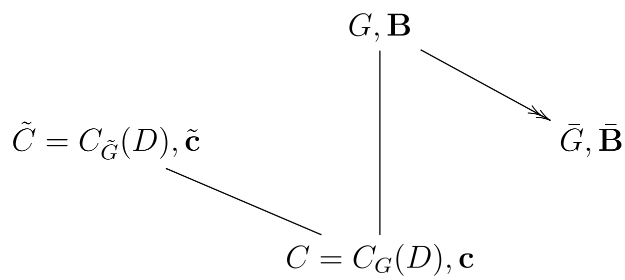
(c) *We have* $O_p(Z) = Z \cap D$*. Moreover,* $D = \langle t, Y \rangle = \langle t, O_p(Z) \rangle$*. In particular,* $r(D) \le 2$*.*

(d) *If* $\langle t \rangle \cap O_p(Z) = \{1\}$*, then* $Y = O_p(Z)$*.*

(e) *If* $|D| = |t| p^a$ *then* $\langle t \rangle \cap O_p(Z) = \{1\}$ *and* $p^a \mid n$*.*

(f) *If* D *is cyclic, then* $D = \langle t \rangle$*.*

FIGURE 1. Some subgroups and blocks of $\mathrm{GL}_n^{\varepsilon}(q)$, I



Proof. (a) and (b) are trivial.

(c) Since D is a defect group, we have $O_p(Z) \leq D$, implying the first assertion. The other assertions follow from $\bar{D} = \langle \bar{t} \rangle$ and $Y \leq O_p(Z)$.

(d) By (c) and the assumption, $D = \langle t \rangle \times O_p(Z)$. Since $Y \leq O_p(Z)$, the quotient D/Y can only be cyclic if $Y = O_p(Z)$.

(e) We have $|t|p^a = |D| \leq |t||Y| \leq |t||O_p(Z)| \leq |t|p^a$, implying the claims.

(f) If D is cyclic and $\bar{D}$ is non-trivial, the image of a proper subgroup of D is a proper subgroup of $\bar{D}$. As the image of $\langle t \rangle$ equals $\bar{D}$, we obtain our claim. □

3.9. **The local configuration.** Let us have a look at the local situation. We begin by introducing further notation.

Notation 3.10. (i) Put $C := C_G(D)$ and $\tilde{C} := C_{\tilde{G}}(D)$. Then $C \trianglelefteq \tilde{C}$.

(ii) Let $\mathbf{c}$ be a Brauer correspondent of $\mathbf{B}$ in C. Then D is a defect group of $\mathbf{c}$.

(iii) Let $\tilde{\mathbf{c}}$ be a block of $\tilde{C}$ covering $\mathbf{c}$ and let $\tilde{D}$ be a defect group of $\tilde{\mathbf{c}}$ with $D = C \cap \tilde{D}$. □

Notice that $C = C_G(t)$ and $\tilde{C} = C_{\tilde{G}}(t)$, as $O_p(Z) \leq Z(\tilde{G})$ and $D = \langle t, O_p(Z) \rangle$. The groups and blocks in question are displayed in the diagram of Figure 1. The invariant $W(\bar{\mathbf{B}})$ will be computed with the help of $\tilde{\mathbf{c}}$.

Lemma 3.11. *The following statements hold.*

(a) *The group $\tilde{C}/C$ is cyclic of order $q - \varepsilon$.*

(b) *The group $\tilde{D}/D$ is cyclic of order dividing p^a. In particular, $\tilde{D}$ is abelian and $r(\tilde{D}) \leq r(D) + 1 \leq 3$.*

(c) *We have $D = G \cap \tilde{D}$ and $D = G \cap O_p(Z(\tilde{C}))$.*

(d) *If $\tilde{D}$ is the Sylow p-subgroup of a maximal torus of $\tilde{G}$, then $|\tilde{D}| = |D|p^a$.*

Proof. (a) This follows from Lemma 3.4(a).

(b) We have $\tilde{D}/D = \tilde{D}/(C \cap \tilde{D}) \cong \tilde{D}C/C \leq \tilde{C}/C$, so that the first claim is a consequence of (a). In particular, $\tilde{D}$ is abelian, as $D \leq Z(\tilde{D})$. The next claim follows from the first and Lemma 2.1(a).

(c) The first assertion is clear, as $C = G \cap \tilde{C}$.

To prove the second assertion, we first show that $D \leq G \cap O_p(Z(\tilde{C}))$. As $D \leq G$, it suffices to show that $D \leq O_p(Z(\tilde{C}))$. Now D is abelian and so $D \leq C_{\tilde{G}}(D) = \tilde{C}$ and $[D, \tilde{C}] = \{1\}$. Hence $D \leq Z(\tilde{C})$, and as D is a p-group, we obtain $D \leq O_p(Z(\tilde{C}))$. To prove the reverse inclusion, first observe that $G \cap Z(\tilde{C}) \leq Z(C)$. As $G \cap O_p(Z(\tilde{C}))$ is a p-group, this implies $G \cap O_p(Z(\tilde{C})) \leq O_p(Z(C))$. Now $O_p(Z(C)) = D$ by [HL24, Lemma 2.1], and our proof is complete.

(d) Assume that $\tilde{D}$ is the Sylow p-subgroup of the maximal torus $\tilde{T}$ of $\tilde{G}$. Then $D = G \cap \tilde{D} = G \cap \tilde{T} \cap \tilde{D}$ is the Sylow p-subgroup of the maximal torus $T := G \cap \tilde{T}$ of G. Since $[\tilde{T} : T] = q - \varepsilon$ by Lemma 3.4(a), the claim follows. □

3.12. **Analyzing the local configuration.** Recall from Notation 3.2 that V is the natural n-dimensional $\mathbb{F}_{q^\delta}$-vector space of $\tilde{G}$. In what follows, we will make use of the corresponding notation introduced in [HL25, Subsections 3.1, 3.2, 3.4].

Lemma 3.13. *The minimal polynomial of t has at most three irreducible factors.*

Proof. Let h denote the number of irreducible factors of the minimal polynomial of t acting on V. By the primary decomposition of V with respect to t, we get $\tilde{C} = C_{\tilde{G}}(t) = \tilde{C}_1 \times \cdots \times \tilde{C}_h$, where each $\tilde{C}_i$ is a general linear or unitary group, possibly over an extension field of $\mathbb{F}_{q^\delta}$. As $p \mid q - \varepsilon$, we have $p \mid |Z(\tilde{C}_i)|$ for each $1 \leq i \leq h$.

By Lemma 2.1(b) we have $r(O_p(Z(\tilde{C})) = h$. As $\tilde{D}$ is a defect group of some p-block of $\tilde{C}$, we have $O_p(Z(\tilde{C})) \leq \tilde{D}$. It follows that $h = r(O_p(Z(\tilde{C})) \leq r(\tilde{D}) \leq 3$, the first inequality arising from Lemma 2.1(c), the second one from Lemma 3.11(b). □

Notation 3.14. (i) Let us write $\Delta_1, \ldots, \Delta_h$ with $h \in \{1, 2, 3\}$ for the monic, irreducible factors of the minimal polynomial of t.

(ii) Fix $j \in \{1, \ldots, h\}$. Let $\xi_j \in \mathbb{F}$ denote a root of Δ_j. Let d_j denote the degree of Δ_j, and n'_j its multiplicity in the characteristic polynomial of t. Write $n'_j = m_j p^{b_j}$ with non-negative integers b_j, m_j and $p \nmid m_j$.

(iii) We choose the notation in such a way that $d_1 \geq \cdots \geq d_h$. If $d_1 = 1$ and $h \geq 2$, we assume $|\xi_j| \geq |\xi_{j+1}|$ for $1 \leq j < h$.

(iv) Put $V_j := \ker(\Delta_j(t))$. Then V_j is $\tilde{C}$-invariant. Let t_j denote the restriction of t to V_j. (Depending on the context, we view t_j as an automorphism of V_j or of V.) Then the minimal polynomial of t_j acting on V_j equals Δ_j. Moreover, $\dim_{\mathbb{F}_{q^\delta}}(V_j) = n_j$ with $n_j := n'_j d_j = m_j d_j p^{b_j}$. □

We record some properties of the quantities introduced above.

Lemma 3.15. *Fix j with $1 \leq j \leq h$. Then the following statements hold.*

(a) *We have $d_j = p^{a_j}$ with a non-negative integer a_j; in particular, $n_j = m_j p^{a_j}$ if $b_j = 0$. Also, $|\xi_j| = p^{a+a_j}$, if $a_j > 0$, and $|\xi_j| \mid p^a$, if $a_j = 0$.*

(b) *If $\varepsilon = -1$, we have $\Delta_j = \Delta_j^\dagger$, i.e. ξ_j^{-q} is a root of Δ_j.*

(c) *The highest power of p dividing $(q^{\delta d_j} - 1)/(q^\delta - 1)$ equals p^{a_j}, and we define the positive integer m'_j by $(q^{\delta d_j} - 1)/(q^\delta - 1) = m'_j p^{a_j}$; then $p \nmid m'_j$.*

(d) *We have $\det(t_j) = \xi_j^{m_j m'_j p^{a_j + b_j}}$. In particular, $|\det(t_j)| = p^{a-b_j}$ if $a_j > 0$ and $b_j \leq a$.*

(e) *Let κ_j denote the restriction of κ to V_j. Then κ_j is a non-degenerate hermitian form if $\varepsilon = -1$. We view $I(V_j, \kappa_j)$ as a subgroup of $\tilde{G}$ in the natural way. Put $\tilde{G}_j := I(V_j, \kappa_j)$ and $\tilde{C}_j := C_{\tilde{G}_j}(t_j)$. Then $\tilde{G}_j \cong \mathrm{GL}^\varepsilon_{n_j}(q)$ and $\tilde{C}_j \cong \mathrm{GL}^\varepsilon_{n'_j}(q^{p^{a_j}})$.*

In particular, $Z(\tilde{C}_j)$ is cyclic and $|Z(\tilde{C}_j)|_p = p^{a+a_j}$.

Proof. (a) See [HL25, Lemma 4.1.1(c)].
(b) This is [HL25, Lemma 3.2.2].
(c) This is standard.
(d) See [HL25, Lemma 4.1.1(e)].
(e) The fact that κ_j is non-degenerate if $\varepsilon = -1$ follows from (b). □

Recall from Notation 3.10(iii) that $\tilde{D}$ denotes a defect group of the block $\tilde{\mathbf{c}}$ of $\tilde{C} = \tilde{C}_1 \times \cdots \times \tilde{C}_h$. There is a decomposition of $\mathbf{c}$ into a tensor products of blocks of $\tilde{C}_j$ with defect groups $\tilde{D}_j \leq \tilde{C}_j$ for $1 \leq j \leq h$ such that $\tilde{D} \cong \tilde{D}_1 \times \cdots \times \tilde{D}_h$.

Lemma 3.16. *Assume the terminology introduced in* Notations 3.2, 3.7, 3.10, 3.14 *and* Lemma 3.15*.*

(a) *We have $|t| \leq p^{a+a_1}$ with equality if $a_1 > 0$.*

(b) *We have*

$$|\tilde{D}| \leq |D| p^a \leq |t| p^{2a} \leq p^{3a+a_1}. \tag{1}$$

If $|\tilde{D}| = p^{3a+a_1}$, then $|D| = |t| p^a$ and $|t| = p^{a+a_1}$ (even if $a_1 = 0$).

(c) *Suppose that the minimal polynomial of* t *is irreducible and that* $D \not\leq Z$. *Then* $\tilde{D}$ *is cyclic, so that* D *is cyclic, and* $\mathbf{c}$ *is covered by a cyclic block. Moreover,* $a_1 > 0$ *and* $b_1 = a$. *In particular,* $n = m_1 p^{a+a_1}$ *with* $p \nmid m_1$.

(d) *Suppose that the minimal polynomial of* t *has exactly two irreducible factors. Then* $\tilde{D}_1$ *and* $\tilde{D}_2$ *are cyclic and* $b_1 = b_2 = 0$. *Moreover, the following statements hold.*

(i) If $a_1 > a_2$, *then* $\langle t \rangle \cap O_p(Z) = \{1\}$ *and* $Y = O_p(Z)$. *In particular,* $c' = 0$. *Moreover,* $a_2 = c$ *and* $|D| = p^{a+a_1+c}$.

(ii) If $a_1 = a_2 > 0$, *then* $|D| = p^{a+2a_1}$, $c' = c - a_1 < a$ *and* $Y = O_p(Z)$.

(iii) If $a_1 = a_2 = 0$, *then* D *is cyclic of order* p^a. *In particular,* $D = \langle t \rangle$ *and* $|t| = p^a$.

(iv) If $a_2 = 0$, *then* D *is cyclic.*

(v) We have $c \geq a_2$.

(vi) Suppose that $a_1 = a_2 = 0$ *and that* $c' < c$. *Then* $n_1 \neq n_2$.

(vii) We have $|t_j| = p^{a+a_j}$ *for* $j = 1, 2$.

(e) *Suppose that the minimal polynomial of* t *has exactly three irreducible factors. Then* $b_1 = a_2 = b_2 = a_3 = b_3 = 0$ *and* $|\tilde{D}| = p^{3a+a_1}$. *In particular,* $p^a \mid n$, $Y = O_p(Z)$ *and* $c' = 0$. *Thus* $|\bar{D}| = |t| = p^{a+a_1}$.

If $a_1 > 0$, *then* $n_2 \neq n_3$ *and* $N_{\tilde{G}}(\tilde{C}) = N_{\tilde{G}_1}(\tilde{C}_1) \times \tilde{G}_2 \times \tilde{G}_3$.

(f) *In all cases,* $|\tilde{D}| = |D|p^a$, $|t| = p^{a+a_1}$ *and* $|\bar{D}| = p^{a+a_1-c'}$.

(g) *We have* $C_{\tilde{G}}(\tilde{D}) \leq C_{\tilde{G}}(D)$ *and* $N_{\tilde{G}}(\tilde{D}) \leq N_{\tilde{G}}(D)$, *with equality in either instance, if the minimal polynomial of* t *is reducible.*

Proof. (a) Lemma 3.15(a) implies that $|t_1| \mid p^{a+a_1}$ with equality if $a_1 > 0$, and $|t_j| \mid p^{a+a_1}$ for $1 \leq j \leq h$.

(b) The first inequality of (1) follows from Lemma 3.11(b), the second one from $D = \langle t, O_p(Z) \rangle$ and $|O_p(Z)| = p^c \leq p^a$, and the last one from (a).

The last assertions are clear from (1).

(c) Here, $D = G \cap O_p(Z(\tilde{C}_1))$. If $a_1 = 0$, then $\tilde{C}_1 = \tilde{G}$ and $D \leq Z$, a case we have excluded. Thus $a_1 > 0$ and D is cyclic of order p^{a+a_1} by (a) and Lemma 3.8(f). If $\tilde{D}_1$ is not cyclic, then $p^{a+a_1} = |D| \geq |\tilde{D}|/p^a \geq p^{a+2a_1}$, where the latter estimate follows from Lemma 3.5, applied to $\tilde{C}_1$. This contradiction shows that $\tilde{D}$ is cyclic. In particular, $|\tilde{D}| = p^{a+a_1+b_1}$, and $\tilde{D}$ is the Sylow p-subgroup of a maximal torus of $\tilde{G}$; see [HL25, Corollary 3.6.2]. Lemma 3.11(d) implies that $p^{a+a_1} = |D| = |\tilde{D}|/p^a = p^{a_1+b_1}$, and thus $b_1 = a$ as claimed.

(d) Recall that $3 \geq r(\tilde{D}) = r(\tilde{D}_1) + r(\tilde{D}_2)$. If $\tilde{D}_j$ is non-cyclic, then $|\tilde{D}_j| \geq p^{2(a+a_j)}$ by Lemma 3.5, applied to $\tilde{C}_j$ for $j = 1, 2$.

Suppose first that $\tilde{D}_1$ is non-cyclic. Then $\tilde{D}_2$ is cyclic, and, using (1) and [HL25, Corollary 3.6.2], we get

$$p^{3a+a_1} \geq |\tilde{D}| \geq p^{2(a+a_1)+a+a_2+b_2} = p^{3a+2a_1+a_2+b_2},$$

which implies $a_1 = a_2 = b_2 = 0$ and also that $|\tilde{D}| = p^{3a}$. Now $D = G \cap O_p(Z(\tilde{C}))$. Since $a_1 = a_2 = 0$, we have $|O_p(Z(\tilde{C}))| = p^{2a}$. Clearly, $O_p(Z(\tilde{C})) \not\leq G$ since $b_2 = 0$, i.e. $p \nmid n_2$. Hence $|D| < p^{2a}$, contradicting $|D| \geq |\tilde{D}|/p^a$.

Suppose then that $\tilde{D}_2$ is non-cyclic. Then

$$p^{3a+a_1} \geq |\tilde{D}| \geq p^{a+a_1+b_1+2(a+a_2)} = p^{3a+a_1+b_1+2a_2},$$

which implies $b_1 = a_2 = 0$ and also that $|\tilde{D}| = p^{3a+a_1}$. If $a_1 = 0$, an analogous argument as above leads to a contradiction. So assume that $a_1 > 0$ in the following. By (b) and Lemma 3.8(e), we obtain $p^a \mid n$. We claim that $b_2 = 0$, i.e. $p \nmid n_2'$. Indeed, $|\det(t_1)| = p^a$ by Lemma 3.15(d), and $\det(t_2) = \xi_2^{m_2 m_2' p^{b_2}} = \xi_2^{n_2' m_2'}$, as $a_2 = 0$. From $\det(t_1)\det(t_2) = \det(t) = 1$, we conclude $p^a = |\det(t_1)| = |\det(t_2)|$, and thus $p \nmid n_2'$, which is our claim. Now $n = n_1' p^{a_1} + n_2'$ and $p \nmid n_2'$. As $a_1 > 0$, we conclude that $p \nmid n$, a contradiction.

We next show that $b_1 = b_2 = 0$, assuming first that $a_1 > 0$. Then $|t| = p^{a+a_1}$ by (a). Now $D = \langle t, O_p(Z)\rangle$ and $|O_p(Z)| = p^c$, so that $|D| \leq |t||O_p(Z)| = p^{a+a_1+c}$. Since $\tilde{D}_1$ and $\tilde{D}_2$ are cyclic, we get $|\tilde{D}| = p^{2a+a_1+b_1+a_2+b_2}$. As $\tilde{D}$ is a Sylow p-subgroup of $\tilde{G}$ by [HL25, Corollary 3.6.2], Lemma 3.11(d) implies that

$$|D| = p^{a+a_1+b_1+a_2+b_2}, \tag{2}$$

and thus

$$c \geq b_1 + a_2 + b_2. \tag{3}$$

Since $a \geq c \geq b_1 + a_2 + b_2$, we obtain $a \geq b_j$ for $j = 1, 2$. If also $a_2 > 0$, we have $|\det(t_j)| = q^{a-b_j}$ for $j = 1, 2$ by Lemma 3.15(d). From $|\det(t_1)| = |\det(t_2)|$, we obtain $b_1 = b_2$, if $a_1, a_2 > 0$.

Assume now that $a_1 > a_2 > 0$. Then $n = m_1 p^{a_1+b_1} + m_2 p^{a_2+b_2}$. As $a_1 + b_1 > a_2 + b_1 = a_2 + b_2$, we obtain $a_2 + b_2 = b \geq c \geq b_1 + a_2 + b_2$. It follows that $b_2 = b_1 = 0$.

Next assume that $a_1 > a_2 = 0$. Then $|\det(t_1)| = p^{a-b_1}$ by Lemma 3.15(d). Since $\det(t_2) = \xi_2^{m_2 m_2' p^{b_2}}$ and $|\xi_2| \mid p^a$, it follows that $b_2 = 0$ if $b_1 = 0$. Now $n = m_1 p^{a_1+b_1} + m_2 p^{b_2}$. If $a_1 + b_1 > b_2$, then $b_2 = b \geq c \geq b_1 + b_2$ and hence $b_1 = 0$. As noticed above, this implies $b_2 = 0$. Otherwise, $a_1 + b_1 \leq b_2$. If $a = b_1$, we obtain $b_2 = 0$ from $c \geq b_1 + b_2 = a + b_2 \geq a \geq c$. But then $p \nmid n$, contradicting $c \geq$

$b_1 + b_2 = a > 0$. Hence $a \neq b_1$, i.e. $|\det(t_1)| = p^{a-b_1} > 1$. Writing $|\xi_2| = p^e$, we obtain $a - b_1 = e - b_2$ from $|\det(t_1)| = |\det(t_2)|$. It follows that $a - b_1 = e - b_2 \leq e - a_1 - b_1$, and so $e \geq a + a_1 > a$, a contradiction.

Finally assume that $a_1 = a_2 > 0$. As $D = G \cap O_p(Z(\tilde{C}))$ and $O_p(Z(\tilde{C})) = \langle t_1 \rangle \times \langle t_2 \rangle$ we get $D \cong \{(t_1^i, t_2^j) \mid \det(t_1^j) = \det(t_2^i)^{-1}\}$. Thus $|D| = p^{a+a_1+a_2+b_1}$, as the kernel of the map $\det\colon \langle t_2 \rangle \to \mathbb{F}^*$ has order $p^{a_2+b_2}$. From $p^{a+a_1+a_2+b_2} = |D| \geq |\tilde{D}|/p^a = p^{a+a_1+b_1+a_2+b_2}$ we obtain $b_1 = 0$, and thus $b_2 = 0$.

Let us now assume that $a_1 = a_2 = 0$. Simultaneously to the proof hat of $b_1 = b_2 = 0$, we also prove statement (iii). Choose the notation such that $b_1 \geq b_2$. Here, $O_p(Z(\tilde{C})) = \langle t_1' \rangle \times \langle t_2' \rangle$ with $|t_j'| = p^a$ for $j = 1, 2$. Since $\det(t_1')$ and $\det(t_2')$ lie in the same subgroup of $\mathbb{F}^*$ and since $b_1 \geq b_2$, we get $|D| = |G \cap O_p(Z(\tilde{C}))| = p^{a+b_2}$. On the other hand $|D| \geq |\tilde{D}|/p^a = p^{a+b_1+b_2}$, which implies that $b_1 = b_2 = 0$. From $p \nmid n_1 n_2$, we find $D = G \cap O_p(Z(\tilde{C})) \cong \langle t_1' \rangle$. This yields (iii).

Let us finally prove the statements (i)–(vii).

We have already proved (iii). Clearly, (v) follows from (i) and (ii). By (iii), it suffices to prove (iv) in the situation of (i). In that case, $c = a_2 = 0$, so that $p \nmid |Z|$. Hence Y is trivial and so $D = \bar{D}$ is cyclic. Let us now prove (i), (ii). In this situation, $|t| = p^{a+a_1}$ and $|D| = p^{a+a_1+a_2}$ by (2). Moreover, $a_2 \leq c$ by (3). Assume now that $a_1 > a_2$. As $n = m_1 p^{a_1} + m_2 p^{a_2}$ with $p \nmid m_1 m_2$ and $a_1 > a_2$, we get $b = a_2$. Since $c \leq b = a_2 \leq c$, we find $a_2 = c$. It follows that

$$p^{a+a_1-c'} = |\bar{D}| = |D/Y| \geq |D/O_p(Z)| = p^{a+a_1} \geq p^{a+a_1-c'}.$$

Hence $c' = 0$, $Y = O_p(Z)$ and $\langle t \rangle \cap O_p(Z) = \{1\}$. This proves (i). Now assume that $a_1 = a_2 > 0$. Here, $|D| = p^{a+2a_1}$ by (2). Since $|t| = p^{a+a_1}$ and $|\langle t \rangle \cap Y| = p^{c'}$ by definition, we get

$$p^{a+a_1-c'} = |\bar{D}| = |D/Y|.$$

In particular, $|Y| = p^{a_1+c'}$. Moreover, $|\langle t \rangle \cap O_p(Z)| = p^{c-a_1}$, since $D = \langle t, O_p(Z) \rangle$. Assume that $|Y| \leq p^{c-a_1}$. Then $Y \leq \langle t \rangle \cap O_p(Z)$, which implies $Y = \langle t \rangle \cap Y$. It follows that

$$p^{a_1+c'} = |Y| = |\langle t \rangle \cap Y| = p^{c'},$$

which implies $a_1 = 0$, a contradiction. Hence $|Y| > p^{c-a_1}$. Then $\langle t \rangle \cap O_p(Z) \leq Y$, which implies $\langle t \rangle \cap Y = \langle t \rangle \cap O_p(Z)$. It follows that $p^{c'} = p^{c-a_1}$, and so $c' = c - a_1$. Moreover,

$$|Y| = p^{a_1+c'} = p^{a_1+c-a_1} = p^c,$$

and thus $Y = O_p(Z)$. This completes the proof of (ii). To prove (vi), assume that $n_1 = n_2$. Then $n = 2n_1$ and $p \nmid n_1$. Hence $c = 0$, contradicting $0 \leq c' < c$. If $a_j > 0$ for $j = 1, 2$, then $|t_j| = p^{a+a_j}$ by Lemma 3.15(a). If $a_1 > a_2 = 0$, then $|\det(t_1)| = p^a$ by Lemma 3.15(d) and thus $|t_2| = p^a$, since $\det(t_1 t_2) = 1$. If $a_1 = a_2 = 0$, then $t = t_1 t_2 = t_2 t_1$, $|t| = p^a$ and $\det(t_1 t_2) = 1$ imply $|t_1| = |t_2| = p^a$. This proves (vii).

(e) Since $r(\tilde{D}) \leq 3$ by Lemma 3.11(b), the factors $\tilde{D}_j$ are cyclic for $1 \leq j \leq 3$. By [HL25, Corollary 3.6.2], we have $|\tilde{D}_j| = p^{a+a_j+b_j}$ for $1 \leq j \leq 3$. The first two claims follow from (1). For the consequences of these consult (b) and Lemma 3.8(d) and (e).

Let us now prove the final claim. We have $n = m_1 p^{a_1} + n_2 + n_3$ and $p \nmid n_2 n_3$. Since $p^a \mid n$ and $a > 0$, we also have $p \mid n_2 + n_3$ since $a_1 > 0$. This implies $n_2 \neq n_3$ since p is odd. Now $\tilde{C} = C_{\tilde{G}}(t)$, and thus $N_{\tilde{G}}(\tilde{C})$ permutes the spaces V_1, V_2 and V_3. Since these have pairwise distinct dimensions, $N_{\tilde{G}}(\tilde{C})$ fixes all of these, which gives our claim.

(f) This is contained in (c)–(e).

(g) Suppose first that the minimal polynomial of t is irreducible. Then $\tilde{D}$ is cyclic by (c), and our claims hold.

Suppose next that the minimal polynomial of t has two irreducible factors. Then, $b_1 = b_2 = 0$ by (d). Thus $|\tilde{D}| = p^{2a+a_1+a_2} = |O_p(Z(\tilde{C}))|$ and so $\tilde{D} = O_p(Z(\tilde{C})) \leq Z(\tilde{C})$. Hence $\tilde{C} \leq C_{\tilde{G}}(\tilde{D}) \leq C_{\tilde{G}}(D) = \tilde{C}$. Since $D = G \cap \tilde{D}$, we obtain $N_{\tilde{G}}(\tilde{D}) \leq N_{\tilde{G}}(D) \leq N_{\tilde{G}}(C_{\tilde{G}}(D)) = N_{\tilde{G}}(\tilde{C}) \leq N_{\tilde{G}}(O_p(Z(\tilde{C}))) = N_{\tilde{G}}(\tilde{D})$. The case when the minimal polynomial of t has three irreducible factors is treated analogously. □

3.17. **The intermediate configuration.** Let us write $h \in \{1, 2, 3\}$ for the number of irreducible factors of the minimal polynomial of t. If $h = 1$, let $\bar{E}$ denote the trivial subgroup of $\bar{D}$; otherwise, let $\bar{E}$ be the unique subgroup of $\bar{D}$ of order p.

Let $\bar{\mathbf{d}}$ denote a Brauer correspondent of $\bar{\mathbf{B}}$ in $C_{\bar{G}}(\bar{E})$. Then $W(\bar{\mathbf{B}}) = W(\bar{\mathbf{d}})$. The latter will be computed by pulling back $(\bar{E}, \bar{\mathbf{d}})$ to G.

Notation 3.18. Assume the terminology of Notation 3.7.

(i) Put $E := \langle t', Y \rangle$ with $t' = 1$, if $h = 1$, and $t' := t^{p^{a+a_1-c'-1}}$, otherwise. (As we have assumed that $\bar{D}$ is non-trivial and $|\bar{D}| = p^{a+a_1-c'}$ by Lemma 3.16(f), we have $a + a_1 - c' \geq 1$.)

(ii) Let $\tilde{\mathbf{H}} := C_{\tilde{\mathbf{G}}}(E) = C_{\tilde{\mathbf{G}}}(t')$ and $\mathbf{H} := \tilde{\mathbf{H}} \cap \mathbf{G} = C_{\mathbf{G}}(E) = C_{\mathbf{G}}(t')$. □

Notice that $\tilde{\mathbf{H}} = \tilde{\mathbf{G}}$ and $\mathbf{H} = \mathbf{G}$ if the minimal polynomial of t is irreducible. Notice also that the inclusion $\mathbf{H} \to \tilde{\mathbf{H}}$ is a regular embedding, and that $\tilde{\mathbf{H}}$ and $\mathbf{H}$ are F-stable.

We collect a few properties of the objects introduced above, assuming the terminology of Notations 3.7, 3.10, 3.14, 3.18 and Lemma 3.15(e). Also, recall that $D = \langle t, O_p(Z)\rangle$ by Lemma 3.8(c), and that $|t| = p^{a+a_1}$ by Lemma 3.16(f).

Lemma 3.19. *Assume that $h \geq 2$. For Parts* (e) *and* (f) *assume in addition that $|\{n_1, n_2, n_3\}| \geq 2$ if $h = 3$ and $a_1 = 0$. Then the following statements hold.*

(a) *We have $|t'| = p^{c'+1}$.*

(b) *We have $N_{\tilde{G}}(D) \leq N_{\tilde{G}}(E)$ and $N_G(D) \leq N_G(E)$.*

(c) *Suppose that $h = 2$. Then $\tilde{H} = \tilde{G}_1 \times \tilde{G}_2$ unless $a_1 = a_2 = 0$ and $c' < c$. In the latter case, $\tilde{H} = \tilde{G}$.*

(d) *Suppose that $h = 3$. If $a_1 = 0$, then $\tilde{H} = \tilde{G}_1 \times \tilde{G}_2 \times \tilde{G}_3$.*

If $a_1 > 0$, then $\tilde{H} = \tilde{G}_1 \times \tilde{G}_{2,3}$ with $\tilde{G}_{2,3} := I(V_2 \oplus V_3, \kappa_{2,3})$, where $\kappa_{2,3}$ denotes the restriction of κ to $V_2 \oplus V_3$.

(e) *We have $[N_G(E)\colon H] = [N_{\tilde{G}}(E)\colon \tilde{H}] \leq 2$.*

(f) *We have $\bar{H} = C_{\bar{G}}(\bar{E})$.*

Proof. An element $\xi \in \mathbb{F}^*$ is called rational, if $\xi \in \mathbb{F}_{q^\delta}$. If ξ is a p-element, this is the case if and only if $|\xi| \leq p^a$.

(a) This follows from $|t| = p^{a+a_1}$.

(b) Notice that Y is a central subgroup of $\tilde{G}$ so that $Y \leq N_{\tilde{G}}(D)$ and $Y \leq N_{\tilde{G}}(E)$. Hence

$$N_{\tilde{G}}(D)/Y = N_{\tilde{G}/Y}(D/Y) \leq N_{\tilde{G}/Y}(E/Y) = N_{\tilde{G}}(E)/Y,$$

where the equalities arise from [HL24, Lemma 2.2(a)], and the inclusion is due to the fact that D/Y is cyclic. Our assertions follow from this.

(c) Suppose first that $a_1 > a_2$, so that Lemma 3.16(d)(i) applies. As $c' = 0$, we have $|t'| = p \leq p^a$, so that the eigenvalues of t' on V_j are rational for $j = 1, 2$. As $\langle t'\rangle \cap O_p(Z) = \{1\}$, the eigenvalues of t' on V_1 and V_2 are distinct.

Now suppose that $a_1 = a_2$, so that Lemma 3.16(d)(ii)(iii) applies. Hence $|D| = p^{a+2a_1}$. From $D = \langle t, O_p(Z)\rangle$ and $|t| = p^{a+a_1}$ we conclude that $|\langle t\rangle \cap O_p(Z)| = p^{c-a_1}$. By (a), we have $t' \in Z \leq Z(\tilde{G})$, if and only if $c' < c - a_1$. The latter can only happen if $a_1 > 0$.

Now suppose that $c' = c - a_1$. If $a_1 > 0$, then $c' + 1 = c - a_1 + 1 \leq c \leq a$, so that the eigenvalues of t' on V_1 and V_2 are rational. If $a_1 = 0$, then $|t| = p^a$ and thus the eigenvalues of t' on on V_1 and V_2 are rational.

(d) Apply Lemma 3.16(e). Suppose first that $a_1 > 0$. Then $|t_1| = p^{a+a_1}$ and $p^a \mid |t_j|$ for $j = 2, 3$. As $|t'| = p$ and $a + a_1 - 1 \geq a$, the eigenvalues of t' on V_1 have order p, and $V_2 \oplus V_3$ is the fixed space of t'. This proves our assertion.

Suppose now that $a_1 = 0$. Then $|t| = p^a$ and all the eigenvalues of t are rational. Moreover, as $Y = O_p(Z)$ and $|O_p(Z)| = p^a$, we may assume that t acts as the identity on V_3, i.e. $\xi_3 = 1$. From $\det(t) = 1$ and $p \nmid n_1 n_2$ we conclude $|\xi_1| = |\xi_1^{n_1}| = |\xi_2^{n_2}| = |\xi_2|$. Thus $|\xi_1| = |\xi_2| = p^a$.

The eigenvalues of t' are $\xi_1^{p^{a-1}}$, $\xi_2^{p^{a-1}}$ and 1. Assume that $\xi_1^{p^{a-1}} = \xi_2^{p^{a-1}}$. Then $1 = \det(t') = \xi_1^{p^{a-1}n_1}\xi_2^{p^{a-1}n_2} = \xi_1^{p^{a-1}(n_1+n_2)}$. As p divides $n = n_1 + n_2 + n_3$ and $p \nmid n_1 n_2 n_3$, the sum $n_1 + n_2$ is prime to p. This contradicts the fact that $|\xi_1| = p^a$ and thus implies that t' has three pairwise distinct eigenvalues. This proves our assertion.

(e) As $[\tilde{H} : H] = q - \varepsilon$ by Lemma 3.4, we obtain $\tilde{H}G = \tilde{G}$. It follows that $N_{\tilde{G}}(E)G = \tilde{G}$ and thus $[N_{\tilde{G}}(E) : N_G(E)] = q - \varepsilon$. Hence $[N_G(E) : H] = [N_{\tilde{G}}(E) : \tilde{H}]$. The structure of $\tilde{H}$ determined in (c) and (d) implies that $[N_{\tilde{G}}(E) : \tilde{H}] \leq 2$, as $N_{\tilde{G}}(E)$ permutes the eigenspaces of t' on V.

(f) This follows from (e) and Lemma 2.2. □

The following example shows that the extra hypothesis for the statements (e) and (f) in Lemma 3.19 is necessary.

Example 3.20. Let $p = n = 3$, and consider $G = \mathrm{SL}_3(7)$. Then $\bar{G} := G/Z = \mathrm{PSL}_3(7)$ has a cyclic 3-block $\bar{\mathbf{B}}$ of defect 1. Now G has a unique conjugacy class of non-central elements of order 3. If we let

$$t := \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\xi \in \mathbb{F}_7$ has order 3, we may assume that $D = \langle t, Z \rangle$ is a defect group of the block $\mathbf{B}$ of G dominating $\bar{\mathbf{B}}$.

We are thus in the situation that $h = 3$, $a_1 = a_2 = a_3 = 0$ and $n_1 = n_2 = n_3 = 1$. Moreover, $D = E$. However, $|C_G(E)| = 36 = |C_{\bar{G}}(\bar{E})|$, so that $\overline{C_G(E)}$ has index 3 in $C_{\bar{G}}(\bar{E})$. □

3.21. **The intermediate blocks.** Keep the notation of Subsection 3.17. In particular, h denotes the number of irreducible factors of the minimal polynomial of t. Recall that $\bar{H} = \overline{C_G(E)} = C_{\bar{G}}(\bar{E})$ by Lemma 3.19(f), unless $h = 3$, $a_1 = 0$ and $n_1 = n_2 = n_3$. In any case, $\bar{H} \leq C_{\bar{G}}(\bar{E})$. Notice that $C_G(D) = C_H(D)$ and $C_{\tilde{G}}(\tilde{D}) = C_{\tilde{H}}(\tilde{D})$.

Notation 3.22. (i) Let $\mathbf{d}$ denote the block of H such that $(E, \mathbf{d}) \leq (D, \mathbf{c})$ as Brauer pairs of H.

(ii) Let $\tilde{\mathbf{d}}$ denote the block of $\tilde{H}$ such that $(\tilde{D}, \tilde{\mathbf{c}})$ is a $\tilde{\mathbf{d}}$-Brauer pair of $\tilde{H}$.

(iii) Let $\bar{\mathbf{d}}$ denote the block of $\bar{H}$ dominated by $\mathbf{d}$.

Lemma 3.23. *If $h = 1$, assume that $D \not\leq Z$. Then $\tilde{\mathbf{d}}$ covers $\mathbf{d}$ and $\bar{\mathbf{d}}$ is a Brauer correspondent of $\bar{\mathbf{B}}$.*

Proof. Observe that $C_{\tilde{H}}(D) = C_{\tilde{G}}(D) = \tilde{C}$. The block $\tilde{\mathbf{c}}$ of $\tilde{C}$ has defect group $\tilde{D}$, and so $\tilde{\mathbf{d}}$ has defect group $\tilde{D}$ by Lemma 2.3. Since $C_G(D) = C_H(D)$, the inclusion $(E, \mathbf{d}) \leq (D, \mathbf{c})$ of Brauer pairs of H is also an inclusion of Brauer pairs of G. Hence $(E, \mathbf{d})$ is a $\mathbf{B}$-Brauer pair by [Lin18, Proposition 6.3.6], and so $\mathbf{d}$ has defect group D, once more by Lemma 2.3.

Notice that $N_{\tilde{H}}(\tilde{D}) \leq N_{\tilde{H}}(D)$ by Lemma 3.16(g). Let $\tilde{\mathbf{b}}$ and $\mathbf{b}$ denote the Brauer correspondent of $\tilde{\mathbf{d}}$, respectively $\mathbf{d}$, in $N_{\tilde{H}}(D)$, respectively $N_H(D)$. By the Harris-Knörr correspondence [HK85, Theorem], it suffices to show that $\tilde{\mathbf{b}}$ covers $\mathbf{b}$. By definition of the inclusion of Brauer pairs, $\tilde{\mathbf{b}}$ covers $\tilde{\mathbf{c}}$, and $\mathbf{b}$ covers $\mathbf{c}$.

As $\tilde{\mathbf{c}}$ covers $\mathbf{c}$ by definition, $\tilde{\mathbf{b}}$ covers some block of $N_H(D)$ covering $\mathbf{c}$. We will show that $N_H(D)/C$ is a p-group. Then $\mathbf{b}$ is the unique block of $N_H(D)$ covering $\mathbf{c}$ and hence $\tilde{\mathbf{b}}$ covers $\mathbf{b}$.

Suppose that $h = 1$, so that $\tilde{H} = \tilde{G}$. By Lemma 3.16(c) we have $C_{\tilde{G}}(D) = \tilde{C} \cong \mathrm{GL}_{n_1}^{\varepsilon}(q^{p^{a_1}})$ and thus $|N_{\tilde{G}}(\tilde{C})/\tilde{C}| = p^{a_1}$. Now consider the chain of maps

$$N_G(D) \hookrightarrow N_{\tilde{G}}(D) \hookrightarrow N_{\tilde{G}}(C_{\tilde{G}}(D)) = N_{\tilde{G}}(\tilde{C}) \to N_{\tilde{G}}(\tilde{C})/\tilde{C},$$

whose kernel equals $N_G(D) \cap \tilde{C} = C$. It follows that $N_G(D)/C$ is a p-group. Suppose now that $h \geq 2$. First, $\overline{N_H(D)} = N_{\bar{H}}(\bar{D})$ by [HL24, Lemma 2.2(a)]. Thus

$$N_H(D)/C \cong \overline{N_H(D)}/\bar{C} \cong N_{\bar{H}}(\bar{D})/\bar{C}.$$

Now $N_{\bar{H}}(\bar{D})/C_{\bar{H}}(\bar{D})$ is a p-group, since $\bar{D}$ is cyclic and $N_{\bar{H}}(\bar{D})$ acts trivially on $\bar{E}$. Also, $C_{\bar{H}}(\bar{D})/\bar{C}$ is a p-group by Lemma 2.2. Since

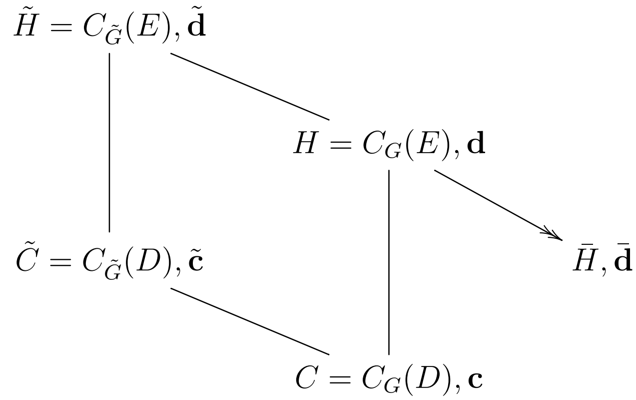
$$|N_{\bar{H}}(\bar{D})/\bar{C}| = |N_{\bar{H}}(\bar{D})/C_{\bar{H}}(\bar{D})| \cdot |C_{\bar{H}}(\bar{D})/\bar{C}|,$$

it follows that $N_H(D)/C$ is a p-group.

We now show that $\bar{\mathbf{d}}$ is a Brauer correspondent of $\bar{\mathbf{B}}$. This is trivial if $h = 1$. Assume then that $h \geq 2$. As $\bar{H} = C_{\bar{G}}(\bar{D})$, this statement makes sense. By Lemma 3.19(b), we have $N_G(D) \leq N_G(E)$. Let $\mathbf{b}'$ and $\bar{\mathbf{b}}'$ denote the Brauer correspondents of $\mathbf{B}$ in $N_G(E)$, respectively of $\bar{\mathbf{B}}$ in $N_{\bar{G}}(\bar{E})$. Then $\overline{N_G(E)} = N_{\bar{G}}(\bar{E})$ by [HL24, Lemma 2.2(a)], and $\mathbf{b}'$ dominates $\bar{\mathbf{b}}'$ by [HL25, Lemma 2.4.1]. Now $(E, \mathbf{d})$ is a $\mathbf{B}$-Brauer pair and so $\mathbf{b}'$ covers $\mathbf{d}$. This easily implies that $\bar{\mathbf{b}}'$ covers $\bar{\mathbf{d}}$, and so $\bar{\mathbf{d}}$ is a Brauer correspondent of $\bar{\mathbf{B}}$. □

A diagram of the relevant groups and blocks is displayed in Figure 2.

FIGURE 2. Some subgroups and blocks of $\mathrm{GL}_n^{\varepsilon}(q)$, II



The maximal tori of $\tilde{C}$ and $\tilde{H}$ are also maximal tori of $\tilde{G}$. The $\tilde{G}$-conjugacy classes of maximal tori of $\tilde{G}$ are labelled by partitions of n. If the $\tilde{G}$-conjugacy class of a maximal torus $\tilde{T}$ is labelled by the partition $\pi = (f_1, f_2, \ldots, f_m)$, we call π the type of $\tilde{T}$. Then

$$\tilde{T} \cong \tilde{T}_{f_1} \times \cdots \times \tilde{T}_{f_m}, \tag{4}$$

and $N_{\tilde{G}}(\tilde{T})$ fixes the sub-products of (4) corresponding to the same f_j.

Lemma 3.24. *If $h = 3$, assume that $a_1 > 0$. Then $\tilde{\mathbf{d}}$ is strictly regular with respect to a maximal torus $\tilde{T}$ of $\tilde{H}$ whose type has the parts $n_1, \ldots, n_h$.*

In particular, there is $\theta \in \mathrm{Irr}(\tilde{T})$ of p'-order and in general position with respect to $\tilde{H}$, such that

$$\tilde{\chi} := \varepsilon_{\tilde{\mathbf{T}}}\varepsilon_{\tilde{\mathbf{H}}} R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{H}}}(\theta)$$

is an irreducible character of $\tilde{\mathbf{d}}$.

Proof. For $j \in \{1, \ldots, h\}$, the block $\tilde{\mathbf{c}}_j$ of $\tilde{C}_j$ is strictly regular with respect to a cyclic maximal torus $\tilde{T}_j$ of $\tilde{C}_j$, of type (n_j) when viewed as a torus of $\tilde{G}_j$, and $\tilde{D}_j$ is a Sylow p-subgroup of $\tilde{T}_j$; see [HL25, Corollary 3.6.2]. Thus $\tilde{T} = \tilde{T}_1 \times \cdots \times \tilde{T}_h$ contains $\tilde{D}$ as a Sylow p-subgroup and $\tilde{\mathbf{c}}$ is a strictly regular block of $\tilde{C} = \tilde{C}_1 \times \cdots \times \tilde{C}_h$ with respect to $\tilde{T}$.

We claim that $N_{\tilde{H}}(\tilde{T})$ fixes each of the factors $\tilde{T}_j$ for $1 \leq j \leq h$. This is trivial if $h = 1$. It is also clear if $h = 2$ and $\tilde{H} = \tilde{G}_1 \times \tilde{G}_2$. If $h = 2$ and $\tilde{H} = \tilde{G}$, then $a_1 = a_2 = 0$ and $c' < c$ by Lemma 3.19(c). But then $n_1 \neq n_2$ by Lemma 3.16(d)(vi), which also implies the claim.

If $h = 3$, we have $a_1 > 0$ by assumption, and then $n_1 = m_1 p^{a_1}, n_2, n_3$ are pairwise distinct. Indeed, $n_2 = n_3$ would give $n = m_1 p^{a_1} + 2n_2$ and thus $p \mid n_2$, a contradiction. This yields our claim.

Now $\tilde{\mathbf{c}}$ contains an irreducible character $\tilde{\psi}$ of the form

$$\tilde{\psi} := \varepsilon_{\tilde{\mathbf{T}}} \varepsilon_{\tilde{\mathbf{C}}} R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{C}}}(\theta),$$

for some irreducible character θ of $\tilde{T}$ of p'-order and in general position with respect to $\tilde{C}$. By the claim, θ is also in general position with respect to $\tilde{H}$, and thus

$$\varepsilon_{\tilde{\mathbf{T}}} \varepsilon_{\tilde{\mathbf{H}}} R_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{H}}}(\theta) = \varepsilon_{\tilde{\mathbf{C}}} \varepsilon_{\tilde{\mathbf{H}}} R_{\tilde{\mathbf{C}}}^{\tilde{\mathbf{H}}}(\tilde{\psi})$$

is an irreducible character of $\tilde{H}$. Moreover, $\tilde{\chi}$ lies in $\tilde{\mathbf{d}}$ by [CaEn99, Theorem 2.5]. This proves our assertions. □

If $h = 1$, we have $\bar{\mathbf{B}} = \bar{\mathbf{d}}$, $\mathbf{B} = \mathbf{d}$, and $\tilde{\mathbf{B}} = \tilde{\mathbf{d}}$. In this case, we compute $W(\bar{\mathbf{B}})$ from $W(\mathbf{B})$ using [HL25, Lemma 2.4.2(c)]. If $h \geq 2$, then, by definition, $\bar{H} \leq C_{\bar{G}}(\bar{E})$, where $\bar{E}$ denotes the unique subgroup of $\bar{D}$ of order p. In this case, [HL25, Remark 2.3.3] shows that $W(\bar{\mathbf{d}})$ can be computed from the sign sequence $\sigma_{\bar{\chi}}^{[l]}(\bar{t})$, where $\bar{\chi}$ denotes the non-exceptional character of $\bar{\mathbf{d}}$, and l is defined by $|\bar{D}| = p^l$. Namely, if $\Lambda = \{0, \ldots, l-1\}$ and $A \subseteq \Lambda \setminus \{0\}$ is such that $\sigma_{\bar{\chi}}^{[l]}(\bar{t}) = \omega_\Lambda(\mathbf{1}_A)$, then $W(\bar{\mathbf{d}}) = W_{\bar{D}}(A)$; for the notation see [HL25, Definitions 2.1.1, 2.1.2].

Lemma 3.25. *If $h = 3$, assume that $a_1 > 0$. Let $\bar{\chi}$ denote the non-exceptional character of* $\operatorname{Irr}(\bar{\mathbf{d}})$*, and let χ denote the inflation of $\bar{\chi}$ to H. Moreover, let $\tilde{\chi} \in \operatorname{Irr}(\tilde{\mathbf{d}})$ be as in* Lemma 3.24*. Then*

$$\sigma_{\chi}^{[a+a_1]}(t) = \omega_{\tilde{\mathbf{H}}}^{[a+a_1]}(t) = \sigma_{\tilde{\chi}}^{[a+a_1]}(t) \tag{5}$$

and

$$\sigma_{\bar{\chi}}^{[a+a_1-c']}(\bar{t}) = \omega_{\tilde{\mathbf{H}}}^{[a+a_1-c']}(t). \tag{6}$$

(For the notation $\omega_{\tilde{\mathbf{H}}}$ see [HL25, Definition 2.5.6]*).*

Proof. By definition, $\mathbf{d}$ dominates the nilpotent cyclic block $\bar{\mathbf{d}}$. By [HL25, Lemma 2.3.4], the character $\bar{\chi}$ is the unique p-rational character in $\bar{\mathbf{d}}$. As the irreducible characters of $\mathbf{d}$ which do not have Y in their kernels are not p-rational, $\operatorname{Irr}(\mathbf{d})$ has a unique p-rational character, namely χ, and χ lifts the unique irreducible Brauer character of $\mathbf{d}$. Thus the hypotheses of [HL25, Lemma 2.5.18] are satisfied, and so $\tilde{\chi}$ lies above χ.

As $\bar{\chi}$ is the non-exceptional character of $\operatorname{Irr}(\bar{\mathbf{d}})$, the values $\bar{\chi}(\bar{u})$ are non-zero integers for $\bar{u} \in \bar{D}$; see [HL24, Lemma 3.3]. The same is then true for the values $\chi(u)$ for $u \in \langle t \rangle$.

By [HL25, Lemma 2.5.18] we have $\sigma_{\tilde\chi}(u) = \sigma_\chi(u)$ for all $u \in \tilde D$, which implies

$$\sigma_{\tilde\chi}^{[a+a_1]}(t) = \sigma_\chi^{[a+a_1]}(t).$$

By [HL25, Lemma 2.5.7],

$$\sigma_{\tilde\chi}^{[a+a_1]}(t) = \omega_{\tilde{\mathbf{H}}}^{[a+a_1]}(t),$$

which yields (5). This trivially implies

$$\sigma_\chi^{[a+a_1-c']}(t) = \omega_{\tilde{\mathbf{H}}}^{[a+a_1-c']}(t).$$

Since χ is the inflation of $\bar\chi$ to H, and as the kernel of the epimorphism $\langle t\rangle \to \langle \bar t\rangle = \bar D$ has order $p^{c'}$, we obtain

$$\sigma_{\bar\chi}^{[a+a_1-c']}(\bar t) = \sigma_\chi^{[a+a_1-c']}(t),$$

which gives (6). □

3.26. **Computing the invariants in case $h = 1$.** Assume that the minimal polynomial of t is irreducible. Recall the results of Lemmas 3.16(c) and 3.23 in this case. If $D \not\leq Z$, then $n = m_1p^{a+a_1}$ with $p \nmid m_1$ and $a_1 > 0$. The defect group D is cyclic of order p^{a+a_1}, and $\mathbf{B}$ is covered by a cyclic block $\tilde{\mathbf{B}}$ of $\tilde G$ with defect group $\tilde D$ of order p^{2a+a_1}. Moreover, $|\bar D| = p^{a+a_1-c'}$.

Proposition 3.27. *Suppose that the minimal polynomial of t is irreducible. If $D \leq Z$, then $W(\bar{\mathbf{B}}) \cong k$.*

Assume in the following that $D \not\leq Z$, so that $a_1 \geq 1$ by Lema 3.16(c). *Then $W(\bar{\mathbf{B}}) \cong k$ if $\varepsilon = 1$, or if $p \equiv 1 \pmod 4$, or if m_1' is even, or if $c' = a$ and $a_1 = 1$. Otherwise, $W(\bar{\mathbf{B}}) = W_{\bar D}([a-c', a+a_1-c'-1])$, if $c' < a$, and $W(\bar{\mathbf{B}}) = W_{\bar D}([1, a_1-1])$, if $c' = a$ and $a_1 > 1$.*

Proof. If $D \leq Z$, then $\bar D \leq Z(\bar G)$, and so $W(\bar{\mathbf{B}}) \cong k$ by [HL24, Lemma 3.6(b)]. Assume then that $D \not\leq Z$.

Put $l := a + a_1$ and $\Lambda := \{0, \ldots, l-1\}$. Since $t^{p^{l-1}}$ has order p, we have $t^{p^{l-1}} \in Z(\tilde H)$. Let $\tilde\chi$ be as in Lemma 3.24. Then [HL25, Lemma 4.1.3] yields $\sigma_{\tilde\chi}^{[a+a_1]}(t) = \omega_\Lambda(I)$ with $I = \emptyset$, unless $\varepsilon = -1$, n is odd, and $p \equiv -1 \pmod 4$, in which case $I = [a, l-1]$. In view of [HL25, Remark 2.3.3] and Equation (5) in Lemma 3.25, this gives $W(\mathbf{B}) \cong W_D(I)$.

Our assertion follows from this and [HL25, Lemma 2.4.2(c)]. □

3.28. **Computing the invariants in case $h = 2$.** We now assume that the minimal polynomial of t has exactly two irreducible factors. Recall the following facts from Lemma 3.16(d). We have $n = n_1 + n_2$ with $n_1 = m_1 p^{a_1}$, $n_2 = m_2 p^{a_2}$ for some non-negative integers $a_1 \geq a_2$, and $p \nmid m_1 m_2$. Also, $c' = 0$ if $a_1 > a_2$, and $c' = c - a_1$ if $a_1 = a_2 > 0$. Moreover, $|\bar{D}| = p^l$ with $l = a + a_1 - c'$.

Recall that $V = V_1 \oplus V_2$ is the primary decomposition of V with respect to t, where $\dim(V_j) = n_j = m_j p^{a_j}$ for $j = 1, 2$. Recall also from Notation 3.2(ii) that $\mathbf{V} = \mathbb{F}^n$ is the natural vector space for $\tilde{\mathbf{G}}$. For $i = 1, 2$, let $\mathbf{V}_i$ denote the $\mathbb{F}$-span of V_i, and let $\tilde{\mathbf{G}}_i$ denote the subgroup of $\tilde{\mathbf{G}}$ induced on $\mathbf{V}_i$. Then $\tilde{\mathbf{G}}_i$ is F-stable and $\tilde{\mathbf{G}}_i^F = \tilde{G}_i$ for $i = 1, 2$.

Proposition 3.29. *Suppose that the minimal polynomial of t has exactly two irreducible factors.*

Then $W(\bar{\mathbf{B}}) = W_{\bar{D}}(I)$, with $I \subseteq \{1, \ldots, l-1\}$ an interval. If $l = 1$, or if $\varepsilon = 1$, or if $a_1 = a_2 = 0$ and $c' = c$, then $I = \emptyset$, i.e. $W(\bar{\mathbf{B}}) \cong k$.

Suppose in the following that $l \geq 2$, that $\varepsilon = -1$, and that $c' < c$ if $a_1 = a_2 = 0$. Then I is non-empty exactly in the following cases.

(a) *We have $a_1 > a_2$, $p \equiv -1 \pmod 4$ and at least one of n_1, n_2 is odd. Then*

$$I = \begin{cases} [a, l-1], & \text{if } n_1 \text{ odd and } (n_2 \text{ even or } a_2 = 0); \\ [a + a_1 - a_2, l-1], & \text{if } n_1 \text{ even, } n_2 \text{ odd, and } a_2 > 0; \\ [a, l - a_2 - 1], & \text{if } n_1 \text{ and } n_2 \text{ odd.} \end{cases}$$

(b) *We have $a_1 = a_2 > 0$, $p \equiv -1 \pmod 4$, and n is odd. Then*

$$I = [a - c', l - 1].$$

(c) *We have $a_1 = a_2 = 0$, $c' < c$ and n_1, n_2 odd. Then*

$$I = \{c - c'\}.$$

Proof. If $l = 1$, then $W(\bar{\mathbf{B}}) \cong k$ by [HL24, Lemma 3.6(b)]. Suppose now that $a_1 = a_2 = 0$ and that $c' = c$. Then $\tilde{H} = \tilde{G}_1 \times \tilde{G}_2$ by Lemma 3.19(c). On the other hand, $\tilde{C} = \tilde{C}_1 \times \tilde{C}_2 = \tilde{G}_1 \times \tilde{G}_2$ since $a_1 = a_2 = 0$. It follows that $\tilde{H} = \tilde{C}$ and thus $H = C$. This implies $C_{\bar{G}}(\bar{E}) = \bar{H} = \bar{C} \leq C_{\bar{G}}(\bar{D}) \leq C_{\bar{G}}(\bar{E})$, the first equality arising from Lemma 3.19(f). Our claim follows from [HL24, Lemma 3.6(b)].

To continue, assume that $l \geq 2$ and that $c' < c$ if $a_1 = a_2 = 0$. Set $\Lambda := \{0, 1, \ldots, l-1\}$. To determine $\omega_{\tilde{\mathbf{H}}}^{[l]}(t)$, which yields $W(\bar{\mathbf{B}})$ by Lemma 3.25 and the remarks preceding it, we first determine $\omega_{\tilde{\mathbf{G}}_j}^{[l]}(t_j)$ for $j = 1, 2$. Recall that $|t_j| = p^{a + a_j}$ for $j = 1, 2$; see Lemma 3.16(d)(vii). Thus $|t_2^{p^{l-1}}| \leq |t_1^{p^{l-1}}| = p^{c'+1} \leq p^a$, where the last inequality arises from

Lemma 3.16(d)(i)(ii), respectively our hypothesis $c' < c$ if $a_1 = a_2 = 0$. Hence $t_j^{p^{l-1}} \in Z(\tilde{G}_j)$ for $j = 1, 2$, and [HL25, Lemmas 2.5.7, 4.1.3] yield

$$\omega^{[l]}_{\tilde{\mathbf{G}}_j}(t_j) = \omega_\Lambda(\mathbf{1}_{I_j})$$

with intervals $I_j \subseteq \Lambda \setminus \{0\}$, determined as follows. If n_j is odd, $\varepsilon = -1$ and $p \equiv -1 \pmod 4$, then

$$I_j = [a + a_1 - c' - a_j, l - 1];$$

otherwise $I_j = \emptyset$. Notice that $I_2 = \emptyset$ if $a_2 = 0$.

If $a_1 > a_2$ or if $a_1 = a_2$ and $c' = c - a_1$, we have $\tilde{\mathbf{H}} = \tilde{\mathbf{G}}_1 \times \tilde{\mathbf{G}}_2$ by Lemma 3.19(c), and thus

$$\omega^{[l]}_{\tilde{\mathbf{H}}}(t) = \omega^{[l]}_{\tilde{\mathbf{G}}_1}(t_1)\omega^{[l]}_{\tilde{\mathbf{G}}_2}(t_2), \tag{7}$$

where the two l-tuples on the right hand side are multiplied componentwise.

We continue the proof under the assumption that (7) holds. Let $I := I_1 \diamond I_2$ denote the symmetric difference of I_1 and I_2. Then $\omega^{[l]}_{\tilde{\mathbf{G}}_1}(t_1)\omega^{[l]}_{\tilde{\mathbf{G}}_2}(t_2) = \omega_\Lambda(I)$; see [HL25, Subsection 2.2]. This yields the assertions in (a) and (b).

Now assume that (7) is not satisfied. Then $a_1 = a_2 = 0$ and $c' < c$. In particular, $D = \langle t \rangle$ and $|t| = p^a$ by Lemma 3.16(d)(iii). Put $l_1 := c - c'$ and $l_2 := l - l_1$. The l-tuple $\omega^{[l]}_{\tilde{\mathbf{H}}}(t)$ contains the values of $\omega_{\tilde{\mathbf{H}}}$ at the elements $t^{p^{l-1}}, t^{p^{l-2}}, \ldots, t^p, t$. As j runs from 1 to l, the order of $t^{p^{l-j}}$ runs from $p^{c'+1}$ to p^a. For $j = l_1$ we get $|t^{p^{l-j}}| = p^c$. As $O_p(Z) \leq D = \langle t \rangle$, we have $t^{p^{l-j}} \in Z$ exactly for $j = 1, \ldots, l_1$. Thus

$$C_{\tilde{\mathbf{H}}}(t^{p^{l-j}}) = \begin{cases} \tilde{\mathbf{H}}, & \text{for } 1 \leq j \leq l_1; \\ C_{\tilde{\mathbf{G}}_1}(t_1^{p^{l-j}}) \times C_{\tilde{\mathbf{G}}_2}(t_2^{p^{l-j}}), & \text{for } l_1 + 1 \leq j \leq l. \end{cases} \tag{8}$$

By Lemma 3.16(c) we have $\tilde{\mathbf{H}} = \tilde{\mathbf{G}}$. Hence

$$\omega_{\tilde{\mathbf{H}}}(t^{p^{l-j}}) = \begin{cases} 1, & \text{for } 1 \leq j \leq l_1; \\ \varepsilon_{\tilde{\mathbf{G}}}\varepsilon_{\tilde{\mathbf{G}}_1}\varepsilon_{\tilde{\mathbf{G}}_2}\omega_{\tilde{\mathbf{G}}_1}(t_1^{p^{l-j}})\omega_{\tilde{\mathbf{G}}_2}(t_2^{p^{l-j}}), & \text{for } l_1 + 1 \leq j \leq l. \end{cases} \tag{9}$$

It follows that $\omega^{[l]}_{\tilde{\mathbf{H}}}(t)$ is the concatenation of the all-1-vector $(1, \ldots, 1)$ of length l_1 with

$$\varepsilon_{\tilde{\mathbf{G}}}\varepsilon_{\tilde{\mathbf{G}}_1}\varepsilon_{\tilde{\mathbf{G}}_2}\omega^{[l_2]}_{\tilde{\mathbf{G}}_1}(t_1)\omega^{[l_2]}_{\tilde{\mathbf{G}}_2}(t_2).$$

By [HL25, Example 2.5.5], we have

$$\varepsilon_{\tilde{\mathbf{G}}}\varepsilon_{\tilde{\mathbf{G}}_1}\varepsilon_{\tilde{\mathbf{G}}_2} = \begin{cases} -1, & \text{if } \varepsilon = -1 \text{ and } n_1, n_2 \text{ odd}; \\ 1, & \text{otherwise}. \end{cases}$$

Suppose in addition that $\varepsilon = -1$ and that n_1 and n_2 are odd. Then $I_1 = I_2$, and so $\omega_{\tilde{\mathbf{G}}_1}(t_1^{p^{l-j}})\omega_{\tilde{\mathbf{G}}_2}(t_2^{p^{l-j}}) = 1$ for all $1 \leq j \leq l$. Thus

$$\omega_{\tilde{\mathbf{H}}}^{[l]}(t) = (1, \ldots, 1, -1, \ldots, -1),$$

where the first entry -1 is at position $l_1 + 1$. Hence $\omega_{\tilde{\mathbf{H}}}^{[l]}(t) = \omega_\Lambda(\{l_1\})$ by [HL25, Lemma 2.2.1]. This yields the instance listed in (c). If $\varepsilon = 1$ or $\varepsilon = -1$ and at least one of n_1, n_2 is even, then $\varepsilon_{\tilde{\mathbf{G}}}\varepsilon_{\tilde{\mathbf{G}}_1}\varepsilon_{\tilde{\mathbf{G}}_2} = 1$, and thus (7) holds by (8) and (9), and the fact that $t_i^{p^{l-j}} \in Z(\tilde{G}_i)$ for $i = 1, 2$ and all $1 \leq j \leq l_1$. This contradiction concludes our proof. □

3.30. **Computing the invariants in case $h = 3$.** We now assume that the minimal polynomial of t has exactly three irreducible factors. Recall the following facts from Lemma 3.16(e). We have $n = n_1 + n_2 + n_3$ with $n_1 = m_1 p^{a_1}$ for a non-negative integer a_1, and $p \nmid m_1 n_2 n_3$. Moreover, $c' = 0$, and $|\bar{D}| = p^l$ with $l = a + a_1$. Notice that $l \geq 2$ if $a_1 > 0$.

Recall that $V = V_1 \oplus V_2 \oplus V_3$ is the primary decomposition of V with respect to t, where $\dim(V_j) = n_j$ for $j = 1, 2, 3$. Recall also from Notation 3.2(ii) that $\mathbf{V} = \mathbb{F}^n$ is the natural vector space for $\tilde{\mathbf{G}}$. For $i = 1, 2, 3$, let $\mathbf{V}_i$ denote the $\mathbb{F}$-span of V_i, and let $\tilde{\mathbf{G}}_i$ denote the subgroup of $\tilde{\mathbf{G}}$ induced on $\mathbf{V}_i$. Put $\mathbf{V}_{2,3} := \mathbf{V}_2 \oplus \mathbf{V}_3$, and let $\tilde{\mathbf{G}}_{2,3}$ denote the subgroup of $\tilde{\mathbf{G}}$ induced on $\mathbf{V}_{2,3}$. Then $\tilde{\mathbf{G}}_i$ is F-stable, and $\tilde{\mathbf{G}}_i^F = \tilde{G}_i$ for $i = 1, 2, 3$. Similarly, $\tilde{\mathbf{G}}_{2,3}$ is F-stable and $\tilde{\mathbf{G}}_{2,3}^F = \tilde{G}_{2,3}$ in the notation of Lemma 3.19(c)(iii).

Proposition 3.31. *Suppose that the minimal polynomial of t has exactly three irreducible factors. Then $W(\bar{\mathbf{B}}) = W_D(A)$, with $A \subseteq \{1, \ldots, l-1\}$ a union of two intervals. If $a_1 = 0$ or if $\varepsilon = 1$, then $A = \emptyset$, i.e. $W(\bar{\mathbf{B}}) \cong k$.*

Suppose in the following that $a_1 > 0$ and $\varepsilon = -1$. Then A is non-empty exactly in the following cases.

(a) *At least one of n_2, n_3 is even, n_1 is odd and $p \equiv -1 \pmod 4$. Then*

$$A = [a, l-1].$$

(b) *Each of n_2 and n_3 is odd and $a_1 \geq a$. Then*

$$A = \begin{cases} \{a_1\}, & \text{if } n_1 \text{ is even or } p \equiv 1 \pmod 4; \\ [a, l-1] \setminus \{a_1\}, & \text{if } n_1 \text{ is odd and } p \equiv -1 \pmod 4. \end{cases}$$

(c) *Each of* n_2 *and* n_3 *is odd and* $a_1 < a$. *Then*

$$A = \begin{cases} \{a_1\}, & \text{if } n_1 \text{ is even or } p \equiv 1 \pmod 4; \\ \{a_1\} \cup [a, l-1], & \text{if } n_1 \text{ is odd and } p \equiv -1 \pmod 4. \end{cases}$$

Proof. Suppose first that $a_1 = 0$ and $n_1 = n_2 = n_3$. Then $n = 3n_1$, and thus $p = 3$ and $a = 1$, as $p \nmid n_1$. In this case, $|\bar{D}| = 3$, hence $W(\bar{\mathbf{B}}) \cong k$ by [HL24, Lemma 3.6(b)].

Suppose next that $a_1 = 0$ and $|\{n_1, n_2, n_3\}| \geq 2$. Then $\tilde{H} = \tilde{G}_1 \times \tilde{G}_2 \times \tilde{G}_3$ by Lemma 3.19(d). Thus $\tilde{H} = \tilde{C}$ and $H = C = C_G(D)$. In particular, $\bar{H} \leq C_{\bar{G}}(\bar{D})$. Now $\bar{H} = C_{\bar{G}}(\bar{E})$ by Lemma 3.19(f), and so $\bar{H} \leq C_{\bar{G}}(\bar{D}) \leq C_{\bar{G}}(\bar{E}) = \bar{H}$. Our assertion follows from [HL24, Lemma 3.6(b)].

Suppose from now on that $a_1 > 0$. Set $t_{2,3} := t_2 t_3$ and $\Lambda := \{0, 1, \ldots, l-1\}$. Since $C_{\tilde{\mathbf{H}}}(t) = C_{\tilde{\mathbf{G}}_1}(t_1) \times C_{\tilde{\mathbf{G}}_{2,3}}(t_{2,3})$ by Lemma 3.19(d), we obtain

$$\omega^{[l]}_{\tilde{\mathbf{H}}}(t) = \omega^{[l]}_{\tilde{\mathbf{G}}_1}(t_1)\omega^{[l]}_{\tilde{\mathbf{G}}_{2,3}}(t_{2,3}),$$

where the multiplication of the l-tuples on the right hand side is defined component-wise.

By [HL25, Corollary 4.1.4] we have

$$\omega^{[l]}_{\tilde{\mathbf{G}}_1}(t_1) = \omega_\Lambda(\mathbf{1}_{I_1})$$

with $I_1 = \emptyset$ unless n_1 is odd, $\varepsilon = -1$ and $p \equiv -1 \pmod 4$, in which case $I_1 = [a, l-1]$.

Let $1 \leq j \leq l$ and put $u := t_{2,3}^{p^{l-j}}$. Lemma 3.16(e) implies that $C_{\tilde{\mathbf{G}}_{2,3}}(u) = \tilde{\mathbf{G}}_{2,3}$ if $j \leq a_1$, and $C_{\tilde{\mathbf{G}}_{2,3}}(u) = \tilde{\mathbf{G}}_2 \times \tilde{\mathbf{G}}_3$ if $a_1 + 1 \leq j \leq a + a_1$. In the former case, $\omega_{\tilde{\mathbf{G}}_{2,3}}(u) = 1$. In the latter case,

$$\omega_{\tilde{\mathbf{G}}_{2,3}}(u) = \varepsilon_{\tilde{\mathbf{G}}_{2,3}} \varepsilon_{\tilde{\mathbf{G}}_2} \varepsilon_{\tilde{\mathbf{G}}_3}.$$

By [HL25, Example 2.5.5], we get

$$\varepsilon_{\tilde{\mathbf{G}}_{2,3}} \varepsilon_{\tilde{\mathbf{G}}_2} \varepsilon_{\tilde{\mathbf{G}}_3} = \begin{cases} -1, & \text{if } \varepsilon = -1 \text{ and } n_2, n_3 \text{ odd}; \\ 1, & \text{otherwise.} \end{cases}$$

It follows that $\omega^{[l]}_{\tilde{\mathbf{G}}_{2,3}}(t_{2,3}) = \omega_\Lambda(\mathbf{1}_{I_{2,3}})$ with $I_{2,3} = \{a_1\}$, if n_2 and n_3 are odd and $\varepsilon = -1$, and $I_{2,3} = \emptyset$, otherwise.

We conclude from the considerations in [HL25, Subsection 2.2] that

$$\omega^{[l]}_{\tilde{\mathbf{G}}_1}(t_1)\omega^{[l]}_{\tilde{\mathbf{G}}_{2,3}}(t_{2,3}) = \omega^{[l]}_\Lambda(\mathbf{1}_A)$$

with $A = I_1 \diamond I_{2,3}$, the symmetric difference of I_1 and $I_{2,3}$. This yields our assertions. □

We record a specific corollary of the above results. If $\tilde{T}$ is a maximal torus of $\tilde{G}$ of type π, we write $T_\pi := \tilde{T} \cap G$. As every maximal torus of $\mathrm{SL}_n^\varepsilon(q)$ is of this form, this yields a labelling of the maximal tori of $\mathrm{SL}_n^\varepsilon(q)$ (up to conjugation) by partitions of n.

Corollary 3.32. *Suppose that $p = 3$ and $n \in \{3, 6, 9\}$ or that $p = 5$ and $n = 5$. Let $\mathbf{B}$ be a p-block of $G = \mathrm{SL}_n^\varepsilon(q)$ with a non-trivial cyclic defect group D. If $n = 9$ assume that $D \not\leq Z$ and that $a \geq 2$. Then the following statements hold.*

(a) *We have $|D| = p$ or $|D| = p^a$. Also, $W(\mathbf{B}) \cong k$ or $W(\mathbf{B}) \cong W_D(\{1\})$. The latter occurs exactly if $\varepsilon = -1$, $|D| = 3^a$ with $a \geq 2$, and $n = 6$ In this case, we have $h = 2$ and $\{n_1, n_2\} = \{5, 1\}$ in the notation of* Proposition 3.29*.*

(b) *Suppose that $p = 3$ and $n = 6$. If $W(\mathbf{B}) \not\cong k$, then $\mathbf{B}$ is strictly regular with respect to $T_{(5,1)}$.*

Proof. (a) In the setup of Notation 3.7 we have $Y = \{1\}$, hence $\bar{\mathbf{B}} = \mathbf{B}$ and $c' = 0$. Also, the group D of Notation 3.7(ii) is cyclic, a defect group of $\mathbf{B}$, and $|D| = |t| = p^{a+a_1}$, the latter by Lemma 3.16(f). In particular, we cannot have $h = 3$, since in that case $Y = O_p(Z) \neq \{1\}$.

If $D \leq Z$, then $W(\mathbf{B}) \cong k$ by [HL24, Lemma 3.6(b)]. Moreover, $n \neq 9$ by hypothesis, and hence $|D| = p$. Assume then that D is non-central in the following. Suppose that $h = 1$. Then $p^{a+a_1} \mid n$ with $a_1 \geq 1$ by Lemma 3.16(c). As we have assumed that $a \geq 2$ if $n = 9$, this case cannot occur.

Suppose finally that $h = 2$. As D is cyclic, $c > 0$, and $|D| = p^{a+a_1}$, we must have $a_1 = a_2 = 0$ by Lemma 3.16(d)(i)(ii). In particular, $|D| = p^a$. Proposition 3.29 shows that $W(\mathbf{B}) \cong k$, unless $\varepsilon = -1$, $a \geq 2$, and n_1 and n_2 are odd. If all these latter conditions are satisfied, $n = 6$ and $\{n_1, n_2\} = \{5, 1\}$ since $3 \nmid n_1 n_2$. Moreover, $W(\mathbf{B}) \cong W_D(\{c - c'\}) = W_D(\{1\})$ by Proposition 3.29(c).

(b) Suppose that $p = 3$, $n = 6$ and $W(\mathbf{B}) \not\cong k$. Without loss of generality we may assume that $n_1 = 5$ and $n_2 = 1$. In the notation of Lemma 3.15(e), we have $\tilde{C}_1 = \tilde{G}_1 = \mathrm{GU}_5(q)$ and $\tilde{C}_2 = \tilde{G}_2 = \mathrm{GU}_1(q)$. Moreover, $D \leq \tilde{D} = \tilde{D}_1 \times \tilde{D}_2$, where $\tilde{D}_j$ is a cyclic defect group of a block of $\tilde{C}_j$, $j = 1, 2$. In particular, $\tilde{D}$ is a Sylow 3-subgroup of a maximal torus $\tilde{T} \leq \tilde{G}$ of type $(5, 1)$; see [HL25, Corollary 3.6.2].

Let $\chi \in \mathrm{Irr}(\mathbf{B})$. Then χ has height 0, and thus $\chi(1)_3 = 3^{4a+2}$, as $|G|_3 = 3^{5a+2}$. The character degrees of G given in [Lue21a] show that $\chi(1) = [G\colon T]_{r'}/f$ with $T = T_{(4,1,1)}$ and $f \in \{1, 2\}$, or $T = T_{(5,1)}$ and $f = 1$. Hence $\mathbf{B}$ is regular with respect to one of these tori. The former case cannot occur, as the torus $T_{(4,1,1)}$ does not have a cyclic Sylow 3-subgroup. Hence $\mathbf{B}$ is strictly regular with repsect to $T_{(5,1)}$. $\square$

Remark 3.33. Suppose that $p = 3$ and $n = 9$, and that $a \geq 2$, so that $c = 2$. Let $Y \leq Z$ with $|Y| = 3$, and let $\bar{\mathbf{B}}$ be a 3-block of $\bar{G} = G/Y$ with a non-central, cyclic defect group $\bar{D} = D/Y$. Then $h = 2$ and $a_1 = a_2 = 0$ by Lemma 3.16(c)(d)(e).

Let $\mathbf{B}$ be a block of G with defect group D dominating $\bar{\mathbf{B}}$. Then $D \not\leq Z$ by hypothesis on $\bar{D}$. In particular, $\mathbf{B}$ is as in Corollary 3.32, so that D is cyclic of order 3^a. Thus $c' = 1$ and $|\bar{D}| = 3^{a-1}$. By Corollary 3.32(a) and [HL25, Lemma 2.4.2(c)], we have $W(\bar{\mathbf{B}}) \cong k$.

4. Synthesis

We now investigate which of the parameter sets exhibited in Section 3 correspond to blocks. Throughout this section, we fix an odd prime p.

Recall that we have fixed a sign $\varepsilon \in \{-1, 1\}$. As in Notation 3.2(iv), we let $\delta = 1$, if $\varepsilon = 1$, and $\delta = 2$, if $\varepsilon = -1$. To construct specific examples, we will vary the parameters, r, q and n, keeping their principal significance. Thus q is a power of the prime r, where $r \neq p$, and n is a positive integer. Whenever we have chosen r, q and n, we adopt the corresponding notation introduced in Section 3. In particular, $\mathbb{F}$ denotes an algebraic closure of the finite field with r elements. Moreover, $\tilde{\mathbf{G}} = \mathrm{GL}_n(\mathbb{F})$, $\tilde{G} = \mathrm{GL}_n^{\varepsilon}(q) = \tilde{\mathbf{G}}^F$, $G = \mathrm{SL}_n^{\varepsilon}(q) = \tilde{\mathbf{G}}^F$ and $Z = Z(G)$, where F is as in Notation 3.2(iii). These definitions are in accordance with Notation 3.2, except that we also allow $n = 1$ here. Furthermore, we put $\mathbf{G}^* := \tilde{\mathbf{G}}/Z(\tilde{\mathbf{G}}) = \mathrm{PGL}_n(\mathbb{F})$, with the Steinberg morphism induced from the one on $\tilde{\mathbf{G}}$. Then $G^* = \tilde{G}/Z(\tilde{G}) = \mathrm{PGL}_n(q)$. Note that the inclusion $i : \mathbf{G} \to \tilde{\mathbf{G}}$ is a regular embedding and that the dual epimorphism $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ is just the canonical map, if we identify $\tilde{\mathbf{G}}$ with its dual group $\tilde{\mathbf{G}}^*$. Our principal aim is to prove Theorem 1.2.

We begin by constructing suitable prime powers q.

Lemma 4.1. *Let a be a positive integer. Then there is a prime r and a power q of r such that $p^a \mid q - \varepsilon$ and $p^{a+1} \nmid q - \varepsilon$. If $p^a = 3$, there is such a q with $q > 2$.*

Proof. Let r be a prime such that $p \mid r - \varepsilon$ but $p^2 \nmid r - \varepsilon$. Then put $q := r^{p^{a-1}}$. The last statement is trivial. $\square$

The following result considers the situation of Lemma 3.16(c).

Proposition 4.2. *Let a, a_1, c' be integers with a, a_1 positive and $0 \leq c' \leq a$. Let q satisfy the conclusion of* Lemma 4.1 *with respect to p*

and a, and put $n := p^{a+a_1}$. Then there is a block $\mathbf{B}$ of $G = \mathrm{SL}_n^\varepsilon(q)$ such that the following statements hold.

(a) *The defect group D of $\mathbf{B}$ is cyclic of order p^{a+a_1}.*

(b) *There is a cyclic block of $\mathrm{GL}_n^\varepsilon(q)$ covering $\mathbf{B}$.*

(c) *There is a subgroup $Y \leq Z$ with $|Y| = p^{c'}$, and a block $\bar{\mathbf{B}}$ of G/Y dominated by $\mathbf{B}$ and with defect group D/Y.*

Proof. Let $\tilde{\mathbf{T}}$ denote a Coxeter torus of $\tilde{\mathbf{G}}$; thus $\tilde{T}$ is a cyclic group of order $q^n - \varepsilon$ (notice that n is odd and $n \geq 3$). Put $\mathbf{T} := \tilde{\mathbf{T}} \cap \mathbf{G}$. By [HL25, Lemma 3.6.3] there exists $\tilde{s} \in \tilde{T}$, whose image in G^* is strictly regular.

Let $\tilde{\chi}$ denote the corresponding irreducible Deligne-Lusztig character of $\tilde{G}$ and let $\tilde{\mathbf{B}}$ denote the p-block of $\tilde{G}$ containing $\tilde{\chi}$. By [HL25, Lemma 2.5.16], a Sylow p-subgroup $\tilde{D}$ of $\tilde{T}$ is a defect group of $\tilde{\mathbf{B}}$. In particular, $\tilde{D}$ is cyclic of order p^{2a+a_1}.

As $s \in G^*$ is strictly regular, [HL25, Corollary 2.5.17] implies that $\chi := \mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is irreducible, $\tilde{\mathbf{B}}$ covers a unique block $\mathbf{B}$ of G and that $D := \tilde{D} \cap G$ is a defect group of $\mathbf{B}$. Clearly, D is cyclic of order p^{a+a_1}. Let $Y \leq G$ with $|Y| = p^{c'}$. The p-block $\bar{\mathbf{B}}$ of G/Y dominated by $\mathbf{B}$ has defect group D/Y. This concludes our proof. □

Proposition 4.2 shows that all possible parameters determined in Lemma 3.16(c) arise from blocks. We next investigate the situation of Lemma 3.16(d), where we restrict to the cases of Proposition 3.29(a) with n_1, n_2 odd, and Proposition 3.29(b) with n_1 even and n_2 odd.

Proposition 4.3. *Let a, a_1, a_2 be integers with a, a_1 positive and $0 \leq a_2 \leq a, a_1$. If $a_1 > a_2$ put $c := a_2$. If $a_1 = a_2$, choose an integer c with $a_1 \leq c \leq a$. If $a_1 = a_2 = c$, let $p > 3$.*

Let q satisfy the conclusion of Lemma 4.1 *with respect to p and a. If $p^a = 3$, assume that $q > 2$. Put $m_2 := 1$. If $a_1 > a_2$, let $m_1 := 1$. If $a_1 = a_2$, let $m_1 := p^{c-a_1} - 1$ unless $c = a_1$; in the latter case, let $m_1 := 2$ if $p \neq 3$, and $m_2 := 4$, if $p = 3$. Finally, let $n := n_1 + n_2$ with $n_j := m_j p^{a_j}$ for $j = 1, 2$.*

Let $Y := O_p(Z)$. Then $|Y| = p^c$ and there is a block $\bar{\mathbf{B}}$ of $\bar{G} = G/Y$ with cyclic defect group of order $p^{a+a_1+a_2-c}$. Moreover, the block $\mathbf{B}$ of G dominating $\bar{\mathbf{B}}$ has abelian defect groups, which are direct products of two cyclic groups of order p^{a+a_1} and p^{a_2}.

Proof. Notice that $n = p^{a_1} + p^{a_2}$ if $a_1 > a_2$, and $n = p^c$ if $a_1 = a_2 < c$. If $a_1 = a_2 = c$, then $n = 3p^{a_2}$ if $p > 3$, and $n = 5p^{a_2}$ if $p = 3$. In any case, $|O_p(Z)| = p^c$.

Let V denote the natural vector space of $\tilde{G}$. Write $V = V_1 \oplus V_2$, an orthogonal decomposition into non-degenerate subspaces if $\varepsilon = -1$, with $\dim V_j = n_j$ for $j = 1, 2$. Fix $j \in \{1, 2\}$. Let $\tilde{G}_j$ denote the subgroup of $\tilde{G}$ induced on V_j, so that $\tilde{G}_j \cong \mathrm{GL}^{\varepsilon}_{n_j}(q)$. Choose a maximal torus $\tilde{T}_j \leq \tilde{G}_j$ such that $\tilde{T}_j$ is cyclic of order $q^{n_j} - \varepsilon^{n_j}$. Let $\tilde{D}_j$ denote the Sylow p-subgroup of $\tilde{T}_j$; then $\tilde{D}_j$ is cyclic of order p^{a+a_j}.

Put $\tilde{D} = \tilde{D}_1 \times \tilde{D}_2$ and $\tilde{T} := \tilde{T}_1 \times \tilde{T}_2 \leq \tilde{G}_1 \times \tilde{G}_2 \leq \tilde{G}$. Then $\tilde{T}$ is a maximal torus of $\tilde{G}$ corresponding to the partition of n with parts n_1, n_2. If $n_2 = 1$, put $f_2 = 1$ and $\tilde{s}_2 = 1$. Otherwise, $n_j \geq 3$ for $j = 1, 2$, and the torus $\tilde{T}_j$ contains a p'-element $\tilde{s}_j$ of prime order f_j, such that $f_j \nmid q^i - \varepsilon^i$ for all $1 \leq i < n_j$; see [HL25, Lemma 3.6.3]. By definition, $n_1 \neq n_2$, and hence $f_1 \neq f_2$. Put $\tilde{s} := \tilde{s}_1\tilde{s}_2 \in \tilde{T}$. Then, $\tilde{s}_1$ and $\tilde{s}_2$ have no common eigenvalue, and thus $C_{\tilde{G}}(\tilde{s}) = C_{\tilde{G}_1}(\tilde{s}_1) \times C_{\tilde{G}_1}(\tilde{s}_1) = \tilde{T}_1 \times \tilde{T}_2 = \tilde{T}$. In other words, $\tilde{s}$ is regular in $\tilde{G}$. Let s denote the image of $\tilde{s}$ in G^*. Then $|s| = f_1 f_2$, and thus s is strictly regular in G^* by [HL25, 2.5.1]. Let $\tilde{\mathbf{B}}$ denote the p-block of $\tilde{G}$ containing the irreducible Deligne-Lusztig character $\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s})$. As in the proof of Proposition 4.2 we find that $\chi := \mathrm{Res}^{\tilde{G}}_G(\tilde{\chi}) \in \mathrm{Irr}(G)$, and the p-block $\mathbf{B}$ of G containing χ is the unique block of G covered by $\tilde{\mathbf{B}}$. Also, $D := \tilde{D} \cap G$ is a defect group of $\mathbf{B}$.

For $j = 1, 2$, let u_j denote a generator of $\tilde{D}_j$. By Lemma 3.15(d), we have $|\det(u_j)| = p^a$ for $j = 1, 2$. If $a_1 = a_2$ we choose u_1 and u_2 in such a way that $(u_1u_2)^{p^{a_1}} \in Z$. This is possible since the eigenvalues of generators of $\tilde{D}_1$ and $\tilde{D}_2$ span the same subgroup of $\mathbb{F}^*$. In any case, there is an integer e, coprime to p, such that $u_1u_2^e$ has determinant 1 and order p^{a+a_1}. By our choice of u_1, u_2, we may and will take $e = -m_1$ in case $a_1 = a_2$. As $u_2^{p^a}$ has determinant 1 and order p^{a_2}, we obtain $D = \langle u_1u_2^e \rangle \times \langle u_2^{p^a} \rangle$. This gives our claim on the structure of D.

We claim that $\bar{D} := D/Y$ is cyclic. If $a_1 > a_2$, then $\langle u_1u_2^e \rangle \cap Y = \{1\}$, since the non-trivial eigenvalues of $u_1^{p^{a_1}}$ and $u_2^{ep^{a_1}}$ have distinct orders. As $|D| = p^{a+a_1+a_2}$ and $a_2 = c$, we get $D = \langle u_1u_2^e \rangle \times Y$. In particular, $\bar{D}$ is cyclic. If $a_1 = a_2$, we have $Y = \langle (u_1u_2)^{p^{a+a_1-c}} \rangle$ by our choice of u_1 and u_2. The Frattini subgroup $\Phi(D)$ of D is generated by $(u_1u_2^e)^p$ and $u_2^{p^{a+1}}$ as a direct product. An elementary calculation shows that $(u_1u_2)^{p^{a+a_1-c}}$ is not contained in $\Phi(D)$. Indeed, $u_1u_2 = (u_1u_2^e)u_2^{1-e}$, and thus $(u_1u_2)^{p^{a+a_1-c}} \in \langle (u_1u_2^e)^p \rangle \times \langle u_2^{p^{a+1}} \rangle$ would imply that

$$u_2^{fp^{a+1}} = u_2^{(1-e)p^{a+a_1-c}} = u_2^{(1+m_1)p^{a+a_1-c}}$$

for some integer f. However, the p-part of $1+m_1$ equals p^{c-a_1}. Hence

$$|u_2^{(1+m_1)p^{a+a_1-c}}| = |u_2^{p^a}| > |u_2^{fp^{a+1}}|,$$

a contradiction. As D is a 2-generator group, this implies that $\bar{D}$ is cyclic.

Let $\bar{\mathbf{B}}$ denote the block of $\bar{G}$ dominated by $\mathbf{B}$. The defect group of $\bar{\mathbf{B}}$ equals $\bar{D}$, which proves our assertions. □

We next investigate the situation of Lemma 3.16(d)(iii).

Proposition 4.4. *Let p be an odd prime and let c', c, a be integers with $0 \leq c' \leq c < a$. If $c = 0$, put $n := 2$. Otherwise, let $n \in \{p^c, 2p^c\}$.*

Let q satisfy the conclusion of Lemma 4.1 *with respect to p and a. Then there is a block $\mathbf{B}$ of $G = \mathrm{SL}_n^\varepsilon(q)$ such that the following statements hold.*

(a) *The defect group of $\mathbf{B}$ is cyclic of order p^a.*

(b) *There is a subgroup $Y \leq Z$ with $|Y| = p^{c'}$, and a block $\bar{\mathbf{B}}$ of G/Y dominated by $\mathbf{B}$.*

Proof. The statements are clear if $c = 0$ and $n = 2$. Thus assume that $c > 0$ in the following. Put $n_1 := n-1$ and $n_2 := 1$. Let $G_1 := \mathrm{GL}_{n_1}^\varepsilon(q)$, naturally embedded into G. Let $D := O_p(Z(G_1))$. Then D is cyclic of order p^a. As $c < a$, we have $C_G(D) = G_1$. If $n_1 > 2$, a cyclic maximal torus of G_1 of order $q^{n_1} - \varepsilon^{n_1}$ contains p'-elements which are regular with respect to G_1; see [HL25, Lemma 3.6.3]. In this case, [HL25, Lemma 2.5.16] guarantees the existence of a block $\mathbf{b}$ of G_1 with defect group D. The same conclusion clearly also holds for $n_1 = 2$. Now $N_G(G_1) = G_1$, and thus the Brauer correspondent $\mathbf{B}$ of $\mathbf{b}$ satisfies (a).

The proof of (b) is analogous to the proof of (c) of Proposition 4.2. □

We finally investigate the situation of Lemma 3.16(e), where we restrict to the cases of Proposition 3.31(b)(c), also assuming that $a \neq a_1$. (If $a = a_1$ in Proposition 3.31(b), the resulting set A is an interval.)

Proposition 4.5. *Let p be an odd prime and let a, a_1 be positive integers with $a \neq a_1$.*

Let q satisfy the conclusion of Lemma 4.1 *with respect to p and a. Put $n_1 := p^{a_1}$ and $n_3 := 1$. If $a_1 > a$, put $n_2 := 2p^a - 1$, and if $a_1 < a$, put $n_2 := p^a - p^{a_1} - 1$. In any case, let $n := n_1 + n_2 + n_3$, so that $p^a = |O_p(Z)|$.*

Put $Y := O_p(Z)$. Then there is a block $\bar{\mathbf{B}}$ of $\bar{G} := G/Y$ with cyclic defect group of order p^{a+a_1}. Moreover, the defect groups of the block $\mathbf{B}$ of G dominating $\bar{\mathbf{B}}$ are direct products of two cyclic groups of orders p^{a+a_1} and p^a, respectively.

Proof. Let V denote the natural vector space of $\tilde{G}$. Write $V = V_1 \oplus V_2 \oplus V_3$, an orthogonal decomposition into non-degenerate subspaces if $\varepsilon = -1$, with $\dim V_j = n_j$ for $j = 1, 2, 3$. Fix $j \in \{1, 2, 3\}$. Let $\tilde{G}_j$ denote the subgroup of $\tilde{G}$ induced on V_j, so that $\tilde{G}_j \cong \mathrm{GL}^{\varepsilon}_{n_j}(q)$. Choose a maximal torus $\tilde{T}_j \leq \tilde{G}_j$ such that $\tilde{T}_j$ is cyclic of order $q^{n_j} - \varepsilon^{n_j}$. Let $\tilde{D}_j$ denote the Sylow p-subgroup of $\tilde{T}_j$; then $\tilde{D}_j$ is cyclic of order p^{a+a_j} (with $a_2 = a_3 = 0$).

Put $\tilde{D} = \tilde{D}_1 \times \tilde{D}_2 \times \tilde{D}_3$ and $\tilde{T} := \tilde{T}_1 \times \tilde{T}_2 \times \tilde{T}_3 \leq \tilde{G}_1 \times \tilde{G}_2 \times \tilde{G}_3 \leq \tilde{G}$. Then $\tilde{T}$ is a maximal torus of $\tilde{G}$ corresponding to the partition of n with parts n_1, n_2, n_3.

For $j = 1, 2$, the torus $\tilde{T}_j$ contains a p'-element $\tilde{s}_j$ of prime order f_j, such that $f_j \nmid q^i - \varepsilon^i$ for all $1 \leq i < n_j$; see [HL25, Lemma 3.6.3]. Now $f_1 \neq f_2$, as $n_1 \neq n_2$. In particular, $\tilde{s}_1$ and $\tilde{s}_2$ have no common eigenvalue, and no eigenvalue 1. Put $\tilde{s} := \tilde{s}_1\tilde{s}_2 \in \tilde{T}$. Then $C_{\tilde{G}}(\tilde{s}) = C_{\tilde{G}_1}(\tilde{s}_1) \times C_{\tilde{G}_2}(\tilde{s}_2) \times \tilde{G}_3 = \tilde{T}_1 \times \tilde{T}_2 \times \tilde{T}_3 = \tilde{T}$. In other words, $\tilde{s}$ is regular in $\tilde{G}$. Let s denote the image of $\tilde{s}$ in G^*. Then $|s| = f_1 f_2$, and thus s is strictly regular in G^* by [HL25, 2.5.1].

Let $\tilde{\mathbf{B}}$ denote the p-block of $\tilde{G}$ containing the corresponding irreducible Deligne-Lusztig character $\tilde{\chi} \in \mathcal{E}(G, s)$. As in the proof of Proposition 4.2 we find that $\chi := \mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi}) \in \mathrm{Irr}(G)$, and the p-block $\mathbf{B}$ of G containing χ is the unique block of G covered by $\tilde{\mathbf{B}}$. Also, $D := \tilde{D} \cap G$ is a defect group of $\mathbf{B}$.

Clearly, $\tilde{D} \cap G \cong \tilde{D}_1 \times \tilde{D}_2$, and thus the structure of D is as claimed. there is an element $t \in D$ of order p^{a+a_1}, which acts trivially on V_2, and whose eigenvalue on V_3 has order p^a. In particular, $\langle t \rangle \cap Y = \{1\}$, so that $D = \langle t \rangle \times Y$. Hence D/Y is cyclic of order p^{a+a_1}.

The block $\bar{\mathbf{B}}$ of $\bar{G}$ dominated by $\mathbf{B}$ has defect group D/Y, which proves our assertions. □

Taking $\varepsilon = -1$, Propositions 4.2–4.5, together with Propositions 3.27–3.31, prove Theorem 1.2.

5. Glossary

To finish with, to facilitate the reading of our results, we summarize the main notation used in this manuscript in form of a glossary. Part of this notation was introduced in Part I and Part II of our work.

General assumptions:

- p is an odd prime number;
- $(K, \mathcal{O}, k)$ is a sufficiently large p-modular system, where $\mathcal{O}$ is a d.v.r. of characteristic zero with residue field $k = \overline{k}$ of characteristic p;

- l is a non-negative integer.

Special functions and intervals:

- $\sigma_\chi := \operatorname{sgn} \circ \chi : X \to \{-1, 0, 1\}$ is the sign function associated to the map $\chi : X \to \mathbb{R}$, where X is a set (see [HL25, Definition 2.1.1]);
- $\rho^{[m]}(t) := (\rho(t^{p^{m-1}}), \rho(t^{p^{m-2}}), \ldots, \rho(t^p), \rho(t)) \in X^m$ for a set X and positive integer m, lists the values of a map $\rho : H \to X$ at the p-elements $t, t^p, t^{p^2}, \ldots, t^{p^{m-1}}$ of a finite group H in reverse order (see [HL25, Definition 2.1.2]);
- $\Lambda := \{x \in \mathbb{Z} \mid 1 \leq x \leq l-1\}$;
- an *interval* is a subset of Λ which is the intersection of Λ with an interval of $\mathbb{R}$, possibly the empty set (see [HL25, Subsection 2.2]);
- a non-empty interval is written as $[i, j]$ with i, respectively j, its smallest, respectively largest, element; (see [HL25, Subsection 2.2]);
- $\mathbb{F}_2 = \{0, 1\}$ is the field with 2 elements;
- $\mathbf{1}_A$ is the characteristic function of $A \subseteq \Lambda$, i.e. the element $\mathbf{1}_A = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1}) \in \mathbb{F}_2^\Lambda$ with $\alpha_j = 1$ if and only if $j \in A$ (see [HL25, Subsection 2.2]);
- $\omega_\Lambda : \mathbb{F}_2^\Lambda \to \{-1, 1\}^l$ is the $\mathbb{F}_2$-isomorphism defined by

$$\omega_\Lambda(\alpha_0, \ldots, \alpha_{l-1})_i = \begin{cases} +1, & \text{if } \sum_{j=0}^{i-1} \alpha_j = 0 \\ -1, & \text{if } \sum_{j=0}^{i-1} \alpha_j = 1 \end{cases} \qquad \text{for each } 1 \leq i \leq l;$$

  (see [HL25, Subsection 2.2]).

Endo-permutation module associated to a cyclic block $\mathcal{B}$ with defect group D of order p^l:

- $W(\mathcal{B})$ is the endo-permutation kD-module associated to $\mathcal{B}$, uniquely determined, as an element of the Dade group of D, by the l-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_{l-1}) \in \mathbb{F}_2^\Lambda$, also written $W(\mathcal{B}) = W_D(\alpha_0, \alpha_1, \ldots, \alpha_{l-1})$ (see [HL24, Subsection 3.1]);
- the label $(\alpha_0, \alpha_1, \ldots, \alpha_{l-1})$ above is identified with a subset of Λ via the $\mathbb{F}_2$-isomorphism $\mathcal{P}(\Lambda) \to \mathbb{F}_2^\Lambda, A \mapsto \mathbf{1}_A$ (see [HL25, Subsections 2.2 and 2.3]);
- $W_D(A) := W_D(\mathbf{1}_A)$ for any subset $A \subseteq \Lambda$ (see [HL25, Definition 2.3.1]);
- $W_D(\emptyset) \cong k$, the trivial module (particular case of the above);
- $\omega_W := \sigma_{\rho_W}$ is the sign function associated to the K-character ρ_W afforded by the unique lift of determinant one of $W := W(\mathcal{B})$ to $\mathcal{O}$ (see [HL24, Subsection 3.1]);
- $\omega_W^{[l]}(t)$ with t a generator of D determines W up to isomorphism (see [HL25, Lemma 2.3.2]);

- $\omega^{[l]}_W(t) = \sigma^{[l]}_\chi(t)$ provided t is as above, $\langle t^{p^{l-1}}\rangle$ is in the center of the group considered, and $\chi \in \mathrm{Irr}(\mathcal{B})$ denotes the unique non-exceptional character (see [HL25, Remark 2.3.3]);
- $\omega_{\mathbf{H}}(s) := \varepsilon_{\mathbf{H}}\varepsilon_{C^\circ_{\mathbf{H}}(s)}$, when it occurs, is a sign associated with the algebraic group $\mathbf{H}$ under consideration and any semisimple element $s \in H$ (see [HL25, Definition 2.5.6]).

The groups considered (Section 1, Section 3 and Section 4):

- $\varepsilon \in \{\pm 1\}$; $\delta := 1$ if $\varepsilon = 1$, and $\delta := 2$ if $\varepsilon = -1$;
- q is a power of a prime number r such that $p \mid q - \varepsilon$;
- $\mathbb{F}$ is an algebraic closure of the finite field with r elements;
- n is a positive integer satisfying: $n \geq 2$ in Section 1 and Section 3, and $n \geq 1$ in Section 4;
- $\tilde{\mathbf{G}} := \mathrm{GL}_n(\mathbb{F})$ and $\mathbf{G} := \{g \in \tilde{\mathbf{G}} \mid \det(g) = 1\}$;
- $F := F_\varepsilon$ is a Steinberg morphism of $\tilde{\mathbf{G}}$ such that $\tilde{G} := \tilde{\mathbf{G}}^F = \mathrm{GL}^\varepsilon_n(q)$;
- $G := \mathbf{G}^F = \mathrm{SL}^\varepsilon_n(q)$;
- $Z := Z(G)$ and satisfies $|Z| = \gcd(q - \varepsilon, n)$;
- a, b, c are non-negative integers such that p^a, p^b, p^c are the highest powers of p dividing $q - \varepsilon$, n and $\gcd(q - \varepsilon, n)$, respectively. Thus $a > 0$ and $c = \min\{a, b\}$;
- $\mathbf{V} := \mathbb{F}^n$ is the natural vector space for $\tilde{\mathbf{G}}$, and $V := \mathbb{F}^n_{q^\delta} \subseteq \mathbf{V}$;
- $\mathbf{G}^* := \tilde{\mathbf{G}}/Z(\tilde{\mathbf{G}}) = \mathrm{PGL}_n(\mathbb{F})$ (with the Steinberg morphism induced from the one on $\tilde{\mathbf{G}}$);
- $G^* := \tilde{G}/Z(\tilde{G}) = \mathrm{PGL}_n(q)$;
- $Y \leq Z$ is a p-subgroup and $\bar{G} := G/Y$ is a central quotient of G.

The blocks and the associated local configuration (Section 3):

- $\bar{\mathbf{B}}$ is a cyclic p-block of $\bar{G}$ and $\mathbf{B}$ is the unique block of G dominating $\bar{\mathbf{B}}$;
- D is a defect group of $\mathbf{B}$ such that $\bar{D} := D/Y$ is a defect group of $\bar{\mathbf{B}}$;
- $t \in D$ is such that $\bar{t} := tY \in \bar{G}$ generates $\bar{D}$;
- $h \in \{1, 2, 3\}$ is the number of irreducible factors of the minimal polynomial of t acting on V;
- c' is the non-negative integer such that $|\bar{D}| = |t|/p^{c'}$.
- $C := C_G(D)$ and $\tilde{C} := C_{\tilde{G}}(D)$;
- $\mathbf{c}$ is a Brauer correspondent of $\mathbf{B}$ in C and $\tilde{\mathbf{c}}$ is a block of $\tilde{C}$ covering $\mathbf{c}$; see Figure 1;
- $\tilde{D}$ is a defect group of $\tilde{\mathbf{c}}$ with $D = C \cap \tilde{D} = G \cap \tilde{D}$.

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