Computational Representation Theory – Lecture I

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Group Theory and Computational Methods
ICTS-TIFR, Bangalore, 05 – 14 November 2016

Contents

Notation

Throughout this lecture, G denotes a finite group and F a field.

Representations: Definitions

An F-representation of G of degree d is a homomorphism

$$\mathfrak{X} : G \to \mathrm{GL}(V),$$

where V is a d-dimensional F-vector space. (This is also called a representation of G on V.)

To accord with GAP, we let $\mathrm{GL}(V)$ act from the right on V.

For computations one chooses a basis of V and obtains a matrix representation $G \to \mathrm{GL}_d(F)$.

Irreducible Representations

$\mathfrak{X} : G \to \mathrm{GL}(V)$ is reducible, if either $V = \{0\}$, or if there exists a subspace $W < V$, $0 \neq W \neq V$, s.t. $w\,\mathfrak{X}(g) \in W$ for all $w \in W$ and $g \in G$. (W is G-invariant.)

Equivalently, there is a basis of V, w.r.t. which $\mathfrak{X}(g)$ has matrix

$$\left[\begin{array}{c|c} \mathfrak{X}_W(g) & 0 \\ \hline * & \mathfrak{X}_{V/W}(g) \end{array}\right]$$

for all $g \in G$.

In this case, $\mathfrak{X}_W$ and $\mathfrak{X}_{V/W}$ are matrix representations of degrees $\dim W$ and $\dim V - \dim W$, respectively.

Otherwise $\mathfrak{X}$ is called irreducible.

Representations: Equivalence

Two representations $\mathfrak{X} : G \to \mathrm{GL}(V)$, and $\mathfrak{Y} : G \to \mathrm{GL}(W)$ on vector spaces V and W are called equivalent,

if there exists an isomorphism $\alpha : V \to W$ such that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\mathfrak{X}(g)} & V \\ \alpha\big\downarrow & & \big\downarrow\alpha \\ W & \xrightarrow[\mathfrak{Y}(g)]{} & W \end{array}$$

(W.r.t. suitable bases of V and W, the matrices for $\mathfrak{X}(g)$ and $\mathfrak{Y}(g)$ are simultaneously similar.)

Representations: Classification

1. There are only finitely many irreducible F-representations of G up to equivalence. Their number is at most equal to the number of conjugacy classes of G containing elements g such that $\mathrm{char}(F) \nmid |g|$.
2. Classify all irreducible representations of G.
3. Describe all irreducible representations of all finite simple groups.
4. Use a computer for sporadic simple groups.

Characters

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be an F-representation of G.

The character afforded by $\mathfrak{X}$ is the map

$$\chi_{\mathfrak{X}} : G \to F, \quad g \mapsto \mathrm{Trace}(\mathfrak{X}(g)).$$

$\chi_{\mathfrak{X}}$ is constant on conjugacy classes: a class function on G.

Equivalent representations have the same character.

An F-character of G is the character of some F-representation.

Irreducible Characters

If $\mathfrak{X}$ is irreducible, $\chi_{\mathfrak{X}}$ is called an irreducible character.

Facts

1. *If $W \leq V$ is G-invariant, then* $\chi_{\mathfrak{X}} = \chi_{\mathfrak{X}_W} + \chi_{\mathfrak{X}_{V/W}}$.
2. *There are only finitely many irreducible characters of G.*
3. *The set of irreducible characters of G is linearly independent (in* $\mathrm{Maps}(G, F)$*).*
4. *Every character is a sum of irreducible characters.*
5. *Two* ***irreducible*** *representations of G are equivalent, if and only if their characters are equal.*
6. *Suppose that* $\mathrm{char}(F) = 0$. *Then* ***any*** *two representations of G are equivalent, if and only if their characters are equal.*

The Ordinary Character Table

From now on let $F = \mathbb{C}$.

Put $\mathrm{Irr}(G) :=$ set of irreducible $\mathbb{C}$-characters of G,
$\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$.

Let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G (same k as above!).

The square matrix

$$\left[\chi_i(g_j)\right]_{1 \leq i,j \leq k}$$

is called the ordinary character table of G.

Example: Alternating Group A_5

Example (Character Table of A_5)

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$A = (1 - \sqrt{5})/2, \qquad *A = (1 + \sqrt{5})/2$$

$$1 \in 1a, \quad (1,2)(3,4) \in 2a, \quad (1,2,3) \in 3a,$$

$$(1,2,3,4,5) \in 5a, \quad (1,3,5,2,4) \in 5b$$

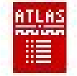
Goals and Results

Aim

Describe all ordinary character tables of all finite simple groups and related finite groups.

Almost done:

1. For alternating groups: Frobenius, Schur
2. For groups of Lie type: Green, Deligne, Lusztig, Shoji, ...
3. For sporadic groups and other "small" groups:



Atlas of Finite Groups, Conway, Curtis, Norton, Parker, Wilson, 1986

The character tables of the ATLAS are also contained in GAP.

Character Tables in GAP and Magma

GAP is a system for computational discrete algebra [...] (http://www.gap-system.org/).

Magma is a [...] software package designed to solve [...] hard problems in algebra, number theory, geometry and combinatorics (http://magma.maths.usyd.edu.au/magma/).

GAP and Magma provide a large number of character tables.

Character tables in GAP and Magma are

1. taken from a character table library (which in GAP currently contains 2279 tables), or
2. computed from scratch (using, e.g., the Burnside-Dixon-Schneider algorithm or lattice reduction), or
3. computed from a generic character table.

The Orthogonality Relations

Let $\mathcal{C}(G)$ denote the set of $\mathbb{C}$-valued class functions on G, and $\mathbb{Z}[\mathrm{Irr}(G)] := \{\sum_{i=1}^{k} z_i\chi_i \mid z_i \in \mathbb{Z} \text{ for all } i\} \subseteq \mathcal{C}(G)$.
Define the inner product $\langle -, - \rangle$ on $\mathcal{C}(G)$ by

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Facts

1. $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ *is an ON basis of* $\mathcal{C}(G)$.
2. $\alpha = \sum_{i=1}^{k} \langle \chi_i, \alpha \rangle \chi_i$ *for* $\alpha \in \mathcal{C}(G)$.
3. $\alpha \in \mathcal{C}(G)$ *is a character if and only if* $\langle \chi_i, \alpha \rangle \in \mathbb{N}$ *for all* i.
4. *Suppose* $\alpha \in \mathbb{Z}[\mathrm{Irr}(G)]$.
   Then $\alpha \in \mathrm{Irr}(G)$ *if and only if* $\langle \alpha, \alpha \rangle = 1$ *and* $\alpha(1) > 0$.

CLASS MULTIPLICATION COEFFICIENTS

Let $C_1, \ldots, C_k$ be the conjugacy classes of G.

Define the class multiplication coefficients c_{ijl} $(1 \leq i, j, l \leq k)$ by

$$c_{ijl} := |\{(x, y) \mid x \in C_i, y \in C_j, xy \in C_l\}|/|C_l|.$$

Put $M_i := [c_{ijl}] \in \mathbb{N}^{k \times k}$, $i = 1, \ldots, k$.

THEOREM (BURNSIDE)

The ordinary character table of G can be computed from

1. *The common column eigenvectors of $M_1, \ldots, M_k$, or*
2. *the common row eigenvectors of $M_1, \ldots, M_k$, or*
3. *the corresponding eigenvalues.*

The Burnside-Dixon-Schneider Algorithm

Let $C_1, \ldots, C_k$ be the conjugacy classes of G, $g_i \in C_i$, $i = 1, \ldots, k$, $g_1 = 1$.

Let $\chi \in \mathrm{Irr}(G)$. Then for all $1 \leq i \leq k$, there are $\omega_{\chi,i} \in \mathbb{C}$ such that

$$\omega_{\chi,i}[\chi(g_1), \ldots, \chi(g_k)] = [\chi(g_1), \ldots, \chi(g_k)]M_i.$$

Algorithm (Burnside-Dixon-Schneider)

1. *Compute the matrices M_i, $1 \leq i \leq k$.*
2. *Find the common row eigenvectors $\chi'_1, \ldots, \chi'_k$ of these.*
3. $\chi_i = c_i \chi'_i$ *and* $\langle \chi'_i, \chi'_i \rangle = 1/c_i^2 \rightsquigarrow \chi_i$.

Computations are done in a finite field and lifted back to $\mathbb{C}$.

Usually, not all of the matrices M_i have to be computed.

Constructions of Characters, I

Product. Let χ, ψ be characters of G. Then the product $\chi \cdot \psi$, defined by

$$[\chi \cdot \psi](g) := \chi(g)\,\psi(g), \quad g \in G$$

is a character as well (proof later).

Symmetrisation. Let χ be a character of G. Then $S^2(\chi)$ and $\Lambda^2(\chi)$ defined by

$$S^2(\chi)(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right), \qquad \Lambda^2(\chi)(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$$

are characters as well.

Restriction. Let $H \leq G$ and χ a character of G. Then the restriction χ_H of χ to H is a character of H.

Constructions of Characters, II

Induction. Let $H \leq G$, and ψ a character of H.

Then ψ^G defined by

$$\psi^G(g) := \sum_{i=1}^{l} \frac{|C_G(g)|}{|C_H(h_i)|} \psi(h_i), \quad g \in G,$$

where $h_1, \ldots, h_l$ are representatives of the H-conjugacy classes contained in the G-conjugacy class of g, is a character of G.

ψ^G is called an induced character.

Brauer's Induction Theorem

Recall $\mathbb{Z}[\mathrm{Irr}(G)] := \{\sum_{i=1}^{k} z_i\chi_i \mid z_i \in \mathbb{Z} \text{ for all } i\}$.

An element of $\mathbb{Z}[\mathrm{Irr}(G)]$ is a generalised character.

Definition

$E \leq G$ is called elementary, if $E = P \times C$, with P a p-group for some prime p, and C a cyclic group.

$\mathcal{E}$: set of elementary subgroups of G.

For $E \in \mathcal{E}$, write $\mathrm{Ind}_E^G(\mathbb{Z}[\mathrm{Irr}(E)]) := \{\psi^G \mid \psi \in \mathbb{Z}[\mathrm{Irr}(E)]\}$.

Theorem (Brauer's Induction Theorem)

$\mathbb{Z}[\mathrm{Irr}(G)] = \sum_{E \in \mathcal{E}} \mathrm{Ind}_E^G(\mathbb{Z}[\mathrm{Irr}(E)])$.

Generate and Split: Overview

Strategy to compute $\mathrm{Irr}(G)$: (Bill Unger, Magma):

In the following, $I \cup B \subseteq \mathbb{Z}[\mathrm{Irr}(G)]$ such that

$$I \subseteq \mathrm{Irr}(G) \qquad \text{and} \qquad \langle I, B \rangle = 0. \tag{1}$$

Repeat the following steps until $|I| + |B| = |\mathrm{Irr}(G)|$ and $\det\langle B, B \rangle = 1$:

1. Compute $L \subseteq \mathbb{Z}[\mathrm{Irr}(G)]$ using above constructions;
2. $L \leftarrow$ projection of L to $\langle I \rangle^{\perp}$ (using inner product);
3. Using the LLL-algorithm (Lenstra-Lenstra-Lovács), compute a basis $I' \cup B'$ of $\langle B \cup L \rangle_{\mathbb{Z}}$ satisfying (1);
4. $I \leftarrow I \cup I'$, $B \leftarrow B'$.

Generate and Split: Remarks

1. By Brauer's induction theorem, the above procedure terminates, if in Step 1 all induced characters of all elementary subgroups are generated.
2. In implementation (Unger), an irredundant subset of these is generated successively.
3. A hybrid method, using a starting set I computed with Burnside-Dixon-Schneider is reasonable.
4. LLL does not guarantee to find all vectors of norm 1 (though in the experiments it does, according to Unger).
5. If it terminates with $B \neq \emptyset$, try to find the factorisations $M = AA^{tr}$ for Gram matrix $M = \langle B, B \rangle$.
6. Recently, Breuer, Malle and O'Brien have recomputed the character tables of the sporadic groups (except for B and M) using Unger's algorithm.

Generic Character Tables

Generic character tables are

1. programs for computing the character table of each individual group of an infinite series of groups, e.g., S_n or A_n for $n \in \mathbb{N}$, or Weyl groups,

   using recursive formulae for the character values (Murnaghan-Nakayama type formulae) or

2. parametrised character tables for an infinite series of groups, e.g., $\mathrm{SL}_2(q)$ or $\mathrm{SU}_3(q)$, q a prime power.

   Conjugacy classes and characters are parametrised; character values are given in terms of the parameters.

   These are computed through Deligne-Lusztig theory.

   Individual tables are obtained by specialisation.

The Generic Character Table for $\mathrm{SL}_2(q)$, q Even

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

$a, m = 1, \ldots, (q-2)/2, \qquad b, n = 1, \ldots, q/2,$

$\zeta := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \qquad \xi := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

$\begin{bmatrix} \mu^a & 0 \\ 0 & \mu^{-a} \end{bmatrix} \in C_3(a)$ ($\mu \in \mathbb{F}_q$ a primitive $(q-1)$th root of 1)

$\begin{bmatrix} \nu^b & 0 \\ 0 & \nu^{-b} \end{bmatrix} \underset{\sim}{\in} C_4(b)$ ($\nu \in \mathbb{F}_{q^2}$ a primitive $(q+1)$th root of 1)

Specialising q to 4, gives the character table of $\mathrm{SL}_2(4) \cong A_5$.

CHEVIE

CHEVIE is a computer algebra project for symbolic calculations with generic character tables of groups of Lie type, Coxeter groups, Iwahori-Hecke algebras and other related structures (http://www.math.rwth-aachen.de/˜CHEVIE/).

It is (currently) based on GAP-3, and MAPLE.
But there will be a GAP-4 version.

Authors: Meinolf Geck, Gerhard Hiss, Frank Lübeck, Gunter Malle, Jean Michel and Götz Pfeiffer

References

1. J. J. Cannon and W. Bosma (Eds.), Handbook of Magma Functions, Edition 2.13 (2006), 4350 pages, (http://magma.maths.usyd.edu.au/magma/).
2. The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.10, 2007, (http://www.gap-system.org).
3. D. F. Holt, B. Eick and E. A. O'Brien, Handbook of Computational Group Theory, Chapman & Hall/CRC, 2005.
4. K. Lux and H. Pahlings, Representations of Groups. A computational approach. Cambridge University Press, 2010.
5. W. R. Unger, Computing the character table of a finite group, *J. Symbol. Comput.* **41** (2006), 847–862.

Thank you for your attention!