Computational Representation Theory – Lecture III

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Contents

Notation

Throughout this lecture, G denotes a finite group and F a field.

FG: group algebra of G over F

1. elements: $\sum_{g \in G} a_g g \quad (a_g \in F)$
2. multiplication: distributive extension of multiplication of G

FG is a finite-dimensional F-algebra.

Representations: According to Aschbacher

A $\mathcal{C}$-representation of G is a group homomorphism

$$G \to \mathrm{Aut}(C),$$

where $\mathcal{C}$ is a category and C is an object of $\mathcal{C}$.

This is rather general. We will mainly look at two categories:

1. f.d. vector spaces over F $\rightsquigarrow$ linear representations
2. finite sets $\rightsquigarrow$ permutation representations

From now on: representation = linear representation

Representations of Groups: Recollection

Recall: An F-representation of G of degree d is a homomorphism

$$\mathfrak{X} : G \to \mathrm{GL}(V),$$

where V is a d-dimensional F-vector space.

For $v \in V$ and $g \in G$, write $v.g := v\mathfrak{X}(g)$.
This makes V into a right FG-module.

$\mathfrak{X}$ is irreducible, if V does not have any proper G-invariant subspaces: V is a simple FG-module.

$\mathfrak{X} : G \to \mathrm{GL}(V)$ and $\mathfrak{Y} : G \to \mathrm{GL}(W)$ are equivalent, if and only if V and W are isomorphic as FG-modules.

Representations of Algebras

Let $\mathfrak{A}$ be an F-algebra, e.g., $\mathfrak{A} = FG$.

An F-representation of $\mathfrak{A}$ of degree d is an F-algebra homomorphism

$$\mathfrak{X} : \mathfrak{A} \to \mathrm{End}(V),$$

where V is a d-dimensional F-vector space.

A group representation $\mathfrak{X} : G \to \mathrm{GL}(V)$ canonically extends to a representation $\mathfrak{X} : FG \to \mathrm{End}(V)$.

For $v \in V$ and $\mathfrak{a} \in \mathfrak{A}$, write $v.\mathfrak{a} := v\mathfrak{X}(\mathfrak{a})$. This makes V into a right $\mathfrak{A}$-module.

Irreducibility and equivalence are defined analogously to the case of group representations.

Representations of Groups: Constructions

Representations of G can be constructed

1. from permutation representations,
2. from two representations through their Kronecker product,
3. from representations through invariant subspaces,
4. in various other ways.

Permutation Representations

A permutation representation of G on the finite set $\Omega = \{\omega_1, \ldots, \omega_n\}$ is a homomorphism

$$\kappa : G \to S_\Omega,$$

where S_Ω denotes the symmetric group on Ω.

Let V denote an F-vector space with basis Ω.

Replacing each $\kappa(g) \in S_\Omega$ by the linear map $\mathfrak{X}(g)$ of V, which permutes its basis like $\kappa(g)$, we obtain an F-representation $\mathfrak{X}$ of degree n of G.

Permutation Representations: Example

Let $G = S_3$, the symmetric group on three letters, acting (from the right) on $\Omega = \{1, 2, 3\}$. Then $G = \langle a, b \rangle$ with $a = (1, 2)$ and $b = (1, 2, 3)$.

Take $F = \mathbb{Q}$; $V \cong \mathbb{Q}^{1\times 3}$. Then G acts on V by $e_i.g = e_{ig}$, where e_1, e_2, e_3 is the standard basis of V.

Then

$$\mathfrak{X}(a) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathfrak{X}(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

are the corresponding permutation matrices. (Note that we use row convention.)

Note: In order to describe (store) a representation $\mathfrak{X}$ of G, it suffices to give the matrices $\mathfrak{X}(a_i)$ for a generating set $\{a_1, \ldots, a_l\}$ of G.

Kronecker Product of Representations

Let $\mathfrak{X} : G \to \mathrm{GL}_d(F)$ and $\mathfrak{Y} : G \to \mathrm{GL}_e(F)$ be two matrix representations of G.

Then $\mathfrak{X} \otimes \mathfrak{Y} : G \to \mathrm{GL}_{de}(F), \quad g \mapsto \mathfrak{X}(g) \otimes \mathfrak{Y}(g)$ is a matrix representation of G.

Here, $\mathfrak{X}(g) \otimes \mathfrak{Y}(g)$ is the Kronecker product of the two matrices $\mathfrak{X}(g)$ and $\mathfrak{Y}(g)$, defined as follows:

$$A \otimes B := \begin{bmatrix} a_{11} B & \cdots & a_{1d} B \\ \vdots & \ddots & \vdots \\ a_{d1} B & \cdots & a_{dd} B \end{bmatrix}$$

for $A = [a_{ij}] \in F^{d \times d}$, and $B \in F^{e \times e}$.

From this: $\chi_{\mathfrak{X} \otimes \mathfrak{Y}} = \chi_{\mathfrak{X}} \cdot \chi_{\mathfrak{Y}}$, i.e. the product of characters is a character.

Invariant Subspaces

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be a representation of G on V.

Let W be a G-invariant subspace of V, i.e.:

$$w.g \in W \quad \text{for all } w \in W, g \in G.$$

(W is an FG-submodule of V.)

We obtain F-representations

$$\mathfrak{X}_W : G \to \mathrm{GL}(W) \quad \text{and} \quad \mathfrak{X}_{V/W} : G \to \mathrm{GL}(V/W)$$

in the natural way.

Invariant Subspaces: Example

Let $G = S_3$ and V be as above.

$W := \langle e_1 + e_2 + e_3 \rangle$ is an invariant subspace.

Choosing the basis $e_1 + e_2 + e_3, e_2, e_3$ of V, and transforming the matrices accordingly, we obtain a representation $\mathfrak{X}'$ with

$$\mathfrak{X}'(a) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 0 & 1 \end{array}\right], \quad \mathfrak{X}'(b) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 1 & -1 & -1 \end{array}\right].$$

Moreover,

$$\mathfrak{X}'_{V/W}(a) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathfrak{X}'_{V/W}(b) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

The Irreducible Constituents

Iterating this process, we obtain a matrix representation $\mathfrak{X}^\infty$ of G, equivalent to $\mathfrak{X}$, s.t.:

$$\mathfrak{X}^\infty(g) = \begin{bmatrix} \mathfrak{X}_1(g) & 0 & \cdots & 0 \\ * & \mathfrak{X}_2(g) & \cdots & 0 \\ & & \ddots & \\ * & * & & 0 \\ * & * & \cdots & \mathfrak{X}_l(g) \end{bmatrix},$$

and all the representations $\mathfrak{X}_i$ are irreducible.

The $\mathfrak{X}_i$ are called the irreducible constituents (or composition factors) of $\mathfrak{X}$ (or of V).

They are unique up to equivalence (Jordan-Hölder theorem).

All Irreducible Representations

Iterating the constructions, e.g.,

1. F-representations from permutation representations,
2. Kronecker products,
3. various others,

and reductions via invariant subspaces,

one obtains all irreducible F-representations of G.

The MeatAxe

The MeatAxe is a collection of programs that perform the above tasks (for finite fields F).

It was invented and developed by Richard Parker and Jon Thackray around 1980.

Since then, it has been improved and enhanced by many people, including Derek Holt, Gábor Ivanyos, Klaus Lux, Jürgen Müller, Sarah Rees, Michael Ringe, and by Richard Parker himself.

The MeatAxe: Basic Problems

1. How does one find a non-trivial proper G-invariant subspace of V?
   - It is enough to find a vector $w \neq 0$ which lies in a proper G-invariant subspace W.
   - Indeed, given $0 \neq w \in W$, the orbit $\{w.g \mid g \in G\}$ spans a G-invariant subspace contained in W.
2. How does one prove that $\mathfrak{X}$ is irreducible?

Norton's Irreducibility Criterion

Let $A_1, \ldots, A_l$ be $(d \times d)$-matrices over F.

Put $\mathfrak{A} := F\langle A_1, \ldots, A_l \rangle$ and $\mathfrak{A}^{tr} := F\langle A_1^{tr}, \ldots, A_l^{tr} \rangle$ (algebra span).

Let $B \in \mathfrak{A}$. Write $\mathcal{N}(B)$ for the (left) nullspace of B.

Then one of the following occurs:

1. B is invertible.
2. There is a non-trivial vector in $\mathcal{N}(B)$ which lies in a proper $\mathfrak{A}$-invariant subspace.
3. Every non-trivial vector in $\mathcal{N}(B^{tr})$ lies in a proper $\mathfrak{A}^{tr}$-invariant subspace.
4. $\mathfrak{A}$ acts irreducibly on $F^{1\times d}$.

The MeatAxe: Basic Strategy

Find singular $B \in \mathfrak{A}$ (by a random search) with nullspace N of small dimension (preferably 1).

For all $0 \neq w \in N$ test if $w.\mathfrak{A} = F^{1\times d}$. (Note that $w.\mathfrak{A}$ is $\mathfrak{A}$-invariant.)

If YES

For one $0 \neq w$ in the nullspace of B^{tr} test if $w.\mathfrak{A}^{tr} = F^{1\times d}$.

If YES, $\mathfrak{X}$ is irreducible.

Note: If $G = \langle g_1, \ldots, g_l \rangle$, and $\mathfrak{X} : G \to \mathrm{GL}_d(F)$ is an F-representation of G, take $A_i := \mathfrak{X}(g_i)$, $1 \leq i \leq l$.

The Spinning Algorithm

The space $w.\mathfrak{A}$ is computed with a linearized version of an orbit algorithm: spin$(A_1, \ldots, A_l, w)$ returns F-basis of $w.\mathfrak{A}$.

Algorithm (spin$(A_1, \ldots, A_l, w)$)

Input: $A_1, \ldots, A_l \in F^{d\times d}$, $0 \neq w \in F^{1\times d}$
Output: *basis* $\mathcal{B}$ *of* $w.\mathfrak{A}$ *with* $\mathfrak{A} = F\langle A_1, \ldots, A_l\rangle$

$\mathcal{B} \leftarrow w$;
for v *in* $\mathcal{B}$ **do**
  for i *from* 1 *to* l **do**
    if $\{vA_i\} \cup \mathcal{B}$ *linearly independent* **then**
      append vA_i *to* $\mathcal{B}$;
    end if;
  end for;
end for;

The MeatAxe: Test for Equivalence

Let $\mathfrak{A}$ be an F-algebra, generated by $\mathfrak{a}_1, \ldots, \mathfrak{a}_l$.

Suppose $\mathfrak{X}, \mathfrak{Y} : \mathfrak{A} \to F^{d \times d}$ are irreducible representations of $\mathfrak{A}$.

Put $A_i := \mathfrak{X}(\mathfrak{a}_i)$, $B_i := \mathfrak{Y}(\mathfrak{a}_i)$, $1 \leq i \leq l$.

Let $\mathfrak{w} \in F\langle X_1, \ldots, X_l \rangle$ such that $A := \mathfrak{w}(A_1, \ldots, A_l)$ has nullity 1.

If nullity of $B := \mathfrak{w}(B_1, \ldots, B_l)$ is not 1, then $\mathfrak{X}$, $\mathfrak{Y}$ are not equivalent. Otherwise, let $0 \neq v \in \mathcal{N}(A)$, $0 \neq w \in \mathcal{N}(B)$.

Let $\mathcal{A} := \mathrm{spin}(A_1, \ldots, A_l, v)$, $\mathcal{B} := \mathrm{spin}(B_1, \ldots, B_l, w)$.

Let $\mathfrak{X}'$, $\mathfrak{Y}'$ be the "transformed" representations (matrices written w.r.t. $\mathcal{A}$, respectively $\mathcal{B}$). Then:

$\mathfrak{X}$ and $\mathfrak{Y}$ are equivalent if and only if $\mathfrak{X}'$ and $\mathfrak{Y}'$ are equal.

The MeatAxe: Remarks

The above strategy works very well if F is small.

For example, 71% of all $(d \times d)$-matrices over $\mathbb{F}_2$ are singular, 57% of all $(d \times d)$-matrices over $\mathbb{F}_2$ have nullspace of dimension 1.

As F gets larger, it gets harder to find a suitable B by a random search.

Holt and Rees use characteristic polynomials of elements of $\mathfrak{A}$ to find suitable Bs and also to reduce the number of tests considerably.

The Holt and Rees Irreducibility Criterion

Let $A_1, \ldots, A_l$, be $(d \times d)$-matrices over F.

Put $\mathfrak{A} := F\langle A_1, \ldots, A_l \rangle$ and $\mathfrak{A}^{tr} := F\langle A_1^{tr}, \ldots, A_l^{tr} \rangle$.

Let $B \in \mathfrak{A}$, χ_B the characteristic polynomial of B.

An irreducible factor p of χ_B is good, if $\deg(p) = \dim(\mathcal{N}(p(B))$.

Theorem (Holt and Rees)

Suppose that p is a good factor of χ_B and let $0 \neq w \in \mathcal{N}(p(B))$, and $0 \neq w' \in \mathcal{N}(p(B)^{tr})$.
Then $\mathfrak{A}$ acts irreducibly on $F^{1\times d}$, if $w.\mathfrak{A} = F^{1\times d}$ and if $w'.\mathfrak{A}^{tr} = F^{1\times d}$.

The MeatAxe according to Holt and Rees

1. Choose a random $B \in \mathfrak{A}$.
2. Compute χ_B.
3. Compute Factors(χ_B). (If this fails go to Step 1.)
4. For each $p \in$ Factors(χ_B) do

   $A := p(B)$;
   if $\dim(\mathcal{N}(A)) = \deg(p)$, then p is good fi;
   take $0 \neq w \in \mathcal{N}(A)$, compute $W := w.\mathfrak{A}$;
   if $W \neq F^{1\times d}$ Return(W) fi;
   take $0 \neq w' \in \mathcal{N}(A^{tr})$, compute $W' := w'.\mathfrak{A}^{tr}$;
   if $W' \neq F^{1\times d}$ Return(W) fi;
   $\#\, W = \{w \in F^{1\times d} \mid w'w^{tr} = 0\ \forall w' \in W'\}$
   if p is good, Return("Irreducible") fi;
5. Go back to Step 1.

Rob Wilson's Atlas

A huge collection of explicit representations of finite groups is contained in Rob Wilson's WWW *Atlas of Finite Group Representations:* (http://brauer.maths.qmul.ac.uk/Atlas/).

These representations have been computed by Wilson and collaborators, e.g.

the representation of M of degree 196 882 over $\mathbb{F}_2$ by Linton, Parker, Walsh, and Wilson.

Much of this information is also available through the GAP-package atlasrep (http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep/).

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Thank you for your attention!